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Transcendence of Periods.

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Abstract

The set of real numbers and the set of complex numbers have the power of continuum. Among these numbers, those which are "*interesting*", which appear "*naturally*", which deserve our attention, form a countable set. In a seminal paper with the title "*Periods*" published in 2000, M. Kontsevich and D. Zagier suggest a suitable definition for that set, by introducing the definition of "*periods*". They propose one conjecture, two principles and five problems. The goal of this talk is to address the question : *what is known on the transcendence of periods ?*

Periods : Maxime Kontsevich and Don Zagier

A *period* is a complex number with real and imaginary parts given by absolutely convergent integrals of rational fractions with rational coefficients on domains of $\mathbf{R}^n$ defined by (in)equalities involving polynomials with rational coefficients

Periods, *Mathematics unlimited—2001 and beyond*, Springer 2001, 771–808.

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The number π

Basic example of a *period* :

$$e^{z+2i\pi} = e^z$$

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The exponential function

$$\frac{d}{dz}e^z = e^z, \qquad e^{z_1+z_2} = e^{z_1}e^{z_2}$$

$$\begin{array}{rccc} \exp: & \mathbf{C} & \to & \mathbf{C}^\times \\ & z & \mapsto & e^z \end{array}$$

$$\ker \exp = 2i\pi\mathbf{Z}.$$

The function $z \mapsto e^z$ is the exponential map of the multiplicative group $\mathbf{G}_m$.
The exponential map of the additive group $\mathbf{G}_a$ is

$$\begin{array}{ccc} \mathbf{C} & \to & \mathbf{C} \\ z & \mapsto & z \end{array}$$

The only period is 0.

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Elliptic curves and elliptic functions

Elliptic curves :

$$E = \left\{(t : x : y)\ ;\ y^2t = 4x^3 - g_2xt^2 - g_3t^3\right\} \subset \mathbf{P}_2(\mathbf{C}).$$

Elliptic functions

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

$$\wp(z_1 + z_2) = R\big(\wp(z_1), \wp(z_2)\big)$$

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Weierstraß elliptic function

$$\Omega = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2 \subset \mathbf{R}^2$$

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

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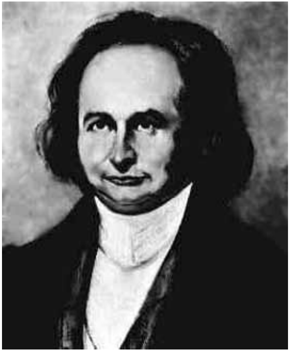
Weierstraß and Jacobi models

Weierstraß :

The function $\wp$

Jacobi :



The functions sn and cn

Periods of an elliptic function

The set of periods of an elliptic function is a *lattice* :

$$\Omega = \{\omega \in \mathbf{C}\ ;\ \wp(z+\omega) = \wp(z)\} = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2.$$

A pair of fundamental periods (ω_1, ω_2) is given by

$$\omega_i = \int_{e_i}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad (i = 1, 2)$$

where

$$4t^3 - g_2 t - g_3 = 4(t - e_1)(t - e_2)(t - e_3).$$

Examples

Example 1 : $g_2 = 4$, $g_3 = 0$, $j = 1728$

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$$y^2 t = 4x^3 - 4xt^2.$$

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$$\omega_1 = \int_1^\infty \frac{dt}{\sqrt{t^3 - t}} = \frac{1}{2} B(1/4, 1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}} = 2.6220575542\ldots$$

and

$$\omega_2 = i\omega_1.$$

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Examples (continued)

Example 2 : $g_2 = 0$, $g_3 = 4$, $j = 0$

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Euler Gamma and Beta functions

$$\begin{aligned}\Gamma(z) &= \int_0^\infty e^{-t} t^z \cdot \frac{dt}{t} \\ &= e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}.\end{aligned}$$

$$\begin{aligned}B(a,b) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \\ &= \int_0^1 x^{a-1}(1-x)^{b-1} dx.\end{aligned}$$

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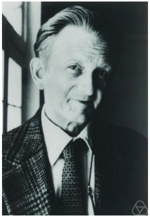

$$\sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} (m+ni)^{-4} = \frac{\Gamma(1/4)^8}{2^6\cdot 3\cdot 5\cdot \pi^2}$$

and

$$\sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} (m+n\varrho)^{-6} = \frac{\Gamma(1/3)^{18}}{2^8\pi^6}$$

Formula of Chowla and Selberg (1966) : *the periods of elliptic curves with complex multiplication are products of values of the Gamma function.*

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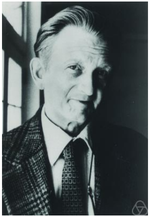

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Elliptic integrals and ellipses

An ellipse with radii a and b has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the length of its perimeter is

$$2\int_{-b}^{b} \sqrt{1 + \frac{a^2x^2}{b^4 - b^2x^2}}\ dx.$$

In the same way, the perimeter of a lemniscate

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

is given by an elliptic integral

$$4a\int_0^1 (1 - t^4)^{-1/2}\ dx.$$

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Hypergeometry and elliptic integrals

Gauss Hypergeometric series

$$ {}_2F_1\left(a,\ b\ ;\ c \mid z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} $$

with (Pochhammer rising factorial power)

$$ (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. $$

$$ \begin{aligned} K(z) &= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-z^2x^2)}} \\ &= \frac{\pi}{2} \cdot {}_2F_1\left(1/2,\ 1/2\ ;\ 1 \mid z^2\right). \end{aligned} $$

Elliptic integrals of the second kind

Quasi–periods of elliptic functions

Let $\Omega = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ be a lattice in $\mathbf{C}$. The *canonical product of Weierstraß* associated with Ω is the sigma function σ_Ω defined by

$$\sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)$$

This function has a simple zero at each point of Ω.

Hadamard canonical products

For $\mathbf{N} = \{0, 1, 2, \ldots\}$:

$$\frac{e^{-\gamma z}}{\Gamma(-z)} = z \prod_{n \geq 1} \left(1 - \frac{z}{n}\right) e^{-z/n}.$$

For $\mathbf{Z}$:

$$\frac{\sin \pi z}{\pi} = z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2}\right).$$

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Wallis formula for π

John Wallis (Arithmetica Infinitorum 1655)

$$\frac{\pi}{2} = \prod_{n\geq 1}\left(\frac{4n^2}{4n^2-1}\right) = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}.$$

Weierstraß sigma function

For $\mathbf{Z} + \mathbf{Z}i$:

$$\sigma_{\mathbf{Z}[i]}(z) = z \prod_{\omega \in \mathbf{Z}[i] \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right) \cdot$$

$$\sigma_{\mathbf{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2} = 0.4749493799\ldots$$

For $\alpha \in \mathbf{Q}(i)$, the number $\sigma_{\mathbf{Z}[i]}(\alpha)$ is algebraic over

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Weierstraß zeta function

The logarithmic derivative of the Weierstraß sigma function is the *Weierstraß zeta function*

$$\frac{\sigma'}{\sigma} = \zeta$$

and the derivative of ζ is $-\wp$. The minus sign is selected so that

$$\wp(z) = \frac{1}{z^2} + \text{a function analytic at } 0.$$

The fonction ζ is therefore *quasi–periodic* : for any $\omega \in \Omega$ there exists $\eta = \eta(\omega)$ such that

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Elliptic integrals of the third kind

Quasi–periodicity of the sigma Weierstraß function :

$$\sigma(z+\omega_i) = -\sigma(z)e^{\eta_i(z+\omega_i/2)} \quad (i=1,2).$$

J-P.Serre (1979) :
the function

$$F_u(z) = \frac{\sigma(z+u)}{\sigma(z)\sigma(u)}e^{-z\zeta(u)}$$

satisfies

$$F_u(z+\omega_i) = F_u(z)e^{\eta_i u-\omega_i\zeta(u)}.$$

Legendre relation

The numbers $\eta(\omega)$ are the *quasi–periods* of the elliptic curve.

When (ω_1, ω_2) is a pair of fundamental periods, we set $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$.

Legendre relation :

$$\omega_2\eta_1 - \omega_1\eta_2 = 2i\pi.$$

this is not Adrien Marie but Louis Legendre

Legendre and Fourier

Peter Duren, Changing Faces : The Mistaken Portrait of Legendre.
Notices of American Mathematical Society, December 2009.

Examples

For the curve $y^2t = 4x^3 - 4xt^2$ the quasi–periods associated to the previous fundamental periods are

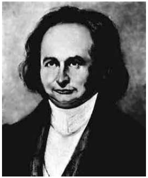
$$\eta_1 = \frac{\pi}{\omega_1} = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \qquad \eta_2 = -i\eta_1,$$

while for the curve $y^2t = 4x^3 - 4t^3$ they are

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} = \frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3}, \qquad \eta_2 = \varrho^2\eta_1.$$

Higher dimensions : abelian varieties

Abelian varieties,
abelian integrals,
theta functions.
Jacobian of an
algebraic curve.



Periods of the jacobian of a Fermat curve : values of Euler Beta function.
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For $n = 1$ and $n = 2$ the genus is 0.

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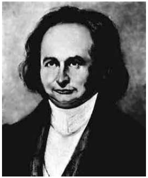

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Fermat curve $x^n + y^n = z^n$

For $n = 3$ the genus is 1 — elliptic curve with complex multiplication by the cubic roots of unity : $\Gamma(1/3)$.

For $n = 4$ the genus is 3 — product of three elliptic curves with complex multiplication by the fourth roots of unity $\mathbf{Q}(i)$: $\Gamma(1/4)$.

For $n = 5$ the genus is 6 — product of three simple abelian surfaces with CM having as field of endomorphisms the field of fifth roots of unity : $\Gamma(1/5)$.

Higher dimensions : commutative algebraic groups

Extensions of abelian varieties by the additive group (abelian integrals of the second kind) and by the multiplicative group (abelian integrals of the third kind).

Lie groups – exponential map, periods.

Further examples of periods

$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$

and all algebraic numbers.

$$\log 2 = \int_{1<x<2} \frac{dx}{x}$$

and all logarithms of algebraic numbers :

$$\log \alpha = \int_{1<x<\alpha,\ xy<1,\ y\geq 0} dxdy.$$

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$$\pi = \int_{x^2+y^2 \leq 1} dxdy,$$

$$\frac{\pi^2}{6} = \zeta(2) = \sum_{n \geq 1} \frac{1}{n^2} = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2}.$$

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$\zeta(s)$ is a period

For s an integer ≥ 2,

$$\zeta(s) = \int_{1>t_1>t_2\cdots>t_s>0} \frac{dt_1}{t_1} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s}.$$

Induction :

$$\int_{t_1>t_2\cdots>t_s>0} \frac{dt_2}{t_2} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s} = \sum_{n\geq 1} \frac{t_1^{n-1}}{n^{s-1}}.$$

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Numbers which are not periods

Problem (Kontsevich–Zagier) : *To produce an explicit example of a number which is not a period.*

Several levels :

$\boxed{1}$ *analog of Cantor* : the set of periods is countable. Hence there are real and complex numbers which are not periods ("most" of them).

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Numbers which are not periods

$\boxed{2}$ *analog of Liouville*

Find a property which should be satisfied by all periods, and construct a number which does not satisfies that property.

Masahiko Yoshinaga, Periods and elementary real numbers
arXiv:0805.0349

Compares the periods with hierarchy of real numbers induced from computational complexities.
In particular, he proves that periods can be effectively approximated by elementary rational Cauchy sequences.

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Prove that given numbers are not periods

Candidates : $1/\pi$, e, Euler constant.

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Relations among periods

$\boxed{1}$ Additivity

$$\int_a^b \big(f(x)+g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

$\boxed{2}$ Change of variables

$$\int_{\varphi(a)}^{\varphi(b)} f(t)dt = \int_a^b f\big(\varphi(u)\big)\varphi'(u)du.$$

Relations among periods

$\boxed{3}$ Newton–Leibniz–Stokes

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Conjecture of Kontsevich and Zagier

Periods,
Mathematics unlimited—
2001 and beyond,
Springer 2001, 771–808.

Conjecture (Kontsevich–Zagier). *If a period has two integral representations, then one can pass from one formula to another using only rules* $\boxed{1}$, $\boxed{2}$ *and* $\boxed{3}$ *in which all functions and domains of integration are algebraic with algebraic coefficients.*

Examples

$$\begin{aligned}
\pi &= \int_{x^2+y^2\leq 1} dxdy & &= 2\int_{-1}^{1} \sqrt{1-x^2}\, dx\\
&= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} & &= \int_{-\infty}^{\infty} \frac{dx}{1+x^2}\\
&= \frac{22}{7} - \int_0^1 \frac{x^4(1-x^4)dx}{1+x^2} & &= 4\int_0^1 \frac{dx}{1+x^2}.
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Dramatic consequences :

There is no new algebraic dependence relation among classical constants from analysis.

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Degree of a period, following Janming Wan

If p is a real period, Janming Wan defines the degree $\deg(p)$ of p as the minimal dimension of a domain Σ such that

$$p = \int_{\Sigma} 1,$$

where Σ is a domain in the Euclidean space given by polynomial inequalities with algebraic coefficients.

For any complex period $p = p_1 + ip_2$, he defines

$$\deg(p) = \max\{\deg(p_1), \deg(p_2)\}.$$

A complex number which is not a period has infinite degree.

Jianming Wan, arXiv:1102.2273 Degrees of periods

Degree of a period, following Janming Wan

Jianming Wan, arXiv:1102.2273 Degrees of periods

Theorem. *Let* p *be a period with* $\deg(p) \leq 2$*. Then the real and imaginary parts of* p *have the forms*

$$a \arctan \xi + b \log \eta + c,$$

where a, b, c, ξ, η *are algebraic numbers.*

Theorem. *Let* p_1, p_2 *be two complex numbers. If* $\deg(p_1) \neq \deg(p_2)$*, then* p_1 *and* p_2 *are linearly independent over the field of algebraic numbers.*

Rational approximation of real periods

Liouville (1844) : *for any algebraic irrational number* α*, there exist two constants* c *and* d *such that, for any rational number* p/q*, we have*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^d}.$$

Liouville numbers

A Liouville number is a number $x \in \mathbf{R}$ such that, for all $\kappa > 0$, there exists $p/q \in \mathbf{Q}$ with $q \geq 2$ satisfying

$$0 < \left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^\kappa}.$$

As a consequence, *a Liouville number is transcendental.*

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Rational approximation of periods

In *dynamical systems* theory, a Liouville number is a real number which does not satisfy a Diophantine condition.

Question. *Let θ be a real irrational period ; does there exist $c(\theta) > 0$ such that, for any rational number p/q with $q \geq 2$, the lower bound*

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{q^{c(\theta)}}$$

holds ?

In other words, it is expected that no period is a Liouville number (i.e. : no Liouville number is a period !).

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Lebesgue measure

A more ambitious goal would be to prove that real or complex periods behave, from the Diophantine approximation point of view, as almost all numbers for Lebesgue measure.

Diophantine approximation of periods

Question. *Given a transcendental period* $\theta \in \mathbf{C}$, *does there exist a constant* $\kappa(\theta)$ *such that, for any nonzero polynomial* $P \in \mathbf{Z}[X]$, *we have*

$$|P(\theta)| \geq H^{-\kappa(\theta)d},$$

where $H \geq 2$ *is an upper bound for the usual height of* P *(maximum of the absolute values of the coefficients) and* d *the degree of* P *?*

Hermite and Lindemann Theorems

Hermite (1873) :
transcendence of e.

Lindemann (1882) :
transcendence of π.

Theorem of Hermite–Lindemann
For any nonzero complex number z, at least one of the two numbers z, e^z is transcendental.

Corollaries : transcendence of $\log \alpha$ and e^β for α and β nonzero algebraic numbers with $\log \alpha \neq 0$.

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Hilbert seventh problem

For α and β algebraic numbers
with $\alpha \neq 0$ and $\beta \notin \mathbf{Q}$
and for any choice of $\log \alpha \neq 0$,
prove that the number

$$\alpha^\beta = \exp(\beta \log \alpha)$$

is transcendental.
Examples : $2^{\sqrt{2}}$, e^π.

http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Hilbert.html

Solution of Hilbert seventh problem

A.O. Gel'fond and Th. Schneider (1934).
Solution of Hilbert seventh problem :
transcendence of α^β

The two algebraically independent functions e^z and $e^{\beta z}$ cannot take algebraic values at the same point $\log \alpha$.

Transcendence of $(\log \alpha_1)/(\log \alpha_2)$ and $e^{\pi\sqrt{d}}$

Equivalent form of Gel'fond-Schneider Theorem :

Let $\log \alpha_1, \log \alpha_2$ *be two nonzero logarithms of algebraic numbers. Assume that the quotient* $(\log \alpha_1)/(\log \alpha_2)$ *is irrational. Then this quotient is transcendental.*

From the Theorem of Gel'fond-Schneider one deduces the transcendence of $2^{\sqrt{2}}$, e^{π}, $\log 2/\log 3$ and $e^{\pi\sqrt{d}}$ when d is a positive integer.

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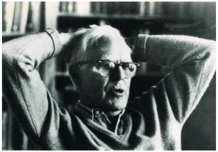
$e^\pi = (-1)^{-i}$

Example :

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743.999\,999\,999\,999\,250\,7\ldots$$

Martin Gardner,
Scientific American,
April 1, 1975.



Imaginary quadratic fields $\mathbf{Q}(\sqrt{-m})$ with class number 1 :

$$m = 1,\ 2,\ 3,\ 7,\ 11,\ 19,\ 43,\ 67,\ 163.$$

For

$$\tau = \frac{1+i\sqrt{163}}{2}, \quad q = e^{2i\pi\tau} = -e^{-\pi\sqrt{163}}$$

we have $j(\tau) = -640\ 320^3$ and

$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$

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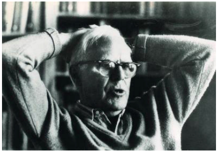

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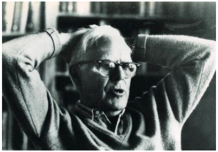

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Baker's Theorem

A. Baker, (1968). *Let* $\log\alpha_1, \ldots, \log\alpha_n$ *be* $\mathbf{Q}$*–linearly independent logarithms of algebraic numbers. Then the numbers* $1, \log\alpha_1, \ldots, \log\alpha_n$ *are linearly independent over the field* $\overline{\mathbf{Q}}$ *of algebraic numbers.*

Consequences of Baker's Theorem

Let $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ be nonzero algebraic numbers and for $1 \leq i \leq n$, let $\log \alpha_i$ be a complex logarithms of α_i. Then the number

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or else transcendental.

Famous example (considered by Siegel in 1949) : from Baker's Theorem, one deduces the transcendence of the number

$$\int_0^1 \frac{dt}{1+t^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right) \cdot$$

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$$\int_0^1 \frac{dt}{1+t^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right)\cdot$$

Genus zero

Corollary. *Let* P *and* Q *be polynomials with algebraic coefficients satisfying* $\deg P < \deg Q$ *and let* γ *be either a closed path, or else a path with limit points either algebraic numbers or infinity. If the integral*

$$\int_\gamma \frac{P(z)}{Q(z)} dz$$

exists, then its value is either rational or transcendental.

Proof.

Decompose the rational fraction $P(z)/Q(z)$ into simple elements.

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Van der Poorten

A. J. Van der Poorten.
On the arithmetic nature of definite integrals of rational functions.
Proc. Amer. Math. Soc. **29** 451–456 (1971).

Periods in genus zero

As a matter of fact, the corollary is equivalent to Baker's Theorem : write the logarithm of an algebraic number as a period. For instance, for the principal value of the logarithm, when α is not a real negative number, we have

$$\log \alpha = \int_0^\infty \frac{(\alpha - 1)dt}{(t+1)(\alpha t + 1)},$$

while

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The corresponding integrals are not Liouville numbers - explicit transcendence measures are also available.

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Transcendence of periods of elliptic integrals

Elliptic analog of Lindemann's Theorem on the transcendence of π.

Theorem (Siegel, 1932) : *If the invariants g_2 and g_3 of $\wp$ are algebraic, then at least one of the two numbers ω_1, ω_2 is transcendental.*

As a consequence, in the CM case, any nonzero period of $\wp$ is transcendental.

A. Thue, C.L. Siegel

Dirichlet's box principle

Thue-Siegel Lemma

Siegel's results on Gamma and Beta values

Consequence of Siegel's 1932 result :
both numbers

$$\Gamma(1/4)^4/\pi \quad \textit{and} \quad \Gamma(1/3)^3/\pi$$

are transcendental.
Ellipse :

$$2\int_{-b}^{b}\sqrt{1+\frac{a^2x^2}{b^4-b^2x^2}}\,dx$$

Transcendence of the perimeter of the lemniscate

$$(x^2+y^2)^2 = 2a^2(x^2-y^2)$$

Elliptic integrals of the first kind

1934 : solution of Hilbert's seventh problem by A.O. Gel'fond and Th. Schneider.

Schneider (1934) : *If the invariants* g_2 *and* g_3 *of* $\wp$ *are algebraic, then any nonzero period* ω *is a transcendental number*
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Transcendence of quasi–periods

Elliptic integrals of the second kind.
Pólya, Popken, Mahler (1935)

Schneider (1934) : *If the invariants g_2 and g_3 of $\wp$ are algebraic, then each of the numbers $\eta(\omega)$ with $\omega \neq 0$ is transcendental.*

Examples : the numbers

$$\Gamma(1/4)^4/\pi^3 \quad \text{and} \quad \Gamma(1/3)^3/\pi^2$$

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Periods of elliptic integrals of the third kind

Theorem (1979). *Assume* g_2, g_3, $\wp(u_1)$, $\wp(u_2)$, β *are algebraic and* $\mathbf{Z}u_1 \cap \Omega = \{0\}$. *Then the number*

$$\frac{\sigma(u_1+u_2)}{\sigma(u_1)\sigma(u_2)} e^{\bigl(\beta-\zeta(u_1)\bigr)u_2}$$

is transcendental.

Corollary. *Transcendence of periods of elliptic integrals of the third kind :*

$$e^{\omega\zeta(u)-\eta u+\beta\omega}.$$

Higher dimensions, several variables

Schneider (1937) : *If the invariants g_2 and g_3 of $\wp$ are algebraic and if α and β are nonzero algebraic numbers, then each of the numbers*

$$2i\pi/\omega_1, \quad \eta_1/\omega_1, \quad \alpha\omega_1 + \beta\eta_1$$

is transcendental.

Schneider (1948) : *for a and b in $\mathbf{Q}$ with a, b and $a+b$ not in $\mathbf{Z}$, the number*

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental.

The proof involves abelian integrals in higher genus, arising from the Jacobian of the Fermat curve.

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Baker's method

A. Baker (1969) :
transcendence of linear combinations with algebraic coefficients in

$$\omega_1, \quad \omega_2, \quad \eta_1 \quad \textit{and} \quad \eta_2.$$

Baker's method

J. Coates (1971) :
transcendence of linear combinations with algebraic coefficients in

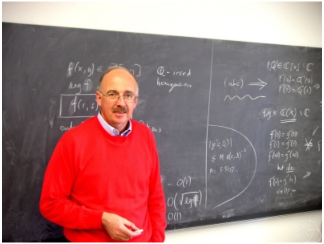
$$\omega_1, \quad \omega_2, \quad \eta_1, \quad \eta_2 \quad \textit{and} \quad 2i\pi.$$

Further, *in the non–CM case, the three numbers*

$$\omega_1, \quad \omega_2 \quad \textit{and} \quad 2i\pi$$

are $\overline{\mathbf{Q}}$*-linearly independent.*

Masser's work



D.W. Masser (1975) : *the six numbers*

$$1,\ \omega_1,\ \omega_2,\ \eta_1,\ \eta_2\ 2i\pi$$

span a $\overline{\mathbf{Q}}$*-vector space of dimension* 6 *in the CM case,* 4 *in the non–CM case :*

$$\dim_{\overline{\mathbf{Q}}}\{1,\omega_1,\omega_2,\eta_1,\eta_2,2i\pi\} = 2 + 2\dim_{\overline{\mathbf{Q}}}\{\omega_1,\omega_2\}.$$

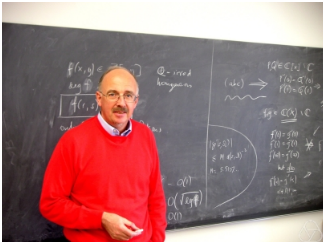
Further : *linear independence measures.*

Elliptic analog of Baker's Theorem

Linear independence over the field of algebraic numbers of elliptic logarithms :

Masser (1974) in the CM case.

Bertrand-Masser (1980) in the general case.



Bertrand–Masser

New proof of Baker's Theorem using functions of several variables in the case of Cartesian products.

The proof rests on Schneider's Criterion (1949), before the solution by Bombieri of a conjecture by Nagata 1970.

Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3. Let $u_1, \ldots, u_n$ be $\mathrm{End}(E)$–linearly independent complex numbers. Assume that for $1 \le i \le n$, either $u_i \in \Omega$ or else $\wp(u_i) \in \overline{\mathbf{Q}}$. Then the numbers $1, u_1, \ldots, u_n$ are $\overline{\mathbf{Q}}$–linearly independent.

Wüstholz's Theorem

G. Wüstholz (1987) – extension of the results by Schneider, Lang, Baker, Coates, Masser, Bertrand to abelian varieties and abelian integrals.

General result of linear independence on commutative algebraic groups (including the result of Baker corresponding to the special case of a product of multiplicative groups).

Wolfart and Wüstholz

Consequences (J. Wolfart and G. Wüstholz) dealing with the values of Euler Beta and Gamma functions : linear independence over the field of algebraic numbers of the values of Euler Beta function at rational points (a, b).

Transcendence of values at algebraic points of hypergeometric functions with rational parameters.

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Elliptic functions and algebraic independence

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The numbers π *and* $\Gamma(1/4)$
are algebraically independent.

Proof :

involves elliptic functions.

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Modular functions

1996, Yu. V. Nesterenko :

The three numbers
π, e^π *and* $\Gamma(1/4)$
are algebraically independent.

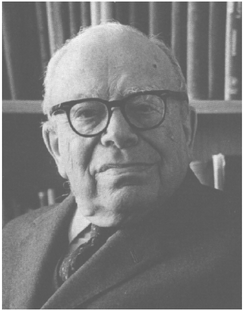
Proof :

involves modular functions.

Open problem :
Show that e *and* π *are algebraically independent.*

Irrationality measure for π

1953 : K. Mahler : π *is not a Liouville number*



1967 : K. Mahler :

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{42}} \quad \text{for } q \geq 2.$$

1974 : M. Mignotte :
exponent 20.6 for $q \geq 2$

1984 : D. and G. Chudnovsky : 14.65 for sufficiently large q.

1992 : M. Hata : 8.0161 for sufficiently large q.

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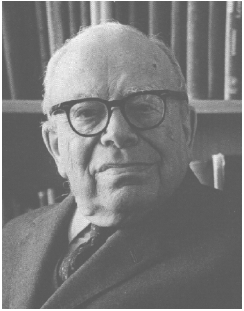

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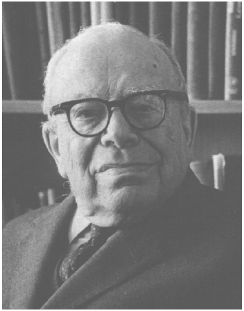

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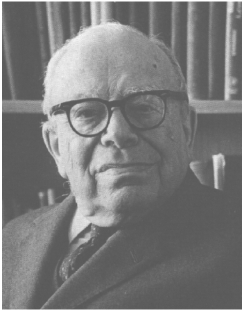

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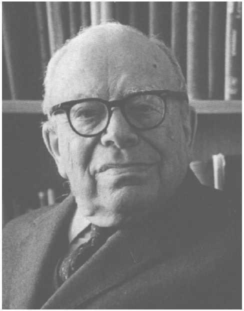

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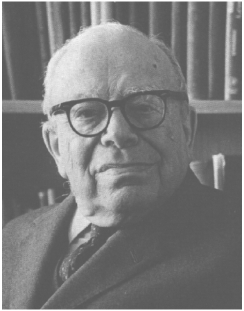

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Irrationality measure for e^π

It is not yet known that e^π is not a Liouville number :

$$\left| e^\pi - \frac{p}{q} \right| > \frac{1}{q^c} ?$$

Best known :

$$\left| e^\pi - \frac{p}{q} \right| > \frac{1}{q^{260 \log \log q}} \quad \text{for } q \geq 3.$$

(Baker's method)

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Irrationality measure for $\Gamma(1/4)$

(Chudnovsky's method)

1999, P. Philippon and S. Bruiltet : *The number* $\Gamma(1/4)$ *is not a Liouville number*

$$\left|\Gamma(1/4) - \frac{p}{q}\right| > \frac{1}{q^{10^{330}}}$$

for sufficiently large q.

Further open problems

Algebraic independence of the three numbers

$$\pi, \quad \Gamma(1/3), \quad \Gamma(1/4).$$

Algebraic independence of at least three numbers among

$$\pi, \quad \Gamma(1/5), \quad \Gamma(2/5), \quad e^{\pi\sqrt{5}}.$$

Faustin Adiceam : consequence of Nestenreko's Theorem using the Formula of Chowla and Selberg.

Algebraic independence of the three numbers π, $e^{\pi\sqrt{5}}$ *and* θ *where*

$$\theta = \Gamma(1/5)\,\Gamma(7/20)\,\Gamma(9/20).$$

Same result with

$$\theta = \frac{\Gamma(1/20)\,\Gamma(3/20)}{\Gamma(1/5)}.$$

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Standard relations among Beta values

(Translation) :

$$\Gamma(a+1) = a\Gamma(a)$$

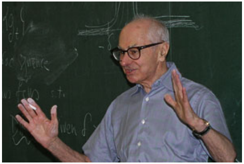
(Reflection) :

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

(Multiplication) : for any positive number n,

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)} \Gamma(na).$$

Conjectures of Rohrlich and Lang



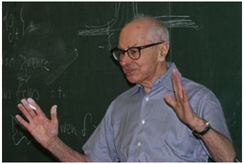
Conjecture (D. Rohrlich) *Any multiplicative relation*

$$\pi^{b/2} \prod_{a \in \mathbf{Q}} \Gamma(a)^{m_a} \in \overline{\mathbf{Q}}$$

with b *and* m_a *in* $\mathbf{Z}$ *is in the ideal generated by the standard relations.*

Conjecture (S. Lang) *Any algebraic dependence relation among* $(2\pi)^{-1/2}\Gamma(a)$ *with* $a \in \mathbf{Q}$ *is in the ideal generated by the standard relations* (universal odd distribution).

Conjectures of Rohrlich and Lang



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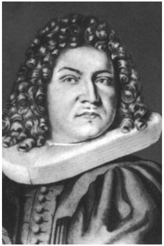
Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}$$

Euler : $s \in \mathbf{R}$.

Riemann : $s \in \mathbf{C}$.

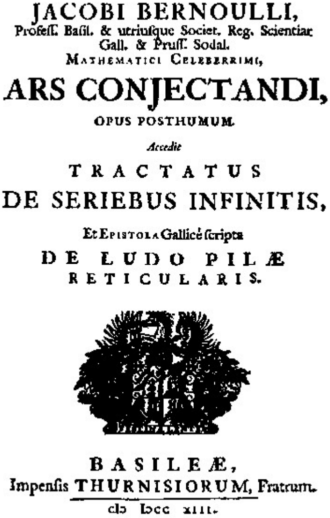
Special values of the Riemann zeta function



$s \in \mathbf{Z}$:
Jacques Bernoulli (1654–1705),
Leonard Euler (1739).

$\pi^{-2k}\zeta(2k) \in \mathbf{Q}$ *for* $k \geq 1$ (Bernoulli numbers).

Jacques Bernoulli (1654–1705)


JACOBI BERNOULLI,
Profeſſ. Baſil. & utriuſque Societ. Reg. Scientiar. Gall. & Pruſſ. Sodal.
MATHEMATICI CELEBERRIMI,
ARS CONJECTANDI,
OPUS POSTHUMUM.
Accedit
TRACTATUS
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DE LUDO PILÆ
RETICULARIS.

BASILEÆ,
Impenſis THURNISIORUM, Fratrum.
cIↄ Iↄcc xiii.

Values of Riemann zeta function at the positive integers

Even positive integers

$$\boxed{\zeta(2n) = (-1)^{n-1} 2^{2n-1} \frac{B_{2n}}{(2n)!} \pi^{2n} \quad (n \geq 1).}$$

Odd positive integers : $\zeta(2n+1)$, $n \geq 1$?

Question : *for $n \geq 1$, is the number*

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Diophantine question

Determine all algebraic relations among the numbers

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7), \ldots$$

Conjecture. *there is no algebraic relation : the numbers*

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are algebraically independent.

As a consequence, one expects the numbers $\zeta(2n+1)$ and $\zeta(2n+1)/\pi^{2n+1}$ for $n \geq 1$ to be transcendental.

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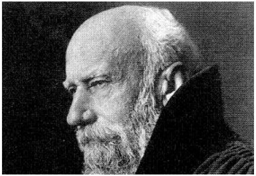
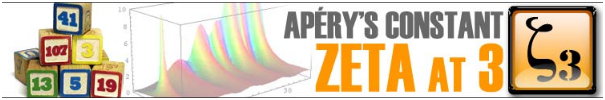
Values of ζ at the even positive integers

• F. Lindemann : π *is a transcendental number, hence* $\zeta(2k)$ *also for* $k \geq 1$.



Values of ζ at the odd positive integers



- Apéry (1978) : *The number*

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511\ \ldots$$

is irrational.

- Rivoal (2000) + Ball, Zudilin, Fischler,... *Infinitely many numbers among* $\zeta(2k+1)$ *are irrational + lower bound for the dimension of the* $\mathbf{Q}$*-space they span.*

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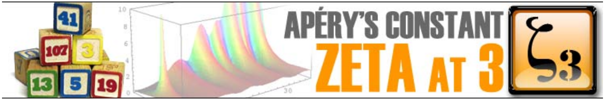

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Tanguy Rivoal

Let $\epsilon > 0$. For any sufficiently large odd integer a, the dimension of the $\mathbf{Q}$–space spanned by the numbers $1,\ \zeta(3),\ \zeta(5),\ \cdots,\zeta(a)$ is at least

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

Wadim Zudilin

- *At least one of the four numbers*
  $$\zeta(5), \quad \zeta(7), \quad \zeta(9), \quad \zeta(11)$$
  is irrational.

- *There exists an odd number j in the interval $[5, 69]$ such that the three numbers*
  $$1, \zeta(3), \zeta(j)$$
  are $\mathbf{Q}$–linearly independent.

Wadim Zudilin

- *At least one of the four numbers*
  $\zeta(5), \quad \zeta(7), \quad \zeta(9), \quad \zeta(11)$
  is irrational.

- *There exists an odd number j in the interval $[5, 69]$ such that the three numbers*
  $1, \zeta(3), \zeta(j)$
  are $\mathbf{Q}$–linearly independent.

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Linearization of the problem *(Euler)*

The product of two special values of the Riemann zeta function is a linear combination of *multizeta values.*

$$\sum_{n_1\geq 1} n_1^{-s_1} \sum_{n_2\geq 1} n_2^{-s_2} = \sum_{n_1>n_2\geq 1} n_1^{-s_1} n_2^{-s_2} + \sum_{n_2>n_1\geq 1} n_2^{-s_2} n_1^{-s_1} + \sum_{n\geq 1} n^{-s_1-s_2}$$

Multizeta values

One deduces, for $s_1 \geq 2$ and $s_2 \geq 2$,

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

with

$$\zeta(s_1, s_2) = \sum_{n_1 > n_2 \geq 1} n_1^{-s_1} n_2^{-s_2}.$$

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

$$\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5)$$
$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$$

Relation among divergent series

$$\zeta(1)\zeta(2) = \zeta(1,2) + \zeta(2,1) + \zeta(3).$$

$\zeta(1)$ and $\zeta(1,2)$ are divergent series

$$\zeta(1) = \sum_{n\geq 1} \frac{1}{n} \quad \text{and} \quad \zeta(1,2) = \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1 n_2^2}.$$

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For k, $s_1, \ldots, s_k$ positive integers satisfying $s_1 \geq 2$, one sets $\underline{s} = (s_1, \ldots, s_k)$ and

$$\zeta(\underline{s}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$

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k is the *depth* and $p = s_1 + \cdots + s_k$ the *weight*.

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The algebra of multizeta

The product of two multizeta values is a multizeta value.

The $\mathbf{Q}$–space spanned by the $\zeta(\underline{s})$ is also a $\mathbf{Q}$–algebra.

The problem of algebraic independence is reduced to a problem of linear independence.

Question : which are the linear relations among these numbers ?

Answer : *there are plenty of linear relations !*

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$\zeta(2, 2, \ldots, 2)$

For $k \geq 1$, set $\{2\}_k = (2, 2, \ldots, 2)$ (with k terms). We have

$$\zeta(\{2\}_k) = \frac{\pi^{2k}}{(2k+1)!}.$$

Hence $\zeta(\{2\}_k)/\zeta(2k) \in \mathbf{Q}$.

Examples.

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(2,2) = \frac{\pi^4}{120}, \quad \zeta(2,2,2) = \frac{\pi^6}{5\,040}.$$

Proof :

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n \geq 1}\left(1 - \frac{z^2}{n^2}\right) = \sum_{k \geq 0} \zeta(\{2\}_k)(-z^2)^k.$$

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The multizeta values are periods

$$\zeta(2,1) = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} \cdot \frac{dt_3}{1-t_3}.$$

Proof.
We have

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Conjecture of Zagier

Let $\mathfrak{Z}_p$ be the $\mathbf{Q}$-subspace of $\mathbf{R}$ spanned by the numbers $\zeta(\underline{s})$ where $\underline{s}$ has weight $s_1 + \cdots + s_k = p$, with $\mathfrak{Z}_0 = \mathbf{Q}$ and $\mathfrak{Z}_1 = \{0\}$.
Let d_p be the dimension of $\mathfrak{Z}_p$.

Conjecture (Zagier). *For $p \geq 3$, we have*

$$d_p = d_{p-2} + d_{p-3}.$$

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Zagier's conjecture can be stated as

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M. Kaneko, M. Noro and K. Tsurumaki. – *On a conjecture for the dimension of the space of the multiple zeta values*, Software for Algebraic Geometry, IMA **148** (2008), 47–58.

It is not yet proved that there exists p with $d_p \geq 2$.

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Upper bound for the dimension

A.B. Goncharov – *Multiple ζ-values, Galois groups and Geometry of Modular Varieties.* Birkhäuser. Prog. Math. **201**, 361-392 (2001).
T. Terasoma – *Mixed Tate motives and Multiple Zeta Values.* Invent. Math. **149**, No. 2, 339-369 (2002).

Theorem. The numbers given by Zagier's Conjecture $d_p = d_{p-2} + d_{p-3}$ with initial conditions $d_0 = 1$, $d_1 = 0$ are actually *upper bounds* for the dimension of $\mathfrak{Z}_p$.

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Francis Brown

arXiv:1102.1310 On the decomposition of motivic multiple zeta values

We review motivic aspects of multiple zeta values, and as an application, we give an exact-numerical algorithm to decompose any (motivic) multiple zeta value of given weight into a chosen basis up to that weight.

arXiv:1102.1312 Mixed Tate motives over $\mathbf{Z}$

We prove that the category of mixed Tate motives over $\mathbf{Z}$ is spanned by the motivic fundamental group of Pro^1 minus three points. We prove a conjecture by M. Hoffman which states that every multiple zeta value is a $\mathbf{Q}$-linear combination of $\zeta(n_1, \ldots, n_r)$ where $n_i \in \{2, 3\}$.

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Transcendence of Periods.

Michel Waldschmidt

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