Ramanujan Institute, Chennai
S.S. Pillai endowment lecture
January 12, 2010

Perfect Powers : Pillai's works and their developments

Michel Waldschmidt

Institut de Mathématiques de Jussieu & Paris VI
http ://www.math.jussieu.fr/~miw/

S.Sivasankaranarayana Pillai (1901–1950)

http ://www.geocities.com/thangadurai_kr/PILLAI.html

Collected works of S. S. Pillai,
ed. R. Balasubramanian and R. Thangadurai, 2010.

On m consecutive integers (number theory)

- Any two consecutive integers are relatively prime.

- Consider three consecutive integers

for 3, 4, 5 : any two of them are relatively prime

for 2, 3, 4 : only 3 is prime to 2 and to 4.

In the general case n, $n+1$, $n+2$, the middle term is relatively prime to each other.

- Given four consecutive integers n, $n+1$, $n+2$, $n+3$, the odd number among $n+1$, $n+2$ is relatively prime to the three remaining integers. Hence one at least of the four numbers is relatively prime to the three others.

On m consecutive integers

- Given five consecutive integers

$$n,\ n+1,\ n+2,\ n+3,\ n+4$$

the only possible common prime factors between two of them are 2 and 3 , and one at least of the odd elements is not divisible by 3. Hence again one at least of the five numbers is relatively prime to the four others.

- After 2, 3, 4, 5, continue with $6, 7, 8\ldots$ up to 16 – done by S.S. Pillai in 1940.

On 17 consecutive integers (S.S. Pillai, 1940)

• In every set of not more than 16 consecutive integers there is a number which is prime to all the others.

• This is not true for 17 consecutive numbers : take $n = 2184$ and consider the 17 consecutive integers $2184, \dots, 2200$. Then any two of them have a gcd > 1.

• One produces infinitely many such sets of 17 consecutive numbers by taking

$$n+N, n+N+1, \dots, n+N+16$$

or

$$N-n-16, n-N-15, \dots, N-n$$

where N is a multiple of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30\,030$.

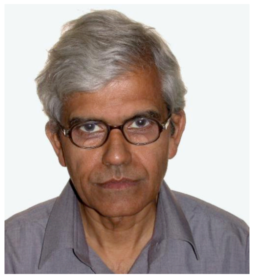
Application to a Diophantine equation

$$n(n+1)\cdots(n+m-1) = y^r$$

No solution n, y when $2 \le m \le 16$ and $r \ge (m+3)/2$.

For any $r \ge 3$ there is at most finitely many solutions.

For $m \ge 2$ and $r \ge c(m)$, there is no solution.



More recent work, esp. by T.N. Shorey

Waring's Problem

Edward Waring (1736 - 1798)

In 1770, a few months before J.L. Lagrange solved a conjecture of Bachet (1621) and Fermat (1640) by proving that every positive integer is the sum of at most four squares of integers, E. Waring wrote :

"Every integer is a cube or the sum of two, three, ...nine cubes ; every integer is also the square of a square, or the sum of up to nineteen such ; and so forth. Similar laws may be affirmed for the correspondingly defined numbers of quantities of any like degree."

Waring's functions $g(k)$ and $G(k)$

• Waring's function g is defined as follows : *For any integer $k \ge 2$, $g(k)$ is the least positive integer s such that any positive integer N can be written $x_1^k + \cdots + x_s^k$.*

• Waring's function G is defined as follows : *For any integer $k \ge 2$, $G(k)$ is the least positive integer s such that any sufficiently large positive integer N can be written $x_1^k + \cdots + x_s^k$.*

David Hilbert (1909)

David Hilbert
(1862 - 1943)

$g(k)$ and $G(k)$ are finite

$G(k) \leq g(k)$.

$g(2) = G(2) = 4$

Joseph-Louis Lagrange
(1736–1813)

Solution of a conjecture of Bachet and Fermat in 1770 :

Every positive integer is the sum of at most four squares of integers.

No integer congruent to -1 modulo 8 can be a sum of three squares of integers.

Sums of squares modulo 8

$x \equiv$	0	1	2	3	4	5	6	7
$x^2 \equiv$	0	1	4	1	0	1	4	1

A square is congruent to 0, 1 or 4 modulo 8.

Sums : $0+0$, $0+1$, $1+1$, $0+4$, $1+4$, $4+4$.

A sum of two squares is congruent to 0, 1 , 2, 4 or 5 modulo 8.

A sum of three squares is not congruent to 7 modulo 8.

$n = x_1^4 + \cdots + x_g^4 : g(4) = 19$

Any positive integer is the sum of at most 19 biquadrates
R. Balasubramanian, J-M. Deshouillers, F. Dress (1986).

Previous estimates for $g(4)$

$g(4) \leq 53$ (J. Liouville, 1859)

$g(4) \leq 47$ (S. Réalis, 1878)

$g(4) \leq 45$ (É. Lucas, 1878)

$g(4) \leq 41$ (É. Lucas, 1878)

$g(4) \leq 39$ (A. Fleck, 1906)

$g(4) \leq 38$ (E. Landau, 1907)

$g(4) \leq 37$ (A. Wieferich, 1909)

$g(4) \leq 35$ (L.E. Dickson, 1933)

$g(4) \leq 22$ (H.E. Thomas, 1973)

$g(4) \leq 21$ (R. Balasubramanian, 1979)

$g(4) \leq 20$ (R. Balasubramanian, 1985)

$n = x_1^4 + \cdots + x_G^4$: $G(4) = 16$

Kempner (1912) $G(4) \geq 16$
$16^m \cdot 31$ need at least 16 biquadrates

Hardy Littlewood (1920)
$G(4) \leq 21$
circle method, singular series

Davenport, Heilbronn, Esterman (1936) $G(4) \leq 17$

Harold Davenport
(1907 - 1969)

Davenport (1939) $G(4) = 16$

Circle method

Srinivasa Ramanujan
(1887 – 1920)

G.H. Hardy
(1877 – 1947)

J.E. Littlewood
(1885 – 1977)

Hardy, ICM Stockholm, 1916
Hardy and Ramanujan (1918) : partitions
Hardy and Littlewood (1920 – 1928) :
Some problems in Partitio Numerorum

On Waring's Problem : $g(6) = 73$

S.S. Pillai, 1940.

- **Any positive integer N is sum of at most 73 sixth powers : $N = x_1^6 + \cdots + x_s^6$ with $s \leq 73$.**

- Since $2^6 = 64$, the integer $N = 63 = 1^6 + \cdots + 1^6$ requires at least 63 terms x_i.

- Any decomposition of an integer $N \leq 728 = 3^6 - 1$ as a sum of sixth powers involves only 1 and 2^6.

- The decomposition as a sum of sixth powers of any integer $N \leq 728$ of the form $63 + k64$ requires at least $63 + k$ terms.

- The number $703 = 63 + 64 \times 10$ requires $63 + 10 = 73$ terms.

Previous estimates for $g(6)$

$g(6) \le 970$ (Kempner, 1912)

$g(6) \le 478$ (Baer, 1913)

$g(6) \le 183$ (James, 1934)

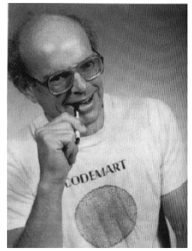
$g(6) \le 73$ (Pillai, 1940)

Results on Waring's Problem

$g(2) = 4$	J-L. Lagrange	(1770)
$g(3) = 9$	A. Wieferich	(1909)
$g(4) = 19$	R. Balasubramanian, J-M. Deshouillers, F. Dress	(1986)
$g(5) = 37$	Chen Jing Run	(1964)
$g(6) = 73$	S.S. Pillai	(1940)
$g(7) = 143$	L.E. Dickson	(1936)

Sequence of values of $g(k)$

1, 4, 9, 19, 37, 73, 143, 279, 548, 1079, 2132, 4223, 8384, 16673, 33203, 66190, 132055, 263619, 526502, 1051899, 2102137, 4201783, 8399828, 16794048, 33579681, 67146738, 134274541, 268520676, 536998744, 1073933573, 2147771272 . . .



Neil J. A. Sloane's encyclopaedia
http ://www.research.att.com/∼njas/sequences/A002804

The ideal Waring's Theorem

For each integer $k \ge 2$, define $I(k) = 2^k + [(3/2)^k] - 2$. It is easy to show that $g(k) \ge I(k)$. Indeed, write

$$3^k = 2^k q + r \quad \text{with} \quad 0 < r < 2^k, \quad q = [(3/2)^k],$$

and consider the integer

$$N = 2^k q - 1 = (q-1)2^k + (2^k - 1)1^k.$$

Since $N < 3^k$, writing N as a sum of k-th powers can involve no term 3^k, and since $N < 2^k q$, it involves at most $(q-1)$ terms 2^k, all others being 1^k ; hence it requires a total number of at least $(q-1) + (2^k - 1) = I(k)$ terms.

The ideal Waring's Theorem

L.E. Dickson and S.S. Pillai proved independently in 1936 that $g(k) = I(k)$, provided that $r = 3^k - 2^k q$ satisfies

$$r \leq 2^k - q - 2.$$

The condition $r \leq 2^k - q - 2$ is satisfied for $3 \leq k \leq 471\,600\,000$.
The conjecture, dating back to 1853, is $g(k) = I(k) = 2^k + [(3/2)^k] - 2$ for any $k \geq 2$. This is true as soon as

$$\left\| \left(\frac{3}{2}\right)^k \right\| \geq \left(\frac{3}{4}\right)^k,$$

where $\|\cdot\|$ denote the distance to the nearest integer.

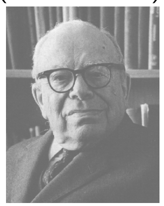
Mahler's contribution

- The estimate

$$\left\| \left(\frac{3}{2}\right)^k \right\| \geq \left(\frac{3}{4}\right)^k$$

is valid for all sufficiently large k.

Kurt Mahler
(1903 - 1988)



Hence the ideal Waring Theorem

$$g(k) = 2^k + [(3/2)^k] - 2$$

holds for all sufficiently large k.

Waring's function $G(k)$

- Recall that Waring's function G is defined as follows : *For any integer $k \geq 2$, $G(k)$ is the least positive integer s such that any sufficiently large positive integer N can be written $x_1^k + \cdots + x_s^k$.*

- $G(k)$ is known only in two cases : $G(2) = 4$ and $G(4) = 16$.

$G(k)$

The only values of $G(k)$ which are known are $G(2) = 4$ and $G(4) = 16$.

Yu. V. Linnik (1943) $g(3) = 9$, $G(3) \leq 7$.

Other estimates for $G(k)$, $k \geq 5$: Davenport, K. Sambasiva Rao, V. Narasimhamurti, K. Thanigasalam , R.C. Vaughan...

The state of the art for $G(k)$

$$G(2) = 4, \; G(4) = 16$$

$4 \le G(3) \le 7$	$14 \le G(13) \le 84$
$6 \le G(5) \le 17$	$15 \le G(14) \le 92$
$9 \le G(6) \le 21$	$16 \le G(15) \le 100$
$8 \le G(7) \le 33$	$64 \le G(16) \le 109$
$32 \le G(8) \le 42$	$18 \le G(17) \le 117$
$13 \le G(9) \le 50$	$27 \le G(18) \le 125$
$12 \le G(10) \le 59$	$20 \le G(19) \le 134$
$12 \le G(11) \le 67$	$25 \le G(20) \le 142$
$16 \le G(12) \le 76$	

On Waring's Problem with exponents $\ge n$

S.S. Pillai, 1940.

• For any integer $n \ge 2$, denote by $g_2(n)$ the least positive integer s such that any positive integer N can be written $x_1^{m_1} + \cdots + x_s^{m_s}$ with $m_i \ge n$.
S.S. Pillai (1940) : explicit formula for $g_2(n)$, $n \ge 32$.

• Proof of the lower bound $g(n) \ge 2^n + h - 1$ if $2^{n+h} \le 3^n$.

Lower bound for $g_2(n)$

• The lower bound $g_2(n) \ge 2^n - 1$ is trivial : take $N = 2^n - 1$.
• Any decomposition $N = x_1^{m_1} + \cdots + x_s^{m_s}$ with $m_i \ge n$ of a positive integer $N < 3^n$ has $x_i \in \{1, 2\}$.
• Let $h \ge 1$ satisfy $2^{n+h} \le 3^n$. Consider the integer $N = 2^{n+h} - 1$. Its binary expansion is

$$N = 2^{n+h-1} + 2^{n+h-2} + \cdots + 2 + 1,$$

hence it can be written

$$N = 2^{n+h-1} + 2^{n+h-2} + \cdots + 2^n + (2^n - 1),$$

which is a sum of h numbers 2^m with $m \ge n$ and $2^n - 1$ powers of 1.

Value of $g_2(n)$ for $n \ge 32$

One easily deduces $g_2(n) \ge 2^n + h - 1$ as soon as h satisfies $2^{n+h} \le 3^n$.
This condition on h is $2^h \le (3/2)^n$, which means $2^h \le I_n$ with $I_n = [(3/2)^n]$.
Define

$$h_n = [\log I_n / \log 2] \quad \text{where} \quad I_n = [(3/2)^n].$$

Pillai's Theorem : *For* $n \ge 32$, $g_2(n) = 2^n + h_n - 1$.

Square, cubes. . .

- A perfect power is an integer of the form a^b where $a \geq 1$ and $b > 1$ are positive integers.
- Squares :

$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196 \ldots\ldots$

- Cubes :

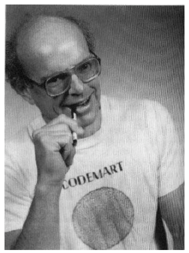
$1, 8, 27, 64, 125, 216, 343, 512, 729, 1\,000, 1\,331 \ldots$

- Fifth powers :

$1, 32, 243, 1\,024, 3\,125, 7\,776, 16\,807, 32\,768 \ldots$

Perfect powers

$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, 196, 216, 225, 243, 256, 289, 324, 343, 361, 400, 441, 484, 512, 529, 576, 625, 676, 729, 784 \ldots$



Neil J. A. Sloane's encyclopaedia
http ://www.research.att.com/~njas/sequences/A001597

Pillai's early work

In 1936 Pillai proved that for any fixed positive integers a and b, both at least 2, the number of solutions (x, y) of the Diophantine inequality $0 < a^x - b^y \leq c$ is asymptotically equal to

$$\frac{(\log c)^2}{2 \log a \log b}$$

as c tends to infinity.

References :

Pillai, S. S. – *On some Diophantine equations*, J. Indian Math. Soc., XVIII (1930), 291-295.
Pillai, S. S. – *On $A^x - B^y = C$*, J. Indian Math. Soc. (N.S.), II (1936), 119–122.

Connexion with some of Ramanujan's work

It is remarkable that this asymptotic value is related to another problem which Pillai studied later and which originates in the following claim by Ramanujan :

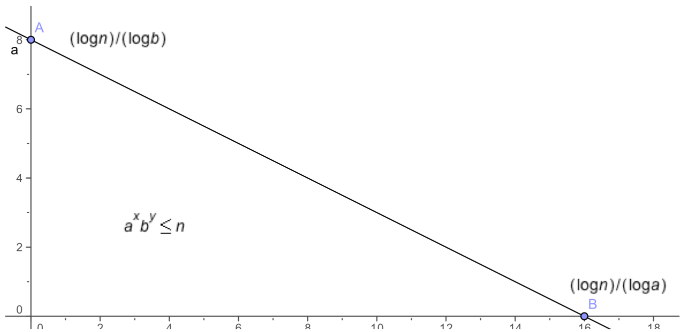
The number of numbers of the form $2^u \cdot 3^v$ less than n is

$$\frac{\log(2n)\log(3n)}{2 \log 2 \log 3}.$$

Number of integers $a^u b^v \leq n$

The number of numbers of the form $a^u \cdot b^v$ less than n is asymptotically

$$\frac{(\log n)^2}{2 \log a \log b}.$$



Perfect powers

The sequence of perfect powers starts with :
Write the sequence of perfect powers

$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125,$
$128, 144, 169, 196, 216, 225, 243, 256, 289, 324, 343,$
$361, 400, 441, 484, 512, 529, 576, 625, 676, 729, 784 \ldots$

as

$a_1 = 1,\ a_2 = 4,\ a_3 = 8,\ a_4 = 9,\ a_5 = 16,\ a_6 = 25,\ a_7 = 27,\ \ldots$

Taking only the squares into account, we deduce

$$a_n \leq n^2 \quad \text{for all} \quad n \geq 1.$$

Lower bound for a_n

We want also a lower bound for a_n. For this we need an upper bound for the number of perfect powers a^x bounded by a_n which are not squares. We do it in a crude way : if $a^x \leq N$ with $a \geq 2$ and $x \geq 3$ then $x \leq (\log N)/(\log 2)$ and $a \leq N^{1/3}$, hence the number of such a^x is less than

$$\frac{1}{\log 2} \cdot N^{1/3} \log N.$$

Hence the number of elements in the sequence of perfect powers which are less than N is at most

$$\sqrt{N} + \frac{1}{\log 2} \cdot N^{1/3} \log N.$$

The sequence of perfect powers

The upper bound

$$n \leq \sqrt{a_n} + \frac{1}{\log 2} \cdot a_n^{1/3} \log a_n$$

together with $a_n \geq n^2$ yields

$$a_n \geq n^2 - \frac{2}{\log 2} \cdot n^{2/3} \log n,$$

and one checks that this estimate is true as soon as $n \geq 8$.
As a consequence

$$\limsup (a_{n+1} - a_n) = +\infty.$$

Consecutive elements in the sequence of perfect powers

- Difference 1 : $(8, 9)$
- Difference 2 : $(25, 27)$
- Difference 3 : $(1, 4)$, $(125, 128)$
- Difference 4 : $(4, 8)$, $(32, 36)$, $(121, 125)$
- Difference 5 : $(4, 9)$, $(27, 32)$

Two conjectures

Eugène Charles Catalan (1814 – 1894)

Subbayya Sivasankaranarayana Pillai (1901-1950)

- Catalan's Conjecture : In the sequence of perfect powers, $8, 9$ is the only example of consecutive integers.
- Pillai's Conjecture : In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.

Pillai's Conjecture :

- Pillai's Conjecture : In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.
- Alternatively : Let k be a positive integer. The equation

$$x^p - y^q = k,$$

where the unknowns x, y, p and q take integer values, all ≥ 2, has only finitely many solutions (x, y, p, q).

Pillai's conjecture

Pillai, S. S. – *On the equation* $2^x - 3^y = 2^X + 3^Y$, Bull. Calcutta Math. Soc. 37, (1945). 15–20.

I take this opportunity to put in print a conjecture which I gave during the conference of the Indian Mathematical Society held at Aligarh.

Arrange all the powers of integers like squares, cubes etc. in increasing order as follows :

$1,\ 4,\ 8,\ 9,\ 16,\ 25,\ 27,\ 32,\ 36,\ 49,\ 64,\ 81,\ 100,\ 121,\ 125,\ 128, \dots$

Let a_n *be the* n*-th member of this series so that* $a_1 = 1$, $a_2 = 4$, $a_3 = 8$, $a_4 = 9$, *etc. Then*

Conjecture :

$$\liminf (a_n - a_{n-1}) = \infty.$$

Indian Science Congress 1949

"The audience may be a little disappointed at the scanty reference to Indian work. ⋯ However, we need not feel dejected. Real research in India started only after 1910 and India has produced Ramanujan and Raman"

This was the statement of Dr. S. Sivasankaranarayana Pillai in the 36th Annual session of the Indian Science Congress on 3rd January, 1949 at Allahabad university.

http ://www.geocities.com/thangadurai_kr/PILLAI.html

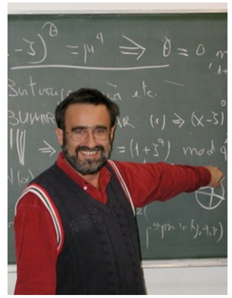
http ://www.geocities.com/thangadurai_kr/PILLAI.html

The tragic end

For his achievements, he was invited to visit the Institute of Advance Studies, Princeton, USA for a year. Also, he was invited to participate in the International Congress of Mathematicians at Harvard University as a delegate of Madras University. So, he proceeded to USA by air in the august 1950. But due to the air crash near Cairo on August 31, 1950, Indian Mathematical Community lost one of the best known mathematicians.

Results

P. Mihăilescu, 2002.



Catalan was right : *the equation $x^p - y^q = 1$ where the unknowns x, y, p and q take integer values, all ≥ 2, has only one solution $(x, y, p, q) = (3, 2, 2, 3)$.*

Previous partial results : J.W.S. Cassels, R. Tijdeman, M. Mignotte. . .

Higher values of k

There is no value of $k \geq 2$ for which one knows that Pillai's equation $x^p - y^q = k$ has only finitely many solutions.

We expect much more than Pillai's Conjecture :

$$|x^p - y^q| \geq c(\epsilon) \max\{x^p, y^q\}^{\kappa-\epsilon}$$

with

$$\kappa = 1 - \frac{1}{p} - \frac{1}{q}.$$

This estimate is a consequence of the *abc* conjecture.

The *abc* Conjecture

- For a positive integer n, we denote by

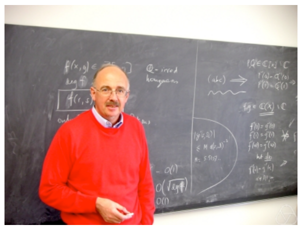
$$R(n) = \prod_{p|n} p$$

the *radical* or *square free part* of n.

- Conjecture (abc Conjecture). *For each $\varepsilon > 0$ there exists $\kappa(\varepsilon)$ such that, if a, b and c in $\mathbf{Z}_{>0}$ are relatively prime and satisfy $a + b = c$, then*

$$c < \kappa(\varepsilon) R(abc)^{1+\varepsilon}.$$

The *abc* Conjecture of Œsterlé and Masser



The *abc* Conjecture resulted from a discussion between D. W. Masser and J. Œsterlé in the mid 1980's.

Beal Equation $x^p + y^q = z^r$

Assume

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

and x, y, z are relatively prime.

Only 10 solutions (up to obvious symmetries) are known

$$1 + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2,$$

$$3^5 + 11^4 = 122^2, \quad 17^7 + 76271^3 = 21063928^2,$$

$$1414^3 + 2213459^2 = 65^7, \quad 9262^3 + 15312283^2 = 113^7,$$

$$43^8 + 96222^3 = 30042907^2, \quad 33^8 + 1549034^2 = 15613^3.$$

Beal Conjecture and prize problem

"Fermat-Catalan" Conjecture H. (Darmon and A. Granville) : *the set of solutions to $x^p + y^q = z^r$ with $(1/p) + (1/q) + (1/r) < 1$ is finite.*

Consequence of the *abc* Conjecture. `Hint` :

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \quad \text{implies} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le \frac{41}{42}.$$

R. Tijdeman, D. Zagier and A. Beal Conjecture : *there is no solution to $x^p + y^q = z^r$ where each of p, q and r is ≥ 3.*

R. D. Mauldin, *A generalization of Fermat's last theorem : the Beal conjecture and prize problem*, Notices Amer. Math. Soc., 44 (1997), pp. 1436–1437.

Collatz equation (Syracuse Problem)

Iterate

$$n \longmapsto \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Lothar Collatz (1937) : does the process converge to the cycle $(4,2,1)$?

Example related to the *abc* conjecture :

$$109 \cdot 3^{10} + 2 = 23^5$$

Continued fraction of $109^{1/5}$: $[2;1,1,4,77733,\dots]$, approximation $23/9$.

N. A. Carella. *Note on the ABC Conjecture*
http ://arXiv.org/abs/math/0606221

Waring's Problem and the *abc* Conjecture

S. David : the estimate

$$\left\| \left(\frac{3}{2}\right)^k \right\| \geq \left(\frac{3}{4}\right)^k$$

for sufficiently large k follows from the *abc* Conjecture.

Hence the ideal Waring Theorem $g(k) = 2^k + [(3/2)^k] - 2$ would follow from an explicit solution of the *abc* Conjecture.

Pillai's work on normal numbers

In 1939 and 1940, S.S. Pillai considered the number obtained by the concatenation of the sequence of integers

$$0.\,1\;10\;11\;100\;101\;110\;111\;1000\;1001\;1010\;1011\;1100\;\dots$$

In other words

$$= \sum_{k\geq 1} k 2^{-c_k} \quad \text{with} \quad c_k = k + \sum_{j=1}^{k} [\log_2 j].$$

He proved that each of the two digit 0 and 1 occurs with frequency $1/2$, each of the four sequences of digits 00, 01, 10 and 11 occurs with frequency $1/4$, and more generally each sequence of n digits occurs with the same frequency $1/2^n$.

Champernowne numbers in binary or decimal basis

In decimal basis, the number

$$0.1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14\,15\,16\,17\,18\,19\,20\,21\,22\,23\,\dots$$

had been studied by Champernowne in 1933 and Mahler proved in 1937 that it is transcendental..

D. G. Champernowne, *The construction of decimals normal in the scale of ten,* Journal of the London Mathematical Society, vol. 8 (1933), p. 254-260

K. Mahler, *Arithmetische Eigenschaften einer Klasse von Dezimalbrüchen,* Proc. Konin. Neder. Akad. Wet. Ser. A. 40 (1937), p. 421-428.

Émile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques,*
Palermo Rend. **27**, 247-271 (1909).
Jahrbuch Database JFM 40.0283.01
`http ://www.emis.de/MATH/JFM/JFM.html`

- *Sur les chiffres décimaux de $\sqrt{2}$ et divers problèmes de probabilités en chaînes,*
C. R. Acad. Sci., Paris **230**, 591-593 (1950).
Zbl 0035.08302

Émile Borel : 1950

- A real number x is called *simply normal in base g* if each digit occurs with frequency $1/g$ in its g–ary expansion.
- A real number x is called *normal in base g* or *g–normal* if it is simply normal in base g^m for all $m \geq 1$.

Normal Numbers

- Hence a real number x is normal in base g if and only if, for any $m \geq 1$, each sequence of m digits occurs with frequency $1/g^m$ in its g–ary expansion.

- A real number is called *normal* if it is normal in any base $g \geq 2$.

- Hence a real number is normal if and only if it is simply normal in any base $g \geq 2$.

Normal numbers

- Almost all real numbers (for Lebesgue's measure) are normal.

- Examples of computable normal numbers have been constructed (W. Sierpinski, H. Lebesgue, V. Becher and S. Figueira) but the known algorithms to compute such examples are fairly complicated ("ridiculously exponential", according to S. Figueira).

- Another example : Chaitin's constant Ω, which represents the probability that a random program will halt.
Ω is definable but not computable.

Further examples of normal numbers

• (Korobov, Stoneham ...) : *if a and g are coprime integers > 1, then*

$$\sum_{n\geq 0} a^{-n} g^{-a^n}$$

is normal in base g.

• A.H. Copeland and P. Erdős (1946) : a normal number in base 10 is obtained by concatenation of the sequence of prime numbers

$$0.2\,3\,5\,7\,11\,13\,17\,19\,23\,29\,31\,37\,41\,43\,47\,53\,59\,61\,67\;\ldots$$

Borel's Conjecture

• **Conjecture.** *Let x be an irrational algebraic real number. Then x is normal.*

• There is no explicitly known example of a triple (g, a, x), where $g \geq 3$ is an integer, a a digit in $\{0, \ldots, g-1\}$ and x an algebraic irrational number, for which one can claim that the digit a occurs infinitely often in the g–ary expansion of x.

• K. Mahler : *For any $g \geq 2$ and any $n \geq 1$, there exist algebraic irrational numbers x such that any block of n digits occurs infinitely often in the g–ary expansion of x.*

Ramanujan Institute, Chennai
S.S. Pillai endowment lecture
January 12, 2010

Perfect Powers : Pillai's works and their developments

Michel Waldschmidt

Institut de Mathématiques de Jussieu & Paris VI
http ://www.math.jussieu.fr/~miw/