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A course on linear recurrent sequences
African Institute for Mathematical Sciences (AIMS)

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Quizz 2 (10') – solution

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Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence :

$$F_0=0,\quad F_1=1,\quad F_n=F_{n-1}+F_{n-2}\quad\text{for}\quad n\geq 2.$$

Let $\Phi=(1+\sqrt{5})/2$ be the golden ratio (golden mean).

Recall

$$\begin{pmatrix}0&1\\1&1\end{pmatrix}^n=\begin{pmatrix}F_{n-1}&F_n\\F_n&F_{n+1}\end{pmatrix}$$

for $n\geq 1$.

(a) Deduce

$$\Phi^n=F_{n-1}+F_n\Phi.$$

(b) What is the value of $F_{n-1}F_{n+1}-F_n^2$ for $n\geq 1$?

Solution.

Since

$$\begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}1\\\Phi\end{pmatrix}=\begin{pmatrix}\Phi\\\Phi^2\end{pmatrix},$$

one deduces by induction

$$\begin{pmatrix}0&1\\1&1\end{pmatrix}^n\begin{pmatrix}1\\\Phi\end{pmatrix}=\begin{pmatrix}\Phi^n\\\Phi^{n+1}\end{pmatrix}$$

for $n\geq 0$. Therefore

$$\begin{pmatrix}F_{n-1}&F_n\\F_n&F_{n+1}\end{pmatrix}\begin{pmatrix}1\\\Phi\end{pmatrix}=\begin{pmatrix}\Phi^n\\\Phi^{n+1}\end{pmatrix},$$

hence the result.

(b) The determinant of

$$\begin{pmatrix}0&1\\1&1\end{pmatrix}^n$$

is $(-1)^n$, while the determinant of

$$\begin{pmatrix}F_{n-1}&F_n\\F_n&F_{n+1}\end{pmatrix}$$

is $F_{n-1}F_{n+1}-F_n^2$. Hence

$$F_{n-1}F_{n+1}-F_n^2=(-1)^n$$

for $n\geq 1$.