Lahore College for Women University (LCWU)

Linear recurrent sequences

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Abstract

Linear recurrence sequences are ubiquitous. They occur in biology, economics, computer science (analysis of algorithms), digital signal processing. We give a survey of this subject, together with connections with linear combinations of powers, with powers of matrices and with linear differential equations.

We give examples : Fibonacci, Lucas, balancing numbers, Perrin, Padovan, Narayana.

Applications of linear recurrence sequences

Combinatorics

Elimination

Symmetric functions

Hypergeometric series

Language

Communication, shift registers

Finite difference equations

Logic

Approximation

Pseudo–random sequences

Applications of linear recurrence sequences

- Biology (Integrodifference equations, spatial ecology).

- Computer science (analysis of algorithms).

- Digital signal processing (infinite impulse response (IIR) digital filters).

- Economics (time series analysis).

https://en.wikipedia.org/wiki/Recurrence_relation

Leonardo Pisano (Fibonacci)

Fibonacci sequence $(F_n)_{n\geq 0}$,

$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,$

$34,\ 55,\ 89,\ 144,\ 233,\dots$

is defined by

$$F_0 = 0,\ F_1 = 1,$$

$$F_{n+2} = F_{n+1} + F_n \quad \text{for} \quad n \geq 0.$$

http://oeis.org/A000045

Leonardo Pisano (Fibonacci)
(1170–1250)

OEIS

http://oeis.org/

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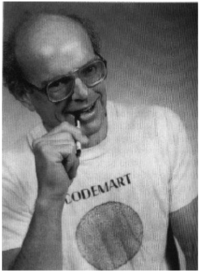
THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane



Neil J. A. Sloane's encyclopaedia

http://oeis.org/

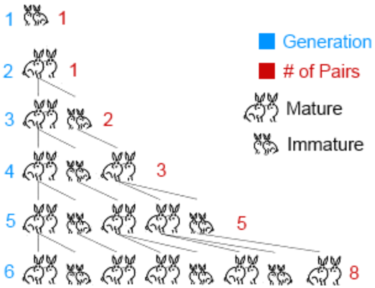
Linear recurrence sequence : 13787 results found.

Fibonacci sequence : http://oeis.org/A000045

Fibonacci rabbits

Fibonacci considered the growth of a rabbit population.

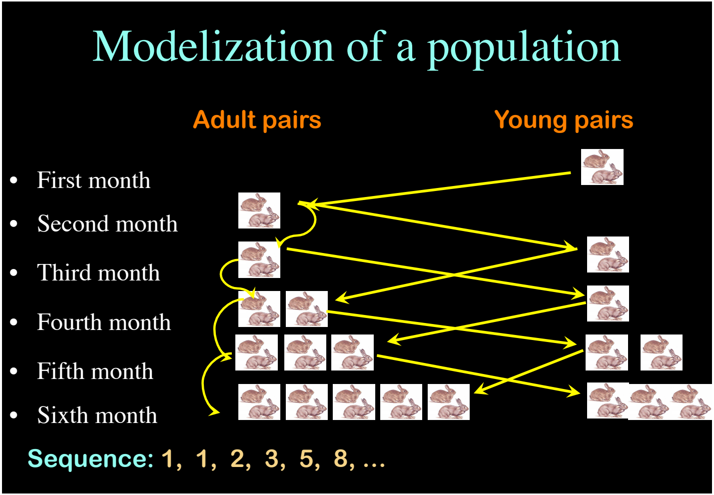
A newly born pair of rabbits, a male and a female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits ; rabbits never die and a mating pair always produces



one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was : how many pairs will there be in one year ?

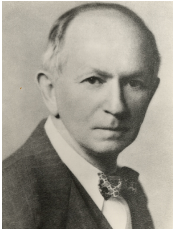
Answer : $F_{12} = 144$.

Fibonacci's rabbits



Alfred Lotka : arctic trees

In cold countries, each branch of some trees gives rise to another one after the second year of existence only.

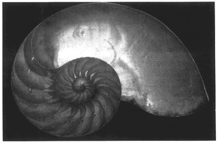


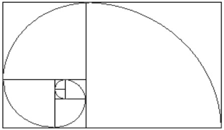
Alfred Lotka
1880 – 1949

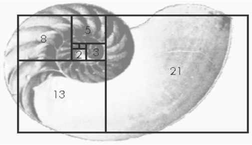
Alfred Lotka : American biophysicist, specialist of population dynamics and energetics. Predator–prey model, developed simultaneously but independently of Vito Volterra.

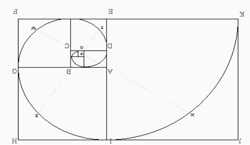
Fibonacci numbers in nature

Ammonite (Nautilus shape)





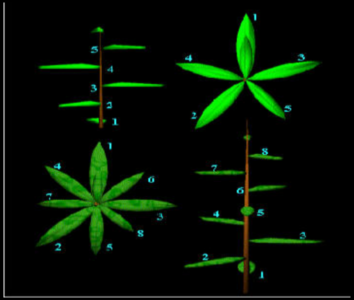




Phyllotaxy

- Study of the position of leaves on a stem and the reason for them
- Number of petals of flowers: daisies, sunflowers, aster, chicory, asteraceae,…
- Spiral patern to permit optimal exposure to sunlight
- Pine-cone, pineapple, Romanesco cawliflower, cactus

Leaf arrangements



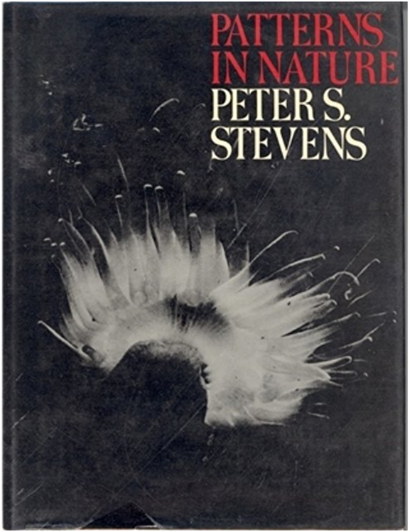
- Université de Nice,
  Laboratoire Environnement Marin Littoral, Equipe d'Accueil "Gestion de la Biodiversité"

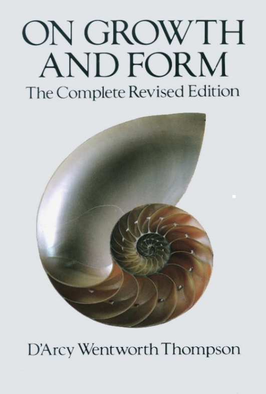
http://www.unice.fr/LEML/coursJDV/tp/tp3.htm

Phyllotaxy

Phyllotaxy

- J. Kepler (1611) uses the Fibonacci sequence in his study of the dodecahedron and the icosaedron, and then of the symmetry of order 5 of the flowers.
- Stéphane Douady and Yves Couder
  Les spirales végétales
  La Recherche *250* (Jan. 1993) vol. **24.**

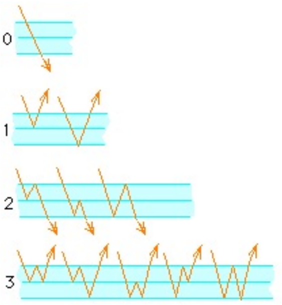

PATTERNS
IN NATURE
PETER S.
STEVENS


ON GROWTH AND FORM
The Complete Revised Edition
D'Arcy Wentworth Thompson

Reflections of a ray of light

Consider three parallel sheets of glass and a ray of light which crosses the first sheet. Each time it touches one of the sheets, it can cross it or reflect on it.

Denote by p_n the number of different paths with the ray going out of the system after n reflections.



$p_0 = 1$,

$p_1 = 2$,

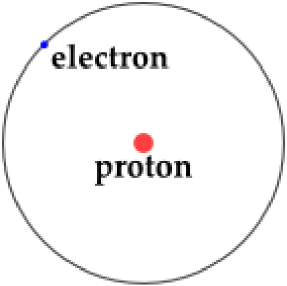
$p_2 = 3$,

$p_3 = 5$.

In general, $p_n = F_{n+2}$.

Levels of energy of an electron of an atom of hydrogen

An atom of hydrogen can have three levels of energy, 0 at the ground level when it does not move, 1 or 2. At each step, it **alternatively** gains and looses some level of energy, either 1 or 2, without going sub 0 nor above 2. Let ℓ_n be the number of different possible scenarios for this electron after n steps.



In general, $\ell_n = F_{n+2}$.

We have $\ell_0 = 1$ (initial state level 0)

$\ell_1 = 2$: state 1 or 2, scenarios (ending with gain) 01 or 02.

$\ell_2 = 3$: scenarios (ending with loss) 010, 021 or 020.

$\ell_3 = 5$: scenarios (ending with gain) 0101, 0102, 0212, 0201 or 0202.

Rhythmic patterns

The Fibonacci sequence appears in Indian mathematics, in connection with Sanskrit prosody. Several Indian scholars, Pingala (200 BC), Virahanka (c. 700 AD), Gopāla (c. 1135), and the Jain scholar Hemachandra (c. 1150). studied rhythmic patterns that are formed by concatenating one beat notes ● and double beat notes ■■.

one-beat note ● : short syllabe (`ti` in Morse Alphabet)

double beat note ■■ : long syllabe (`ta ta` in Morse)

1 beat, 1 pattern : ●

2 beats, 2 patterns : ● ● and ■■

3 beats, 3 patterns : ● ● ●, ● ■■ and ■■ ●

4 beats, 5 patterns :

● ● ● ●, ■■ ● ●, ● ■■ ●, ● ● ■■, ■■ ■■

n beats, F_{n+1} patterns.

Fibonacci sequence and the Golden ratio

For $n \geq 0$, the Fibonacci number F_n is the nearest integer to

$$\frac{1}{\sqrt{5}}\Phi^n,$$

where Φ is the *Golden Ratio* : http://oeis.org/A001622

$$\Phi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1.6180339887499\dots$$

which satisfies

$$\Phi = 1 + \frac{1}{\Phi}.$$

Binet's formula

For $n \geq 0$,

$$F_n = \frac{\Phi^n - (-\Phi)^{-n}}{\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}},$$

Jacques Philippe Marie Binet
1786–1856

$$\Phi = \frac{1+\sqrt{5}}{2}, \quad -\Phi^{-1} = \frac{1-\sqrt{5}}{2},$$

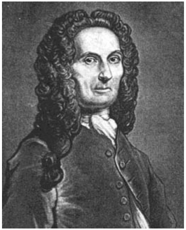
$$X^2 - X - 1 = (X - \Phi)(X + \Phi^{-1}).$$

The so–called Binet Formula

Formula of A. De Moivre (1718, 1730), Daniel Bernoulli (1726), L. Euler (1728, 1765), J.P.M. Binet (1843) : for $n \geq 0$,

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

Abraham de Moivre (1667–1754)



Daniel Bernoulli (1700–1782)

Leonhard Euler (1707–1783)

Jacques P.M. Binet (1786–1856)

Generating series

A single series encodes all the Fibonacci sequence :

$$\sum_{n\geq 0} F_n X^n = X + X^2 + 2X^3 + 3X^4 + 5X^5 + \cdots + F_n X^n + \cdots$$

Fact : this series is the Taylor expansion of a rational fraction :

$$\sum_{n\geq 0} F_n X^n = \frac{X}{1 - X - X^2}.$$

`Proof` : the product

$$(X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \cdots)(1 - X - X^2)$$

is a telescoping series

$$\begin{aligned} & X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \cdots \\ & \quad - X^2 - X^3 - 2X^4 - 3X^5 - 5X^6 - \cdots \\ & \qquad - X^3 - X^4 - 2X^5 - 3X^6 - \cdots \\ = {} & X. \end{aligned}$$

Generating series of the Fibonacci sequence

Remark. The denominator $1 - X - X^2$ in the right hand side of

$$X + X^2 + 2X^3 + 3X^4 + \cdots + F_n X^n + \cdots = \frac{X}{1 - X - X^2}$$

is $X^2 f(X^{-1})$, where $f(X) = X^2 - X - 1$ is the irreducible polynomial of the Golden ratio Φ.

Homogeneous linear differential equation

Consider the homogeneous linear differential equation

$$y'' - y' - y = 0.$$

If $y = \mathrm{e}^{\lambda x}$ is a solution, from $y' = \lambda y$ and $y'' = \lambda^2 y$ we deduce

$$\lambda^2 - \lambda - 1 = 0.$$

The two roots of the polynomial $X^2 - X - 1$ are Φ (the Golden ration) and Φ' with

$$\Phi' = 1 - \Phi = -\frac{1}{\Phi}\cdot$$

A basis of the space of solutions is given by the two functions $\mathrm{e}^{\Phi x}$ and $\mathrm{e}^{\Phi' x}$. Since (Binet's formula)

$$\sum_{n\geq 0} F_n \frac{x^n}{n!} = \frac{1}{\sqrt{5}}\left(\mathrm{e}^{\Phi x} - \mathrm{e}^{\Phi' x}\right),$$

this exponential generating series of the Fibonacci sequence is a solution of the differential equation.

Fibonacci and powers of matrices

The Fibonacci linear recurrence relation $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ can be written

$$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}.$$

By induction one deduces, for $n \geq 0$,

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

An equivalent formula is, for $n \geq 1$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

Characteristic polynomial

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is

$$\det(XI - A) = \det \begin{pmatrix} X & -1 \\ -1 & X - 1 \end{pmatrix} = X^2 - X - 1,$$

which is the irreducible polynomial of the Golden ratio Φ.

Fibonacci sequence and the Golden ratio (continued)

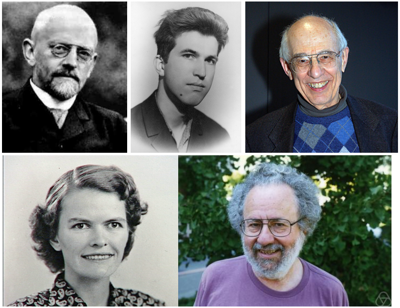
For $n \geq 1$, $\Phi^n \in \mathbb{Z}[\Phi] = \mathbb{Z} + \mathbb{Z}\Phi$ is a linear combination of 1 and Φ with integer coefficients, namely

$$\Phi^n = F_{n-1} + F_n\Phi.$$

Fibonacci sequence and Hilbert's 10th problem

Yuri Matiyasevich (1970) showed that there is a polynomial P in n, m, and a number of other variables $x, y, z, \ldots$ having the property that $n = F_{2m}$ iff there exist integers $x, y, z, \ldots$ such that $P(n, m, x, y, z, \ldots) = 0$.

This completed the proof of the impossibility of the tenth of Hilbert's problems (*does there exist a general method for solving Diophantine equations?*) thanks to the previous work of Hilary Putnam, Julia Robinson and Martin Davis.



The Fibonacci Quarterly

The Fibonacci sequence satisfies a lot of very interesting properties. Four times a year, the *Fibonacci Quarterly* publishes an issue with new properties which have been discovered.

The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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Why are there so many occurrences of Fibonacci numbers and Golden ratio in the nature ?

According to Leonid Levin, objects with a small algorithmic Kolmogorov complexity (generated by a short program) occur more often than others.

Another example is given by Sierpinski triangles.

Reference : J-P. Delahaye.
http://cristal.univ-lille.fr/~jdelahay/pls/

Lucas sequence

http://oeis.org/000032

The Lucas sequence $(L_n)_{n \geq 0}$ satisfies the same recurrence relation as the Fibonacci sequence, namely

$$L_{n+2} = L_{n+1} + L_n \quad \text{for} \quad n \geq 0,$$

only the initial values are different :

$$L_0 = 2, \ L_1 = 1.$$

The sequence of Lucas numbers starts with

$$2, \ 1, \ 3, \ 4, \ 7, \ 11, \ 18, \ 29, \ 47, \ 76, \ 123, \ 199, \ 322, \dots$$

A closed form involving the Golden ratio Φ is

$$L_n = \Phi^n + (-\Phi)^{-n},$$

from which it follows that for $n \geq 2$, L_n is the nearest integer to Φ^n.

François Édouard Anatole Lucas

Edouard Lucas is best known for his results in number theory. He studied the Fibonacci sequence and devised the test for Mersenne primes still used today.

Édouard Lucas
1842 - 1891

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lucas.html

Generating series of the Lucas sequence

The generating series of the Lucas sequence

$$\sum_{n\geq 0} L_n X^n = 2 + X + 3X^2 + 4X^3 + \cdots + L_n X^n + \cdots$$

is nothing else than

$$\frac{2-X}{1-X-X^2}.$$

Homogeneous linear differential equation

We have seen that

$$\sum_{n\geq 0} F_n \frac{x^n}{n!} = \frac{1}{\sqrt{5}}\left(\mathrm{e}^{\Phi x} - \mathrm{e}^{\Phi' x}\right)$$

is a solution of the homogeneous linear differential equation

$$y'' - y' - y = 0.$$

Since

$$\sum_{n\geq 0} L_n \frac{x^n}{n!} = \mathrm{e}^{\Phi x} + \mathrm{e}^{\Phi' x},$$

we deduce that a basis of the space of solutions is given by the two generating series

$$\sum_{n\geq 0} F_n \frac{x^n}{n!} \quad \text{and} \quad \sum_{n\geq 0} L_n \frac{x^n}{n!}.$$

The Lucas sequence and power of matrices

From the linear recurrence relation $L_{n+2} = L_{n+1} + L_n$ one deduces, (as we did for the Fibonacci sequence), for $n \geq 0$,

$$\begin{pmatrix} L_{n+1} \\ L_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} L_n \\ L_{n+1} \end{pmatrix},$$

hence

$$\begin{pmatrix} L_n \\ L_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The four sequences

$$(F_n)_{n\geq 0}, \quad (L_n)_{n\geq 0}, \quad (\Phi^n)_{n\geq 0}, \quad \left((-\Phi)^{-n}\right)_{n\geq 0}$$

span a vector space of dimension 2, any two of these four sequences give a basis of this space.

An interesting street number

The puzzle itself was about a street in the town of Louvain in Belgium, where houses are numbered consecutively. One of the house numbers had the peculiar property that the total of the numbers lower than it was exactly equal to the total of the numbers above it. Furthermore, the mysterious house number was greater than 50 but less than 500.

Prasanta Chandra Mahalanobis

1893 – 1972

Srinivasa Ramanujan

1887 – 1920

http://mathshistory.st-andrews.ac.uk/Biographies/Mahalanobis.html

https://www.math.auckland.ac.nz/~butcher/miniature/miniature2.pdf

Street number : examples

Examples :

- House number 6 in a street with 8 houses :

$$1+2+3+4+5=15, \quad 7+8=15.$$

- House number 35 in a street with 49 houses. To compute

$$S := 1+2+3+\cdots+32+33+34$$

write

$$S = 34+33+32+\cdots+3+2+1$$

so that $2S = 34\times 35$:

$$1+2+3+\cdots+34 = \frac{34\times 35}{2} = 595.$$

On the other side of the house,

$$36+37+\cdots+49 = \frac{49\times 50}{2} - \frac{35\times 36}{2} = 1225-630 = 595.$$

Other solutions to the puzzle

- House number 1 in a street with 1 house.
- House number 0 in a street with 0 house.

Ramanujan : *if no banana is distributed to no student, will each student get a banana ?*

The puzzle requests the house number between 50 and 500.

Street number

Let m be the house number and n the number of houses :

$$1 + 2 + 3 + \cdots + (m - 1) = (m + 1) + (m + 2) + \cdots + n.$$

$$\frac{m(m-1)}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2}.$$

This is $2m^2 = n(n+1)$. Complete the square on the right :

$$8m^2 = (2n+1)^2 - 1.$$

Set $x = 2n + 1$, $y = 2m$. Then

$$x^2 - 2y^2 = 1.$$

Infinitely many solutions to the puzzle

Ramanujan said he has infinitely many solutions (but a single one between 50 and 500).
Sequence of balancing numbers (number of the house)
https://oeis.org/A001109

$$0, 1, 6, 35, \mathbf{204}, 1189, 6930, 40391, 235416, 1372105, 7997214\ldots$$

This is a linear recurrence sequence $u_{n+1} = 6u_n - u_{n-1}$ with the initial conditions $u_0 = 0$, $u_1 = 1$.

The number of houses is https://oeis.org/A001108

$$0, 1, 8, 49, \mathbf{288}, 1681, 9800, 57121, 332928, 1940449, \ldots$$

Balancing numbers

A balancing number is an integer $B \geq 0$ such that there exists C with

$$1 + 2 + 3 + \cdots + (B-1) = (B+1) + (B+2) + \cdots + C.$$

Same as $B^2 = C(C+1)/2$: a balancing number is an integer B such that B^2 is a triangular number (and a square!).
Sequence of balancing numbers : https://oeis.org/A001109

$0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, 7997214\dots$

This is a linear recurrence sequence

$$B_{n+1} = 6B_n - B_{n-1}$$

with the initial conditions $B_0 = 0$, $B_1 = 1$.

Sequence $(B_n)_{n\geq 0}$ of balancing numbers :

$$2{B_n}^2 = C_n(C_n+1)$$

The corresponding sequence $(C_n)_{n\geq 0}$ is
https://oeis.org/A001108

$$0, 1, 8, 49, 288, 1681, 9800, 57121, 332928, 1940449, \ldots$$

The solutions of $x^2 - 2y^2 = 1$ are given by

$$x_n = 2B_n, \quad y_n = 2C_n + 1.$$

Both sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ satisfy

$$u_{n+1} = 6u_n - u_{n-1}.$$

with $x_0 = 0$, $x_1 = 2$, $y_0 = 1$, $y_1 = 3$.
Hence

$$C_{n+1} = 6C_n - C_{n-1} + 2.$$

The sequence of balancing numbers

Characteristic polynomial :
$f(X) = X^2 - 6X + 1 = (X - 3 - 2\sqrt{2})(X - 3 + 2\sqrt{2})$.
Closed formula :

$$B_n = \frac{1}{4\sqrt{2}}\big((3+2\sqrt{2})^n - (3-2\sqrt{2})^n\big).$$

Generating series :

$$\varphi(X) = \sum_{n\geq 0} B_n X^n = X + 6X^2 + 35X^3 + \cdots = \frac{X}{1-6X+X^2}.$$

`Exercise` :

$$X^2\varphi' = (1-X^2)\varphi^2.$$

Takao Komatsu & Prasanta Kumar Ray. *Higher-order identities for balancing numbers*. `arXiv:1608.05925 [math.NT]`

Exponential generating series of the sequence of balancing numbers

$$y(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!}$$
$$= x + 3x^2 + \frac{35}{6}x^3 + \cdots$$
$$= \frac{1}{4\sqrt{2}}\left(e^{(3+2\sqrt{2})x} - e^{(3-2\sqrt{2})x}\right).$$

This is a solution of the homogeneous linear differential equation of order 2

$$y'' = 6y' - y$$

with the initial conditions $y(0) = 0$, $y'(0) = 1$.

Balancing numbers and the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix}$

$$\begin{pmatrix} B_{n+1} \\ B_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} B_n \\ B_{n+1} \end{pmatrix} \quad (n \geq 0).$$

Powers of A :

$$\begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix}^n = \begin{pmatrix} -B_n & B_{n+1} \\ -B_{n+1} & B_{n+2} \end{pmatrix} \quad (n \geq 0).$$

Characteristic polynomial :

$$\det(XI - A) = \det \begin{pmatrix} X & -1 \\ 1 & X-6 \end{pmatrix} = X^2 - 6X + 1.$$

Perrin sequence

http://oeis.org/A001608

The Perrin sequence (also called *skiponacci sequence*) is the linear recurrence sequence $(P_n)_{n\geq 0}$ defined by

$$P_{n+3} = P_{n+1} + P_n \quad \text{for} \quad n \geq 0,$$

with the initial conditions

$$P_0 = 3,\ P_1 = 0,\ P_2 = 2.$$

It starts with

$$3,\ 0,\ 2,\ 3,\ 2,\ 5,\ 5,\ 7,\ 10,\ 12,\ 17,\ 22,\ 29,\ 39,\ 51,\ 68, \dots$$

François Olivier Raoul Perrin (1841-1910) :
https://en.wikipedia.org/wiki/Perrin_number

Plastic (or silver) constant

https://oeis.org/A060006

The ratio of successive terms in the Perrin sequence tends to the plastic number

$$\varrho = 1.324\,717\,957\,244\,746\ldots$$

which is the minimal Pisot–Vijayaraghavan number, real root of

$$x^3 - x - 1.$$

This constant is equal to

$$\varrho = \frac{\sqrt[3]{108 + 12\sqrt{69}} + \sqrt[3]{108 - 12\sqrt{69}}}{6}.$$

Perrin sequence and the plastic constant

Decompose the polynomial $X^3 - X - 1$ into irreducible factors over $\mathbb{C}$

$$X^3 - X - 1 = (X - \varrho)(X - \rho)(X - \overline{\rho})$$

and over $\mathbb{R}$

$$X^3 - X - 1 = (X - \varrho)(X^2 + \varrho X + \varrho^{-1}).$$

Hence ρ and $\overline{\rho}$ are the roots of $X^2 + \varrho X + \varrho^{-1}$. Then, for $n \geq 0$,

$$P_n = \varrho^n + \rho^n + \overline{\rho}^n.$$

It follows that, for $n \geq 0$, P_n is the nearest integer to ϱ^n.

Generating series of the Perrin sequence

The generating series of the Perrin sequence

$$\sum_{n\geq 0} P_n X^n = 3 + 2X^2 + 3X^3 + 2X^4 + \cdots + P_n X^n + \cdots$$

is nothing else than

$$\frac{3 - X^2}{1 - X^2 - X^3}.$$

The denominator $1 - X^2 - X^3$ is $X^3 f(X^{-1})$ where $f(X) = X^3 - X - 1$ is the irreducible polynomial of ϱ.

Exponential generating series of the Perrin sequence

The power series

$$y(x) = \sum_{n \geq 0} P_n \frac{x^n}{n!}$$

is a solution of the differential equation

$$y''' - y' - y = 0$$

with the initial conditions $y(0) = 3$, $y'(0) = 0$, $y''(0) = 2$.

Perrin sequence and power of matrices

From

$$P_{n+3} = P_{n+1} + P_n$$

we deduce

$$\begin{pmatrix} P_{n+1} \\ P_{n+2} \\ P_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}.$$

Characteristic polynomial

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is

$$\det(XI - A) = \det \begin{pmatrix} X & -1 & 0 \\ 0 & X & -1 \\ -1 & -1 & X \end{pmatrix} = X^3 - X - 1,$$

which is the irreducible polynomial of the plastic constant ϱ.

Perrin's remark

1484. [I9c] La curieuse proposition d'origine chinoise qui fait l'objet de la question 1401 fournirait, si elle était exacte, un criterium plus pratique que le théorème de Wilson pour vérifier si un nombre donné m est premier ou non ; il suffirait de calculer les résidus par rapport à m des termes successifs de la suite récurrente

$$u_n = 3u_{n-1} - 2u_{n-2}$$

avec les valeurs initiales $u_0 = -1$, $u_1 = 0$.

J'ai rencontré une autre suite récurrente qui paraît jouir de la même propriété ; c'est celle dont le terme général est

$$v_n = v_{n-2} + v_{n-3}$$

— 77 —

avec les valeurs initiales $v_0 = 3$, $v_1 = 0$, $v_2 = 2$. Il est facile de démontrer que v_n est divisible par n, si n est premier; j'ai vérifié qu'il ne l'est pas dans le cas contraire, jusqu'à des valeurs assez élevées de n; mais il serait intéressant de savoir ce qu'il en est réellement, d'autant plus que la suite v_n fournit des nombres bien moins rapidement croissants que la suite u_n (pour $n = 17$, par exemple, on trouve $u_n = 131070$, $v_n = 119$), et se prête à des simplifications de calcul lorsque n est un grand nombre.

La même méthode de démonstration, applicable à l'une des suites, le sera sans doute à l'autre, si la propriété énoncée est exacte pour toutes les deux : il ne s'agit que de la découvrir.

R. Perrin.

R. Perrin *L'intermédiaire des mathématiciens,* Query 1484, v.6, 76–77 (1899).

The website www.Perrin088.org maintained by Richard Turk is devoted to Perrin numbers. See OEISA113788.

Perrin pseudoprimes

https://oeis.org/A013998

If p is prime, then p divides P_p.

The smallest composite n such that n divides P_n is $521^2 = 271441$.
The number P_{271441} has $33\,150$ decimal digits (the number c which satisfies $10^c = \varrho^{271441}$ is $c = 271441(\log \varrho)/(\log 10)$).

Also for the composite number $n = 904631 = 7 \times 13 \times 9941$, the number n divides P_n.

Jon Grantham has proved in 2010 that there are infinitely many Perrin pseudoprimes.

Padovan sequence

https://oeis.org/A000931

The Padovan sequence $(p_n)_{n\geq 0}$ satisfies the same recurrence

$$p_{n+3} = p_{n+1} + p_n$$

as the Perrin sequence but has different initial values :

$$p_0 = 1, \quad p_1 = p_2 = 0.$$

It starts with

$$1,\ 0,\ 0,\ 1,\ 0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 3,\ 4,\ 5,\ 7,\ 9,\ 12,\ 16, \dots$$

Richard Padovan

http://mathworld.wolfram.com/LinearRecurrenceEquation.html

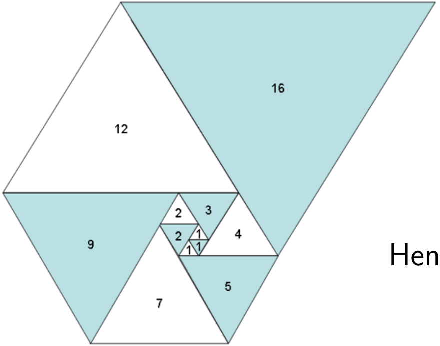
Generating series and power of matrices

$$1 + X^3 + X^5 + \cdots + p_n X^n + \cdots = \frac{1 - X^2}{1 - X^2 - X^3}.$$

For $n \geq 0$,

$$\begin{pmatrix} p_n \\ p_{n+1} \\ p_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Padovan triangles



$$p_n = p_{n-2} + p_{n-3}$$

$$p_{n-1} = p_{n-3} + p_{n-4}$$

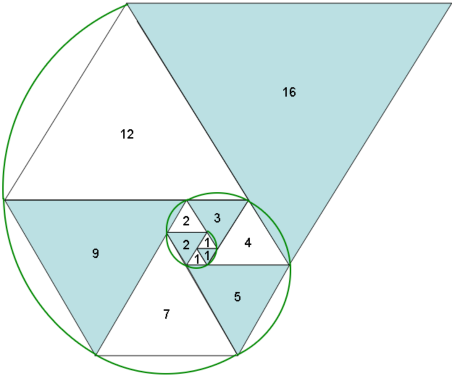
$$p_{n-2} = p_{n-4} + p_{n-5}$$

Hence

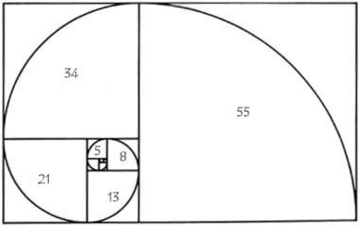
$$p_n - p_{n-1} = p_{n-5}$$

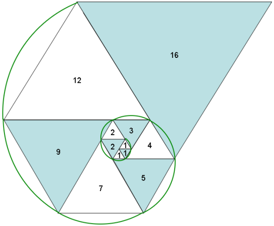
$$p_n = p_{n-1} + p_{n-5}$$

Padovan triangles



Fibonacci squares vs Padovan triangles





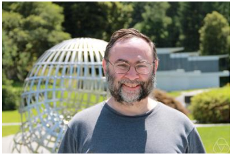
Both are C^1 curve, not C^2

Padovan, Euler, Zagier, Goncharov and Brown

For $n \geq 0$, the number of compositions $\underline{s} = (s_1, \dots, s_k)$ with $s_i \in \{2, 3\}$ and $s_1 + \cdots + s_k = n$ is p_{n+3}. This is (an upper bound for) the dimension of the space spanned by the multiple zeta values of weight n of Euler and Zagier.

Leonhard Euler

Don Zagier



Alexander Goncharov

Francis Brown

Narayana sequence

https://oeis.org/A000930

Narayana sequence is defined by the recurrence relation

$$C_{n+3} = C_{n+2} + C_n$$

with the initial values $C_0 = 2$, $C_1 = 3$, $C_2 = 4$.
It starts with

$$2,\ 3,\ 4,\ 6,\ 9,\ 13,\ 19,\ 28,\ 41,\ 60,\ 88,\ 129,\ 189,\ 277, \ldots$$

Real root of $x^3 - x^2 - 1$

$$\frac{\sqrt[3]{\dfrac{29+3\sqrt{93}}{2}} + \sqrt[3]{\dfrac{29-3\sqrt{93}}{2}} + 1}{3} = 1.465571231876768\ldots$$

Generating series and power of matrices

$$2 + 3X + 4X^2 + 6X^3 + \cdots + C_n X^n + \cdots = \frac{2 + X + X^2}{1 - X - X^3}.$$

Differential equation : $y''' - y'' - y = 0$;
initial conditions : $y(0) = 2$, $y'(0) = 3$, $y''(0) = 4$.

For $n \geq 0$,

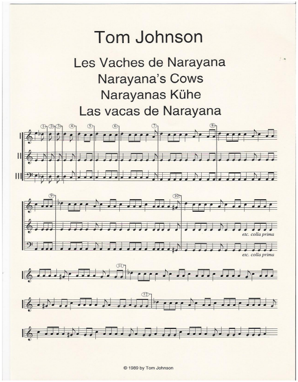
$$\begin{pmatrix} C_n \\ C_{n+1} \\ C_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

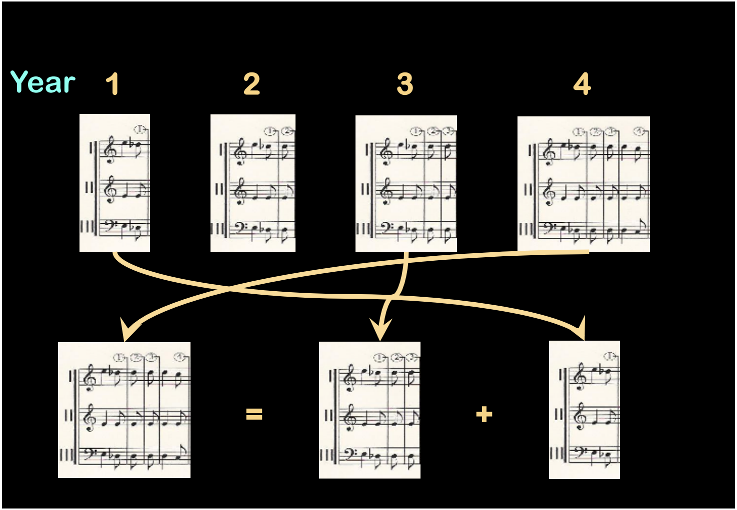
Narayana's cows

Narayana was an Indian mathematician in the 14th century who proposed the following problem :
A cow produces one calf every year. Beginning in its fourth year each calf produces one calf at the beginning of each year. How many calves are there altogether after, for example, 17 *years ?*

Music : http://www.pogus.com/21033.html

In working this out, Tom Johnson found a way to translate this into a composition called *Narayana's Cows*.
Music : Tom Johnson
Saxophones : Daniel Kientzy




Year
1
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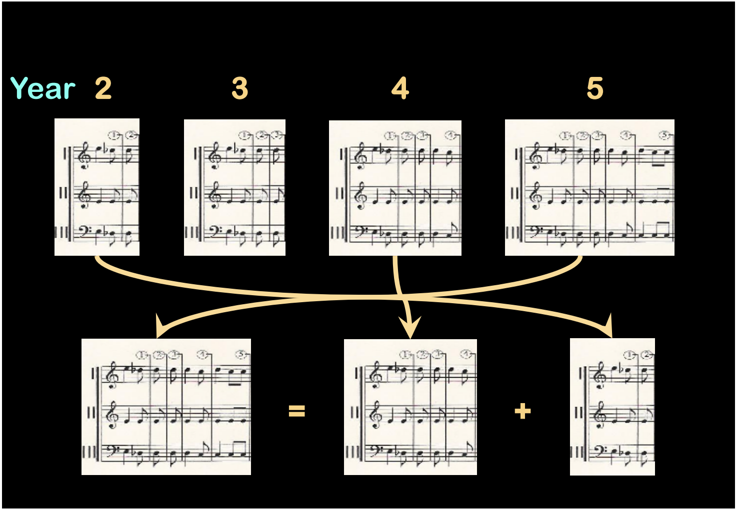
Year 2
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Narayana's cows

http://www.math.jussieu.fr/~michel.waldschmidt/

Year	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Original Cow	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Second generation	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Third generation	0	0	0	1	3	6	10	15	21	28	36	45	55	66	78	91	105
Fourth generation	0	0	0	0	0	0	1	4	10	20	35	56	84	120	165	220	286
Fifth generation	0	0	0	0	0	0	0	0	0	1	5	15	35	70	126	210	330
Sixth generation	0	0	0	0	0	0	0	0	0	0	0	0	1	6	21	56	126
Seventh generation	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	7
Total	**2**	**3**	**4**	**6**	**9**	**13**	**19**	**28**	**41**	**60**	**88**	**129**	**189**	**277**	**406**	**595**	**872**

17th year: 872 cows

Jean-Paul Allouche and Tom Johnson

 

http://www.math.jussieu.fr/~jean-paul.allouche/bibliorecente.html
http://www.math.jussieu.fr/~allouche/johnson1.pdf

Cows, music and morphisms

Jean-Paul Allouche and Tom Johnson

- Narayana's Cows and Delayed Morphisms
In 3èmes Journées d'Informatique Musicale (JIM '96), Ile de Tatihou, Les Cahiers du GREYC (1996 no. 4), pages 2-7, May 1996.
http://kalvos.org/johness1.html

- Finite automata and morphisms in assisted musical composition,
Journal of New Music Research, no. 24 (1995), 97 – 108.
http://www.tandfonline.com/doi/abs/10.1080/09298219508570676
http://web.archive.org/web/19990128092059/www.swets.nl/jnmr/vol24_2.html

Music and the Fibonacci sequence

- Dufay, XV$^{\text{ème}}$ siècle
- Roland de Lassus
- Debussy, Bartok, Ravel, Webern
- Stockhausen
- Xenakis
- **Tom Johnson** *Automatic Music for six percussionists*

Some recent work

Christian Ballot
On a family of recurrences that includes the Fibonacci and the Narayana recurrences.
arXiv:1704.04476 [math.NT]

We survey and prove properties a family of recurrences bears in relation to integer representations, compositions, the Pascal triangle, sums of digits, Nim games and Beatty sequences.

Linear recurrence sequences : examples

$q \geq 1$; initial conditions $u_0 = u_1 = \cdots = u_{q-2} = 0$, $u_{q-1} = 1$.

$$\boxed{X^q - X^{q-1} - 1 :}$$

$q = 1$, $X - 2$, exponential $u_n = 2^n$
$q = 2$, $X^2 - X - 1$, Fibonacci $u_n = F_n$
$q = 3$, $X^3 - X^2 - 1$, Narayana $u_n = C_n$

$$\boxed{X^q - X^{q-1} - X^{q-2} - \cdots - X - 1 :}$$

$q = 1$, $X - 1$, constant sequence $u_n = 1$
$q = 2$, $X^2 - X - 1$, Fibonacci $u_n = F_n$
$q = 3$, $X^3 - X^2 - X - 1$, Tribonacci

$$\boxed{X^q - X - 1 :}$$

$q = 2$, $X^2 - X - 1$, Fibonacci $u_n = F_n$
$q = 3$, $X^3 - X - 1$, Padovan $u_n = p_n$

Summary

The same mathematical object occurs in a different guise :

- Linear recurrence sequences

$$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

- Linear combinations with polynomial coefficients of powers

$$p_1(n){\gamma_1}^n + \cdots + p_\ell(n)\gamma_\ell^n.$$

- Taylor coefficients of rational functions.

- Coefficients of power series which are solutions of homogeneous linear differential equations.

- Sequence of coefficients of powers of a matrix.

Polynomial combinations of powers

Given polynomials $p_1, \dots, p_\ell$ in $\mathbb{C}[X]$ and elements $\gamma_1, \dots, \gamma_\ell$ in $\mathbb{C}^\times$, the sequence

$$\left(p_1(n){\gamma_1}^n + \cdots + p_\ell(n)\gamma_\ell^n\right)_{n\geq 0}$$

is a linear recurrence sequence, the minimal polynomial of which is of the form

$$X^d - a_1X^{d-1} - \cdots - a_d = \prod_{i=1}^{\ell}(X-\gamma_i)^{t_i},$$

Fact : any linear recurrence sequence is of this form.

Consequence : the sum and the product of any two linear recurrence sequences are linear recurrence sequences.
The set of all linear recurrence sequences with coefficients in $\mathbb{C}$ is a sub–$\mathbb{C}$–algebra of $\mathbb{C}^{\mathbb{N}}$.

Sum of polynomial combinations of powers

For $\mathbf{u} = (u_n)_{n\geq 0}$ and $\mathbf{v} = (v_n)_{n\geq 0}$,

$$\mathbf{u} + \mathbf{v} = (u_n + v_n)_{n\geq 0}.$$

If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two linear recurrence sequences of characteristic polynomials f_1 and f_2 respectively, then $\mathbf{u}_1 + \mathbf{u}_2$ satisfies the linear recurrence, the characteristic polynomial of which is

$$\frac{f_1 f_2}{\gcd(f_1, f_2)}.$$

Product of polynomial combinations of powers

For $\mathbf{u} = (u_n)_{n\geq 0}$ and $\mathbf{v} = (v_n)_{n\geq 0}$,

$$\mathbf{uv} = (u_n v_n)_{n\geq 0}.$$

If the characteristic polynomials of the two linear recurrence sequences $\mathbf{u}_1$ and $\mathbf{u}_2$ are respectively

$$f_1(T) = \prod_{j=1}^{\ell} (T - \gamma_j)^{t_j} \quad \text{and} \quad f_2(T) = \prod_{k=1}^{\ell'} (T - \gamma'_k)^{t'_k},$$

then $\mathbf{u}_1\mathbf{u}_2$ satisfies the linear recurrence, the characteristic polynomial of which is

$$\prod_{j=1}^{\ell} \prod_{k=1}^{\ell'} (T - \gamma_j \gamma'_k)^{t_j + t'_k - 1}.$$

Linear recurrence sequences and Brahmagupta–Pell–Fermat Equation

Let d be a positive integer, not a square. The solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of the Brahmagupta–Pell–Fermat Equation

$$x^2 - dy^2 = \pm 1$$

form a sequence $(x_n, y_n)_{n \in \mathbb{Z}}$ defined by

$$x_n + \sqrt{d} y_n = (x_1 + \sqrt{d} y_1)^n.$$

From

$$2x_n = (x_1 + \sqrt{d} y_1)^n + (x_1 - \sqrt{d} y_1)^n$$

we deduce that $(x_n)_{n \geq 0}$ is a linear recurrence sequence. Same for y_n, and also for $n \geq 0$.

Doubly infinite linear recurrence sequences

A sequence $(u_n)_{n \in \mathbb{Z}}$ indexed by $\mathbb{Z}$ is a linear recurrence sequence if it satisfies

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

for all $n \in \mathbb{Z}$.

Recall $a_d \neq 0$.

Such a sequence is determined by d consecutive values.

Signed Fibonacci numbers

http://oeis.org/A039834
The Fibonacci sequence extended to negative indices.

$$\ldots, -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

The sequence

$$1, 1, 0, 1, -1, 2, -3, 5, -8, 13, -21, 34, -55, 89, -144, 233, \ldots$$

is a linear recurrent sequence $u_{n+2} = -u_{n+1} + u_n$.
The ratio of successive terms converges to $-1/\phi$.

Example of a doubly infinite sequence

The sequence of Lucas numbers beginning at $(2, 1)$ is given by
$u_n = \Phi^n + \tilde{\Phi}^n$, $n \geq 0$ http://oeis.org/A000032

$$2, 1, 3, 4, 7, 11, \ldots$$

The *Fibonacci-type sequence based on subtraction* of
http://oeis.org/A061084

$$u_{-n} = \Phi^{-n} + \tilde{\Phi}^{-n} = (-1)^n(\Phi^n + \tilde{\Phi}^n), \quad n \geq 0$$

starts with

$$2, -1, 3, -4, 7, -11, \ldots$$

This gives the doubly infinite linear recurrence sequence

$$\ldots, -11, 7, -4, 3, -1, 2, 1, 3, 4, 7, 11, \ldots$$

Discrete version of linear differential equations

A sequence $\mathbf{u} \in \mathbb{C}^{\mathbb{N}}$ can be viewed as a linear map $\mathbb{N} \longrightarrow \mathbb{C}$. Define the discrete derivative $\mathcal{D}$ by

$$\mathcal{D}\mathbf{u} : \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{C} \\ n & \longmapsto & u_{n+1} - u_n. \end{array}$$

A sequence $\mathbf{u} \in \mathbb{C}^{\mathbb{N}}$ is a linear recurrence sequence if and only if there exists $Q \in \mathbb{C}[T]$ with $Q(1) \neq 1$ such that

$$Q(\mathcal{D})\mathbf{u} = 0.$$

Linear recurrence sequences are a discrete version of linear differential equations with constant coefficients.

The condition $Q(1) \neq 0$ reflects $a_d \neq 0$ – otherwise one gets *ultimately* recurrent sequences.

Skolem – Mahler – Lech Theorem

Theorem (Skolem 1934 – Mahler 1935 – Lech 1953). *Given a linear recurrence sequence, the set of indices* $n \geq 0$ *such that* $u_n = 0$ *is a finite union of arithmetic progressions.*

Thoralf Albert Skolem (1887 – 1963)

Kurt Mahler (1903 – 1988)

Christer Lech

An *arithmetic progression* is a set of positive integers of the form $\{n_0, n_0 + k, n_0 + 2k, \ldots\}$. Here, we allow $k = 0$.

A dynamical system

Let V be a finite dimensional vector space over a field of zero characteristic, H an hyperplane of V, $f : V \to V$ an endomorphism (linear map) and x an element in V.

Theorem. *If there exist infinitely many* $n \geq 1$ *such that* $f^n(x) \in H$, *then there is an (infinite) arithmetic progression of* n *for which it is so.*

A. J. Parameswaran

S.G. Dani

A dynamical system

Let V be a finite dimensional vector space over a field of zero characteristic, W a subspace of V, $f : V \to V$ an endomorphism (linear map) and x an element in V.

Corollary of the Skolem – Mahler – Lech Theorem. *The set of* $n \geq 0$ *such that* $f^n(x) \in W$ *is a finite union of arithmetic progressions.*

By induction, it suffices to consider the case where $W = H$ is an hyperplane of V.

Proof of the corollary

Choose a basis of V. The endomorphism f is given by a square $d \times d$ matrix A, where d is the dimension of V. Consider the characteristic polynomial of A, say

$$X^d - a_1 X^{d-1} - \cdots - a_{d-1} X - a_d.$$

By the Theorem of Cayley – Hamilton,

$$A^d = a_1 A^{d-1} + \cdots + a_{d-1} A + a_d I_d$$

where I_d is the identity $d \times d$ matrix.

Hyperplane membership

Let $b_1x_1 + \cdots + b_dx_d = 0$ be an equation of the hyperplane H in the selected basis of V. Let ${}^t\underline{b}$ denote the $1 \times d$ matrix $(b_1, \ldots, b_d)$ (transpose of a column matrix $\underline{b}$). Using the notation $\underline{v}$ for the $d \times 1$ (column) matrix given by the coordinates of an element v in V, the condition $v \in H$ can be written ${}^t\underline{b}\,\underline{v} = 0$.

Let x be an element in V and $\underline{x}$ the $d \times 1$ (column) matrix given by its coordinates. The condition $f^n(x) \in H$ can now be written

$$ {}^t\underline{b}A^n\underline{x} = 0. $$

The entry u_n of the 1×1 matrix ${}^t\underline{b}A^n\underline{x}$ satisfies a linear recurrence relation, hence, the Skolem – Mahler – Lech Theorem applies.

Remark on the theorem of Skolem–Mahler–Lech

T.A. Skolem treated the case $K = \mathbb{Q}$ of in 1934.

K. Mahler the case $K = \overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, in 1935.

The general case was settled by C. Lech in 1953.

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Linear recurrent sequences

Michel Waldschmidt

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http://www.imj-prg.fr/~michel.waldschmidt/