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in honour of Professor R. Balasubramanian's 70th Birthday

**Schanuel's Conjecture:
algebraic independence
of transcendental numbers**

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Abstract

Schanuel's conjecture asserts that for \mathbb{Q} -linearly independent complex numbers x_1, \dots, x_n , there are at least n algebraically independent numbers among the $2n$ numbers

$$x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_n).$$

This simple statement has many remarkable consequences; we will explain some of them. We will also present the state of the art on this topic.

Note : We write $\exp(z)$ for e^z .

Linear independence over \mathbb{Q}

Given complex numbers, we may ask whether they are linearly independent over \mathbb{Q} .

For instance given a number x , the linear independence of $1, x$ over \mathbb{Q} is equivalent to the irrationality of x .

As an example, the numbers

$$\log 2, \log 3, \log 5, \dots, \log p, \dots$$

are linearly independent over \mathbb{Q} : for $b_i \in \mathbb{Z}$,

$$b_1 \log p_1 + \dots + b_n \log p_n = 0 \implies b_1 = \dots = b_n = 0.$$

$$p_1^{b_1} \dots p_n^{b_n} = 1 \implies b_1 = \dots = b_n = 0.$$

Linear independence over $\overline{\mathbb{Q}}$

The set of algebraic numbers is a subfield of \mathbb{C} (sums and products of algebraic numbers are algebraic).

Given complex numbers, we may ask whether they are linearly independent over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

For instance, given a number x , the linear independence of $1, x$ over $\overline{\mathbb{Q}}$ is equivalent to the transcendence of x .

It has been proved by A. Baker in 1968 that the numbers

$$1, \log 2, \log 3, \log 5, \dots, \log p, \dots$$

are linearly independent over $\overline{\mathbb{Q}}$: for $\beta_i \in \overline{\mathbb{Q}}$,

$$\beta_0 + \beta_1 \log p_1 + \dots + \beta_n \log p_n = 0 \implies \beta_0 = \dots = \beta_n = 0.$$

Algebraic independence

Given complex numbers x_1, \dots, x_n , we may ask whether they are algebraically independent over \mathbb{Q} : this means that there is no nonzero polynomial $P \in \mathbb{Q}[X_1, \dots, X_n]$ such that $P(x_1, \dots, x_n) = 0$.

This is equivalent to saying that x_1, \dots, x_n are algebraically independent over $\overline{\mathbb{Q}}$: if a nonzero polynomial $Q \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ satisfies $Q(x_1, \dots, x_n) = 0$, then by taking for P the product of the *conjugates* of Q over \mathbb{Q} one gets a nonzero polynomial $P \in \mathbb{Q}[X_1, \dots, X_n]$ such that $P(x_1, \dots, x_n) = 0$.

For $n = 1$, x_1 is algebraically independent over \mathbb{Q} if and only if x_1 is transcendental over \mathbb{Q} .

If x_1, \dots, x_n are algebraically independent, each of these numbers x_i is transcendental.

Transcendence degree

The transcendence degree of a field extension $K_1 \subset K_2$, denoted either $\text{trdeg}(K_2/K_1)$ or $\text{trdeg}_{K_1}(K_2)$, is the maximal number of elements in K_2 which are algebraically independent over K_1 . The transcendence degree $\text{trdeg } K$ of a field K of characteristic zero is the transcendence degree of K over \mathbb{Q} .

Given complex numbers x_1, \dots, x_m , the maximal number of algebraically independent elements in the set $\{x_1, \dots, x_m\}$ is the same as the transcendence degree of the field $\mathbb{Q}(x_1, \dots, x_m)$ (over \mathbb{Q}).

The transcendence degree of the field $\mathbb{Q}(x_1, \dots, x_m)$ is m if and only if x_1, \dots, x_m are algebraically independent.

For $m = 1$, the transcendence degree of the field $\mathbb{Q}(x)$ is 0 if x is algebraic, 1 if x is transcendental.

Additivity of the transcendence degree

For $K_1 \subset K_2 \subset K_3$, we have

$$\text{trdeg}(K_3/K_1) = \text{trdeg}(K_3/K_2) + \text{trdeg}(K_2/K_1).$$

Also K_2 is an algebraic extension of K_1 if and only if $\text{trdeg}(K_2/K_1) = 0$.

Lindemann–Weierstraß Theorem (1885)

Let β_1, \dots, β_n be algebraic numbers which are linearly independent over \mathbb{Q} . Then the numbers $e^{\beta_1}, \dots, e^{\beta_n}$ are algebraically independent (over \mathbb{Q} or over $\overline{\mathbb{Q}}$).

Ferdinand von Lindemann

(1852 – 1939)



Karl Weierstraß

(1815 - 1897)

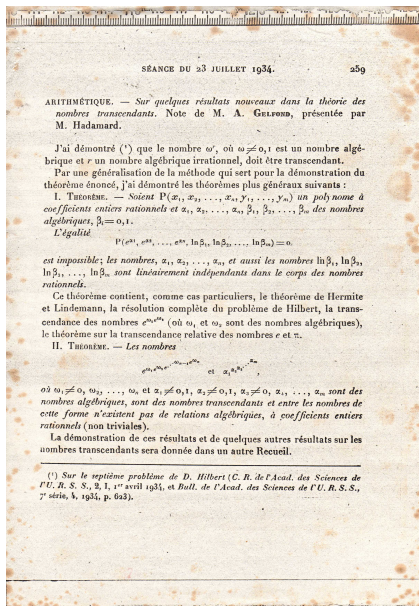


Equivalent forms of the Lindemann–Weierstraß Theorem

Let β_1, \dots, β_n be algebraic. Then the numbers $e^{\beta_1}, \dots, e^{\beta_n}$ are algebraically independent (over \mathbb{Q} or over $\overline{\mathbb{Q}}$) if and only if β_1, \dots, β_n are linearly independent over \mathbb{Q} .

Let $\gamma_1, \dots, \gamma_n$ be algebraic numbers. Then the numbers $e^{\gamma_1}, \dots, e^{\gamma_n}$ are linearly independent (over \mathbb{Q} or over $\overline{\mathbb{Q}}$) if and only if $\gamma_1, \dots, \gamma_n$ are pairwise distinct.

$$(e^{\beta_1})^{b_1} \dots (e^{\beta_n})^{b_n} = e^{b_1\beta_1 + \dots + b_n\beta_n}.$$



Statement by Gel'fond (1934)

Let $P(X_1, \dots, X_n, Y_1, \dots, Y_m)$ be a polynomial with rational integer coefficients and $\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m$ algebraic numbers, $\beta_i \neq 0, 1$.

The equality

$$P(e^{\alpha_1}, e^{\alpha_2}, \dots, e^{\alpha_n}, \ln \beta_1, \ln \beta_2, \dots, \ln \beta_m) = 0$$

is impossible; the numbers $\alpha_1, \alpha_2, \dots, \alpha_n$, as well as the numbers $\ln \beta_1, \ln \beta_2, \dots, \ln \beta_m$ are linearly independent in the field of rational numbers.

Statement by Gel'fond (1934)

This theorem includes as special cases, the theorems of Hermite and Lindemann, the complete solution of Hilbert's 7th problem, the transcendence of numbers $e^{\omega_1 e^{\omega_2}}$ (where ω_1 and ω_2 are algebraic numbers), the theorem on the relative transcendence of the numbers e and π .

Second statement by A.O. Gel'fond

The numbers

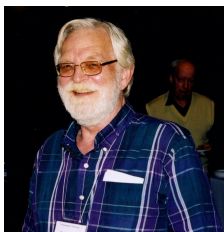
$$e^{\omega_1 e^{\omega_2 e^{\dots \omega_{n-1} e^{\omega_n}}}} \quad \text{and} \quad \alpha_1^{\alpha_2^{\alpha_3^{\dots \alpha_m}}},$$

where $\omega_1 \neq 0, \omega_2, \dots, \omega_n$ and $\alpha_1 \neq 0, 1, \alpha_2 \neq 0, 1, \alpha_3 \neq 0, \dots, \alpha_m$ are algebraic numbers, are transcendental numbers, and among numbers of this form there is no nontrivial algebraic relations with rational integer coefficients.

The proof of this result and a few other results on transcendental numbers will be given in another journal.

Remark by [Mathilde Herblot](#) : the condition on α_2 should be that it is irrational.

Schanuel's Conjecture



If x_1, \dots, x_n are \mathbb{Q} -linearly independent complex numbers, then n at least of the $2n$ numbers $x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}$ are algebraically independent.

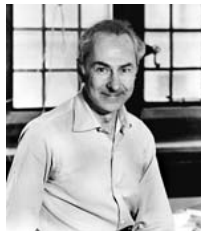
Equivalently :


If x_1, \dots, x_n are \mathbb{Q} -linearly independent complex numbers, then

$$\text{trdeg } \mathbb{Q}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}) \geq n.$$

Origin of Schanuel's Conjecture

Course given by [Serge Lang](#)
(1927–2005) at Columbia in
the 60's



 [S. LANG](#) – *Introduction to transcendental numbers*,
Addison-Wesley 1966.

Power series



James Ax
1937 – 2006

J. Ax's Theorem (1968) :
Version of Schanuel's
Conjecture for power series
over \mathbb{C} .

R. Coleman (1980) for power
series over $\overline{\mathbb{Q}}$.

Samit Dasgupta, *Introduction to Transcendence Theory*.
Math 790, 3/2/2021. Lecture 3: Schanuel's Conjecture and Ax's Theorem
<https://services.math.duke.edu/~dasgupta/Transcendence/index.html>

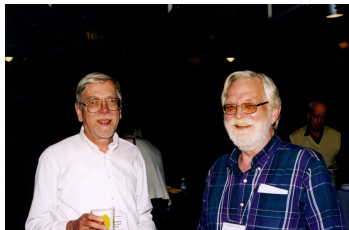
Formal analogs



W.D. Brownawell

Joint work by W.D. Brownawell and K. Kubota (1975) on the elliptic analog of Ax's Theorem.

Dale Brownawell and Stephen Schanuel



Methods from logic

Ehud Hrushovski



Boris Zilber



Jonathan Kirby



pre-dimension function (E. Hrushovski)

B. Zilber : *pseudo-exponentiation*

Also : A. Macintyre, D.E. Marker, G. Terzo, A.J. Wilkie,
D. Bertrand...

Kirby, Zilber, Wilkie

Kirby, Jonathan; Zilber, Boris

The uniform Schanuel conjecture over the real numbers.
Bull. London Math. Soc. **38** (2006), no. 4, 568–570.

Wilkie, A. J.

Schanuel's conjecture and the decidability of the real exponential field.

Algebraic model theory (Toronto, ON, 1996), 223–230,
NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 496, Kluwer Acad.
Publ., Dordrecht, 1997.



Daniel Bertrand



Daniel Bertrand,
*Schanuel's conjecture for
non-isoconstant elliptic curves
over function fields.*

Model theory with applications to algebra and analysis. Vol. 1,
41–62, London Math. Soc. Lecture Note Ser., **349**, Cambridge
Univ. Press, Cambridge, 2008.

Ax-Schanuel

- for Shimura varieties,
 - for variations of Hodge structures.,
 - for the modular invariant,
 - for linear differential equations,
 - and o-minimality
- ...

Lindemann–Weierstraß Theorem (1885)

According to the Lindemann–Weierstraß Theorem, Schanuel's Conjecture is true for algebraic x_1, \dots, x_n : in this case the transcendence degree of the field $\mathbb{Q}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n})$ is n .

Ferdinand von Lindemann

(1852 – 1939)



Karl Weierstraß

(1815 - 1897)



Transcendence degree $\leq n$

If we select e^{x_1}, \dots, e^{x_s} to be algebraic (this means that the x_i 's are logarithms of algebraic numbers) and x_{s+1}, \dots, x_n also to be algebraic, then the transcendence degree of the field

$$\mathbb{Q}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n})$$

is the same as the transcendence degree of the field

$$\mathbb{Q}(x_1, \dots, x_s, e^{x_{s+1}}, \dots, e^{x_n}),$$

hence is $\leq n$.

The conjecture (A.O. Gel'fond) is that it is n : the numbers

$$\log \alpha_1, \dots, \log \alpha_s, e^{\beta_{s+1}}, \dots, e^{\beta_n}$$

are expected to be algebraically independent.

Baire and Lebesgue

René Baire

1874 – 1932



Henri Léon Lebesgue

1875 – 1941

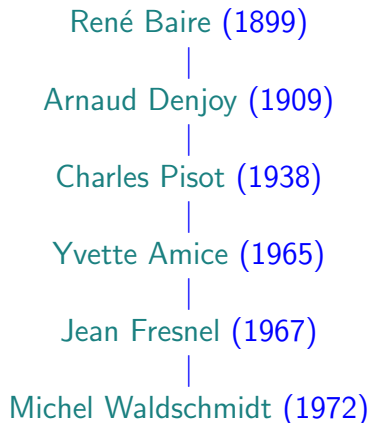


The set of tuples (x_1, \dots, x_n) in \mathbb{C}^n such that the $2n$ numbers $x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}$ are algebraically independent

- is a G_δ set (countable intersection of dense open sets) in Baire's classification (a *generic set* for dynamical systems)
- and has full Lebesgue measure.

True for any transcendental function in place of the exponential function.

Mathematical genealogy



<http://genealogy.math.ndsu.nodak.edu>

Joint work with Senthil Kumar and Thangadurai



Given two integers m and n with $1 \leq m \leq n$, there exist uncountably many tuples (x_1, \dots, x_n) in \mathbb{R}^n such that x_1, \dots, x_n and e^{x_1}, \dots, e^{x_n} are all **Liouville** numbers and the transcendence degree of the field

$$\mathbb{Q}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n})$$

is $n + m$.

$$m = 0?$$

$$1 \leq m \leq n :$$

$$\text{trdeg } \mathbb{Q}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}) = n + m.$$

We do not know whether there are **Liouville** numbers x such that e^x is also a **Liouville** number and the two numbers x and e^x are algebraically dependent.

Schanuel's Conjecture for $n = 1$

For $n = 1$, Schanuel's Conjecture is the Hermite–Lindemann Theorem :

If x is a non-zero complex numbers, then one at least of the two numbers x , e^x is transcendental.

Equivalently, *if x is a non-zero algebraic number, then e^x is a transcendental number.*

Another equivalent statement is that *if α is a non-zero algebraic number and $\log \alpha$ any non-zero logarithm of α , then $\log \alpha$ is a transcendental number.*

Consequence : transcendence of numbers like

$$e, \pi, \log 2, e^{\sqrt{2}}.$$

Proof: take for x respectively

$$1, i\pi, \log 2, \sqrt{2}.$$

Schanuel's Conjecture for $n = 2$

For $n = 2$, Schanuel's Conjecture is not yet known :

? If x_1, x_2 are \mathbb{Q} -linearly independent complex numbers, then among the 4 numbers $x_1, x_2, e^{x_1}, e^{x_2}$, at least two are algebraically independent.

A few consequences (open problems) :

With $x_1 = 1, x_2 = i\pi$: algebraic independence of e and π .

With $x_1 = 1, x_2 = e$: algebraic independence of e and e^e .

With $x_1 = \log 2, x_2 = (\log 2)^2$: algebraic independence of $\log 2$ and $2^{\log 2}$.

With $x_1 = \log 2, x_2 = \log 3$: algebraic independence of $\log 2$ and $\log 3$.

Easy consequence of Schanuel's Conjecture

A consequence of Schanuel's Conjecture is that the following numbers are algebraically independent :

$$e + \pi, e\pi, \pi^e, e^{\pi^2}, e^e, e^{e^2}, \dots, e^{e^e}, \dots, \pi^\pi, \pi^{\pi^2}, \dots, \pi^{\pi^\pi} \dots$$

$$\log \pi, \log(\log 2), \pi \log 2, (\log 2)(\log 3), 2^{\log 2}, (\log 2)^{\log 3} \dots$$

Lang's exercise



Define $E_0 = \mathbb{Q}$. Inductively, for $n \geq 1$, define E_n as the algebraic closure of the field generated over E_{n-1} by the numbers $\exp(x) = e^x$, where x ranges over E_{n-1} . Let E be the union of E_n , $n \geq 0$.

Then Schanuel's Conjecture implies that the number π does not belong to E .

More precisely : Schanuel's Conjecture implies that the numbers $\pi, \log \pi, \log \log \pi, \log \log \log \pi, \dots$ are algebraically independent over E .

A variant of Lang's exercise

Define $L_0 = \mathbb{Q}$. Inductively, for $n \geq 1$, define L_n as the algebraic closure of the field generated over L_{n-1} by the numbers y , where y ranges over the set of complex numbers such that $e^y \in L_{n-1}$. Let L be the union of L_n , $n \geq 0$. Then Schanuel's Conjecture implies that the number e does not belong to L .

More precisely : Schanuel's Conjecture implies that the numbers $e, e^e, e^{e^e}, e^{e^{e^e}} \dots$ are algebraically independent over L .

Arizona Winter School AWS2008, Tucson

Theorem [Chuangxun Cheng, Brian Dietel, Mathilde Herblot, Jingjing Huang, Holly Krieger, Diego Marques, Jonathan Mason, Martin Mereb and Robert Wilson.]

Schanuel's Conjecture implies that the fields E and L are linearly disjoint over $\overline{\mathbb{Q}}$.

Definition Given a field extension F/K and two subextensions $F_1, F_2 \subseteq F$, we say F_1, F_2 are linearly disjoint over K when the following holds : any set $\{x_1, \dots, x_n\} \subseteq F_1$ of K -linearly independent elements is linearly independent over F_2 .

Some consequences of Schanuel's conjecture. J. Number Theory **129** (2009), no. 6, 1464–1467. [arXiv.0804.3550](https://arxiv.org/abs/0804.3550)

Contribution of Georges Racinet.

Abelian analog

Philippon, Patrice ; Saha, Biswajyoti ; Saha, Ekata.

An abelian analogue of Schanuel's conjecture and applications.

Ramanujan J. **52** (2020), no. 2, 381–392.



Semi-abelian analog

Cristiana Bertolin, Patrice Philippon, Biswajyoti Saha, Ekata Saha

Semi-abelian analogues of Schanuel Conjecture and applications, 2020.

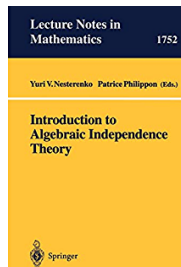
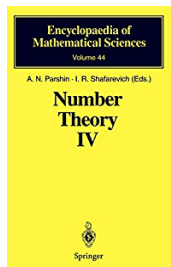
<https://arxiv.org/abs/2010.15170>



Algebraic independence

Towards Schanuel's Conjecture :

Ch. Hermite, F. Lindemann, C.L. Siegel, A.O. Gel'fond,
Th. Schneider, A. Baker, S. Lang, W.D. Brownawell,
D.W. Masser, D. Bertrand, G.V. Chudnovsky, P. Philippon,
G. Wüstholz, Yu.V. Nesterenko, D. Roy...



N.I. Feldman and Yu. V. Nesterenko Yu. V. Nesterenko and P. Philippon

Algebraic independence : A.O. Gel'fond 1948



The two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[4]{2}}$ are algebraically independent.

More generally, if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an algebraic number of degree $d \geq 3$, then two at least of the numbers

$$\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}}$$

are algebraically independent.

Problem of Gel'fond and Schneider

Raised by A.O. Gel'fond in 1948 and Th. Schneider in 1952.

Conjecture : if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an irrational algebraic number of degree d , then the $d - 1$ numbers

$$\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}}$$

are algebraically independent.

Special case of Schanuel's Conjecture : take

$x_i = \beta^{i-1} \log \alpha$. The conclusion is, for β algebraic number of degree d ,

$$\text{trdeg } \mathbb{Q}(\log \alpha, \alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}}) = d.$$

Large transcendence degree

Let α be a nonzero complex algebraic number, $\log \alpha$ a nonzero logarithm of α and β an algebraic number of degree $d \geq 2$. For $z \in \mathbb{C}$, define $\alpha^z := \exp(z \log \alpha)$. Then

$$\text{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha^{\beta}, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}}) \geq \left\lfloor \frac{d+1}{2} \right\rfloor.$$

This result is due to the work of [A.O. Gel'fond](#), [G.V. Chudnovsky](#), [P. Philippon](#), [Yu. V. Nesterenko](#), [G. Diaz](#) (1989).

References :

[N.I. Fel'dman](#) and [Yu.V. Nesterenko](#). *Transcendental numbers. Number theory, IV*, Encyclopaedia Math. Sci., **44**, Springer, Berlin, 1998.

[Yu.V. Nesterenko](#) and [P. Philippon](#) (ed). *Introduction to algebraic independence theory*. Lecture Notes in Mathematics, **1752**. Springer-Verlag, Berlin, 2001.

[Yu.V. Nesterenko](#). *Algebraic independence*. Published for the Tata Institute of Fundamental Research, Bombay ; by Narosa Publishing House, New Delhi, 2009.

p -adic transcendental numbers

Two open problems : the radius of convergence of \exp_p is finite :

$$n! \rightarrow \infty \text{ but } |n!|_p \rightarrow 0.$$

Problem 1 : p -adic analogue of the *Lindemann – Weierstraß Theorem*.

Problem 2 : p -adic analogue of *Gel'fond's Theorem* on the algebraic independence of α^β and α^{β^2} for β cubic irrational.

W.W. Adams (1966) :

For $[\mathbb{Q}(\beta) : \mathbb{Q}] = d \geq 4$, two of the numbers

$$\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}}$$

are algebraically independent.

For $[\mathbb{Q}(\beta) : \mathbb{Q}] = 3$, two of the numbers $\log \alpha$, α^β and α^{β^2} are algebraically independent.

Transcendence in finite characteristic



Federico Pellarin

On a variant of Schanuel conjecture for the Carlitz exponential.

J. Théor. Nombres Bordeaux
29 (2017), no. 3, 845–873.

Criteria for transcendence

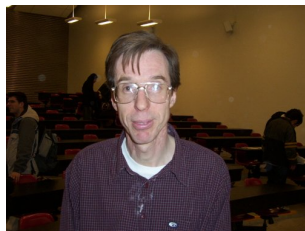
A.O. Gel'fond, W.D. Brownawell, P. Philippon,
G.V. Chudnovsky, Yu.V. Nesterenko, M. Laurent, D. Roy. . .



Patrice Philippon



Michel Laurent



Damien Roy

e^{π^2} , e and π (1972)

W.D. Brownawell

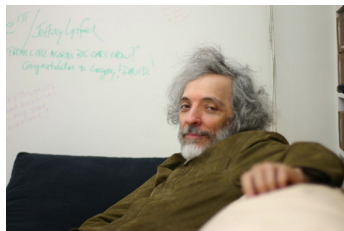


One at least of the two following statements is true :

- the number e^{π^2} is transcendental
- the two numbers e and π are algebraically independent.

Schanuel's Conjecture implies that both statements are true !

Algebraic independence



G.V. Chudnovsky (1978)

The numbers π and $\Gamma(1/4) = 3.625\ 609\ 908\ 2\dots$ are algebraically independent.

Also π and $\Gamma(1/3) = 2.678\ 938\ 534\ 7\dots$ are algebraically independent.

On the number e^π



Yu.V.Nesterenko (1996)
Algebraic independence of
 $\Gamma(1/4)$, π and e^π .
Also : Algebraic
independence of
 $\Gamma(1/3)$, π and $e^{\pi\sqrt{3}}$.

Corollary : *The numbers $\pi = 3.141\ 592\ 653\ 5\dots$ and $e^\pi = 23.140\ 692\ 632\ 7\dots$ are algebraically independent.*

The proof uses modular functions.

On the number e^π

Nesterenko : algebraic independence of π and e^π .

Open problem : e^π is not a **Liouville** number :

$$\left| e^\pi - \frac{p}{q} \right| > \frac{1}{q^\kappa}.$$

G.V. Chudnovsky : algebraic independence of π and $\Gamma(1/4)$.

Yu.V. Nesterenko : Algebraic independence of π , $\Gamma(1/4)$ and e^π .

Open problem : algebraic independence of π and e .

Expected : e , π and e^π are algebraic independent.

Conjecture of algebraic independence of logarithms of algebraic numbers

The most important special case of **Schanuel's Conjecture** is :

Conjecture. *Let $\lambda_1, \dots, \lambda_n$ be \mathbb{Q} -linearly independent complex numbers. Assume that the numbers $e^{\lambda_1}, \dots, e^{\lambda_n}$ are algebraic. Then the numbers $\lambda_1, \dots, \lambda_n$ are algebraically independent over \mathbb{Q} .*

Not yet known that the transcendence degree is ≥ 2 .

Baker's linear independence Theorem

Let $\lambda_1, \dots, \lambda_n$ be \mathbb{Q} -linearly independent logarithms of algebraic numbers. Then the numbers $1, \lambda_1, \dots, \lambda_n$ are linearly independent over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

Schanuel's Conjecture deals with algebraic independence (over \mathbb{Q} or $\overline{\mathbb{Q}}$), Baker's Theorem deals with linear independence. Baker's Theorem is a special case of Schanuel's Conjecture.

Serre's reformulation of Baker's Theorem

Denote by \mathcal{L} the set of complex numbers λ for which e^λ is algebraic (set of logarithms of algebraic numbers). Hence \mathcal{L} is a \mathbb{Q} -vector subspace of \mathbb{C} .

J-P. Serre

(Bourbaki seminar) :

the injection of \mathcal{L} into \mathbb{C}

extends to a $\overline{\mathbb{Q}}$ -linear map

$\iota : \overline{\mathbb{Q}} + \mathcal{L} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \rightarrow \mathbb{C}$, and

Baker's Theorem means that

ι is an injective map.



Reformulation by D. Roy

Instead of taking logarithms of algebraic numbers and looking for the algebraic independence relations, D. Roy fixes a polynomial and looks at the points, with coordinates logarithms of algebraic numbers, on the corresponding hypersurface.

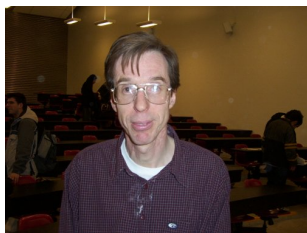
Recall that \mathcal{L} denotes the \mathbb{Q} -vector subspace of \mathbb{C} of all logarithms of all nonzero algebraic numbers.

The Conjecture on (homogeneous) algebraic independence of logarithms of algebraic numbers is equivalent to :

Conjecture (Roy). *For any algebraic subvariety V of \mathbb{C}^n defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers, the set $V \cap \mathcal{L}^n$ is the union of the sets $E \cap \mathcal{L}^n$, where E ranges over the set of vector subspaces of \mathbb{C}^n which are contained in V .*

Points with coordinates logarithms of algebraic numbers

Damien Roy : Grassmanian varieties.



Stéphane Fischler : orbit of an affine algebraic group G over $\overline{\mathbb{Q}}$ related to a linear representation of G on a vector space with a $\overline{\mathbb{Q}}$ -structure.

Quadratic relations among logarithms of algebraic numbers

One does not know yet how to prove that there is no nontrivial quadratic relations among logarithms of algebraic numbers, like

$$(\log \alpha_1)(\log \alpha_2) = \log \beta.$$

Example: Assume $e^{\pi^2} = \beta$ is algebraic. Then

$$(-i\pi)(i\pi) = \log \beta.$$

- **Open problem** : *is the number e^{π^2} transcendental?*

Homogeneous quadratic relations among logarithms of algebraic numbers

Any homogeneous quadratic relation among logarithms of algebraic numbers

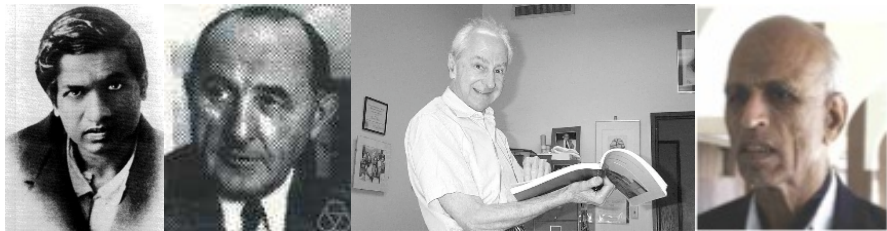
$$(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$$

should be trivial.

Example of a trivial relation : $(\log 2)(\log 9) = (\log 4)(\log 3)$.

The *Four Exponentials Conjecture* can be stated as : *any quadratic relation $(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$ among logarithms of algebraic numbers is trivial : either $\log \alpha_1 / \log \alpha_2$ is rational, or $\log \alpha_1 / \log \alpha_3$ is rational.*

S. Ramanujan,
C.L. Siegel, S. Lang, K. Ramachandra



Ramanujan : Highly composite numbers.

Alaoglu and Erdős (1944), Siegel.

Four exponentials conjecture (special case)

Let t be a positive real number. Assume 2^t and 3^t are both integers. Prove that t is an integer.

Set $n = 2^t$. Then $t = (\log n)/(\log 2)$ and

$$3^t = e^{t \log 3} = e^{(\log n)(\log 3)/(\log 2)} = n^{(\log 3)/(\log 2)}.$$

Equivalently :

If n is a positive integer such that

$$n^{(\log 3)/(\log 2)}$$

is an integer, then n is a power of 2 :

$$2^{k(\log 3)/(\log 2)} = 3^k.$$

Siegel : if 2^t , 3^t and 5^t are integers, then t is an integer.

The four exponentials Conjecture

Let x_1, x_2 be two linearly independent complex numbers and also y_1, y_2 be two linearly independent complex numbers. Then one at least of the four numbers $e^{x_i y_j} = \alpha_{ij}$, $i = 1, 2$, $j = 1, 2$ is transcendental.

A 2×2 matrix has rank 1 if and only if it can be written

$$\begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}$$

The four exponentials conjecture is equivalent to : Let

$$\begin{pmatrix} \log \alpha_{11} & \log \alpha_{12} \\ \log \alpha_{21} & \log \alpha_{22} \end{pmatrix}$$

be a 2×2 matrix with \mathbb{Q} -linearly independent rows and \mathbb{Q} -linearly independent columns. If the entries are logarithms of algebraic numbers, then the rank of this matrix is 2.

Four exponentials and six exponentials

Let $t \in \mathbb{C}$; the matrix

$$\begin{pmatrix} \log 2 & \log 3 \\ t \log 2 & t \log 3 \end{pmatrix}$$

has rank one. Hence if t is irrational, the Four exponentials Conjecture implies that one at least of 2^t , 3^t is transcendental.

Six exponentials Theorem : $e^{x_i y_j} = \alpha_{ij}$, $i = 1, 2$, $j = 1, 2, 3$.

Matrix $(\log \alpha_{ij})_{i=1,2; j=1,2,3}$.

Example : let $t \in \mathbb{C}$ be irrational; the matrix

$$\begin{pmatrix} \log 2 & \log 3 & \log 5 \\ t \log 2 & t \log 3 & t \log 5 \end{pmatrix}$$

has rank one. Hence one at least of 2^t , 3^t , 5^t is transcendental.

Structural rank of a matrix with entries in \mathcal{L}

Let M be a matrix with entries in \mathcal{L} (logarithms of algebraic numbers). Take a basis of the \mathbb{Q} -space spanned by the entries – let m be the dimension. Write the entries of M as linear combinations with rational coefficients of the elements of the basis. Replace the basis by variables X_1, \dots, X_m ; one gets a matrix with entries in $\mathbb{Q}(X_1, \dots, X_m)$. The rank of that matrix is the *structural rank* of M . It does not depend on the choice of the basis.

The conjecture of algebraic independence of logarithms (the homogeneous form suffices) implies that both matrices have the same rank.

Structural rank of a matrix

Let K be a field, k a subfield and \mathbf{M} a matrix with entries in K . Following D. Roy, we define *structural rank of \mathbf{M} with respect to k*

Consider the k -vector subspace \mathcal{E} of K spanned by the entries of \mathbf{M} . Choose an injective morphism φ of \mathcal{E} into a k -vector space $kX_1 + \cdots + kX_n$. The image $\varphi(\mathbf{M})$ of \mathbf{M} is a matrix with entries in the field $k(X_1, \dots, X_n)$ of rational fractions. Its rank does not depend on the choice of φ .

This is the *structural rank* of \mathbf{M} with respect to k .

Homogeneous algebraic independence of logarithms

According to [D. Roy](#), the homogeneous case of the conjecture on algebraic independence of logarithms of algebraic numbers is **equivalent** to :

Conjecture. *Let M be a matrix with logarithms of algebraic numbers entries. Then the rank of M is equal to its structural rank with respect to \mathbb{Q} .*

Half of this conjecture is known :

The rank of a matrix with logarithms of algebraic numbers entries is at least half its structural rank with respect to \mathbb{Q}

[Smit Dasgupta](#), *Introduction to Transcendence Theory*.

Math 790, 3/4/2021. Lecture 4: Ranks of matrices of logarithms

<https://services.math.duke.edu/~dasgupta/Transcendence/index.html>

Algebraic independence of logarithms

According to [D. Roy](#), the conjecture on algebraic independence of logarithms of algebraic numbers is equivalent to :

Conjecture. *Any matrix*

$$(b_{ij} + \lambda_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq \ell}}$$

with $b_{ij} \in \mathbb{Q}$ and $\lambda_{ij} \in \mathcal{L}$ has a rank equal to its structural rank.

Any Polynomial is the Determinant of a Matrix

Roy's proof of the equivalence between the two conjectures uses the nice auxiliary result :

For any $P \in k[X_1, \dots, X_n]$ there exists a square matrix with entries in the k -vector space $k + kX_1 + \dots + kX_n$, the determinant of which is P .

The Strong Four Exponentials Conjecture

Denote by $\tilde{\mathcal{L}}$ the $\overline{\mathbb{Q}}$ -vector space spanned by 1 and \mathcal{L} : hence $\tilde{\mathcal{L}}$ is the set of linear combinations with algebraic coefficients of logarithms of algebraic numbers :

$$\tilde{\mathcal{L}} = \{\beta_0 + \beta_1\lambda_1 + \cdots + \beta_n\lambda_n ; n \geq 0, \beta_i \in \overline{\mathbb{Q}}, \lambda_i \in \mathcal{L}\}.$$

Conjecture. If $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ are elements in $\tilde{\mathcal{L}}$ such that Λ_1/Λ_2 and Λ_1/Λ_3 are transcendental, then

$$\Lambda_1\Lambda_4 \neq \Lambda_2\Lambda_3.$$

Means :

$$\det \begin{vmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{vmatrix} \neq 0.$$

Consequence of the Strong Four Exponentials Conjecture

Assume the strong Four Exponentials Conjecture.

- If Λ is in $\tilde{\mathcal{L}} \setminus \overline{\mathbb{Q}}$ then the quotient $1/\Lambda$ is not in $\tilde{\mathcal{L}}$.
- If Λ_1 and Λ_2 are in $\tilde{\mathcal{L}} \setminus \overline{\mathbb{Q}}$, then the product $\Lambda_1\Lambda_2$ is not in $\tilde{\mathcal{L}}$.
- If Λ_1 and Λ_2 are in $\tilde{\mathcal{L}}$ with Λ_1 and Λ_2/Λ_1 transcendental, then this quotient Λ_2/Λ_1 is not in $\tilde{\mathcal{L}}$.

Partial result

D. Roy proved results related to the *Strong four exponentials Conjecture* including the *Strong six exponentials Theorem* as well as higher dimensional results like :

The rank of a matrix, with entries which are linear combinations of logarithms of algebraic numbers with algebraic coefficients, is at least half its structural rank with respect to the field $\overline{\mathbb{Q}}$ of algebraic numbers.

Quadratic independence of logarithms of algebraic numbers

One does not know yet how to prove that there is no nontrivial quadratic relations among logarithms of algebraic numbers, like

$$(\log \alpha_1)(\log \alpha_2) = \gamma.$$

Diaz' Conjecture. *Let $u \in \mathbb{C}^\times$. Assume $|u|$ is algebraic. Then e^u is transcendental.*

If $|u|^2 = \gamma$ and $e^u = \alpha$ are algebraic numbers, then

$$(\log \alpha)(\log \bar{\alpha}) = \gamma.$$

For instance with $|u| = 1$, the curve $x \mapsto e^{e^{ix}}$ should not meet any algebraic point.

Similarly, the curve $x \mapsto e^{2i\pi e^{ix}}$ should not meet any algebraic point.

Conjectures of G. Diaz

- For $\lambda \in \mathcal{L} \setminus \{0\}$, is it true that $\lambda\bar{\lambda} \notin \mathcal{L}$?

Previous example : $\lambda = i\pi$ (transcendence of e^{π^2}).

- **More generally** : for λ_1 and λ_2 in $\mathcal{L} \setminus \{0\}$, is it true that $\lambda_1\lambda_2 \notin \mathcal{L}$?

- For λ_1 and λ_2 in $\mathcal{L} \setminus \{0\}$, is it true that $\lambda_1\lambda_2 \notin \tilde{\mathcal{L}}$?



G. DIAZ – *Utilisation de la conjugaison complexe dans l'étude de la transcendance de valeurs de la fonction exponentielle*, J. Théor. Nombres Bordeaux **16** (2004), p. 535–553.



G. DIAZ – *Produits et quotients de combinaisons linéaires de logarithmes de nombres algébriques : conjectures et résultats partiels* », J. Théor. Nombres Bordeaux **19** (2007), no. 2, 373–391.

Damien Roy

Strategy suggested by D. Roy
in 1999, Journées
Arithmétiques, Roma :
Conjecture equivalent to
Schanuel's Conjecture.



Roy's approach to Schanuel's Conjecture (1999)

Let \mathcal{D} denote the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

over the ring $\mathbb{C}[X_0, X_1]$. The *height* of a polynomial $P \in \mathbb{C}[X_0, X_1]$ is defined as the maximum of the absolute values of its coefficients.

Let k be a positive integer, y_1, \dots, y_k complex numbers which are linearly independent over \mathbb{Q} , $\alpha_1, \dots, \alpha_k$ non-zero complex numbers and s_0, s_1, t_0, t_1, u positive real numbers satisfying

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}$$

and

$$\max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Roy's Conjecture

Assume that, for any sufficiently large positive integer N , there exists a non-zero polynomial $P_N \in \mathbb{Z}[X_0, X_1]$ with partial degree $\leq N^{t_0}$ in X_0 , partial degree $\leq N^{t_1}$ in X_1 and height $\leq e^N$ which satisfies

$$\left| (\mathcal{D}^k P_N) \left(\sum_{j=1}^k m_j y_j, \prod_{j=1}^k \alpha_j^{m_j} \right) \right| \leq \exp(-N^u)$$

for any non-negative integers k, m_1, \dots, m_k with $k \leq N^{s_0}$ and $\max\{m_1, \dots, m_k\} \leq N^{s_1}$. Then

$$\text{trdeg } \mathbb{Q}(y_1, \dots, y_k, \alpha_1, \dots, \alpha_k) \geq k.$$

Equivalence between Schanuel and Roy

Let $(y, \alpha) \in \mathbb{C} \times \mathbb{C}^\times$, and let s_0, s_1, t_0, t_1, u be positive real numbers satisfying the inequalities of Roy's Conjecture. Then the following conditions are equivalent :

(a) The number αe^{-y} is a root of unity.

(b) For any sufficiently large positive integer N , there exists a nonzero polynomial $Q_N \in \mathbb{Z}[X_0, X_1]$ with partial degree $\leq N^{t_0}$ in X_0 , partial degree $\leq N^{t_1}$ in X_1 and height $H(Q_N) \leq e^N$ such that

$$|(\partial^k Q_N)(my, \alpha^m)| \leq \exp(-N^u)$$

for any $k, m \in \mathbb{N}$ with $k \leq N^{s_0}$ and $m \leq N^{s_1}$.

Further progress by D. Roy

$$\mathbb{G}_a, \quad \mathbb{G}_m, \quad \mathbb{G}_a \times \mathbb{G}_m.$$

Small value estimates for the additive group. Int. J. Number Theory **6** (2010), 919–956.

Small value estimates for the multiplicative group. Acta Arith. **135** (2008), 357–393.

A small value estimate for $\mathbb{G}_a \times \mathbb{G}_m$. Mathematika **59** (2013), 333–363.

Further developments

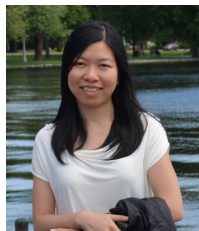
Roy's Conjecture deals with polynomials vanishing on some subsets of $\mathbb{C} \times \mathbb{C}^\times$ with multiplicity along the space associated with the derivation $\partial/\partial X + Y\partial/\partial Y$.

Roy's conjecture depends on parameters s_0, s_1, t_0, t_1, u in a certain range. D. Roy proved that if his conjecture is true for one choice of values of these parameters in the given range, then Schanuel's Conjecture is true, and that conversely, if Schanuel's Conjecture is true, then his conjecture is true for all choices of parameters in the same range.

Nguyen Ngoc Ai Van

extended the range of these parameters.

A refined criterion for Schanuel's conjecture. Chamchuri J. Math. **1** (2009), no. 2, 25–29.



Ubiquity of Schanuel's Conjecture

Other contexts : p -adic numbers, Leopoldt's Conjecture on the p -adic rank of the units of an algebraic number field.

Non-vanishing of Regulators.

Non-degenerescence of heights.

Conjecture of B. Mazur on rational points.

Diophantine approximation on tori.

Dipendra Prasad



Gopal Prasad



Dipendra Prasad. *An analogue of a conjecture of Mazur : a question in Diophantine approximation on tori.* Contributions to automorphic forms, geometry, and number theory, 699– 709, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

Gopal Prasad, Andrei S. Rapinchuk. *Zariski-dense subgroups and transcendental number theory.* Math. Res. Lett. **12** (2005), no. 2-3, 239–249.

Peter Bundschuh (1979)



For $p/q \in \mathbb{Q}$ with
 $0 < |p/q| < 1$, the sum of the
series

$$\sum_{n=2}^{\infty} \zeta(n)(p/q)^n$$

is a transcendental number.

For $p/q \in \mathbb{Q} \setminus \mathbb{Z}$,

$$\frac{\Gamma'}{\Gamma} \left(\frac{p}{q} \right) + \gamma$$

is transcendental (where γ is Euler's constant).

Golomb arithmetic function (1973)



Solomon W. Golomb
1932–2016

$\gamma(n)$: number of distinct representations of n in the form a^b with positive integers a and b .

Peter Bundschuh (1979). A consequence of Schanuel's Conjecture is the transcendence of

$$\sum_{n=2}^{\infty} \gamma(n)n^{-s}$$

for $s \geq 4$.

Peter Bundschuh (1979)

(P. Bundschuh) : As a consequence of *Nesterenko's Theorem*, the number

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + 1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} = 2.076\,674\,047\,4\dots$$

is transcendental, while

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4}$$

(telescoping series).

Hence the number

$$\sum_{n=2}^{\infty} \frac{1}{n^s - 1}$$

is transcendental over \mathbb{Q} for $s = 4$. The transcendence of this number for even integers $s \geq 4$ would follow as a consequence of *Schanuel's Conjecture*.

Sum of values of a rational function

Arithmetic nature of

$$\sum_{n \geq 1} \frac{P(n)}{Q(n)}$$

where

$$P/Q \in \mathbb{Q}(X).$$

In case Q has distinct zeroes, by decomposing P/Q in simple fractions one gets linear combinations of logarithms of algebraic numbers (Baker's method).

The example $P(X)/Q(X) = 1/X^3$ shows that the general case is hard :

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3}.$$

S. Gun, R. Murty, P. Rath

Multiple Gamma function of Barnes, defined by $\Gamma_0(z) = 1/z$,
 $\Gamma_1(z) = \Gamma(z)$,

$$\Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)},$$

with $\Gamma_n(1) = 1$.

Assuming *Schanuel's Conjecture*, one at least of the two following statements is true :

- (i) The two numbers π and $\zeta(3)$ are algebraically independent.
- (ii) The number $\Gamma_2(1/2)/\Gamma_3(1/2)$ is transcendental.



Sum of values of a rational function (2001)

Let P and Q be non-zero polynomials having rational coefficients and $\deg Q \geq 2 + \deg P$. Consider

$$\sum_{\substack{n \geq 0 \\ Q(n) \neq 0}} \frac{P(n)}{Q(n)}.$$

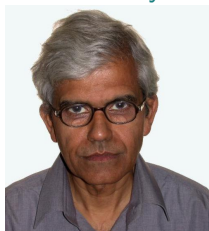
S.D. Adhikari



N. Saradha



T.N. Shorey



R. Tijdeman



Transcendental infinite sums. Indag. Math. (N.S.) **12** (2001), no. 1, 1–14.

Sum of values of a rational function

More recent references :

- [N. Saradha](#) and [R. Tijdeman](#),

On the transcendence of infinite sums of values of rational functions.

J. London Math. Soc. (2) 67 (2003), no. 3, 580–592.

- [R. Tijdeman](#),

On irrationally and transcendency of infinite sums of rational numbers.

in Diophantine equations, 279–296, Tata Inst. Fund. Res. Stud. Math., 20, Tata Inst. Fund. Res., Mumbai, 2008.

- [Ram Murty](#); [Chester J. Weatherby](#).

On the transcendence of certain infinite series.

Int. J. Number Theory 7 (2011), no. 2, 323–339.

Telescoping series

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1, \quad \sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4},$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{5n+2} - \frac{3}{5n+7} + \frac{1}{5n-3} \right) = \frac{5}{6}$$

Transcendental values

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}},$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3}$$

are transcendental.

Transcendental values

$$\sum_{n=0}^{\infty} \frac{1}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)}$$
$$= \frac{1}{4320} (192 \log 2 - 81 \log 3 - 7\pi\sqrt{3})$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} = 2.076\,674\,047\,4\dots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{2\pi}{e^{\pi} - e^{-\pi}} = 0.272\,029\,054\,982\dots$$

Catalan's Constant

Catalan's constant is

$$G = \sum_{n \geq 1} \frac{(-1)^n}{(2n+1)^2}$$
$$= 0.915\,965\,594\,177\,219\,015\,0\dots$$

Is it an irrational number?



Eugène Catalan
(1814 - 1894)

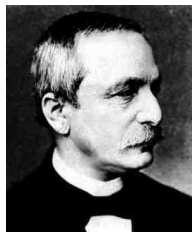
Catalan's constant, Dirichlet and Kronecker

Catalan's constant is the value at $s = 2$ of the Dirichlet L -function $L(s, \chi_{-4})$ associated with the Kronecker character

$$\chi_{-4}(n) = \left(\frac{n}{4}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod{4}, \\ -1 & \text{if } n \equiv -1 \pmod{4}. \end{cases}$$



Johann Peter Gustav Lejeune Dirichlet
1805 – 1859



Leopold Kronecker
1823 – 1891

Catalan's constant, Dedekind and Riemann

The Dirichlet L -function $L(s, \chi_{-4})$ associated with the Kronecker character χ_{-4} is the quotient of the Dedekind zeta function of $\mathbb{Q}(i)$ and the Riemann zeta function :

$$\zeta_{\mathbb{Q}(i)}(s) = L(s, \chi_{-4})\zeta(s)$$

$$G = L(2, \chi_{-4}).$$



Julius Wilhelm Richard
Dedekind
1831 – 1916



Georg Friedrich Bernhard
Riemann
1826 – 1866



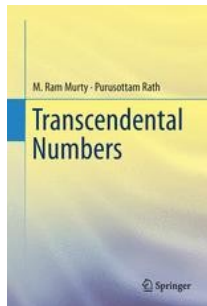
Assuming *Schanuel's Conjecture*, one at least of the two next statements is true :

- (i) The two numbers π and G are algebraically independent.
- (ii) The number $\Gamma_2(1/4)/\Gamma_2(3/4)$ is transcendental.

Further applications of Schanuel's Conjecture

Ram Murty ; Purusottam Rath.

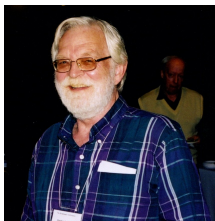
Transcendental numbers. Springer, New York, 2014.



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23 The Baker–Birch–Wirsing Theorem	131
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25 Linear Independence of Values of Dirichlet L -Functions	153
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28 Periods, Multiple Zeta Functions and $\zeta(3)$	185

- Transcendental sums related to the zeros of zeta functions.
- Linear and algebraic independence of generalized Euler-Briggs constants.
- Special values of L -functions.
- Transcendence of the log gamma function and some discrete periods.

Conjectures by S. Schanuel and A. Grothendieck



- **Schanuel** : if x_1, \dots, x_n are \mathbb{Q} -linearly independent complex numbers, then *n* at least of the $2n$ numbers $x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}$ are algebraically independent.
- Periods conjecture by **Grothendieck** : Dimension of the Mumford–Tate group of a smooth projective variety.

Motives



Y. André : generalization of Grothendieck's conjecture to motives.
Implies Schanuel's Conjecture.



Case of 1-motives :
Elliptico-Toric Conjecture of C. Bertolin.
(+ Philippon & Saha²)

Francis Brown

Irrationality proofs for zeta values, moduli spaces and dinner parties

arXiv:1412.6508 <http://www.ihes.fr/~brown/IrratModuliMotivesv8.pdf>

Slides : <http://www.ihes.fr/~brown/IrrationalitySlidesPrintable.pdf>

A simple geometric construction on the moduli spaces $\mathcal{M}_{0,n}$ of curves of genus 0 with n ordered marked points is described which gives a common framework for many irrationality proofs for zeta values. This construction yields Apéry's approximations to $\zeta(2)$ and $\zeta(3)$, and for larger n , an infinite family of small linear forms in multiple zeta values with an interesting algebraic structure. It also contains a generalisation of the linear forms used by Ball and Rivoal to prove that infinitely many odd zeta values are irrational.

Francis Brown

For k, s_1, \dots, s_k positive integers with $s_1 \geq 2$, we set $\underline{s} = (s_1, \dots, s_k)$ and

$$\zeta(\underline{s}) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} \dots n_k^{s_k}}.$$

The \mathbb{Q} -vector space \mathfrak{Z} spanned by the numbers $\zeta(\underline{s})$ is also a \mathbb{Q} -algebra. For $n \geq 2$, denote by \mathfrak{Z}_n the \mathbb{Q} -subspace of \mathfrak{Z} spanned by the real numbers $\zeta(\underline{s})$ where \underline{s} has weight $s_1 + \dots + s_k = n$.

The numbers $\zeta(s_1, \dots, s_k)$, $s_1 + \dots + s_k = n$, where each s_i is 2 or 3, span \mathfrak{Z}_n over \mathbb{Q} .



Chester Weatherby

Let q_1, \dots, q_k be natural numbers. For a complex valued function f on the group
 $G = \mathbb{Z}/q_1\mathbb{Z} \times \dots \times \mathbb{Z}/q_k\mathbb{Z}$,
consider the multiple Dirichlet series



$$F(s_1, \dots, s_k) = \sum_{n_1, \dots, n_k=1}^{\infty} \frac{f(n_1, \dots, n_k)}{n_1^{s_1} \dots n_k^{s_k}}.$$

Assuming the conjecture on algebraic independence of logarithms of algebraic numbers, [Chester Weatherby](#) proves that $F(1, \dots, 1)$ is either zero or transcendental.

[Ram Murty](#); [Chester J. Weatherby](#). On the transcendence of certain infinite series. *Int. J. Number Theory* **7** (2011), no. 2, 323–339.

Periods : Maxime Kontsevich and Don Zagier



Periods,
Mathematics
unlimited—2001
and beyond,
Springer 2001,
771–808.



A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in \mathbb{R}^n given by polynomial inequalities with rational coefficients.

The number π

Basic example of a *period* :

$$e^{z+2i\pi} = e^z$$

$$2i\pi = \int_{|z|=1} \frac{dz}{z}$$

$$\begin{aligned}\pi &= \iint_{x^2+y^2 \leq 1} dx dy = \int_{-\infty}^{\infty} \frac{dx}{1-x^2} \\ &= \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = 2 \int_{-1}^1 \sqrt{1-x^2} dx.\end{aligned}$$

Further examples of periods

$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$

and all algebraic numbers.

$$\log 2 = \int_{1 < x < 2} \frac{dx}{x}$$

and all logarithms of algebraic numbers.

$$\pi = \int_{x^2 + y^2 \leq 1} dx dy,$$

A product of periods is a period (subalgebra of \mathbb{C}), but $1/\pi$ is expected not to be a period.

Numbers which are not periods

Problem (Kontsevich–Zagier) : *To produce an explicit example of a number which is not a period.*

Several levels :

1 *analog of Cantor* : the set of periods is countable.
Hence there are real and complex numbers which are not periods (*most* of them).

Numbers which are not periods

2 *analog of Liouville*

Find a property which should be satisfied by all periods, and construct a number which does not satisfy that property.

Masahiko Yoshinaga, *Periods and elementary real numbers*
[arXiv:0805.0349](https://arxiv.org/abs/0805.0349)

Compares the periods with hierarchy of real numbers induced from computational complexities.

In particular, he proves that periods can be effectively approximated by elementary rational Cauchy sequences.

As an application, he exhibits a computable real number which is not a period.

Numbers which are not periods

3 analog of **Hermité**

Prove that given numbers are not periods

Candidates : $1/\pi$, **e**, **Euler** constant.

M. Kontsevich : exponential periods

The last chapter, which is at a more advanced level and also more speculative than the rest of the text, is by the first author only.

M. Kontsevich and **D. Zagier**. *Periods*, Mathematics unlimited—2001 and beyond, Springer 2001, 771–808.

Relations among periods

1 Additivity

(in the integrand and in the domain of integration)

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx,$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

2 Change of variables :

if $y = f(x)$ is an invertible change of variables, then

$$\int_{f(a)}^{f(b)} F(y) dy = \int_a^b F(f(x)) f'(x) dx.$$

Relations among periods (continued)



3 Newton–Leibniz–Stokes Formula

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Conjecture of Kontsevich and Zagier



A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following



Conjecture (Kontsevich–Zagier). *If a period has two integral representations, then one can pass from one formula to another by using only rules $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ in which all functions and domains of integration are algebraic with algebraic coefficients.*

Period Conjecture of Kontsevich and Zagier

We do not expect any miraculous coincidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.

*This conjecture, which is similar in spirit to the **Hodge** conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.*

Advice : *if you wish to prove a number is transcendental, first prove it is a period.*

இனிய பிறந்த நாள்
நல்வாழ்த்துக்கள்



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