January 25 - February 13, 2021. Limbe (Cameroun) - online

A course on linear recurrent sequences
African Institute for Mathematical Sciences (AIMS)

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Tutorial 1

Tuesday, January 26, 2021

• **1.** Let d be a positive integer which is not the square of an integer. The goal is to give two proofs that d is not the square of a rational number. Assume $\sqrt{d} = n/m$ with n, m positive integers and $n/m \notin \mathbb{Z}$.
(a) Prove that there exists an integer k in the interval

$$\sqrt{d} - 1 < k < \sqrt{d}.$$

Define n' and m' by

$$n' = dm - kn = n(\sqrt{d} - k) \quad \text{and} \quad m' = n - km = m(\sqrt{d} - k).$$

Check $0 < n' < n$, $0 < m' < m$ and $n/m = n'/m'$.
Conclude.
(b) Prove that there exists an integer ℓ in the interval

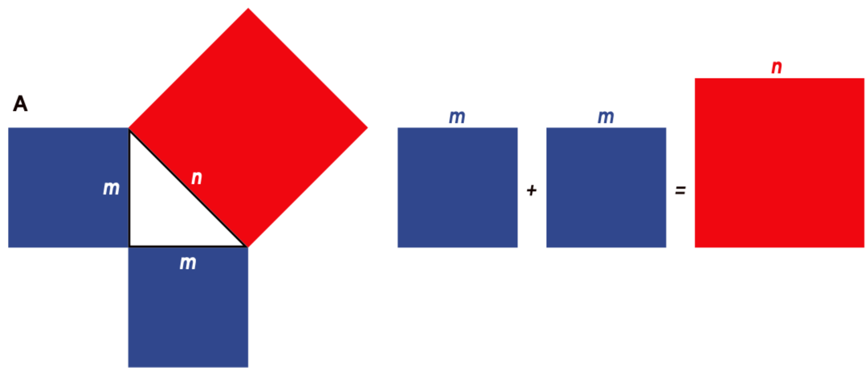
$$\sqrt{d} < \ell < \sqrt{d} + 1.$$

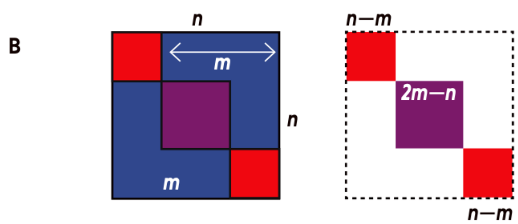
Check that the numbers n' and m' defined by

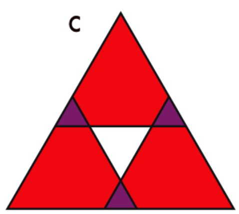
$$n' = \ell n - dm = n(\ell - \sqrt{d}) \quad \text{and} \quad m' = \ell m - n = m(\ell - \sqrt{d})$$

satisfy $0 < n' < n$, $0 < m' < m$ and $n/m = n'/m'$. Conclude.

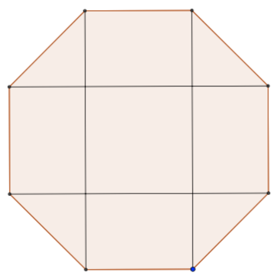
• **2.** Prove the irrationality of $\sqrt{2}$ using the pictures A and B below, and the irrationality of $\sqrt{3}$ using the picture C below. Explain the connection with question 1 (a) for $\sqrt{2}$ and 1(b) for $\sqrt{3}$.

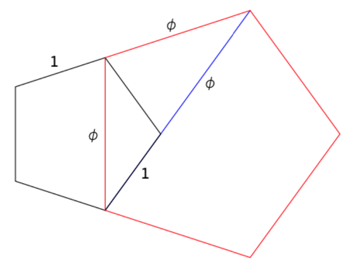






• **3.** Prove the irrationality of the diagonal of a regular octogon and of the diagonal of a regular pentagon :





• **4.** Consider an equilateral triangle having its vertices on a regular square grid with squares of side 1.
(a) Prove that the area of this triangle is a rational number.
(b) Let a be the length of the side of the triangle. Check that a^2 is an integer. Compute the area of the triangle in terms of a.
(c) Check that 3 does not divide a sum of two squares of relatively prime integers.
(d) Can you draw an equilateral triangle on the screen of a computer ?

`Hint.` The following pictures may help you :