Integer–valued functions, Hurwitz functions and related topics: a survey

by

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Abstract

An integer–valued function is an entire function which maps the nonnegative integers $\mathbb{N}$ to the integers. An example is 2^z. A Hurwitz function is an entire function having all derivatives taking integer values at 0. An example is e^z.

Lower bound for the growth order of such functions have a rich history. Many variants have been considered: for instance, assuming that the first k derivatives at the integers are integers, or assuming that the derivatives at k points are integers. We survey some of these variants.

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1 On the sequence of values $(f(0), f(1), f(2), \ldots)$

The topic of integer-valued entire functions was initiated by Pólya's fundamental result on transcendental entire functions taking their values in $\mathbb{Z}$ at each point in $\{0, 1, 2, \ldots\}$

1.1 Order and type of an entire function

Let f be an entire function. For $r \geq 0$ we define $|f|_r = \sup_{|z| \leq r|} |f(z)|$. From the maximum modulus principle we deduce $|f|_r = \sup_{|z|=r|} |f(z)|$.

The order of an entire function f is

$$\varrho(f) = \limsup_{r \to \infty} \frac{\log \log |f|_r}{\log r},$$

while the exponential type of an entire function is

$$\tau(f) = \limsup_{r \to \infty} \frac{\log |f|_r}{r} = \limsup_{n \to \infty} |f^{(n)}(z_0)|^{\frac{1}{n}} \quad (z_0 \in \mathbb{C}).$$

We use the notation:

$$f^{(n)}(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} f(z).$$

If the exponential type is finite, then f has order ≤ 1. If f has order < 1, then the exponential type is 0. For $\tau \in \mathbb{C} \setminus \{0\}$, the function $\mathrm{e}^{\tau z}$ has order 1 and exponential type $|\tau|$.

For $\varrho > 0$, we define

$$\tau_\varrho(f) = \limsup_{r \to \infty} \frac{\log |f|_r}{r^\varrho},$$

so that $\tau_1(f) = \tau(f)$. If f has order $< \varrho$, then $\tau_\varrho(f) = 0$. If f has order $> \varrho$, then $\tau_\varrho(f) = +\infty$.

1.2 Functions vanishing at $0, 1, 2, \dots$

Let us consider the entire functions which vanish at each point in $\mathbb{N} = \{0, 1, 2, \dots\}$ An example of such a function is the Weierstrass canonical product for $\mathbb{N}$, namely (Titchmarsh, 1939, § 4.41 and § 8.4.(vi)),

$$\frac{1}{\Gamma(-z)} = -z\mathrm{e}^{-\gamma z} \prod_{m=1}^{\infty} \left(1 - \frac{z}{m}\right) \mathrm{e}^{\frac{z}{m}}$$

(Hadamard product for the Gamma function). This is an entire function of order 1 and infinite type (Gel'fond, 1952, Chap. 2 § 1.3 (17)). The "smallest" entire function vanishing at each point in $\mathbb{N}$ is

$$\sin(\pi z) = \pi z \prod_{m \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{m}\right) \mathrm{e}^{\frac{z}{m}}$$

which in fact vanishes at each point in $\mathbb{Z}$ (Weierstrass canonical product for $\mathbb{Z}$, Hadamard product for the sine function), which has order 1 and exponential type π. See (Titchmarsh, 1939, § 3.23 and § 8.4.(v)), and (Boas, 1954, § 9.4). Indeed, a well–known theorem from the thesis of F. Carlson in 1914 using the Phragmén–Lindelöf principle states that there is no nonzero entire function f of exponential type $< \pi$ satisfying $f(\mathbb{N}) = \{0\}$ – see for instance (Titchmarsh, 1939, § 5.81), (Gel'fond, 1952, Chap. 2, § 3 Th. VII and Chap. 3, § 3.1), (Boas, 1954, Th. 9.2.1), (Boas and Buck, 1964, Chap. IV §19).

1.3 Integer–valued functions

An *integer–valued function* is an entire function f such that $f(n) \in \mathbb{Z}$ for all $n \geq 0$. A basis of the $\mathbb{Z}$–module of polynomials in $\mathbb{C}[z]$ which map $\mathbb{N}$ to $\mathbb{Z}$ is given by the polynomials

$$1, z, \frac{z(z-1)}{2}, \dots, \frac{z(z-1)\cdots(z-m+1)}{m!}, \dots \tag{1}$$

(Narkiewicz, 1995), (Cahen and Chabert, 1997), which are sometimes called (in papers from transcendental number theory) the Feld'man's polynomials, after N.I. Feldman introduced them for improving Baker's lower bounds for linear forms in logarithms of algebraic numbers – see for instance (Bugeaud, 2018).

The simplest example of a transcendental integer–valued function is 2^z. In 1915, in his seminal paper (Pólya, 1915), G. Pólya proved that if f is an integer–valued entire function such that

$$\lim_{r \to \infty} |f|_r 2^{-r} \sqrt{r} = 0,$$

then f is a polynomial.

The method of Pólya is to expand f as an interpolation series

$$f(z) = a_0 + a_1 z + a_2 \frac{z(z-1)}{2} + \cdots + a_m \frac{z(z-1)\cdots(z-m+1)}{m!} + \cdots \tag{2}$$

and to estimate the coefficients a_m. See for instance (Boas, 1954, Th. 9.12.1) and (Masser, 2016, Chap. 9).

Pólya conjectured that the conclusion of his theorem should be true under the weaker assumption

$$\lim_{r\to\infty} |f|_r 2^{-r} = 0.$$

This refinement was achieved by (Hardy, 1917). See also (Landau, 1920) and (Whittaker, 1935, § 11 Th. 11). In (Pólya, 1920), the same conclusion is obtained under the weaker assumption

$$\limsup_{r\to\infty} |f|_r 2^{-r} < 1;$$

further, if $|f|_r = O(2^r r^k)$ for some $k > 0$, then $f(z)$ is of the form $P(z)2^z + Q(z)$ where P and Q are polynomials with rational coefficients. It is interesting to note that there are only countably many such functions.

Twenty years later, (Selberg, 1941b) proved that if an integer–valued function f satisfies

$$\tau(f) \le \log 2 + \frac{1}{1500},$$

then $f(z)$ is of the form $P(z)2^z + Q(z)$. (Pisot, 1942) went one step further: if an integer–valued function f has exponential type ≤ 0.8, then f is of the form

$$P_0(z) + 2^z P_1(z) + \gamma^z P_2(z) + \overline{\gamma}^z P_3(z),$$

where P_0, P_1, P_2, P_3 are polynomials and

$$\gamma = \frac{3 + i\sqrt{3}}{2}, \quad \overline{\gamma} = \frac{3 - i\sqrt{3}}{2}$$

are the roots of the polynomial $z^2 - 3z + 3$. See (Boas, 1954, Th. 9.12.2), (Buck, 1948a). This contains the result of Selberg, since $|\log \gamma| = 0.758\cdots > \log 2 = 0.693\ldots$, Pisot (Pisot, 1946b) obtained more general results for functions of exponential type $< 0.9934\ldots$ with additional terms; he also investigated the growth of transcendental entire functions f with values $f(n)$ close to integers: $f(n) = u_n + \epsilon_n$ with $u_n \in \mathbb{Z}$, $|\epsilon_n| < \kappa^n$ for sufficiently large n and $0 < \kappa < 1$. Functions which are almost integer-valued were considered by (Gel'fond, 1929c) and by (Buck, 1946, Th. 6.3) and more recently by (Hirata, 1985). Besides, (Buck, 1946, Th. 7.1) considered the problem of transcendental entire functions taking prime values at the integers.

1.4 Completely integer–valued functions

A *completely integer–valued function* is an entire function which takes values in $\mathbb{Z}$ at all points in $\mathbb{Z}$.

Let $u > 1$ be a quadratic unit, root of a polynomial $X^2 + aX + 1$ for some $a \in \mathbb{Z}$. Then the functions

$$u^z + u^{-z} \text{ and } \frac{u^z - u^{-z}}{u - u^{-1}}$$

are completely integer–valued functions of exponential type $\log u$.

Examples of such quadratic units are the roots of the polynomial $X^2 - 3X + 1$:

$$\theta = \frac{3+\sqrt{5}}{2} = 2.354\ldots, \quad \theta^{-1} = \frac{3-\sqrt{5}}{2} = 0.424\ldots$$

Hence, examples of completely integer–valued functions are

$$\theta^z + \theta^{-z} \quad \text{and} \quad \frac{1}{\sqrt{5}}(\theta^z - \theta^{-z}),$$

both of exponential type $\log \theta = 0.962\ldots$

Notice that $\theta = \phi^2$ where ϕ is the Golden ratio $\frac{1+\sqrt{5}}{2}$. Let $\tilde{\phi} = -\phi^{-1}$, so that

$$X^2 - X - 1 = (X - \phi)(X - \tilde{\phi}).$$

For any $m \in \mathbb{Z}$ we have

$$\phi^m + \tilde{\phi}^m \in \mathbb{Z}.$$

The function $\phi^z = \exp(z \log \phi)$ has exponential type $\log \phi < \log 2$ and we have $\log |\tilde{\phi}| = -\log \phi$. However, $\phi^z + \tilde{\phi}^z$ is not a counterexample to Pólya's result on the growth of transcendental integer-valued entire function (Masser, 2016, Chap. 9 p. 108). Indeed, while $\phi^z = \exp(z \log \phi)$ is well defined since $\phi > 0$, the definition of $\tilde{\phi}^z$ requires to choose a logarithm of the negative number $\tilde{\phi} = -\phi^{-1}$. With $\log \tilde{\phi} = -\log \phi + i\pi$, the function $\tilde{\phi}^z = \exp(z \log \tilde{\phi})$ has exponential type $((\log \phi)^2 + \pi^2)^{\frac{1}{2}} = 3.178\cdots > \log 2$.

According to (Pólya, 1915), a completely integer–valued function f which satisfies

$$\lim_{r\to\infty} |f|_r \theta^{-r} r^{\frac{3}{2}} = 0$$

is a polynomial. In (Pólya, 1920), it is proved that a completely integer–valued function f which satisfies

$$\limsup_{r\to\infty} |f|_r \theta^{-r} r^k < \infty$$

for some $k > 0$ is of the form

$$P_0(z) + P_1(z)\theta^z + P_2(z)\theta^{-z}$$

where P_0, P_1, P_2 are polynomials.

(Carlson, 1921) gave refined results by means of the Laplace transform – see (Pólya, 1929), (Whittaker, 1935, § 10), (Buck, 1948a, Th. 5.2, Cor. 2), (Gel'fond, 1952, Chap. 3, § 2), (Boas, 1954, § 9.12), (Robinson, 1971). According to (Selberg, 1941b), if a completely integer–valued function satisfies

$$\tau(f) \leq \log\theta + 2 \cdot 10^{-6},$$

then f is of the form

$$P_0(z) + P_1(z)\theta^z + P_2(z)\theta^{-z}$$

where P_0, P_1, P_2 are polynomials.

1.5 Further results

Among the surveys on these topics, let us quote (Gramain, 1978), (Gramain, 1986), (Gramain and Schnitzer, 1989), (Gramain, 1990) and (Masser, 2016, Chap. 9 and 10).

Further results have been proved on the growth of entire functions satisfying $f(\mathbb{N}) \subset \mathbb{K}$ or $f(\mathbb{Z}) \subset \mathbb{K}$, where $\mathbb{K}$ is the field of algebraic numbers. Assuming suitable growth conditions on $|f|_r$, on the algebraic numbers $f(m)$ and all of their conjugates, one deduces that f is a polynomial. Interpolation formulae yield such results, and also methods from transcendental number theory. We come back to this topic in § 5.2.

2 On the sequence of values $(f(0), f'(0), f''(0), \dots)$

In place of the values of f at the integers, we now consider the derivatives of f at 0. The sequence of polynomials (1) is replaced by the polynomials $\frac{z^n}{n!}$, while the interpolation series (2) is replaced by Taylor expansion.

2.1 Hurwitz functions

A *Hurwitz function* is an entire function f such that $f^{(n)}(0) \in \mathbb{Z}$ for all $n \geq 0$. Polynomials in $\mathbb{Q}[z]$ of the form

$$\sum_{n=0}^{N} a_n \frac{z^n}{n!}$$

with $a_0, a_1, \dots, a_N$ in $\mathbb{Z}$ are Hurwitz functions, while the exponential function

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + \frac{z^n}{n!} + \cdots$$

is a transcendental Hurwitz function. A basis of the $\mathbb{Z}$–module of polynomials in $\mathbb{C}[z]$ which are Hurwitz functions is given by $1, z, \frac{z^2}{2}, \dots, \frac{z^n}{n!}, \dots$

The first lower bound for the growth of a transcendental Hurwiz function is due to (Kakeya, 1916), who proved that a Hurwitz function satisfying

$$\limsup_{r\to\infty} |f|_r \mathrm{e}^{-r}\sqrt{r} = 0$$

is a polynomial. This was refined by (Pólya, 1921) – see also (Pólya and Szegő, 1998, Part VIII, Chap. 3, § 6, n°187): a Hurwitz function satisfying

$$\limsup_{r\to\infty} |f|_r \mathrm{e}^{-r}\sqrt{r} < \frac{1}{\sqrt{2\pi}} = 0.398\dots$$

is a polynomial. This is best possible for uncountably many functions, as shown by the functions

$$f(z) = \sum_{n\geq 0} \frac{e_n}{2^n!} z^{2^n}$$

with $e_n \in \{1, -1\}$ which satisfy

$$\limsup_{r\to\infty} |f|_r \mathrm{e}^{-r}\sqrt{r} = \frac{1}{\sqrt{2\pi}}\cdot$$

2.2 Refined estimates by Sato and Straus

More precise results are achieved by D. Sato and E.G. Straus thanks to a careful study of the function

$$\phi(r) = \max_{n\geq 0} \frac{r^n}{n!}\cdot$$

In (Sato and Straus, 1964, Corollary 1 p. 304) and (Sato and Straus, 1965, Corollary p. 20) (see also (Sato, 1971)), they proved that for every $\epsilon > 0$, there exists a transcendental Hurwitz function with

$$\limsup_{r\to\infty} |f|_r \mathrm{e}^{-r}\sqrt{2\pi r}\left(1 + \frac{1+\epsilon}{24r}\right)^{-1} < 1,$$

while every Hurwitz function for which

$$\limsup_{r\to\infty} |f|_r \mathrm{e}^{-r}\sqrt{2\pi r}\left(1 + \frac{1-\epsilon}{24r}\right)^{-1} \leq 1$$

is a polynomial.

3 Several points or several derivatives

3.1 Introduction

There are several natural ways to mix integer–valued functions and Hurwitz functions: one may include finitely many derivatives in the study of integer–valued functions, yielding to the study of *k–times integer–valued functions*. One may consider entire functions having all their derivatives at k points taking integer values, yielding to the study of *k–point Hurwitz functions*. One may consider entire functions f which satisfy $f^{(n)}(m) \in \mathbb{Z}$ for all $n \geq 0$ and $m \geq 0$ (resp. $m \in \mathbb{Z}$), which are the *utterly* (resp. *completely utterly*) *integer–valued functions*. Finally, the study of entire functions f such that $f^{(n)}(n) \in \mathbb{Z}$ is related with Abel's interpolation.

In Figures 1 to 6, we display horizontally the rational integers and vertically the derivatives.

Integer–valued functions: horizontal
Figure 1

$$\begin{array}{ccccccc} f & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ & 0 & 1 & 2 & \cdots & m & \cdots \end{array}$$

Hurwitz functions: vertical
Figure 2

$$\begin{array}{cc} & \vdots \\ f^{(n)} & \bullet \\ & \vdots \\ f' & \bullet \\ f & \bullet \\ & 0 \end{array}$$

2–times integer–valued functions
Figure 3

$$\begin{array}{ccccccc} f' & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ f & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ & 0 & 1 & 2 & \cdots & m & \cdots \end{array}$$

2–point Hurwitz functions
Figure 4

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ f^{(n)} & \bullet & \bullet \\ \vdots & \vdots & \vdots \\ f' & \bullet & \bullet \\ f & \bullet & \bullet \\ & 0 & 1 \end{array}$$

Utterly integer–valued
entire functions
Figure 5

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & & \vdots & \\
f^{(n)} & \bullet & \bullet & \cdots & \bullet & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
f' & \bullet & \bullet & \cdots & \bullet & \cdots \\
f & \bullet & \bullet & \cdots & \bullet & \cdots \\
 & 0 & 1 & \cdots & m & \cdots
\end{array}
$$

Abel interpolation
Figure 6

$$
\begin{array}{cccccc}
\vdots & & & & & \iddots \\
f^{(n)} & & & & \bullet & \\
\vdots & & & \iddots & & \\
f' & & \bullet & & & \\
f & \bullet & & & & \\
 & 0 & 1 & \cdots & n & \cdots
\end{array}
$$

3.2 k–times integer–valued functions

The first natural way to mix integer–valued functions and Hurwitz functions is *horizontal*, including finitely many derivatives in the study of integer–valued functions, like in (Gel'fond, 1929a), (Selberg, 1941a). Let us call *k–times integer–valued function* an entire function f such that $f^{(n)}(m) \in \mathbb{Z}$ for all $m \geq 0$ and $n = 0, 1, \ldots, k-1$. According to (Gel'fond, 1929b), a k–times integer–valued function of exponential type $< k \log \left(1 + \mathrm{e}^{-\frac{k-1}{k}}\right)$ is a polynomial. A proof is given in (Fridman, 1968). Improvements are due to (Selberg, 1941b), and (Bundschuh and Zudilin, 2004) who also investigated what could be the best possible results. The best known results so far are due to (Welter, 2005b).

3.3 k–point Hurwitz functions

The second solution is *vertical*, considering *k–point Hurwitz functions*, namely entire functions having all their derivatives at $0, 1, \ldots, k-1$ taking integer–values. This question was first considered by (Gel'fond, 1934). It has been proved by (Straus, 1950) that the order of such a function is $\geq k$, and this is best possible, as shown by the function $\mathrm{e}^{z(z-1)\cdots(z-k+1)}$.

Precise results are known for $k = 2$. (Sato, 1971) proved that there exist transcendental two point Hurwitz entire functions with

$$|f|_r \leq \exp\left(r^2 + r - \log r + O(1)\right),$$

while every two point Hurwitz entire functions with

$$|f|_r \le C\exp\left(r^2 - r - \log r\right)$$

for some positive constant C must be a polynomial.

Another example is (Straus, 1950, Th. 3): if f is a transcendental entire function and $f^{(n)}(z)$ is in $\mathbb{Z}$ for $z = 0$ and $z = \frac{p}{q}$ with $\gcd(p, q) = 1$, then f is at least of order 2 and type $\tau_2(f) \ge \frac{q}{p}$. A similar estimate from (Straus, 1950, Th. 2) holds when $f^{(n)}(z)$ is in $\mathbb{Z}$ for $z = 0, \frac{p_1}{q_1}, \dots, \frac{p_{k-1}}{q_{k-1}}$.

For $k \ge 3$ our knowledge is more limited. (Bieberbach, 1953) stated that if a transcendental entire function f of order ϱ is a k–point Hurwitz entire function, then either $\varrho > k$, or $\varrho = k$ and the type $\tau_k(f)$ of f satisfies $\tau_k(f) \ge 1$. However, as noted by (Fridman, 1968) and (Sato, 1971, p. 2), this result is not true. Indeed, the polynomial

$$a(z) = \frac{1}{2}z(z-1)(z-2)(z-3)$$

can be written

$$a(z) = \frac{1}{2}z^4 - 3z^3 - \frac{11}{2}z^2 - 3z,$$

and hence it satisfies $a'(z) \in \mathbb{Z}[z]$; it follows that the function $f(z) = \mathrm{e}^{a(z)}$ is a transcendental 4-point Hurwitz function of order $\varrho = 4$ and $\tau_4(f) = \frac{1}{2}$. More generally, for $k \ge 1$, the polynomial

$$a_k(z) = \frac{1}{2}z(z-1)(z-2)\cdots(z-4k+1),$$

has $a_k'(z) \in \mathbb{Z}[z]$, hence the function $f_k(z) = \mathrm{e}^{a_k(z)}$ is a transcendental $4k$-point Hurwitz function of order $\varrho = 4k$ and $\tau_\varrho(f_k) = \frac{1}{2}$.

3.4 Utterly integer–valued functions

An *utterly integer–valued function* is an entire function f which satisfies $f^{(n)}(m) \in \mathbb{Z}$ for all $n \ge 0$ and $m \in \mathbb{Z}$ with $m \ge 0$. A *completely utterly integer–valued function* is an entire function f which satisfies $f^{(n)}(m) \in \mathbb{Z}$ for all $n \ge 0$ and all $m \in \mathbb{Z}$. From either the results on k–point Hurwitz functions or the results on k–times integer–valued functions, it follows that a transcendental utterly integer–valued function must be of infinite order. One also deduces the irrationality of any nonzero period of a nonconstant Hurwitz function of finite order – compare with § 5.2.

(Straus, 1951) suggested that transcendental utterly integer–valued functions may not exist. He proposed:

> *Show that the only (entire) analytic functions which have integral (Gaussian integral) derivatives at all (infinitely many) integers (Gaussian integers) are polynomials.*

(Fridman, 1968) showed that there exist transcendental utterly integer–valued functions f with

$$\limsup_{r\to\infty} \frac{\log\log |f|_r}{r} \le \pi$$

and proved that an utterly integer–valued function f satisfying

$$\limsup_{r\to\infty} \frac{\log\log |f|_r}{r} < \log\left(1+\frac{1}{\mathrm{e}}\right) = 0.313\ldots$$

is a polynomial.

(Sato, 1985) constructed a nondenumerable set of completely utterly integer–valued functions. He selected inductively the coefficients a_n with

$$\frac{1}{n!(2\pi)^n} \le |a_n| \le \frac{3}{n!(2\pi)^n}$$

and defined

$$f(z) = \sum_{n\ge 0} a_n \sin^n(2\pi z).$$

The above-mentioned bound $\log\left(1+\frac{1}{\mathrm{e}}\right)$ of (Fridman, 1968) was improved by (Welter, 2005a) to $\log 2 = 0.693\ldots$ Hence a transcendental utterly integer–valued function grows at least like the double exponential e^{2^z}. For completely utterly integer–valued functions, the bound in (Welter, 2005a) is $\log\left(\frac{3+\sqrt{5}}{2}\right) = 0.962\ldots$

3.5 Abel interpolation

There is also a *diagonal* way of mixing the study of integer–valued functions and Hurwitz functions by considering entire functions f such that $f^{(n)}(n) \in \mathbb{Z}$.

The source of this question goes back to (Abel, 1881a), (Abel, 1881b). The related interpolation problem was studied by (Halphén, 1882a), (Halphén, 1882b). See also (Pareto, 1892), (Gontcharoff, 1930), (Buck, 1946) and (Buck, 1948b, § 7 and § 10 Corollary 5). Further references are (Whittaker, 1935, Chap. III), (Gel'fond, 1952, Chap. III, § 3.2), (Boas, 1954, § 9.10), (Bézivin, 1992).

The sequence of polynomials $(P_n)_{n\ge 0}$ defined by

$$P_0 = 1, \quad P_n(z) = \frac{1}{n!} z(z-n)^{n-1} \qquad (n \ge 1)$$

was introduced by (Abel, 1881b). These polynomials satisfy

$$P_n'(z) = P_{n-1}(z-1) \quad (n \ge 1).$$

Therefore $P_n^{(k)}(k) = \delta_{k,n}$ for k and $n \geq 0$, where $\delta_{k,n}$ the Kronecker symbol:

$$\delta_{k,n} = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

One deduces that any polynomial f has a finite expansion

$$f(z) = \sum_{n \geq 0} f^{(n)}(n) P_n(z).$$

It was proved by (Halphén, 1882b) (see also (Gontcharoff, 1930, p. 31)) that such an expansion (with a series in the right hand side which is absolutely and uniformly convergent on any compact of $\mathbb{C}$) holds also for any entire function f of finite exponential type $< \omega$, where $\omega = 0.278\ldots$ is the positive real number defined by $\omega \mathrm{e}^{\omega+1} = 1$. The proof rests on Laplace transform – see (Gel'fond, 1952, Chap. 3 § 2 p.209).

Lower bounds for the growth of entire functions satisfying $f^{(n)}(n) \in \mathbb{Z}$ were investigated by (Bertrandias, 1958). See also (Wallisser, 1969). The method arises from (Pisot, 1946a), and (Pisot, 1946b).

For $t \in \mathbb{C}$, the function $f_t(z) := \mathrm{e}^{tz}$ satisfies the functional differential equation

$$f_t'(z) = t\mathrm{e}^t f_t(z-1)$$

with the initial condition $f_t(0) = 1$, and hence $f_t^{(n)}(n) = (t\mathrm{e}^t)^n$ for all $n \geq 0$. Let $\tau_0 = 0.567\ldots$ be the positive real number defined by $\tau_0 \mathrm{e}^{\tau_0} = 1$. We have $f_{\tau_0}^{(n)}(n) = 1$ for all $n \geq 0$.

Following (Bertrandias, 1958), an entire function f of exponential type $< \tau_0$ such that $f^{(n)}(n) \in \mathbb{Z}$ for all sufficiently large integers $n \geq 0$ is a polynomial. More precisely, let τ_1 be the complex number defined by $\tau_1 \mathrm{e}^{\tau_1} = \frac{1+i\sqrt{3}}{2}$; its modulus is $|\tau_1| = 0.616\ldots$ Then an entire function f of exponential type $< |\tau_1|$ such that $f^{(n)}(n) \in \mathbb{Z}$ for all sufficiently large integers $n \geq 0$ is of the form $P(z) + Q(z)\mathrm{e}^{\tau_0 z}$, where P and Q are polynomials.

4 Variants

We do not claim to quote all variations and related works around this theme; we only give a few examples among those which would deserve further surveys.

• Several variables: (Lang, 1965), (Gross, 1965), (Baker, 1967), (Gross, 1969), (Avanissian and Gay, 1975), (Gramain, 1977), (Gramain, 1978), (Bundschuh, 1980), (Bézivin, 1984), (Gramain, 1986), (Gramain, 1990), (Rivoal and Welter, 2009) ...

• Analogs in finite characteristic: (Car, 1997), (Car, 2001), (Delamette, 2003), (Adam, 2004), (Adam and Hirata-Kohno, 2006), (Adam, 2011), (Adam and Welter, 2015), ...

• q analogues and multiplicative versions (geometric progressions): (Gel'fond, 1933), (Gel'fond, 1952, Chap. 2, § 3.4, Th. VIII), (Gel'fond, 1967), (Kaz'min, 1973), (Bézivin, 1984), (Wallisser, 1985), (Gramain, 1990), (Bézivin, 1990a), (Bézivin, 1990b), (Bézivin, 1991), (Bundschuh, 1992), (Bézivin, 1994a), (Bézivin, 1994b), (Bundschuh and Shiokawa, 1995), (Bézivin, 1998), (Welter, 2000), (Welter, 2004), (Welter, 2005a), (Welter, 2005b), (Welter, 2005c), (Bézivin, 2014) ...

• An integer-valued function $f(n)$ is said to be *congruence preserving* if, for all natural numbers a, b and m, the condition $a \equiv b \pmod{m}$ implies that $f(a) \equiv f(b) \pmod{m}$ (Ruzsa, 1971), (Hall, 1971), (Perelli and Zannier, 1981), (Perelli and Zannier, 1984), (Zannier, 1996). For a q–analogue, see (Bézivin, 1998). The notion of *concordant sequences* was introduced in (Pila and Rodriguez Villegas, 1999); see (Pila, 2002), (Pila, 2003), (Pila, 2005), (Pila, 2008), and (Pila, 2009).

• Definable functions in o-minimal structures. The seminal paper is (Bombieri and Pila, 1989). As an example, (Wilkie, 2016) proves that *if $f : \mathbb{R} \to \mathbb{R}$ is an $\mathbb{R}_{\mathrm{an,exp}}$ – definable function such that, for some $r \in \mathbb{R}$ with $0 < r < 1$, $|f(x)| \leq 2^{rx}$ for all sufficiently large x and such that $f(n) \in \mathbb{Z}$ for all sufficiently large $n \in \mathbb{N}$, then there exists a polynomial P with rational coefficients such that $f(x) = P(x)$ for all sufficiently large x.* Variations on this result are given in (Jones et al., 2012), (Jones and Qiu, 2020) and (Chalebgwa, 2020b). For a survey of related results, including works by E. Bombieri and J. Pila, J. Pila, D.W. Masser, E. Besson, A. Surroca, G. Boxall and G. Jones. see (Chalebgwa, 2020a). Similar results are obtained for meromorphic functions in (Chalebgwa, 2020c), involving Nevanlinna theory.

• A number of results have been extended to non entire functions. Among a wealth of references we only quote (Yoshino, 1984), (Langley, 2007) and (Rivoal and Welter, 2009).

• p-adic functions: (Hilliker, 1968), (Hilliker and Straus, 1970), (Laohakosol et al., 1990) [See the review of the last item by Takashi Harase in `MR1058234`].

• As shown in (Ferguson, 2021), there exist entire functions $f(z)$, with Taylor expansion at the origine having computable rational coefficients, for which the following decision problem is undecidable:

> `Input`*: An integer $n \geq 0$.* `Question`*: Is $f(n) \in \mathbb{Q}$?*

5 Connection with transcendental number theory

5.1 From $\mathbb{Z}$ to $\mathbb{Z}[i]$

Historically, the very first steps towards a solution of Hilbert's seventh on the transcendence of a^b came from the extension by (Fukasawa, 1928) and (Gel'fond, 1929a) of Pólya's result on integer–valued functions on $\mathbb{N}$ to integer–valued functions on $\mathbb{Z}[i]$.

According to A.O. Gel'fond, there exists an absolute constant $\gamma > 0$ such that a transcendental entire function f such that $f(a + ib) \in \mathbb{Z}[i]$ for all $a + ib \in \mathbb{Z}[i]$ satisfies

$$\limsup_{R\to\infty} \frac{1}{R^2} \log |f|_R \geq \gamma.$$

For the proof, Gel'fond expands $f(z)$ into a Newton interpolation series at the Gaussian integers. He obtained $\gamma \geq 10^{-45}$.

Since the canonical product associated with the lattice $\mathbb{Z}[i]$, namely the Weierstrass sigma function

$$\sigma(z) = z \prod_{\omega \in \mathbb{Z}[i] \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right),$$

is an entire function vanishing on $\mathbb{Z}[i]$ of order 2 with $\tau_2(\sigma) = \frac{\pi}{2}$ (Hurwitz and Courant, 1944, Part 2, Chap. I, § 13), (Pólya and Szegő, 1998, Part IV, Chap. 1, § 3, n°49), (Masser, 2016, Exercises 10.4 and 20.98b), the best (largest) admissible value for γ satisfies

$$10^{-45} \leq \gamma \leq \frac{\pi}{2}\cdot$$

The study of entire functions f satisfying $f(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$ was pursued by (Cayford, 1969) and (Gruman, 1980). The exact value is $\gamma = \dfrac{\pi}{2e}$: (Masser, 1980) proved the upper bound and (Gramain, 1981) the lower bound. See (Masser, 2016, Chap. 10).

A side effect of these works is the introduction of the so–called Masser–Gramain–Weber constant (Masser, 1980), (Gramain and Weber, 1985), (Masser, 2016, Exercise 10.10), an analog of Euler's constant for $\mathbb{Z}[i]$, which arises in a 2–dimensional analogue of Stirling's formula:

$$\delta = \lim_{n \to \infty} \left(\sum_{k=2}^{n} (\pi r_k^2)^{-1} - \log n \right),$$

where r_k is the radius of the smallest disc in $\mathbb{R}^2$ that contains at least k integer lattice points inside it or on its boundary. In (Melquiond et al., 2013) the first four digits are computed:

$$1.819776 < \delta < 1.819833,$$

disproving a conjecture of (Gramain, 1982).

5.2 Transcendence

One of the main motivations of studying this kind of problems arises from transcendental number theory. See for instance the application to the Hermite–Lindemann Theorem in (Gel'fond, 1952, Chap. 2, § 3.4, Th. IX).

(Straus, 1950) developed the subject of integer–valued functions in connection with transcendental number theory; he deduces the Hermite–Lindemann Theorem from his (Straus, 1950, Th. 4). The footnote in (Straus, 1950, p. 188) reads: *Since this paper was written there has appeared the paper* (Schneider, 1949) *which contains an approach very similar*, which *is more profound*, and whose *results are much more complete.* Schneider's work gave rise to the so–called Schneider–Lang Criterion (Lang, 1962).

One may notice that a main assumption in the final version of the Schneider–Lang criterion is that the functions satisfy a differential equation; hence this criterion does not contain (Schneider, 1949, Satz III) nor (Straus, 1950, Th. 4) where no such condition occurs. For instance, one easily deduces from either of these two results *the transcendence of any nonzero period of a nonconstant Hurwitz function of finite order*. The main example of course is the transcendence of π.

When there is no assumption on the derivatives, one may apply the transcendence method introduced by Th. Schneider for the solution of Hilbert's seventh problem – see for instance (Waldschmidt, 1978). Examples are given in (Gramain and Mignotte, 1983), (Gramain et al., 1986), (Rochev, 2007), (Rochev, 2011) and (Ably, 2011). A connection with the six exponentials Theorem is introduced in (Pila, 2008).

Chapters 9 and 10 of (Masser, 2016) are devoted to integer valued entire functions in the context of transcendental number theory.

6 Lidstone interpolation and generalizations

We conclude by stating some new results related with integer–valued functions. The proofs are given in (Waldschmidt, 2020a) and (Waldschmidt, 2020b).

The definition of Hurwitz functions involves the sequence of derivatives of an entire function f at the origin, that is $\bigl(f^{(n)}(0)\bigr)_{n\geq 0}$. The Taylor expansion of f introduces the sequence of polynomials $\bigl(\frac{z^n}{n!}\bigr)_{n\geq 0}$. There are other similar expansions for entire functions of finite exponential type. If the exponential type of an entire function f is $<\pi$, then f is determined by the two sequences $\bigl(f^{(2n)}(0)\bigr)_{n\geq 0}$ and $\bigl(f^{(2n)}(1)\bigr)_{n\geq 0}$; more precisely, f is given by a series involving the sequence of Lidstone polynomials $\bigl(\Lambda_n(z)\bigr)_{n\geq 0}$. If the exponential type of an entire function f is $<\frac{\pi}{2}$, then f is determined by the two sequences $\bigl(f^{(2n)}(1)\bigr)_{n\geq 0}$ and $\bigl(f^{(2n+1)}(0)\bigr)_{n\geq 0}$; more precisely, f is given by a series involving the sequence of Whittaker polynomials $\bigl(M_n(z)\bigr)_{n\geq 0}$. References for two points are given in (Waldschmidt, 2020a). These studies have been extended to m points, by Poritsky for the sequences $\bigl(f^{(mn)}(s_j)\bigr)_{n\geq 0}$ $(j=0,1,\dots,m-1)$ and Gontcharoff for sequences like $\bigl(f^{(mn+j)}(s_j)\bigr)_{n\geq 0}$ $(j=0,1,\dots,m-1)$. References for several points are given in (Waldschmidt, 2020b). Building on these expansions, we are able to investigate the arithmetic situation as follows.

6.1 Arithmetic result for Poritsky and Lidstone interpolation

Theorem 1. *Let $s_0, s_1, \dots, s_{m-1}$ be distinct complex numbers and f an entire function of sufficiently small exponential type. If*

$$f^{(mn)}(s_j)\in\mathbb{Z}$$

for all sufficiently large n and for $0\leq j\leq m-1$, then f is a polynomial.

Figure 7 displays the case $m = 2$, Figure 9 the case $m = 3$.

For $m = 2$ (Lidstone interpolation), with $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$, the assumption on the exponential type $\tau(f)$ of f is

$$\tau(f) < \min\left\{1, \frac{\pi}{|s_1 - s_0|}\right\},$$

and this is best possible, as shown by the functions

$$f(z) = \frac{\sinh(z - s_1)}{\sinh(s_0 - s_1)} \quad \text{and} \quad f(z) = \sin\left(\pi \frac{z - s_0}{s_1 - s_0}\right),$$

which have exponential type 1 and $\frac{\pi}{|s_1 - s_0|}$ respectively.

When $|s_1 - s_0| \leq 2$, there is a nondenumerable set of entire functions f of exponential type ≤ 1 satisfying $f^{(2n)}(s_0) = 0$ and $f^{(2n)}(s_1) \in \{-1, 0, 1\}$ for all $n \geq 0$, for which the set $\{n \geq 0 \mid f^{(2n)}(s_1) \neq 0\}$ is infinite.

6.2 Arithmetic result for Gontcharoff and Whittaker interpolation

Theorem 2. *Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and f an entire function of sufficiently small exponential type. Assume that for each sufficiently large n, one at least of the numbers*

$$f^{(n)}(s_j) \quad j = 0, 1, \ldots, m-1$$

is in $\mathbb{Z}$. Then f is a polynomial.

Figure 8 displays the case $m = 2$, $f^{(2n+1)}(s_0) \in \mathbb{Z}$, $f^{(2n)}(s_1) \in \mathbb{Z}$. Figure 10 displays the case $m = 3$, $f^{(3n)}(s_0) \in \mathbb{Z}$, $f^{(3n+1)}(s_1) \in \mathbb{Z}$, $f^{(3n+2)}(s_2) \in \mathbb{Z}$.

In the case $m = 2$ with $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ (Whittaker interpolation), the assumption on the exponential type of f is

$$\tau(f) < \min\left\{1, \frac{\pi}{2|s_1 - s_0|}\right\},$$

and this is best possible, as shown by the functions

$$f(z) = \frac{\sinh(z - s_1)}{\cosh(s_0 - s_1)} \quad \text{and} \quad f(z) = \cos\left(\frac{\pi}{2} \cdot \frac{z - s_0}{s_1 - s_0}\right),$$

which have exponential type 1 and $\frac{\pi}{2|s_1 - s_0|}$ respectively.

When $|s_1 - s_0| < \log(2 + \sqrt{3}) = 1.316\ldots$, there is a nondenumerable set of entire functions f of exponential type ≤ 1 satisfying $f^{(2n+1)}(s_0) = 0$ and $f^{(2n)}(s_1) \in \{-1, 0, 1\}$ for all $n \geq 0$ and such that the set of $n \geq 0$ with $f^{(2n)}(s_1) \neq 0$ is infinite.

As before, let us display horizontally the points (they are no longer assumed to be the consecutive integers) and vertically the derivatives.

- $\bullet$ interpolation values $\quad\quad \circ$ no condition

Lidstone interpolation

Figure 7

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
f^{(2n+1)} & \circ & \circ \\
f^{(2n)} & \bullet & \bullet \\
\vdots & \vdots & \vdots \\
f'' & \bullet & \bullet \\
f' & \circ & \circ \\
f & \bullet & \bullet \\
 & s_0 & s_1
\end{array}$$

Whittaker interpolation

Figure 8

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
f^{(2n+1)} & \bullet & \circ \\
f^{(2n)} & \circ & \bullet \\
\vdots & \vdots & \vdots \\
f'' & \circ & \bullet \\
f' & \bullet & \circ \\
f & \circ & \bullet \\
 & s_0 & s_1
\end{array}$$

Poritsky interpolation (3 points)

Figure 9

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
f^{(3n+2)} & \circ & \circ & \circ \\
f^{(3n+1)} & \circ & \circ & \circ \\
f^{(3n)} & \bullet & \bullet & \bullet \\
\vdots & \vdots & \vdots & \vdots \\
f^{(\mathrm{iv})} & \circ & \circ & \circ \\
f''' & \bullet & \bullet & \bullet \\
f'' & \circ & \circ & \circ \\
f' & \circ & \circ & \circ \\
f & \bullet & \bullet & \bullet \\
 & s_0 & s_1 & s_2
\end{array}$$

Gontcharoff interpolation (3 points)
Figure 10

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
f^{(3n+2)} & \circ & \circ & \bullet \\
f^{(3n+1)} & \circ & \bullet & \circ \\
f^{(3n)} & \bullet & \circ & \circ \\
\vdots & \vdots & \vdots & \vdots \\
f^{(\mathrm{iv})} & \circ & \bullet & \circ \\
f''' & \bullet & \circ & \circ \\
f'' & \circ & \circ & \bullet \\
f' & \circ & \bullet & \circ \\
f & \bullet & \circ & \circ \\
 & s_0 & s_1 & s_2
\end{array}
$$

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