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Number Theory
I : Linear Recurrent Sequences
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Quizz 1 *12/11/2021* – Solution

Let a and b be two nonzero complex numbers. Set $s = a + b$, $p = ab$. Let E be the complex vector space of sequences $(u_n)_{n\geq 0}$ satisfying

$$u_{n+2} = su_{n+1} - pu_n.$$

(1). Assume $a \neq b$. Show that the two sequences $(a^n)_{n\geq 0}$ and $(b^n)_{n\geq 0}$ give a basis for E.
(2). Assume $a = b$. Show that the two sequences $(a^n)_{n\geq 0}$ and $(na^n)_{n\geq 0}$ give a basis for E.

Solution

Recall that two element $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ in E are linearly independent if and only the determinant of the 2×2 matrix $\begin{pmatrix} u_0 & u_1 \\ v_0 & v_1 \end{pmatrix}$, namely $u_0v_1 - u_1v_0$, is not 0.

(1) The two roots of $X^2 - sX + p$ are a and b :

$$X^2 - sX + p = (X - a)(X - b).$$

Hence $a^2 = sa - p$, $b^2 = sb - p$. Multiplying by a^n and b^n respectively yields

$$a^{n+2} = sa^{n+1} - pa^n, \quad b^{n+2} = sb^{n+1} - pb^n,$$

from which one deduces that two sequences $(a^n)_{n\geq 0}$ and $(b^n)_{n\geq 0}$ belong to E.

These two sequences are linearly independent since $a^n = (1, a, \dots)$, $b^n = (1, b, \dots)$ and $a \neq b$, so the determinant of the 2×2 matrix $\begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}$ is not 0.

(2). When $a = b$ we have $s = 2a$, $p = a^2$. The polynomial $X^2 - sX + p$ has a double root at $X = a$:

$$X^2 - sX + p = X^2 - 2aX + a^2 = (X - a)^2.$$

As in (1), we have $a^{n+2} = 2a^{n+2} - a^{n+2} = sa^{n+1} - pa^n$. Also for $u_n = na^n$ we have

$$u_{n+2} = (n + 2)a^{n+2} = 2(n + 1)a^{n+2} - na^{n+2} = 2au_{n+1} - a^2u_n.$$

Hence the two sequences $(a^n)_{n\geq 0}$ and $(u_n)_{n\geq 0} = (na^n)_{n\geq 0}$ belong to E.

The two sequences $(a^n)_{n\geq 0}$ and $(na^n)_{n\geq 0}$ are linearly independent since $a^n = (1, a, \dots)$ and $na^n = (0, a, \dots)$ with $a \neq 0$, so the determinant of the 2×2 matrix $\begin{pmatrix} 1 & a \\ 0 & a \end{pmatrix}$ is not 0. .