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**Number Theory**
**II. Prime Numbers**
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**Quizz 3** Date: *19/11/2021*

(a) Check that $x^2 + 4y^2 = (x + 2y)^2 - 4xy$.

(b) Show that for $n \geq 2$ the number $n^4 + 4^n$ is not prime.
Hint: Consider the case $n$ is even and the case $n$ is odd (in which case you can use (a))

(c) i) Write the characteristic polynomial and a linear recurrence relation satisfied by the sequences $(n^4)_{n\geq 0}$ and $(4^n)_{n\geq 0}$.

ii) Deduce the characteristic polynomial and a linear recurrence relation satisfied by the sequence $(n^4 + 4^n)_{n\geq 0}$.

**Solution:**

(a) We have
$$(x + 2y)^2 - 4xy = x^2 + 4xy + 4y^2 - 4xy = x^2 + 4y^2.$$
(b) If $n$ is even, then $n^4 + 4^n$ is even and $> 2$, hence is not prime.
Assume $n$ is odd, $n = 2c + 1$. Using (a) with $x = a^2$ and $y = b^2$, we deduce
$$a^4 + 4b^4 = (a^2 + 2b^2)^2 - 4a^2b^2 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2).$$
With $a = n$ and $b = 2^c$ we deduce that $n^4 + 4^{2c+1}$ is divisible by $n^2 + 2nb + 2b^2$ and by $n^2 - 2nb + 2b^2$, which are both $> 1$ for $c \geq 1$, hence for $n \geq 3$.

(c)
(i) For $0 \leq i < d-1$, the sequence $(n^i)_{n\geq 0}$ satisfies the linear recurrence, the characteristic polynomial of which is $(X - 1)^d$. Here $i = 4$, we can take $d = 5$ with the characteristic polynomial
$$(X - 1)^5 = X^5 - 5X^4 + 10X^3 - 10X^2 + 5X - 1;$$
the sequence $(n^4)_{n\geq 0}$ satisfies the linear recurrence
$$u_{n+5} = 5u_{n+4} - 10u_{n+3} + 10u_{n+2} - 5u_{n+1} + u_n.$$
Also for $\gamma$ a nonzero complex number the sequence $(\gamma^n)_{n\geq 0}$ satisfies the linear recurrence $u_{n+1} = \gamma u_n$, the characteristic polynomial of which is $X - \gamma$. Here $\gamma = 4$ and a linear recurrence satisfied by the sequence $(4^n)_{n\geq 0}$ is
$$u_{n+1} = 4u_n.$$
(ii) Since
$$(X - 1)^5(X - 4) = X^6 - 9X^5 + 30X^4 - 50X^3 + 45X^2 - 21X + 4,$$
a linear recurrence relation satisfied by $u_n = n^4 + 4^n$ is
$$u_{n+6} = 9u_{n+5} - 30u_{n+4} + 50u_{n+3} - 45u_{n+2} + 21u_{n+1} - 4u_n.$$
`Remark`. The sequence $(n^4 + 4^n)_{n\geq 0}$ starts with
$$1, 5, 32, 145, 512, 1649, 5392, 18785, 69632, 268705, 1058576, 4208945, 16797952,$$
The only prime number in this sequence is 5.

See `https://oeis.org/A001589` where
$$a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$$
is called *Sophie Germain's identity*.