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Diophantine approximations and continued fractions

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Abstract

We give an introduction to the theory of Diophantine approximation of power series. We present results in the classical Diophantine approximations and state their analogues in the case of power series.

Introduction

In the introduction of his paper in 1873, where he proved the transcendence of e, Ch. Hermite starts by recalling the theory of simultaneous Diophantine approximation to several real numbers by rational tuples. He points out that the case of a single number is nothing else than the algorithm of continued fractions. He claims that he will do something similar with functions. This is the birth of the theory of Padé approximation, and Hermite pursues by giving an explicit solution for what is now called Padé approximants of type II for the exponential function.

Rational approximations to a real number

We start with one simple fact. If x is a rational number, there is a constant $c > 0$ such that for any $\frac{p}{q} \in \mathbb{Q}$ with $\frac{p}{q} \neq x$, we have $|x - \frac{p}{q}| \geq \frac{c}{q}$. This statement can easily be proven, we just need to write x as $x = \frac{a}{b}$ and set $c = \frac{1}{b}$.

On the other hand, if x is a real irrational number, there are infinitely many $\frac{p}{q} \in \mathbb{Q}$, with $\gcd(p, q) = 1$ such that $|x - \frac{p}{q}| < \frac{1}{q^2}$. This is a consequence of the Dirichlet's theorem, which states that for any positive real number α and any positive integer N, there exist coprime integers p and q such that $1 \leq q \leq N$ and $|q\alpha - p| \leq \dfrac{1}{N}$.

It is known that the best rational approximations $\frac{p}{q}$ are given by the algorithm of continued fractions. We will give a short survey on this algorithm.

Before doing so, we give two generalisations of the problem in higher dimension. Let $x_1, \ldots, x_m$ be given real numbers, we may either consider

$$\max_{1 \leq i \leq m} \left| x_i - \frac{p_i}{q} \right|,$$

for $p_1, \ldots, p_m, q$ in $\mathbb{Z}$ with $q > 0$, which is the simultaneous approximation of the tuple $(x_1, \ldots, x_m)$ by rational numbers with the same denominator, or else

$$|p_1 x_1 + \cdots + p_m x_m - q|$$

$p_1, \ldots, p_m, q$ in $\mathbb{Z}$ not all zero.

As we want to give analogues for power series, let us note that the first one corresponds to Padé approximants of type II and the second one corresponds to Padé approximants of type I.

The continued fractions

We explain very briefly the algorithm of continued fractions. Let $x \in \mathbb{R}$. Imitating Euclidean division, we can "divide" x by 1

$$x = \lfloor x \rfloor + \{x\},$$

with $\lfloor x \rfloor \in \mathbb{Z}$ and $0 \leq \{x\} < 1$. If x is not an integer, then $\{x\} \neq 0$. Set $x_1 = \frac{1}{\{x\}}$, so that

$$x = \lfloor x \rfloor + \frac{1}{x_1},$$

and $x_1 > 1$. If x_1 is not an integer, set $x_2 = \frac{1}{\{x_1\}}$ and write

$$x = \lfloor x \rfloor + \frac{1}{\lfloor x_1 \rfloor + \{x_1\}} = \lfloor x \rfloor + \frac{1}{\lfloor x_1 \rfloor + \frac{1}{x_2}}$$

with $x_2 > 1$. Set $a_0 = \lfloor x \rfloor$ and $a_i = \lfloor x_i \rfloor$ for $i \geq 1$. Then

$$x = \lfloor x \rfloor + \frac{1}{\lfloor x_1 \rfloor + \frac{1}{\lfloor x_2 \rfloor + \frac{1}{\ddots}}} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

So, we can see the explanation of the name continued fraction. We use the notation $x = [a_0,\ a_1,\ a_2,\ a_3, \ldots]$. As a matter of fact, the algorithm stops after finitely many steps if and only if x is rational.

If x is rational, we can recognize the Euclidean division. Write $x = \frac{p}{q}$, then divide p by q: $p = a_0 q + r_0$, where $0 \leq r_0 < q$. If $r_0 \neq 0$, then consider $x_1 = \frac{q}{r_0} > 1$ and do the same: $q = a_1 r_0 + r_1$, $0 \leq r_1 < r_0$, then $r_1 = 0$ or put $x_2 = \frac{r_0}{r_1}$ etc. Since Euclidean algorithm stops, this one will too, which confirms our claim on finite continued fraction of a rational number.

Let $a_0, a_1, \ldots, a_n$ be given integers with $a_i \geq 1$ for $i \geq 1$. Then the finite continued fraction $[a_0,\ a_1,\ a_2,\ a_3, \ldots, a_n]$ can be written as

$$\frac{P_n(a_0, a_1, \ldots, a_n)}{Q_n(a_1, a_2, \ldots, a_n)},$$

where P_n and Q_n are polynomials with integer coefficients. We would like to write these polynomials explicitly.

Let $\mathbb{F}$ be a field, $Z_0, Z_1, \ldots$ variables. We will define polynomials P_n and Q_n in $\mathbb{F}[Z_0, \ldots, Z_n]$ and $\mathbb{F}[Z_1, \ldots, Z_n]$ respectively such that

$$[Z_0, Z_1, \ldots, Z_n] = \frac{P_n}{Q_n}.$$

Here are the first values:

$$P_0 = Z_0, \quad Q_0 = 1, \quad \frac{P_0}{Q_0} = Z_0;$$

$$P_1 = Z_0 Z_1 + 1, \quad Q_1 = Z_1, \quad \frac{P_1}{Q_1} = Z_0 + \frac{1}{Z_1};$$

$$P_2 = Z_0 Z_1 Z_2 + Z_2 + Z_0, \quad Q_2 = Z_1 Z_2 + 1, \quad \frac{P_2}{Q_2} = Z_0 + \frac{1}{Z_1 + \frac{1}{Z_2}}$$

$$P_3 = Z_0 Z_1 Z_2 Z_3 + Z_2 Z_3 + Z_0 Z_3 + Z_0 Z_1 + 1, \quad Q_3 = Z_1 Z_2 Z_3 + Z_3 + Z_1,$$

$$\frac{P_3}{Q_3} = Z_0 + \frac{1}{Z_1 + \frac{1}{Z_2 + \frac{1}{Z_3}}}.$$

We can easily observe the relations between these polynomials:

$$P_2 = Z_2 P_1 + P_0, \quad Q_2 = Z_2 Q_1 + Q_0.$$

$$P_3 = Z_3 P_2 + P_1, \quad Q_3 = Z_3 Q_2 + Q_1.$$

We infer that a more general rule holds. Let us define two sequences of polynomials in the following way:

$$P_n = Z_n P_{n-1} + P_{n-2}, \quad Q_n = Z_n Q_{n-1} + Q_{n-2}.$$

One can check that it is really true that $[Z_0, Z_1, \ldots, Z_n] = \dfrac{P_n}{Q_n}$ for all $n \geq 0$. Indeed, we can prove this by induction. As we saw, the statement is true for small numbers $n = 1, 2, 3$. To show the inductive step, we need to write

$$[Z_0, Z_1, \ldots, Z_n] = \frac{P_{n-1}(Z_0, \ldots, Z_{n-2}, T)}{Q_{n-1}(Z_1, \ldots, Z_{n-2}, T)},$$

where a variable T represents $Z_{n-1} + \frac{1}{Z_n}$. Then, the proof follows by using inductive hypothesis (twice), replacement $T = Z_{n-1} + \frac{1}{Z_n}$ and simple calculations.
The matrix form is

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix} \begin{pmatrix} Z_n \\ 1 \end{pmatrix}.$$

It is a better idea to consider 2×2 matrices

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix} \begin{pmatrix} Z_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence, there is a nice formula for the polynomials P_n and Q_n

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} Z_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Z_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} Z_n & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } n \geq -1,$$

if we define

$$\begin{pmatrix} P_{-1} & P_{-2} \\ Q_{-1} & Q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and treat an empty product (the case $n = -1$) as the identity matrix.

There are a few definitions we will need to work with continued fractions. Let $x = [a_0,\ a_1,\ a_2, \dots, a_n]$. As we have seen, $x = \frac{p_n}{q_n}$, with $p_n = P_n(a_0, a_1, \dots, a_n)$ and $q_n = Q_n(a_1, \dots, a_n)$. For $x = [a_0,\ a_1,\ a_2, \dots, a_n, \dots]$, the rational numbers in the sequence

$$\frac{p_n}{q_n} = [a_0,\ a_1,\ a_2, \dots, a_n] \qquad (n = 1, 2, \dots)$$

give rational approximations for x which are the best ones when comparing the quality of the approximation and the size of the denominator. We call $a_0, a_1, a_2, \dots$ the *partial quotients*. Additionally, $\frac{p_n}{q_n}$ $(n \geq 0)$ are the *convergents*, whereas $x_n = [a_n, a_{n+1}, \dots]$ $(n \geq 0)$ are the *complete quotients*. Also, note

$$x = [a_0, a_1, \dots, a_{n-1}, x_n] = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}}.$$

There are some inequalities that are worth mentioning, such as

$$a_n q_{n-1} \leq q_n \leq (a_n + 1) q_{n-1}$$

and

$$\frac{1}{(a_{n+1} + 2) q_n} < \frac{1}{q_{n+1} + q_n} < |q_n x - p_n| < \frac{1}{q_{n+1}} < \frac{1}{a_{n+1} q_n}.$$

The most important property of convergents is that they are the best rational approximations (in a size of a denominator). Indeed, we have the following theorem and an easy consequence.

Theorem 1. *Let $\frac{p_n}{q_n}$ be the n–th convergent of the continued fraction expansion of an irrational number x. Let $\frac{a}{b}$ be any rational number, such that $1 \leq b \leq q_n$. Then*

$$|q_n x - p_n| \leq |bx - a|,$$

with equality if and only if $(a, b) = (p_n, q_n)$.

Corollary 1.1. *For $1 \le b \le q_n$ we have*

$$\left|x - \frac{p_n}{q_n}\right| \le \left|x - \frac{a}{b}\right|,$$

with equality if and only if $(a, b) = (p_n, q_n)$.

Analogies in the power series case

Let us now describe the analogy with power series. Let $\mathbb{F}$ be a field. For $\frac{P}{Q} \in \mathbb{F}(T)$, define

$$\left|\frac{P}{Q}\right| = e^{\deg P - \deg Q}$$

with $|0| = 0$. We can easily check that this function satisfies all of the properties of absolute value on the field $\mathbb{F}(T)$: positive-definiteness, multiplicativity, the triangle inequality. The field $\mathbb{F}(T)$ is not complete with respect to this absolute value, and its completion is $\mathbb{F}((1/T))$. The absolute value is extended in the following way - for $x \in \mathbb{F}((1/T))$ with $x \neq 0$ and

$$x = a_{k_0}T^{k_0} + a_{k_0-1}T^{k_0-1} + \cdots = \sum_{k \le k_0} a_k T^k$$

where $k_0 \in \mathbb{Z}$, $a_k \in \mathbb{F}$ for all $k \le k_0$ and $a_{k_0} \neq 0$, we define $|x| = e^{k_0}$.

Here we can see the analogues between the real numbers and the power series case:

$$\begin{array}{ccccc} \mathbb{Z} & \subset & \mathbb{Q} & \subset & \mathbb{R} \\ \updownarrow & & \updownarrow & & \updownarrow \\ \mathbb{F}[T] & \subset & \mathbb{F}(T) & \subset & \mathbb{F}((1/T)) \end{array}$$

$$\mathbb{Q}: \text{norm} \quad \left|\frac{a}{b}\right| = \max\{|a|, |b|\}, \quad \text{completion} \quad \sum_{n \ge -k} a_n 10^{-n} \longrightarrow \mathbb{R},$$

$$\mathbb{F}(T) : \text{norm } \left|\frac{P}{Q}\right| = e^{\deg P - \deg Q}, \text{ completion } \sum_{n \geq -k} a_n T^{-n} \longrightarrow \mathbb{F}((1/T)).$$

Notice that any element in $\mathbb{F}(T)$ has a unique continued fraction expansion $[A_0, A_1, \ldots, A_n]$ with $A_i \in \mathbb{F}[T]$ for $i \geq 0$ and $\deg A_i \geq 1$ for $i \geq 1$.

For $x \in \mathbb{F}((1/T))$, we have the same definitions as before. Let

$$x = [A_0, A_1, \ldots]$$

be a continued fraction. Then polynomials A_n are called *partial quotients* , rational functions $\frac{P_n}{Q_n}$, with $P_n = P_n(A_0, A_1, \ldots, A_n)$ and $Q_n = Q_n(A_1, \ldots, A_n)$ are called *convergents* and members of the sequence $x_n = [A_n, A_{n+1}, \ldots]$ are *complete quotients*.

Also, it is true that

$$x = [A_0, A_1, \ldots, A_{n-1}, x_n] = \frac{x_n P_{n-1} + P_{n-2}}{x_n Q_{n-1} + Q_{n-2}}.$$

For $x = [A_0, A_1, \ldots] \in \mathbb{F}((1/T))$,

$$P_n = P_n(A_0, A_1, \ldots, A_n), \quad Q_n = Q_n(A_1, \ldots, A_n),$$

one can prove that

$$|Q_n| = |A_n| \cdot |A_{n-1}| \cdots |A_1| \quad (n \geq 1)$$

and

$$\left|x - \frac{P_n}{Q_n}\right| = \frac{1}{|Q_n|\,|Q_{n+1}|} = \frac{1}{|A_{n+1}|\,|Q_n|^2} \quad (n \geq 0).$$

In this case it is also true that the convergents are the best rational approximations, namely the following is true.

Theorem 2. *Let $\frac{P_n}{Q_n}$ be the n–th convergent of the continued fraction expansion of $x \in \mathbb{F}((T^{-1})) \setminus \mathbb{F}(T)$. Let $\frac{A}{B}$ be any element in $\mathbb{F}(T)$ such that $|B| \leq |Q_n|$. Then*

$$|Q_n x - P_n| \leq |Bx - A|$$

with equality if and only if $(A, B) = (P_n, Q_n)$.

There is a straightforward corollary.

Corollary 2.1. *For $|B| \leq |Q_n|$ we have*

$$\left| x - \frac{P_n}{Q_n} \right| \leq \left| x - \frac{A}{B} \right|$$

with equality if and only if $(A, B) = (P_n, Q_n)$.

Let us state a few theorems in the case of real numbers and function fields.

Theorem 3. *(Legendre Theorem)*
(1) **Real numbers:** *If $x \in \mathbb{R} \setminus \mathbb{Q}$ and*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$ is a convergent of x.
(2) **Power series***: If $x \in \mathbb{F}((T^{-1})) \setminus \mathbb{F}(T)$ and*

$$\left| x - \frac{P}{Q} \right| < \frac{1}{|Q|^2},$$

then $\frac{P}{Q}$ is a convergent of x.

Theorem 4. *(Lagrange Theorem)*
(1) **Real numbers***: The continued fraction expansion of a real irrational number x is ultimately periodic if and only if x is quadratic over $\mathbb{Q}$.*

(2) **Power series***: If the continued fraction expansion of an element $x \in \mathbb{F}((T^{-1})) \smallsetminus \mathbb{F}(T)$ is ultimately periodic, then x is quadratic over $\mathbb{F}(T)$. The converse is true when the field has nonzero characteristic and is an algebraic extension of its prime field $\mathbb{F}_p$ (but not otherwise).*

Let us recall that ultimately periodic sequence is a sequence that is periodic starting from some index. An element $x \in \mathbb{F}((T^{-1})) \smallsetminus \mathbb{F}(T)$ has a *pseudo periodic expansion*, namely an expansion of the form

$$[A_0, A_1, \dots, A_{n-1}, B_1, \dots, B_{2t}, aB_1, a^{-1}B_2, aB_3, \dots, a^{-1}B_{2t}, \\ a^2B_1, a^{-2}B_2, \dots, a^{-2}B_{2t}, a^3B_1, a^{-3}B_2, \dots],$$

if and only if there exist R, S, T, U in $\mathbb{F}[T]$ with

$$x = \frac{Rx + S}{Tx + U}$$

where $\begin{pmatrix} R & S \\ T & U \end{pmatrix}$ has determinant 1 and is not a multiple of the identity matrix.
If D is polynomial which is irreducible over any quadratic extension of $\mathbb{F}$ then the regular continued fraction expansion of $\sqrt{D}$ is not pseudo–periodic.

We conclude by mentioning that there is a formal analogy between Nevanlinna theory and Diophantine approximation. Via Vojta's dictionary, the Second Main Theorem in Nevanlinna theory corresponds to Schmidt's Subspace Theorem in Diophantine approximation.

References

[1] P. Flajolet, B. Vallée, I. Vardi, *Continued fractions from Euclid to the present day*, 44p.
http://www.lix.polytechnique.fr/Labo/Ilan.Vardi/continued_fractions.ps

[2] A. Lasjaunias, *A survey of Diophantine approximation in fields of power series*. *Monatsh. Math.*, 130(3):211–229, 2000.

[3] A. Lasjaunias, *Diophantine approximation and continued fractions in power series fields*. https://www.math.u-bordeaux.fr/~alasjaun/survey.pdf

[4] W. M. Schmidt, *On continued fractions and Diophantine approximation in power series fields*. *Acta Arith.*, 95(2):139–166, 2000.

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