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Workshop Diophantine approximation
and dynamical systems

Some open problems

by

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1 An open problem related with the Skolem Mahler Lech Theorem

One main open problem related with the Theorem of Skolem–Mahler–Lech is that it is not effective: explicit upper bounds for the number of arithmetic progressions, depending only on the order of the linear recurrence sequence, are known (W.M. Schmidt, U. Zannier), but no upper bound for the arithmetic progressions themselves is known. A related open problem raised by T.A. Skolem and C. Pisot [T1] is:

Problem. *Given an integer linear recurrence sequence $(x_n)_{n\geq 0}$, is the truth of the statement "$x_n \neq 0$ for all n" decidable in finite time?*

References

[T1] Terence Tao. *Effective Skolem Mahler Lech theorem.* In "Structure and Randomness: pages from year one of a mathematical blog", American Mathematical Society (2008), 298 pages.
`http://terrytao.wordpress.com/2007/05/25/open-question-effective-skolem-mahler-lech-theorem/`

2 On the Littlewood Conjecture for power series over function fields

The classical Littlewood Conjecture asserts that given two real numbers x and y, we have

$$\inf_{q\geq 1} q \cdot \|qx\| \cdot \|qy\| = 0,$$

where $\|\cdot\|$ is the distance to the nearest integer.

Let $\mathbf{k}$ be a field and $A = \mathbf{k}[X]$ the ring of polynomials in one variable X with coefficients in $\mathbf{k}$ and K the field $\mathbf{k}((X^{-1}))$ of formal Laurent series. A norm on K is defined by setting $|0| = 0$ and for any non–zero power series

$$f = \sum_{h=-m}^{\infty} a_h X^{-h}$$

with $a_{-m} \neq 0$, by setting $|f| = 2^m$. The integral part $\lfloor f \rfloor$ and fractional part $\{f\}$ of f are defined by

$$\lfloor f \rfloor = \sum_{-m \leq h \leq 0} a_h X^{-h} \quad \text{and} \quad \{f\} = \sum_{h=0}^{\infty} a_h X^{-h},$$

so that $f = \lfloor f \rfloor + \{f\}$. Set $\|f\| = |\{f\}|$. The analog of the Littlewood Conjecture for function fields [AB] is whether for any f and g in K,

$$\inf_{q \in A \setminus \{0\}} |q| \cdot \|qf\| \cdot \|qg\| = 0.$$

When $\mathbf{k}$ is an infinite field, a negative answer to this question has been obtained by Davenport and Lewis (1963), while explicit counterexamples have been given by A. Baker (1964), R.T. Bumby (1967), T.W. Cusik (1967, 1971) and T. Komatsu (1991).

Problem. *Prove or disprove the analog of the Littlewood Conjecture for function fields over a finite field* $\mathbf{k}$.

References

[AB] Boris Adamczewski and Yann Bugeaud. *On the Littlewood Conjecture in fields of power series.* Advanced Studies in Pure Mathematics, **49** (2007), Probability and Number Theory — Kanazawa 2005, 1– 20 `http://adamczewski.perso.math.cnrs.fr/LSF.pdf`.

3 An open problem related with parametric geometry of numbers, finite fields and Hankel determinants

Here is an open problem from [RW] concerning parametric geometry of numbers in function fields.

Let $\mathbf{k}$ be a field, $A = \mathbf{k}[T]$ the ring of polynomials in one variable with coefficients in $\mathbf{k}$ and $\mathbf{k}[[T]]$ the ring of formal power series in one variable with coefficients in $\mathbf{k}$.

Let $\mathbf{f} = (f_1, \ldots, f_n)$ be an n-tuple of elements of $\mathbf{k}[[T]]$. A linear algebra argument shows that, for any non-zero $(\varrho_1, \ldots, \varrho_n) \in \mathbb{N}^n$, there exists a non-zero point $\mathbf{a} = (a_1, \ldots, a_n)$ in A^n such that

(1) $\quad \deg(a_i) \leq \varrho_i - 1 \quad (1 \leq i \leq n) \quad \text{and} \quad \operatorname{ord}_0(\mathbf{a} \cdot \mathbf{f}) \geq \varrho_1 + \cdots + \varrho_n - 1.$

Following Mahler and Jager (see [RW]), we say that $\mathbf{f}$ is *normal* for $(\varrho_1, \dots, \varrho_n)$ if any non-zero solution $\mathbf{a}$ of (1) in A^n has $\mathrm{ord}_0(\mathbf{a}\cdot\mathbf{f}) = \varrho_1 + \cdots + \varrho_n - 1$. Then, those solutions together with 0 constitute, over $\mathbf{k}$, a one dimensional subspace of A^n. We also say that $\mathbf{f}$ is a *perfect system* if it is normal for any $(\varrho_1, \dots, \varrho_n) \in \mathbb{N}^n \setminus \{0\}$.

It is easy to see that if $\mathbf{k}$ is finite with at most 3 elements, there is no perfect n–system of series of $\mathbf{k}[[T]]$ with $n \geq 2$.

Question. *Assume the field* $\mathbf{k}$ *is finite with at least* 4 *elements. Does there exist a perfect* n*–system of series of* $\mathbf{k}[[T]]$ *with* $n \geq 2$*?*

This question amounts to decide whether there exists a sequence $(a_0, a_1, a_2, \dots)$ of elements in $\mathbf{k}$ such that, for any $m \geq 0$ and any $h \geq 1$, the Hankel determinant

$$\begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+h-1} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+h} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+h-1} & a_{m+h} & \cdots & a_{m+2h} \end{vmatrix}$$

does not vanish.

Hankel determinants play a role in several problems related with dynamical systems (see for instance [AS]).

References

[AS] Jean-Paul Allouche and Jeffrey Shallit. *Automatic Sequences: Theory, Applications, Generalizations.* Cambridge University Press, 2003.
https://cs.uwaterloo.ca/~shallit/asas.html

[RW] Damien Roy and Michel Waldschmidt. *Parametric geometry of numbers in function fields.* Mathematika, **63** 3 (2017) 1114–1135.
https://arxiv.org/abs/1704.00291 [math.NT].

4 Interval contracted rotations and Phragmén election method

Let $0 < \lambda < 1$ and $0 \leq \delta < 1$. Consider the circle λ–affine contraction $f : x \in [0, 1) \to \lambda x + \delta \mod 1$. Let ρ be the rotation number of the map f. In [LN], using Hecke–Mahler series and a tree structure, Laurent and Nogueira prove that when both parameters λ and δ are algebraic numbers, then ρ is a rational number. In the case λ and δ are rational, they propose the following conjecture.

Conjecture. *Assume* $\lambda = a/b$ *and* $\delta = r/s$ *are rational numbers. Let* $\gamma > 1$*. Then there exists a constant* $c(\gamma, a, b, s)$ *such that, if* $b > a^\gamma$*, then* $\rho = p/q$ *with*

$$0 \leq p < q \leq c(\gamma, a, b, s).$$

They prove this result in the special case where $\gamma = (1+\sqrt{5})/2$ is the Golden ratio, with

$$c(\gamma, a, b, s) = \gamma^{2+\frac{\gamma \log(sb)}{\log b - \gamma \log a}}.$$

For a connection with an election method suggested by Lars Edvard Phragmén, see [JÖ].

References

[JÖ] Svante Janson and Anders Öberg. *A piecewise contractive dynamical system and Election methods.*
https://arxiv.org/abs/1709.06398 [math.DS] [v1] Tue, 19 Sep 2017.

[LN] Michel Laurent and Arnaldo Nogueira. *Rotation number of interval contracted rotations.*
https://arxiv.org/abs/1704.05130 [math.DS] [v2] Wed, 26 Apr 2017.

5 The Chowla Conjecture and the Sarnak Conjecture

The Möbius arithmetic function μ is defined for $n \geq 1$ by $\mu(n) = (-1)^r$ if n is squarefree and product of r distinct primes, and $\mu(n) = 0$ otherwise. The estimate

$$\sum_{n \leq x} \mu(n) = O(x^{\frac{1}{2}+o(1)})$$

is equivalent to the Riemann Hypothesis. Here is Chowla's Conjecture:

Conjecture (Sarvadaman Chowla)**.** *Let $a_1, \ldots, a_m$ be non negative integers with at least one of the a_m odd. Then*

$$\sum_{n \leq x} \mu(n+1)^{a_1} \cdots \mu(n+m)^{a_m} = o_m(x)$$

as $x \to \infty$.

Here is Sarnak's Conjecture:

Conjecture (Peter Sarnak)**.** *Let $f : \mathbb{N} \to \mathbb{C}$ be deterministic bounded sequence. Then*

$$\sum_{n \leq x} \mu(n) f(n) = o_f(x)$$

as $x \to \infty$.

Peter Sarnak proved that the Chowla Conjecture implies the Sarnak Conjecture (see the post [T2] by Terence Tao), and conversely el Houcein el Abdalaoui [A] proved that the Sarnak Conjecture implies the Chowla Conjecture.

References

[T2] Terence Tao. *The Chowla Conjecture and the Sarnak Conjecture* https://terrytao.wordpress.com/2012/10/14/the-chowla-conjecture-and-the-sarnak-conjecture/

[A] El Houcein el Abdalaoui. *On Veech's proof of Sarnak's Theorem on the Möbius flow* https://arxiv.org/abs/1711.06326 arXiv:1711.06326v1 [math.DS]

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