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Multiple Zeta Values

Michel Waldschmidt

Université Pierre et Marie Curie (Paris 6) France

http://www.imj-prg.fr/~michel.waldschmidt/

Abstract

L. Euler (1707–1783) investigated the values of the numbers

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

for s a rational integer, and B. Riemann (1826–1866) extended this function to complex values of s. For s a positive even integer, $\zeta(s)/\pi^s$ is a rational number. Our knowledge on the values of $\zeta(s)$ for s a positive odd integer is extremely limited. Recent progress involves the wider set of numbers

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

for $s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$.

Abstract (Continued)

Some Bourbaki lectures (by Pierre Cartier in March 2001 and by Pierre Deligne in January 2012) have been devoted to this question. As a matter of fact, there are three $\mathbf{Q}$–algebras which are intertwined : the first one is the subalgebra of the complex numbers spanned by these multiple zeta values (MZV). Another one is the algebra of formal MZV arising from the known combinatorial relations among the multiple zeta values. The main conjecture is to prove that these two algebras are isomorphic. The solution is likely to come from the study of the third algebra, which is the algebra of motivic zeta values, arising from the pro–unipotent fundamental group, involving cohomology, mixed Tate motives. Outstanding progress (mainly by Francis Brown) has been made recently on motivic zeta values.

Harmonic series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

$$\frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$$

Nicolas Oresme (1320 – 1382)

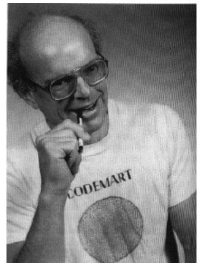
Nicolas Oresme (1320 – 1382)

$$\frac{1}{1}+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}+\cdots$$

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2^n}>\frac{n}{2}$$

Euler–Mascheroni constant

$$\gamma = \lim_{N\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}-\log N\right) = 0.577\,215\,664\,9\ldots$$

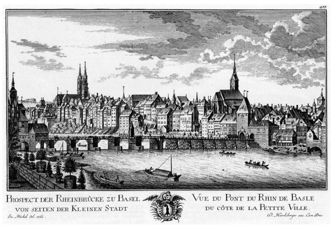


Neil J. A. Sloane – The On-Line Encyclopedia of Integer Sequences
http://oeis.org/A001620

The Basel Problem (1644) : $\sum_{n\geq 1} 1/n^2$

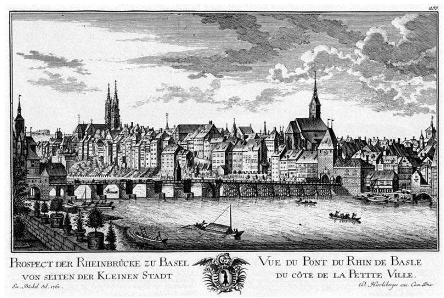
In 1644, Pietro Mengoli (1626 – 1686) asked the exact value of the sum

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots = 1.644934\ldots$$



Basel in 1761

The Bernoulli family was originally from Antwerp, at that time in the Spanish Netherlands, but emigrated to escape the Spanish persecution of the Huguenots. After a brief period in Frankfurt the family moved to Basel, in Switzerland.



The Bernoulli family

Jacob Bernoulli (1654–1705 ; also known as James or Jacques)
Mathematician after whom Bernoulli numbers are named.

Johann Bernoulli (1667–1748 ; also known as Jean)
Mathematician and early adopter of infinitesimal calculus.

The Bernoulli family (continued)

Nicolaus II Bernoulli (1695–1726) Mathematician ;
worked on curves, differential equations, and probability.

Daniel Bernoulli (1700–1782) Developer of
Bernoulli's principle and *St. Petersburg paradox*.

Johann II Bernoulli (1710–1790 ; also known as Jean)
Mathematician and physicist.

Johann III Bernoulli (1744–1807 ; also known as Jean)
Astronomer, geographer, and mathematician.

Jacob II Bernoulli (1759–1789 ; also known as Jacques)
Physicist and mathematician.

Nicolaus II

Daniel

Johan III

Jacob II

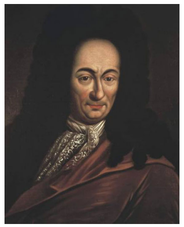
Similar series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots=1.$$

Telescoping series :

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1.$$

Known by Gottfried Wilhelm von Leibniz (1646 – 1716) and
Johann Bernoulli (1667–1748)



Another similar series

Example

$$\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}\cdots=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\log 2.$$

$$\log(1+t)=\sum_{n\geq 1}(-1)^{n-1}\frac{t^n}{n} \qquad -1<t\leq 1.$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)(2n+2)}=\log 2.$$

The Basel Problem : $\sum_{n\geq 1} 1/n^2$

1728 Daniel Bernoulli : approximate value $8/5 = 1.6$

1728 Christian Goldbach : 1.6445 ± 0.0008

1731 Leonard Euler : $1.644934\cdots$

$\zeta(2) = \pi^2/6$ by L. Euler (1707 – 1783)

The Basel problem, first posed by Pietro Mengoli in 1644, was solved by Leonhard Euler in 1735, when he was 28 only.

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n\geq 1} \frac{1}{n^2}.$$

$$\zeta(2) = \frac{\pi^2}{6}.$$

"Proof" of $\zeta(2) = \pi^2/6$, following Euler

The sum of the inverses of the roots of a polynomial f with $f(0) = 1$ is $-f'(0)$: for

$$1 + a_1 z + a_2 z^2 + \cdots + a_n z^n = (1 - \alpha_1 z)\cdots(1 - \alpha_n z)$$

we have $\alpha_1 + \cdots + \alpha_n = -a_1$.
Write

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Set $z = x^2$. The zeroes of the function

$$\frac{\sin\sqrt{z}}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \cdots$$

are π^2, $4\pi^2$, $9\pi^2$, ... *hence* the sum of the inverses of these numbers is

$$\sum_{n\geq 1} \frac{1}{n^2\pi^2} = \frac{1}{6}.$$

Remark

Let $\lambda \in \mathbf{C}$. The functions

$$f(z) = 1 + a_1 z + a_2 z^2 + \cdots$$

and

$$e^{\lambda z} f(z) = 1 + (a_1 + \lambda) z + \cdots$$

have the same zeroes, say $1/\alpha_i$.

The sum $\sum_i \alpha_i$ cannot be at the same time $-a_1$ and $-a_1 - \lambda$.

Completing Euler's proof

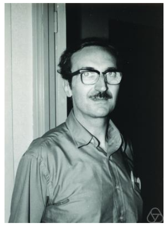
$$\frac{\sin x}{x} = \prod_{n\geq 1}\left(1 - \frac{x^2}{n^2\pi^2}\right).$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \cdots \Longrightarrow \sum_{n\geq 1}\frac{1}{n^2\pi^2} = \frac{1}{6}.$$

http://en.wikipedia.org/wiki/Basel_problem

Evaluating $\zeta(2)$. Fourteen proofs compiled by Robin Chapman.

Another proof (Calabi)



Eugenio Calabi

Pierre Cartier

P. Cartier. – *Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents.* Sém. Bourbaki no. 885 Astérisque **282** (2002), 137-173.

Another proof (Calabi)

$$\frac{1}{1-x^2y^2} = \sum_{n\geq 0} x^{2n}y^{2n}.$$

$$\int_0^1 x^{2n}dx = \frac{1}{2n+1}.$$

$$\int_0^1\int_0^1 \frac{dxdy}{1-x^2y^2} = \sum_{n\geq 0}\frac{1}{(2n+1)^2}.$$

$$x = \frac{\sin u}{\cos v}, \qquad y = \frac{\sin v}{\cos u},$$

$$\int_0^1\int_0^1 \frac{dxdy}{1-x^2y^2} = \int_{0\leq u\leq \pi/2,\ 0\leq v\leq \pi/2,\ u+v\leq \pi/2} dudv = \frac{\pi^2}{8}.$$

Completing Calabi's proof of $\zeta(2) = \pi^2/6$

From

$$\sum_{n\geq 0}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

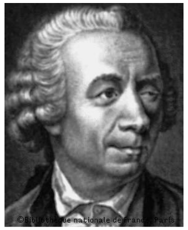
one deduces

$$\sum_{n\geq 1}\frac{1}{n^2} = \sum_{n\geq 1}\frac{1}{(2n)^2} + \sum_{n\geq 0}\frac{1}{(2n+1)^2}.$$

$$\sum_{n\geq 1}\frac{1}{(2n)^2} = \frac{1}{4}\sum_{n\geq 1}\frac{1}{n^2}.$$

$$\zeta(2) = \sum_{n\geq 1}\frac{1}{n^2} = \frac{4}{3}\sum_{n\geq 0}\frac{1}{(2n+1)^2} = \frac{\pi^2}{6}.$$

Introductio in analysin infinitorum



Leonhard Euler

(1707 – 1783)

Introductio in analysin infinitorum

Special values of the Zeta function

$\zeta(s)$ for $s \in \mathbf{Z}$, $s \geq 2$
Jacques Bernoulli (1654–1705),
Leonard Euler (1739).

$\pi^{-2k}\zeta(2k) \in \mathbf{Q}$ *for* $k \geq 1$ (Bernoulli numbers).

Bernoulli numbers

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{n \geq 1} (-1)^{n+1} B_n \frac{t^{2n}}{(2n)!}.$$

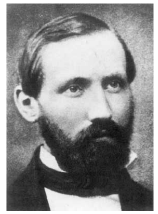
$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66} \dots$$

$$\boxed{\zeta(2n) = 2^{2n-1} \frac{B_n}{(2n)!} \pi^{2n} \quad (n \geq 1).}$$

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}.$$

Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}$$



Euler : $s \in \mathbf{R}$.

Riemann : $s \in \mathbf{C}$.

Analytic continuation of the Riemann zeta function

The complex function which is defined for $\Re e s > 1$ by the Dirichlet series

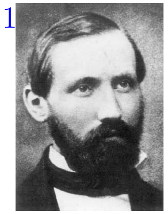
$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

has a meromorphic continuation to $\mathbf{C}$ with a unique pole in $s = 1$ of residue 1.

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma.$$

Euler Constant :

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right)$$
$$= 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots$$



B. Riemann
(1826–1866)

Functional equation of the Riemann zeta function

Connection between $\zeta(s)$ and $\zeta(1-s)$:

$$\zeta(s) = 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

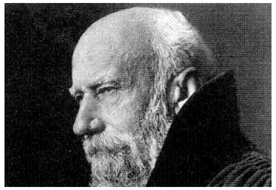
Euler Gamma function

$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty}\frac{(1+1/n)^s}{1+s/n} = \int_0^{\infty} x^{s-1}e^{-x}dx.$$

Trivial zeroes of the Riemann zeta function -2, -4, $-6\ldots$
Riemann hypothesis :
The non trivial zeroes of the Riemann zeta function have real part $1/2$.

Values of ζ at the positive even integers

- *F. Lindemann : π is a transcendental number, hence $\zeta(2k)$ also for $k \geq 1$.*



- *Hermite–Lindemann : transcendence of $\log\alpha$ and e^β for α and β nonzero algebraic numbers with $\log\alpha \neq 0$.*

Diophantine question

Determine all algebraic relations among the numbers

$$\zeta(2),\quad \zeta(3),\quad \zeta(5),\quad \zeta(7),\ldots$$

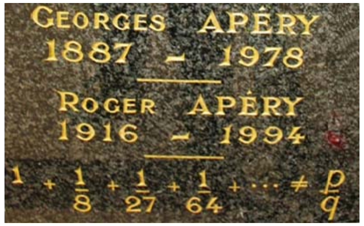
Conjecture. *There is no algebraic relation among these numbers : the numbers*

$$\zeta(2),\quad \zeta(3),\quad \zeta(5),\quad \zeta(7),\ldots$$

are algebraically independent.

In particular the numbers $\zeta(2n+1)$ and $\zeta(2n+1)/\pi^{2n+1}$ for $n \geq 1$ are expected to be transcendental.

$$\zeta(3) = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots \notin \mathbf{Q}$$



- Roger Apéry (1978) : *The number*

$$\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3} = 1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511\,\ldots$$

is irrational.

Infinitely many odd zeta are irrational

Tanguy Rivoal (2000)

Let $\epsilon > 0$. *For any sufficiently large odd integer* a, *the dimension of the* $\mathbf{Q}$*–vector space spanned by the numbers* $1,\ \zeta(3),\ \zeta(5),\ \cdots, \zeta(a)$ *is at least*

$$\frac{1-\epsilon}{1+\log 2} \log a.$$

Wadim Zudilin

- *At least one of the four numbers*
  $\zeta(5),\quad \zeta(7),\quad \zeta(9),\quad \zeta(11)$
  is irrational.

Stéphane Fischler and Wadim Zudilin

There exist odd integers j_1 *and* j_2 *with* $5 \leq j_1 \leq 139$ *and* $5 \leq j_2 \leq 1961$ *such that the four numbers* 1, $\zeta(3)$, $\zeta(j_1)$, $\zeta(j_2)$ *are linearly independent over* $\mathbf{Q}$.

Linearization of the problem *(Euler)*

The problem of *algebraic independence* of values of the Riemann zeta function is difficult. We show that it can be reduced to a problem of *linear independence*.

The product of two special values of the zeta function is a sum of *multiple zeta values.*

$$\sum_{n_1\ge 1}\frac{1}{n_1^{s_1}}\sum_{n_2\ge 1}\frac{1}{n_2^{s_2}}=\sum_{n_1>n_2\ge 1}\frac{1}{n_1^{s_1}n_2^{s_2}} + \sum_{n_2>n_1\ge 1}\frac{1}{n_2^{s_2}n_1^{s_1}}+\sum_{n\ge 1}\frac{1}{n^{s_1+s_2}}$$

Multiple zeta values *(Euler)*

For $s_1 \ge 2$ and $s_2 \ge 2$, we have

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

with

$$\zeta(s_1, s_2) = \sum_{n_1>n_2\ge 1} n_1^{-s_1} n_2^{-s_2}.$$

Examples :

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$$
$$\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5)$$

Multiple zeta values (MZV)

For k, $s_1, \dots, s_k$ positive integers with $s_1 \ge 2$, we set $\underline{s} = (s_1, \dots, s_k)$ and

$$\zeta(\underline{s}) = \sum_{n_1>n_2>\cdots>n_k\ge 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}.$$

For $k = 1$ we recover the special values of ζ.

k is the *depth* while $n = s_1 + \cdots + s_k$ is the *weight*.

The algebra of multiple zeta values

The product of two multiple zeta values is a linear combination of multiple zeta values.

Hence, the $\mathbf{Q}$–vector space $\mathfrak{Z}$ spanned by the numbers $\zeta(\underline{s})$ is also a $\mathbf{Q}$–algebra.

The problem of algebraic independence becomes a problem of linear independence.

Question : which are the linear relations among these numbers $\zeta(\underline{s})$?

Answer : *there are many of them !* This algebra $\mathfrak{Z}$ has a rich algebraic structure, not yet fully understood.

Two main conjectures

First Conjecture : there is no linear relation among multiple zeta values of different weights.

Recall that $\mathfrak{Z}$ denotes the $\mathbf{Q}$-subspace of $\mathbf{R}$ spanned by the real numbers $\zeta(\underline{s})$ with $\underline{s} = (s_1, \ldots, s_k)$, $k \geq 1$ and $s_1 \geq 2$.

Further, for $n \geq 2$, denote by $\mathfrak{Z}_n$ the $\mathbf{Q}$-subspace of $\mathfrak{Z}$ spanned by the real numbers $\zeta(\underline{s})$ where $\underline{s}$ has weight $s_1 + \cdots + s_k = n$.

Define also $\mathfrak{Z}_0 = \mathbf{Q}$ and $\mathfrak{Z}_1 = \{0\}$.

The *First Conjecture* is

$$\mathfrak{Z} = \bigoplus_{n \geq 0} \mathfrak{Z}_n.$$

The second main Conjecture

Denote by d_n the dimension of $\mathfrak{Z}_n$.

Conjecture (Zagier). *For $n \geq 3$, we have*

$$d_n = d_{n-2} + d_{n-3}.$$

$$(d_0,\ d_1,\ d_2, \ldots) = (1, 0, 1, 1, 1, 2, 2, \ldots).$$

Zagier's Conjecture can be written

$$\sum_{n \geq 0} d_n X^n = \frac{1}{1 - X^2 - X^3}.$$

Small weight : $k = 0,\ 1,\ 2,\ 3,\ 4$

Weight 0 $\quad d_0 = 1 \quad \zeta(s_1, \ldots, s_k) = 1$ for $k = 0$, $\mathfrak{Z}_0 = \mathbf{Q}$.

Weight 1 $\quad d_1 = 0 \quad k = 1$, $\mathfrak{Z}_1 = \{0\}$.

Weight 2 $\quad d_2 = 1 \quad \zeta(2) \neq 0$

Weight 3 $\quad d_3 = 1 \quad \zeta(2,1) = \zeta(3) \neq 0$

Weight 4 $\quad d_4 = 1 \quad \zeta(3,1) = \frac{1}{4}\zeta(4), \qquad \zeta(2,2) = \frac{3}{4}\zeta(4),$

$$\zeta(2,1,1) = \zeta(4) = \frac{2}{5}\zeta(2)^2$$

Weight 5

$d_5 = 2$?

One can check :

$$\begin{aligned}
\zeta(2,1,1,1) &= \zeta(5), \\
\zeta(3,1,1) &= \zeta(4,1) = 2\zeta(5) - \zeta(2)\zeta(3), \\
\zeta(2,1,2) &= \zeta(2,3) = \frac{9}{2}\zeta(5) - 2\zeta(2)\zeta(3), \\
\zeta(2,2,1) &= \zeta(3,2) = 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5),
\end{aligned}$$

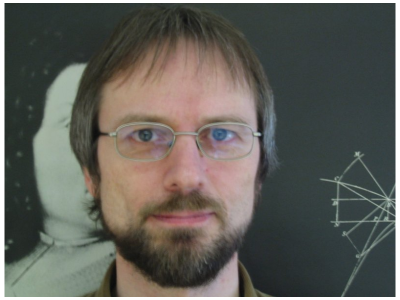
Hence $d_5 \in \{1, 2\}$. Moreover $d_5 = 2$ if and only if the number

$$\zeta(2)\zeta(3)/\zeta(5)$$

is irrational.

A *modular* relation in weight 12

$$5\,197\,\zeta(12) = 19\,348\,\zeta(9,3) + 103\,650\,\zeta(7,5) + 116\,088\,\zeta(5,7).$$



Herbert Gangl

EZ Face

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http://oldweb.cecm.sfu.ca/cgi-bin/EZFace/zetaform.cgi
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Broadhurst and Kreimer

A filtration of $\mathfrak{Z}_n$ is $(\mathfrak{Z}_n^\ell)_{\ell\geq 0}$ where $\mathfrak{Z}_n^\ell$ is the space of MZV of weight n and depth $\leq \ell$

Denote by $d_{n\ell}$ the dimension of $\mathfrak{Z}_n^\ell/\mathfrak{Z}_n^{\ell-1}$.

The Conjecture of Broadhurst and Kreimer is :

$$\sum_{n\geq 0}\sum_{\ell\geq 1} d_{n\ell}X^nY^\ell = \frac{1+\mathbb{E}(X)Y}{1-\mathbb{O}(X)Y+\mathbb{S}(X)(Y^2-Y^4)},$$

where

$$\mathbb{E}(X) = \frac{X^2}{1-X^2}, \quad \mathbb{O}(X) = \frac{X^3}{1-X^2},$$

$$\mathbb{S}(X) = \frac{X^{12}}{(1-X^4)(1-X^6)}.$$

Broadhurst and Kreimer imply Zagier

For $Y=1$, the Conjecture of Broadhurst and Kreimer

$$\sum_{n\geq 0}\sum_{\ell\geq 1} d_{n\ell}X^nY^\ell = \frac{1+\mathbb{E}(X)Y}{1-\mathbb{O}(X)Y+\mathbb{S}(X)(Y^2-Y^4)}$$

is Zagier's conjecture

$$\sum_{n\geq 0} d_nX^n = \frac{1}{1-X^2-X^3}.$$

Modular relations

Notice that

$$\mathbb{E}(X) = \frac{X^2}{1-X^2} = \sum_{k\geq 1} X^{2k},$$

$$\mathbb{O}(X) = \frac{X^3}{1-X^2} = \sum_{k\geq 1} X^{2k+1},$$

$$\mathbb{S}(X) = \frac{X^{12}}{(1-X^4)(1-X^6)} = \sum_{k\geq 0} \dim_{\mathbf{C}}(S_k)X^k,$$

where S_k is the space of parabolic modular forms of weight k.

Hoffman's remark

The number d_n of tuples $(s_1, \ldots, s_k)$, where each s_i is 2 or 3 and $s_1 + \cdots + s_k = n$, satisfies Zagier's recurrence relation

$$d_n = d_{n-2} + d_{n-3}$$

with $d_1 = 0$, $d_2 = d_3 = 1$.

Hoffman's Conjecture

Michael Hoffman conjectures :
A basis of $\mathfrak{Z}_n$ over $\mathbf{Q}$ is given by the numbers $\zeta(s_1, \ldots, s_k)$, $s_1 + \cdots + s_k = n$, where each s_i is 2 or 3.

Hoffman's Conjecture for $n \leq 20$

For $n \leq 20$, Hoffman's Conjecture is compatible with known relations among MZV.

Masanobu Kaneko

M. Kaneko, M. Noro and K. Tsurumaki. – *On a conjecture for the dimension of the space of the multiple zeta values*, Software for Algebraic Geometry, IMA **148** (2008), 47–58.

Francis Brown

The numbers $\zeta(s_1, \ldots, s_k)$, $s_1 + \cdots + s_k = n$, where each s_i is 2 or 3, span $\mathfrak{Z}_n$ over $\mathbf{Q}$.

Previous upper bound for the dimension

Zagier's numbers d_n are *upper bounds* for the dimension of $\mathfrak{Z}_n$.


Alexander Goncharov


Tomohide Terasoma

A.B. Goncharov – *Multiple ζ-values, Galois groups and Geometry of Modular Varieties.* Birkhäuser. Prog. Math. **201**, 361-392 (2001).
T. Terasoma – *Mixed Tate motives and Multiple Zeta Values.* Invent. Math. **149**, No.2, 339-369 (2002).

Motivic zeta values

From Brown's results, it follows that the algebraic independence of the numbers

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7), \dots$$

is a consequence of the two main Conjectures.

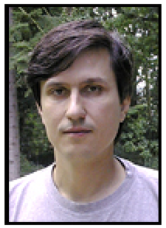
There is a combinatorial description of linear relations among MZV, we do not know yet whether they provide a complete picture of the situation.

Problem : lower bound for the dimension

The Diophantine problem is now to prove lower bounds for the dimension.

We do not even know how to prove $d_n \geq 2$ for at least one value of n !

Periods, following Kontsevich and Zagier



Periods,
Mathematics unlimited— 2001 and beyond,
Springer 2001, 771–808.

A *period* is a complex number with real and imaginary parts given by absolutely convergent integrals of rational fractions with rational coefficients on domains of $\mathbf{R}^n$ defined by (in)equalities involving polynomials with rational coefficients.

$\zeta(s)$ is a period

$$\frac{1}{1-u} = \sum_{n\geq 1} u^{n-1}, \qquad \int_0^1 u^{n-1} du = \frac{1}{n}\cdot$$

$$\frac{1}{1-u_1\cdots u_s} = \sum_{n\geq 1} (u_1\cdots u_s)^{n-1},$$

$$\int_{[0,1]^s} \frac{du_1\cdots du_s}{1-u_1\cdots u_s} = \sum_{n\geq 1} \left(\int_0^1 u^{n-1} du\right)^s = \sum_{n\geq 1} \frac{1}{n^s} = \zeta(s).$$

$\zeta(2)$ is a period

$$\zeta(2) = \int_0^1 \int_0^1 \frac{dudv}{1-uv}\cdot$$

Another integral for $\zeta(2)$:

$$\frac{\pi^2}{6} = \zeta(2) = \sum_{n\geq 1} \frac{1}{n^2} = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2}\cdot$$

$\zeta(2)$ is a period

$$\begin{aligned}\int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} &= \int_0^1 \left(\int_0^{t_1} \frac{dt_2}{1-t_2}\right) \frac{dt_1}{t_1}\\ &= \int_0^1 \left(\int_0^{t_1} \sum_{n\geq 1} t_2^{n-1} dt_2\right) \frac{dt_1}{t_1}\\ &= \sum_{n\geq 1} \frac{1}{n} \int_0^1 t_1^{n-1} dt_1\\ &= \sum_{n\geq 1} \frac{1}{n^2} = \zeta(2).\end{aligned}$$

Kontsevich–Zagier philosophy of periods

There should be a direct proof of

$$\int_0^1 \int_0^1 \frac{du_1 du_2}{1-u_1u_2} = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2}\cdot$$

Change of variables $t_1 = u_1$, $t_2 = u_1u_2$,

$$0 \leq u_1, u_2 \leq 1, \quad 0 \leq t_2 \leq t_1 \leq 1,$$

$$dt_1 dt_2 = u_1 du_1 du_2, \qquad \frac{du_1 du_2}{1-u_1u_2} = \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2}\cdot$$

$\zeta(s)$ is a period

For s integer ≥ 2,

$$\zeta(s) = \int_{1>t_1>t_2\cdots>t_s>0} \frac{dt_1}{t_1} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s}.$$

Induction

$$\int_{t_1>t_2\cdots>t_s>0} \frac{dt_2}{t_2} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s} = \sum_{n\geq 1} \frac{t_1^{n-1}}{n^{s-1}}.$$

MZV are periods

$$\zeta(2,1) = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} \cdot \frac{dt_3}{1-t_3}.$$

Proof.
We have

$$\int_0^{t_2} \frac{dt_3}{1-t_3} = \sum_{n\geq 1} \frac{t_2^{n-1}}{n}, \quad \text{next} \quad \int_0^{t_1} \frac{t_2^{n-1}dt_2}{t_2-1} = \sum_{m>n} \frac{t_1^m}{m},$$

and

$$\int_0^1 t_1^{m-1} dt_1 = \frac{1}{m},$$

hence

$$\int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} \cdot \frac{dt_3}{1-t_3} = \sum_{m>n\geq 1} \frac{1}{m^2 n} = \zeta(2,1)$$

Notation

Set

$$\omega_0 = \frac{dt}{t}, \qquad \omega_1 = \frac{dt}{1-t}.$$

For $s \geq 2$ we write the relation

$$\zeta(s) = \int_{1>t_1>\cdots>t_s>0} \frac{dt_1}{t_1} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s}$$

as

$$\zeta(s) = \int_0^1 \omega_0^{s-1} \omega_1.$$

This leads to a definition of a (non–commutative) product of differential forms.

Chen iterated integrals

When φ is a holomorphic 1-form,

$$\int_0^z \varphi$$

is the primitive of φ which vanishes at $z = 0$.
When $\varphi_1, \ldots, \varphi_k$ are holomorphic 1-forms, we define inductively

$$\int_0^z \varphi_1 \cdots \varphi_k := \int_0^z \varphi_1(t) \int_0^t \varphi_2 \cdots \varphi_k.$$

Chen iterated integrals

$$\int_0^z \varphi_1 \cdots \varphi_k := \int_0^z \varphi_1(t) \int_0^t \varphi_2 \cdots \varphi_k.$$

If $\varphi_1(t) = \psi_1(t)dt$, then

$$\frac{d}{dz} \int_0^z \varphi_1 \cdots \varphi_k = \psi_1(z) \int_0^z \varphi_2 \cdots \varphi_k.$$

for $\underline{s} = (s_1, \ldots, s_k)$, set

$$\omega_{\underline{s}} = \omega_{s_1} \cdots \omega_{s_k} = \omega_0^{s_1-1}\omega_1 \cdots \omega_0^{s_k-1}\omega_1.$$

Then

$$\zeta(\underline{s}) = \int_0^1 \omega_{\underline{s}}.$$

Coding MZV

$$\underline{s} = (s_1, \ldots, s_k) \qquad \omega_{\underline{s}} = \omega_0^{s_1-1}\omega_1 \cdots \omega_0^{s_k-1}\omega_1$$

- ends with ω_1
- starts with ω_0 $\quad (s_1 \geq 2)$.

Weight : $n = s_1 + \cdots + s_k$ is the number of factors
Depth : k is the number of ω_1

Depth 1 : for $s \geq 2$, $\omega_s = \omega_0^{s-1}\omega_1$ weight s
Examples in depth 2 : $\omega_{2,1} = \omega_0\omega_1^2$ weight 3
$\omega_{4,3} = \omega_0^3\omega_1\omega_0^2\omega_1$ weight 7

Multiple zeta values are periods

$\underline{s} = (s_1, \ldots, s_k)$, $s_1 \geq 2$, $p = s_1 + \cdots + s_k$

$$\boxed{\zeta(\underline{s}) = \int_{1>t_1>t_2>\cdots>t_p>0} \omega_0^{s_1-1}\omega_1 \cdots \omega_0^{s_k-1}\omega_1}$$

Example

$$\zeta(2,1) = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} \cdot \frac{dt_3}{1-t_3} = \int_0^1 \omega_0\omega_1^2.$$

Quadratic relations

The product of two multiple zeta values is a linear combination, with positive integer coefficients, of multiple zeta values.

Besides, there are two essentially different ways of writing such a product as a linear combination of MZV : one of them arises from the product as series

$$\zeta(\underline{s}) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}},$$

the other one arises from the integral representation

$$\zeta(\underline{s}) = \int_0^1 \omega_{\underline{s}}.$$

Products of integrals

$$\zeta(2) = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2}.$$

$$\zeta(2)^2 = \int_{\substack{1>t_1>t_2>0\\1>u_1>u_2>0}} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} \cdot \frac{du_1}{u_1} \cdot \frac{du_2}{1-u_2}.$$

We decompose the cartesian product of two simplices

$$\{1 > t_1 > t_2 > 0\} \times \{1 > u_1 > u_2 > 0\}$$

as a union, essentially disjoint (up to subsets of zero measure), of 6 simplices, which yields

$$\zeta(2)^2 = 4\zeta(3,1) + 2\zeta(2,2).$$

$\{1 > t_1 > t_2 > 0\} \times \{1 > u_1 > u_2 > 0\}$

$1 > t_1 > t_2 > u_1 > u_2 > 0$	$\frac{1}{t_1} \cdot \frac{1}{1-t_2} \cdot \frac{1}{u_1} \cdot \frac{1}{1-u_2}$	$\zeta(2,2)$
$1 > t_1 > u_1 > t_2 > u_2 > 0$	$\frac{1}{t_1} \cdot \frac{1}{u_1} \cdot \frac{1}{1-t_2} \cdot \frac{1}{1-u_2}$	$\zeta(3,1)$
$1 > t_1 > u_1 > u_2 > t_2 > 0$	$\frac{1}{t_1} \cdot \frac{1}{u_1} \cdot \frac{1}{1-u_2} \cdot \frac{1}{1-t_2}$	$\zeta(3,1)$
$1 > u_1 > t_1 > t_2 > u_2 > 0$	$\frac{1}{u_1} \cdot \frac{1}{t_1} \cdot \frac{1}{1-t_2} \cdot \frac{1}{1-u_2}$	$\zeta(3,1)$
$1 > u_1 > t_1 > u_2 > t_2 > 0$	$\frac{1}{u_1} \cdot \frac{1}{t_1} \cdot \frac{1}{1-u_2} \cdot \frac{1}{1-t_2}$	$\zeta(3,1)$
$1 > u_1 > u_2 > t_1 > t_2 > 0$	$\frac{1}{u_1} \cdot \frac{1}{1-u_2} \cdot \frac{1}{t_1} \cdot \frac{1}{1-t_2}$	$\zeta(2,2)$

Linear relations among MZV

As a consequence, multiple zeta values satisfy a lot of independent linear relations with integer coefficients.

Example

Product of series :

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$$

Product of integrals :

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1)$$

Hence

$$\zeta(4) = 4\zeta(3,1).$$

The algebras $\mathcal{P}$ of multiple zeta periods

Recall that $\mathfrak{Z}$ is the subalgebra of $\mathbf{R}$ over $\mathbf{Q}$ spanned by the numbers $\zeta(\underline{s})$, where $\underline{s} = (s_1, \dots, s_k)$, $s_1 \geq 2$.

Let $\mathcal{P}$ be the $\mathbf{Q}$–algebra defined by generators $Z_{\underline{s}}$, $\underline{s} = (s_1, \dots, s_k)$ with $s_1 \geq 2$, and the relations among MZV arising from the products of series and integrals.

There is a homomorphism $ev : \mathcal{P} \to \mathbf{R}$ (think of elements of $\mathcal{P}$ as equivalence classes of programs and ev as the "exec" command). It should be expected that ev is an injective map.

The algebras $\mathfrak{M}$ of motivic zeta values

The third algebra is the algebra $\mathfrak{M}$ of *motivic zeta values*.
$\mathfrak{M}$ is a graded algebra generated by homogeneous elements $\zeta^{\mathfrak{m}}(\underline{s})$.

There is also an evaluation map $ev^{\mathfrak{m}} : \mathfrak{M} \to \mathbf{R}$, such that $ev^{\mathfrak{m}}\big(\zeta^{\mathfrak{m}}(\underline{s})\big) = \zeta(\underline{s})$, and a commutative diagram

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{ev^{\mathfrak{m}}} & \mathfrak{Z} \\ \downarrow & \nearrow_{ev} & \\ \mathcal{P} & & \end{array}$$

F. Brown has shown that a basis of $\mathfrak{M}$ as a $\mathbf{Q}$–vector space is given by the $\zeta^{\mathfrak{m}}(\underline{s})$ where $s_i \in \{2,3\}$ $(i = 1, \dots, k)$.

The motivic Galois group

Thanks to the work of F. Brown, we control the automorphism group of $\mathfrak{M}$.

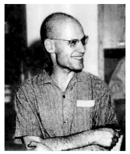
F. Brown deduces that the category of mixed Tate motives of $\mathbf{Z}$ is generated by the fundamental group of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$.
`Ref.:` Bourbaki seminar by P. Deligne in 2012.

We expect the evaluation map from $\mathfrak{M}$ to $\mathbf{R}$ to be injective.
This would imply for instance that the numbers

$$\pi,\ \zeta(3),\ \zeta(5)\dots$$

are transcendental and algebraically independent. According to P. Cartier, this wild dream is to be fulfilled around 2040 !.

Connection with works by



Alexandre Grothendieck

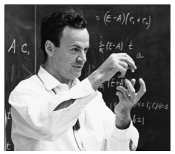
Pierre Deligne

Maxime Kontsevich

Connection with works by

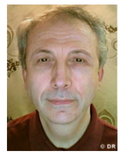
Alain Connes

Dirk Kreimer



Richard Feynman

Yuri Manin



Vladimir Drinfeld

Connection with works by

Sarah Car and Leila Schneps

Pierre Lochak

Jacky Cresson

Jean Ecalle

Hidekazu Furusho

Connection with works by

Hoang Ngoc Minh

Michel Petitot

David Bradley

Georges Racinet

Kentaro Ihara

Clément Dupont

References on MZV

- Link to Michael Hoffman's MZV website
  `http://www.usna.edu/Users/math/meh/biblio.html`

- Thesis of Samuel Baumard
  `http://tel.archives-ouvertes.fr/docs/01/01/70/22/PDF/these.pdf`