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Irrationality and transcendence of values of special functions.

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Abstract

There is no general result on the irrationality or transcendence of values of analytic functions : one needs to restrict to special functions. A number of results are known concerning analytic functions satisfying differential equations. We survey this topic.

We consider the values of the E and G functions introduced by Siegel in 1929, the values of the exponential function, of elliptic or more generally abelian functions, of modular functions, and also the values of functions satisfying functional equations.

Extended abstract

- Irrationality of the values of special functions, transcendence, algebraic independence.
- The exponential function. Hermite–Lindemann Theorem. Trigonometric functions, logarithms.
- Weierstrass dream : transcendental functions take transcendental values. Need to restrict to special functions.
- Siegel E and G–functions. Bessel's functions. Hypergeometric functions.
- Differential equations : the Schneider–Lang Theorem. Elliptic functions ; elliptic integrals ; Weierstrass $\wp$ function, zeta function, sigma function ; Jacobi functions sn and cn. Modular invariants j and J, related by $j(\tau) = J(e^{2i\pi\tau})$. Theta functions. Special values of Euler Gamma and Beta function. Abelian functions and integrals.
- Riemann zeta function
- Functional equations. Mahler's functions.

Numbers : irrationality, transcendence, algebraic independence

We are interested with arithmetic properties of values of special functions.

Irrationality : *is* $f(z_0)$ *rational or irrational ?*

Transcendence : *is* $f(z_0)$ *algebraic (root of a nonzero polynomial with integer coefficients) ?*

Algebraic independence : *are* $f(z_1), f(z_2), \dots, f(z_m)$ *algebraically dependent ?*
i.e. *does there exist a nonzero polynomial* P *with integer coefficients such that* $P\bigl(f(z_1), f(z_2), \dots, f(z_m)\bigr) = 0$ *?*

Functions : rational, algebraic, transcendental, algebraically independent

Rational functions : $\mathbb{C}(z)$.
Algebraic functions : $P\bigl(z, f(z)\bigr) = 0$
Example :

$$\frac{1}{\sqrt{1-4z}} = \sum_{n\ge 0} \binom{2n}{n} z^n \qquad (|z| < 1).$$

Transcendental functions. Examples :

$$e^z = \sum_{n\ge 0} \frac{z^n}{n!}.$$

$$\frac{1}{z}\log(1-z) = -\sum_{n\ge 0} \frac{z^n}{n+1} \qquad (|z| < 1).$$

The exponential function

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + \frac{z^n}{n!} + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}z} e^z = e^z, \qquad e^0 = 1.$$

L. Euler

$$e^{i\pi} = -1.$$

Leonhard Euler
(1707 – 1783)

Charles Hermite and Ferdinand Lindemann

Hermite (1873)

Transcendence of e
$e = 2.718\,281\,828\,4\ldots$

Lindemann (1882)

Transcendence of π
$\pi = 3.141\,592\,653\,5\ldots$

Hermite – Lindemann Theorem (1882)

Ch. Hermite
(1822 – 1901)

von Lindemann
(1852 – 1939)

Theorem. *If w is a nonzero complex number, one at least of the two numbers w, e^w is transcendental.*

Consequences : transcendence of e, π, $\log\alpha$, e^β, for algebraic α and β assuming $\alpha \ne 0$, $\beta \ne 0$, $\log\beta \ne 1$.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hermite.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lindemann.html

Transcendental functions

A complex function is called transcendental if it is transcendental over the field $\mathbb{C}(z)$, which means that the functions z and $f(z)$ are algebraically independent : if $P \in \mathbb{C}[X, Y]$ is a non-zero polynomial, then the function $P\bigl(z, f(z)\bigr)$ is not 0.

Exercise. *An entire function (analytic in $\mathbb{C}$) is transcendental if and only if it is not a polynomial.*
A meromorphic function in $\mathbb{C}$ is transcendental if and only if it is not rational.

Example. *The transcendental entire function e^z takes an algebraic value at an algebraic argument z only for $z = 0$.*

Weierstrass question

Is it true that a transcendental entire function f takes usually transcendental values at algebraic arguments ?

Karl Weierstrass

Examples : for $f(z) = e^z$, there is a single exceptional point α algebraic with e^α also algebraic, namely $\alpha = 0$.

For $f(z) = e^{P(z)}$ where $P \in \mathbb{Z}[z]$ is a non–constant polynomial, there are finitely many exceptional points α, namely the roots of P.

The exceptional set of $e^z + e^{1+z}$ is empty (Lindemann–Weierstrass).

The exceptional set of functions like 2^z or $e^{i\pi z}$ is $\mathbb{Q}$, (Gel'fond and Schneider).

Exceptional sets

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain, Surroca...
If S is a countable subset of $\mathbb{C}$ and T is a dense subset of $\mathbb{C}$, there exist transcendental entire functions f mapping S into T, as well as all its derivatives.

Any set of algebraic numbers is the exceptional set of some transcendental entire function.
Also multiplicities can be included.

van der Poorten : *there are transcendental entire functions f such that $D^k f(\alpha) \in \mathbb{Q}(\alpha)$ for all $k \geq 0$ and all algebraic α.*

Further results on exceptional sets

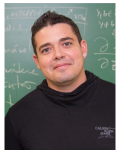
For each countable subset A of $\mathbb{C}$ and each family of dense subsets $E_{\alpha,s}$ of $\mathbb{C}$ indexed by $(\alpha, s) \in A \times \mathbb{N}$, there exists a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f^{(s)}(\alpha) \in E_{\alpha,s}$ for each $(\alpha, s) \in A \times \mathbb{N}$.

Jingjing Huang, Diego Marques and Martin Mereb ;
Algebraic values of transcendental functions at algebraic points.
Bull. Aust. Math. Soc. **82** (2010), 322–327

Recent results on exceptional sets

There exists uncountably many transcendental entire functions f with the property that both f and its inverse function assume algebraic values at algebraic points.

Diego Marques and Carlos Gustavo Moreira ;
A positive answer for a question proposed by K. Mahler.
Math. Annalen, to appear.
`http://arxiv.org/abs/1510.06122`



Diego Marques

Gustavo Moreira

Differential equations, functional equations

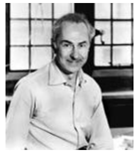
Siegel : E and G functions : linear differential equations ;
hypergeometric functions, Bessel's functions.

Schneider – Lang : nonlinear differential equations ;
exponential function, elliptic functions, abelian functions.

Mahler : functional equations

C.L. Siegel

Th. Schneider



S. Lang

L. Mahler

Siegel E–functions

Let

$$f(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n \in \mathbb{Q}[[z]]$$

be such that

- a_n increases at most exponentially in n (hence f is an entire function)
- f satisfies a linear differential equation with coefficients in $\mathbb{Q}(z)$
- The common denominator of $a_0, a_1, \ldots, a_n$ increases at most exponentially in n.

Examples. Algebraic constants, polynomials with algebraic coefficients, the exponential function e^z, the trigonometric functions $\cos z$ and $\sin z$.

Bessel function

The Bessel function

$$J_0(z) = \sum_{n\geq 0} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}$$

$$= 1 - \left(\frac{z}{2}\right)^2 + \frac{1}{4}\left(\frac{z}{2}\right)^4 - \frac{1}{36}\left(\frac{z}{2}\right)^6 + \cdots,$$

F.W. Bessel

is an E function, solution of the Bessel differential equation

$$y'' + \frac{1}{z} y' + y = 0.$$

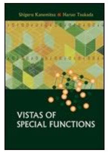
References on special functions

Shigeru Kanemitsu, Haruo Tsukada,
Vistas of Special Functions, World Scientific, 2007

Kalyan Chakraborty, Shigeru Kanemitsu and Haruo Tsukada,
Vistas of Special Functions II World Scientific, Singapore, 2009.

Kalyan Chakraborty, Shigeru Kanemitsu and T. Kuzumaki,
A Quick Introduction to Complex Analysis, World Scientific, 2016.


K. Chakraborty




S. Kanemitsu

Srinivasa Ramanujan and Friedrich Wilhelm Bessel

Henri Cohen,
Some formulas of Ramanujan involving Bessel functions
Publications mathématiques de Besançon (2010), 59–68.
`http://pmb.univ-fcomte.fr/2010/Cohen.pdf`


F.W. Bessel


S. Ramanujan


H. Cohen

Ramanujan also investigated Bessel q series.

Pochhammer symbol : *rising factorial power*

$$(x)_n = x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}.$$


Leo August Pochhammer
(1841-1920)

`http://scienceworld.wolfram.com/biography/Pochhammer.html`

Euler Gamma and Beta functions

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \cdot \frac{\mathrm{d}t}{t} = e^{-\gamma z} z^{-1} \prod_{n\geq 1} \left(1+\frac{z}{n}\right)^{-1} e^{z/n}.$$


L. Euler

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x.$$

Siegel hypergeometric E functions

Let $a_1,\dots,a_\ell$, $b_1,\dots,b_m$ be rational numbers with $m>\ell$ and $b_1,\dots,b_m$ not in $\{0,-1,-2,\dots\}$ and $b_m=1$. Define

$$c_n=\frac{(a_1)_n\cdots(a_\ell)_n}{(b_1)_n\cdots(b_m)_n}$$

Set $t=m-\ell$.

Then

$$f(z)=\sum_{n\geq 1} c_n z^{tn}$$

is an E–function.

Siegel's Theorem (1929)

For $\lambda\in\mathbb{Q}\setminus\{-1,-2,\dots\}$, consider the E–function

$$K_\lambda(z)=\sum_{n\geq 0}\frac{(-1)^n}{(\lambda+1)_n n!}\left(\frac{z}{2}\right)^{2n},$$

solution of the second order differential equation

$$y''+\frac{2\lambda+1}{z}y'+y=0.$$

For $\lambda\in\mathbb{Q}$ not in $\{\pm\frac{1}{2},-1,\pm\frac{3}{2},-2,\dots\}$, for any algebraic number $\alpha\neq 0$, the two numbers $K_\lambda(\alpha)$ and $K'_\lambda(\alpha)$ are algebraically independent.

Bessel functions

Bessel functions of the first kind :

$$J_\lambda(z)=\sum_{n\geq 1}\frac{(-1)^n(z/2)^{2n+\lambda}}{n!\Gamma(n+1+\lambda)} =\frac{1}{\Gamma(\lambda+1)}\left(\frac{z}{2}\right)^{\lambda}K_\lambda(z),$$

solution of the differential equation

$$z^2y''+zy'+(z^2-\lambda^2)y=0.$$

Also $J_{-\lambda}(z)$ is a solution of the same differential equation.
Modified Bessel functions of the first kind :

$$I_\lambda(z)=\sum_{n\geq 1}\frac{(z/2)^{2n+\lambda}}{n!\Gamma(n+1+\lambda)}=i^{-\lambda}J_\lambda(iz).$$

Bessel functions and continued fractions

From Siegel's 1929 Theorem, it follows that the number

$$\frac{I_1(2)}{I_0(2)}=[0;1,2,3,\dots]=0.697774658\dots$$

(Sloane's `A052119`, `A001053` and `A001040`)
is transcendental.

Weinstein, Eric W. *Continued Fraction Constant.* From MathWorld–A Wolfram Web Resource.
http://mathworld.wolfram.com/ContinuedFractionConstant.html

Bessel functions and continued fractions

$$I_{1/2}(1) = \frac{\sqrt{2}}{\pi} \cdot \frac{e+e^{-1}}{2}, \quad I_{1/2}(1) = \frac{\sqrt{2}}{\pi} \cdot \frac{e-e^{-1}}{2}.$$

$$[1;3,5,7\dots] = \frac{e^2+1}{e^2-1} = \frac{I_{-1/2}(1)}{I_{1/2}(1)},$$

$$[2;6,10,14\dots] = \frac{e+1}{e-1} = \frac{I_{-1/2}(1/2)}{I_{1/2}(1/2)}$$

B. Sury

B. Sury, Bessels contain continued fractions of progressions ; Resonance **10** 3 (2005) 80–87.
http://www.ias.ac.in/article/fulltext/reso/010/03/0080-0087

Siegel–Shidlovskii theory

Generalization of C.L. Siegel 1929 results by C.L. Siegel himself in 1949, by A.B. Shidlovskii in 1953 – 1955.

Given a set $\{f_1,\dots,f_n\}$ of E–functions satisfying a system of linear differential equations and an algebraic number α, the transcendence degree of the field $\mathbb{Q}\big(f_1(\alpha), f_2(\alpha),\dots,f_n(\alpha)\big)$ over $\mathbb{Q}$ is equal to the transcendence degree of the field $\mathbb{C}\big(z, f_1, f_2,\dots,f_n\big)$ over $\mathbb{C}(z)$.

A.B. Shidlovskii
Transcendental numbers.
Studies in mathematics, **12**,
Walter de Gruyter (1989).

Siegel G–functions

Let

$$g(z) = \sum_{n\geq 0} a_n z^n \in \mathbb{Q}[[z]]$$

be such that
• g has a positive radius of convergence
• g satisfies a linear differential equation with coefficients in $\mathbb{Q}(z)$
• The common denominator of $a_0, a_1, \dots, a_n$ increases at most exponentially in n.

Examples.
• Algebraic functions
• Hypergeometric functions with rational parameters
• Solutions of Picard–Fuchs equations over $\mathbb{Q}(z)$.

Gauss Hypergeometric function

$${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n n!} z^n.$$

It satisfies second order linear differential equation

$$z(z-1)y'' + \big((a+b+1)z - c\big)y' + aby = 0.$$

Examples of hypergeometric functions

$$ {}_2F_1\left(\begin{matrix}1 & 1\\ & 2\end{matrix}\middle| z\right) = -\frac{1}{z}\log(1-z) $$

$$ {}_2F_1\left(\begin{matrix}1/2 & 1\\ & 1\end{matrix}\middle| z\right) = (1-z)^{-1/2} $$

$$ {}_2F_1\left(\begin{matrix}1/2 & 1/2\\ & 1\end{matrix}\middle| z\right) = \frac{2}{\pi}\int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-zt^2)}}. $$

Eisenstein hypergeometric function

The G function

$$ \sum_{n\geq 0}(-1)^n \frac{\binom{5n}{n}}{4n+1} z^{4n+1} $$

F.G.M. Eisenstein

$$ = z - z^5 + 10\frac{z^9}{2!} - 15\cdot 14\frac{z^{13}}{3!} + \cdots, $$

which converges for $|z| < 5^{-5/4}$, is a solution of the quintic equation $x^5 + x = z$.

J. Stillwell, Eisenstein's footnote, Math. Intelligencer **17** (1995), no. 2, 58–62.

F.G.M. Eisenstein, Allgemeine Auflösung der Gleichungen von den ersten vier Graden. J. Reine Angew. Math. 27 (1844), 81–83. Mathematische Werke I, 7–9.

Connections between E and G functions

If

$$ g(z) = \sum_{n\geq 0} a_n z^n $$

is a G–function, then

$$ f(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n $$

is an E–function and conversely.

Remarks:

• Siegel's definitions involve algebraic coefficients, not only rational coefficients.

• Apéry's proof of the irrationality of $\zeta(3)$ is related with the theory of G–functions.

S. Fischler and T. Rivoal, Approximants de Padé et séries hypergéométriques équilibrées, J. Math. Pures Appl. **82** (2003), no. 10, 1369–1394.

The ring $\mathbb{G}$

S. Fischler and T. Rivoal introduce the set $\mathbb{G}$ of all values taken by any analytic continuation of any G–function at any algebraic point.

S. Fischler

T. Rivoal

They prove that $\mathbb{G}$ is a countable subring of $\mathbb{C}$ which contains the field $\overline{\mathbb{Q}}$ of algebraic numbers and the logarithms of algebraic numbers.

Conjecturally, $\mathbb{G}$ is not a field.

S. Fischler and T. Rivoal, On the values of G-functions, Comment. Math. Helv. **89** (2014), 313–341.

Conjecture of Bombieri and Dwork

According to a conjecture of Bombieri and Dwork, $\mathbb{G}$ should coincide with the set of periods of algebraic varieties defined over $\overline{\mathbb{Q}}$.


E. Bombieri


B. Dwork

Connection with the periods of Kontsevich and Zagier


M. Kontsevich


D. Zagier

Euler's constant γ

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$
$$= 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots$$

$$\gamma = \sum_{k\geq 1}\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right)$$
$$= \int_1^\infty\left(\frac{1}{\lfloor x\rfloor}-\frac{1}{x}\right)\mathrm{d}x$$
$$= -\int_0^1\int_0^1\frac{(1-x)\mathrm{d}x\mathrm{d}y}{(1-xy)\log(xy)}.$$

Euler's constant γ

J. Sondow double integral was inspired by F. Beukers's work on Apéry's proof of the irrationality of

$$\zeta(3) = \sum_{n\geq 1}\frac{1}{n^3} = 1.202\,056\,903\,159\,594\,285\,399\,738\,161\,511\ \ldots$$

in 1978.


J. Sondow


F. Beukers

Quoting Tanguy Rivoal

It is believed that $\gamma\notin\mathbb{Q}$. Why ? Because if $\gamma = p/q\in\mathbb{Q}$ with $\gcd(p,q)=1$, then $|q|\geq 10\,242\,080$.

It is also believed that $\gamma\notin\mathbb{G}$. Why ? Because Euler and Ramanujan would have found various formulas proving this fact.
It is also plausible that γ does not even belong to the field of fractions of $\mathbb{G}$.

Further, it is expected that $e=\exp(1)$ does not belong to the field of fractions of $\mathbb{G}$.

Catalan's constant G and Euler's Gamma function

Nothing is known on the arithmetic nature of *Catalan's constant*

$$G = \sum_{n\geq 1} \frac{(-1)^n}{(2n+1)^2} = 0.915\,965\,594\,177\,219\,015\,0\ldots$$

and of the value

$$\Gamma(1/5) = 4.590\,843\,711\,998\,803\,053\,204\,758\,275\,929\,152\,0\ldots$$

of Euler's Gamma function.

The units of the ring $\mathbb{G}$

The group of units of $\mathbb{G}$ contains $\overline{\mathbb{Q}}^\times$ and the values $\mathrm{B}(a,b)$, (a, b in $\mathbb{Q}$) of Euler Beta function.

The numbers $\Gamma(a/b)^b$, $a/b \in \mathbb{Q} \setminus \{0,-1,-2,\ldots\}$, are units in the ring $\mathbb{G}$.

For instance, $\pi = \Gamma(1/2)^2$ is a unit.
Proof :

$$\pi = \sum_{n\geq 1} \frac{4(-1)^n}{2n+1}, \quad \frac{1}{\pi} = \sum_{n\geq 1} \frac{(42n+5)\binom{2n}{n}^3}{2^{12n+4}}.$$

Walt Disney Productions and $1/\pi$

The formula

$$\frac{16}{\pi} = \sum_{n\geq 0} (42n+5)\frac{(1/2)_n^3}{n!^3 2^{6n}}$$

appeared in the Walt Disney film *High School Musical*, starring Vanessa Anne Hudgens, who plays an exceptionally bright high school student named Gabriella Montez. Gabriella points out to her teacher that she had incorrectly written the left-hand side as $\frac{8}{\pi}$ instead of $\frac{16}{\pi}$ on the blackboard. After first claiming that Gabriella is wrong, her teacher checks (possibly Ramanujan's Collected Papers ?) and admits that Gabriella is correct.

N.D. Baruah, B. Berndt and H.H. Chan.
Ramanujan's Series for $1/\pi$: A Survey.
American Mathematical Monthly **116** (2009) 567–587.

Ramanujan series for $1/\pi$

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2} \sum_{k\geq 0} \frac{(4k)!}{k!^4} \cdot \frac{26390k+1103}{396^{4k}}$$


N.D. Baruah


B. Berndt


Chan Heng Huat

Nayandeep Deka Baruah, Bruce C. Berndt and Heng Huat Chan.
Ramanujan's Series for $1/\pi$: A Survey.
American Mathematical Monthly **116** (2009) 567–587.

https://en.wikipedia.org/wiki/Ramanujan-Sato_series

Algebraic values of Siegel G functions

Let f be a G–function which is not algebraic. Is–it true that $f(\alpha)$ is algebraic for at most finitely many algebraic α ?

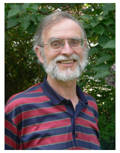
F. Beukers

Wolfart's work (1988)

Let $f(z) = {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right)$ with a, b, c in $\mathbb{Q}$. Let Δ be the monodromy group and

$$E = \left\{\alpha \in \overline{\mathbb{Q}} \mid f(\alpha) \in \overline{\mathbb{Q}}\right\}.$$

(1) If f is algebraic (Δ finite), then $E = \overline{\mathbb{Q}}$.
(2) If f is *arithmetic*, then E is dense in $\overline{\mathbb{Q}}$.
(3) Otherwise, E is finite.

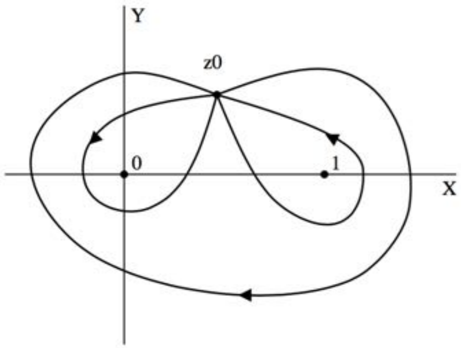


Jürgen Wolfart

Yafaev and Edixhoven (2003) : complete the proof of (3).

Monodromy

Singular points 0, 1, ∞



Monodromy matrices : M_0, M_1, M_∞ with $M_0 M_1 M_\infty = \mathrm{Id}$.

Arithmetic groups

We assume that $0 < a, b, c \leq 1$ and either $a, b < c$ or $c < a, b$. Then the monodromy modulo scalars embeds in $\mathrm{PSL}(2, \mathbb{R})$.

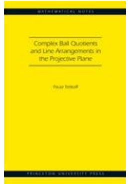
Let $g_i \in \mathrm{SL}(2, \mathbb{R})$ be a lift of the monodromy around $i = 0, 1, \infty$. Then $\mathrm{Id}_2, g_0^2, g_1^2, g_\infty^2$ generate a quaternion algebra H defined over

$$k = \mathbb{Q}\big(\cos^2 \pi c, \cos^2 \pi(a-b), \cos^2 \pi(c-a-b), \cos \pi c \cos \pi(a-b) \cos \pi(c-a-b)\big).$$

We say that our monodromy is *arithmetic* if H is split at exactly one infinite places of k.

A recent reference

Paula Tretkoff,
Complex ball quotients and line arrangements in the projective plane.
Mathematical Notes **51**,
Princeton University Press,
2016.



Paula Tretkoff

http://press.princeton.edu/titles/10782.html

Modular group

Let $a = 1/12$, $b = 5/12$, $c = 1/2$. Then the quaternion algebra is $M(2, \mathbb{Q})$ and the monodromy group is $\mathrm{SL}(2, \mathbb{Z})$. It can be shown that

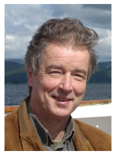
$$ {}_2F_1\left(\begin{matrix}1/12 & 5/12\\ & 1/2\end{matrix}\middle| 1 - \frac{1}{J(\tau)}\right)^4 = \frac{E_4(\tau)}{E_4(i)} $$

where

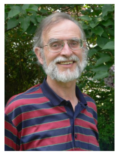
$$ E_4(\tau) = 1 + 240\sum_{n\geq 1}\frac{n^3 q^n}{1-q^n}, \quad \Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24} $$

and $q = e^{2i\pi\tau}$, $J(\tau) = E_4(\tau)^3/1728\Delta(\tau)$. In particular $J(i) = 1$. From CM-theory it follows that if $\tau_0 \in \mathbb{Q}(i)$, $\mathrm{Im}(\tau_0) > 0$, then both $J(\tau_0)$ and $E_4(\tau_0)/E_4(i)$ are algebraic.

Example



Frits Beukers



Jürgen Wolfart

$$ {}_2F_1\left(\begin{matrix}1/12 & 5/12\\ & 1/2\end{matrix}\middle| \frac{1323}{1331}\right) = \frac{3}{4}\sqrt[4]{11}. $$

Other examples

$$ {}_2F_1\left(\begin{matrix}1/4 & 1/2\\ & 3/4\end{matrix}\middle| \frac{80}{81}\right) = \frac{9}{5}. $$

$$ {}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 5/6\end{matrix}\middle| \frac{27}{32}\right) = \frac{8}{5}. $$

$$ {}_2F_1\left(\begin{matrix}1/12 & 1/4\\ & 5/6\end{matrix}\middle| \frac{135}{256}\right) = \frac{2}{5}\sqrt[6]{270}. $$

Akihito Ebisu (2014)

$$ {}_2F_1\left(\begin{matrix}1/24 & 7/24\\ & 5/6\end{matrix}\middle| -\frac{2^{10}3^35}{11^4}\right) = \sqrt{6}\sqrt[6]{\frac{11}{5^5}}. $$

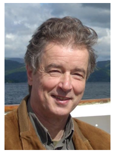
Yifan Yang (2015)

Non arithmetic examples

$${}_2F_1\left(\begin{matrix}1-3a & 3a\\ a\end{matrix}\middle|\frac{1}{2}\right)=2^{3-2a}\cos\pi a.$$

$${}_2F_1\left(\begin{matrix}2a & 1-4a\\ 1-a\end{matrix}\middle|\frac{1}{2}\right)=4^{a}\cos\pi a.$$

$${}_2F_1\left(\begin{matrix}7/48 & 31/48\\ 29/24\end{matrix}\middle|-\frac{1}{3}\right)=2^{5/24}3^{-11/12}5\cdot\sqrt{\frac{\sin\pi/24}{\sin 5\pi/24}}.$$



F. Beukers

Schneider – Lang Theorem (1949, 1966)

Theodor Schneider
(1911 – 1988)

Serge Lang
(1927 – 2005)

Let $f_1,\dots,f_m$ be meromorphic functions in $\mathbb{C}$. Assume f_1 and f_2 are algebraically independent and of finite order. Let $\mathbb{K}$ be a number field. Assume f_j' belongs to $\mathbb{K}[f_1,\dots,f_m]$ for $j=1,\dots,m$. Then the set

$$S=\{w\in\mathbb{C}\ \mid\ w \text{ not pole of } f_j,\ f_j(w)\in\mathbb{K} \text{ for } j=1,\dots,m)\}$$

is finite.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Schneider.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lang.html

Hermite – Lindemann Theorem (again)

Charles Hermite
(1822 – 1901)

von Lindemann
Carl Louis Ferdinand
(1852 – 1939)

Corollary. *If w is a nonzero complex number, one at least of the two numbers w, e^w is transcendental.*

Proof. Let $\mathbb{K}=\mathbb{Q}(w,e^w)$. The two functions $f_1(z)=z$, $f_2(z)=e^z$ are algebraically independent, of finite order, and satisfy the differential equations $f_1'=1$, $f_2'=f_2$. The set S contains $\{\ell w\ \mid\ \ell\in\mathbb{Z}\}$. Since $w\neq 0$, this set is infinite ; it follows that $\mathbb{K}$ is not a number field. $\square$

The exponential function (again)

$$\frac{d}{dz}e^z=e^z,\qquad e^{z_1+z_2}=e^{z_1}e^{z_2}$$

$$\begin{array}{rccc}\exp: & \mathbb{C} & \rightarrow & \mathbb{C}^\times\\ & z & \mapsto & e^z\end{array}$$

$$\ker\exp=2i\pi\mathbb{Z}.$$

The function $z\mapsto e^z$ is the exponential map of the multiplicative group $\mathbb{G}_m$.
The exponential map of the additive group $\mathbb{G}_a$ is

$$\begin{array}{ccc}\mathbb{C} & \rightarrow & \mathbb{C}\\ z & \mapsto & z\end{array}$$

The only period is 0.

Elliptic curves and elliptic functions

Elliptic curves : $\Delta = g_2^3 - 27g_3^2 \neq 0$.

$$E = \left\{(t : x : y)\ ;\ y^2t = 4x^3 - g_2xt^2 - g_3t^3\right\} \subset \mathbb{P}_2(\mathbb{C}).$$

Elliptic functions

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

$$\wp(z_1 + z_2) = R\big(\wp(z_1), \wp(z_2)\big)$$

$$\begin{array}{rccc} \exp_E : & \mathbb{C} & \to & E(\mathbb{C}) \\ & z & \mapsto & \big(1, \wp(z), \wp'(z)\big) \end{array}$$

$$\ker \exp_E = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Weierstraß elliptic function

$$\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{R}^2$$

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

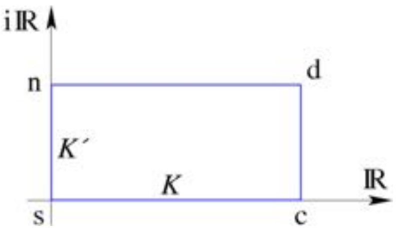
Karl Weierstrass

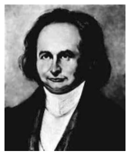
(1815–1897)

$$\wp'(z) = \sum_{\omega \in \Omega} \frac{-2}{(z-\omega)^3}.$$

Jacobi 12 elliptic functions

Elliptic curve as an intersection of quadrics : the functions sn and cn.





Karl Jacobi
(1804–1851)

sn sc sd ns nc nd cs cn cd ds dn dc

https://en.wikipedia.org/wiki/Jacobi_elliptic_functions

Periods of a Weierstrass elliptic function

The set of periods of an elliptic function is a *lattice* :

$$\Omega = \{\omega \in \mathbb{C}\ ;\ \wp(z+\omega) = \wp(z)\} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

A pair of fundamental periods (ω_1, ω_2) is given by

$$\omega_i = 2\int_{e_i}^{\infty} \frac{\mathrm{d}t}{\sqrt{4t^3 - g_2t - g_3}}, \qquad (i = 1, 2)$$

where

$$4t^3 - g_2t - g_3 = 4(t - e_1)(t - e_2)(t - e_3).$$

Examples

Example 1 : $g_2 = 4,\ g_3 = 0,\ j = 1728$

A pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4xt^2.$$

is given by

$$\omega_1 = \int_1^\infty \frac{\mathrm{d}t}{\sqrt{t^3 - t}} = \frac{1}{2}\mathrm{B}(1/4, 1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}} = 2.6220575542\ldots$$

and

$$\omega_2 = i\omega_1.$$

Examples (continued)

Example 2 : $g_2 = 0,\ g_3 = 4,\ j = 0$

A pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4t^3.$$

is

$$\omega_1 = \int_1^\infty \frac{\mathrm{d}t}{\sqrt{t^3 - 1}} = \frac{1}{3}\mathrm{B}(1/6, 1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\ldots$$

and

$$\omega_2 = \varrho\omega_1$$

where $\varrho = e^{2i\pi/3}$.

Chowla–Selberg Formula

S. Chowla

A. Selberg

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m + ni)^{-4} = \frac{\Gamma(1/4)^8}{2^6 \cdot 3 \cdot 5 \cdot \pi^2}$$

and

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m + n\varrho)^{-6} = \frac{\Gamma(1/3)^{18}}{2^8 \pi^6}$$

Formula of Chowla and Selberg (1966) : *the periods of elliptic curves with complex multiplication are products of values of the Gamma function at rational points.*

Chowla–Selberg Formula : an example

F. Adiceam

Faustin Adiceam (2011) :

$$\Gamma\left(\frac{1}{5}\right) = \sqrt{\frac{\pi}{2^{19/5}} \cdot \frac{1}{\sin\left(\frac{3\pi}{5}\right)\sin\left(\frac{9\pi}{20}\right)\sin\left(\frac{7\pi}{20}\right)\sin\left(\frac{\pi}{10}\right)} \cdot \frac{\Gamma\left(\frac{1}{20}\right)\times\Gamma\left(\frac{3}{20}\right)}{\Gamma\left(\frac{9}{20}\right)\times\Gamma\left(\frac{7}{20}\right)}}$$

$$= \sqrt{\frac{\pi}{2^{9/5}} \cdot \frac{(5+\sqrt{5})\left(2\sqrt{5} - \sqrt{2(5+\sqrt{5})}\right)}{5} \cdot \frac{\Gamma\left(\frac{1}{20}\right)\times\Gamma\left(\frac{3}{20}\right)}{\Gamma\left(\frac{9}{20}\right)\times\Gamma\left(\frac{7}{20}\right)}}.$$

Elliptic integrals and ellipses

An ellipse with radii a and b has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the length of its perimeter is

$$2\int_{-b}^{b} \sqrt{1 + \frac{a^2x^2}{b^4 - b^2x^2}}\, \mathrm{d}x.$$

In the same way, the perimeter of a lemniscate

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

is given by an elliptic integral

$$4a\int_0^1 (1 - t^4)^{-1/2}\, \mathrm{d}x.$$

Hypergeometry and elliptic integrals

Recall Gauss Hypergeometric series

C.F. Gauss

$${}_2F_1\left(a,\ b\ ;\ c \mid z\right) = \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

$$\begin{aligned} K(z) &= \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-z^2x^2)}} \\ &= \frac{\pi}{2} \cdot {}_2F_1\left(1/2,\ 1/2\ ;\ 1 \mid z^2\right). \end{aligned}$$

Weierstrass sigma function

K. Weierstrass

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$.
The canonical product of Weierstraß associated with Ω is the sigma function σ_Ω defined by

$$\sigma_\Omega(z) = z \prod_{\omega\in\Omega\setminus\{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)$$

This function has a simple zero at each point of Ω.

Hadamard canonical products

J. Hadamard

For $\mathbb{N} = \{0, 1, 2, \ldots\}$:

$$\frac{e^{-\gamma z}}{\Gamma(-z)} = z \prod_{n\geq 1} \left(1 - \frac{z}{n}\right) e^{-z/n}.$$

For $\mathbb{Z}$:

$$\frac{\sin \pi z}{\pi} = z \prod_{n\geq 1} \left(1 - \frac{z^2}{n^2}\right).$$

Wallis formula for π

John Wallis (Arithmetica Infinitorum 1655)

$$\frac{\pi}{2} = \prod_{n\geq 1}\left(\frac{4n^2}{4n^2-1}\right) = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}.$$

J. Wallis

Weierstraß sigma function : an example

For $\Omega = \mathbb{Z} + \mathbb{Z}i$:

$$\sigma_{\mathbb{Z}[i]}(z) = z \prod_{\omega\in\mathbb{Z}[i]\setminus\{0\}} \left(1-\frac{z}{\omega}\right)\exp\left(\frac{z}{\omega}+\frac{z^2}{2\omega^2}\right)\cdot$$

$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2} = 0.4749493799\ldots$$

For $\alpha\in\mathbb{Q}(i)$, the number $\sigma_{\mathbb{Z}[i]}(\alpha)$ is algebraic over

$$\mathbb{Q}\big(\pi,\ e^{\pi},\ \Gamma(1/4)\big).$$

Weierstraß zeta function

The logarithmic derivative of the Weierstraß sigma function is the *Weierstraß* zeta function

$$\frac{\sigma'}{\sigma} = \zeta$$

and the derivative of ζ is $-\wp$. The minus sign is selected so that

$$\wp(z) = \frac{1}{z^2} + \text{a function analytic at } 0.$$

The function ζ is therefore *quasi–periodic* : for any $\omega\in\Omega$ there exists $\eta = \eta(\omega)$ such that

$$\zeta(z+\omega) = \zeta(z) + \eta.$$

Legendre relation

The numbers $\eta(\omega)$ are the *quasi–periods* of the elliptic curve.
When (ω_1,ω_2) is a pair of fundamental periods, we set $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$.
Legendre relation :

$$\omega_2\eta_1 - \omega_1\eta_2 = 2i\pi.$$

this is not Adrien Marie but Louis Legendre

Legendre and Fourier

Peter Duren, Changing Faces : The Mistaken Portrait of Legendre.
Notices of American Mathematical Society, **56** (2009) 1440–1443.

Examples

For the curve $y^2t = 4x^3 - 4xt^2$ the quasi–periods associated to the previous fundamental periods are

$$\eta_1 = \frac{\pi}{\omega_1} = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \qquad \eta_2 = -i\eta_1,$$

while for the curve $y^2t = 4x^3 - 4t^3$ they are

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} = \frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3}, \qquad \eta_2 = \varrho^2\eta_1.$$

Transcendence and elliptic functions

Siegel (1932) : elliptic analog of Lindemann's Theorem on the transcendence of π.

Schneider (1937) : elliptic analog of Hermite–Lindemann Theorem. General transcendence results on values of elliptic functions, on periods, on elliptic integrals of the first and second kind.

C.L. Siegel

Th. Schneider

Elliptic analog of Hermite–Lindemann Theorem

Let $w \in \mathbb{C}$, not pole of $\wp$. Then one at least of the numbers g_2, g_3, w, $\wp(w)$ is transcendental.

`Proof` as a consequence of the Schneider–Lang Theorem.
Let $\mathbb{K} = \mathbb{Q}(g_2, w, \wp(w), \wp'(w))$. The two functions $f_1(z) = z$, $f_2(z) = \wp(z)$ are algebraically independent, of finite order. Set $f_3(z) = \wp'(z)$. From $\wp'^2 = 4\wp^3 - g_2\wp - g_3$ one deduces

$$f_1' = 1, \quad f_2' = f_3, \quad f_3' = 6f_2^2 - (g_2/2).$$

The set S contains

$$\{\ell w \mid \ell \in \mathbb{Z},\ \ell w \text{ not pole of } \wp\}$$

which is infinite. Hence $\mathbb{K}$ is not a number field. $\square$

Elliptic integrals of the third kind

Quasi–periodicity of the Weierstraß sigma function :

$$\sigma(z+\omega_i) = -\sigma(z)e^{\eta_i(z+\omega_i/2)} \quad (i=1,2).$$

The function

$$F_u(z) = \frac{\sigma(z+u)}{\sigma(z)\sigma(u)} e^{-z\zeta(u)}$$

satisfies

$$F_u(z+\omega_i) = F_u(z)e^{\eta_i u - \omega_i \zeta(u)}.$$

J-P. Serre (1979)

Periods of elliptic integrals of the third kind

Theorem (1979). *Assume g_2, g_3, $\wp(u_1)$, $\wp(u_2)$, β are algebraic and $\mathbb{Z}u_1 \cap \Omega = \{0\}$. Then the number*

$$\frac{\sigma(u_1+u_2)}{\sigma(u_1)\sigma(u_2)} e^{\left(\beta - \zeta(u_1)\right)u_2}$$

is transcendental.

Corollary. *Transcendence of periods of elliptic integrals of the third kind :*

$$e^{\omega\zeta(u) - \eta u + \beta\omega}.$$

Schneider's Theorem on Euler's Beta function

Th. Schneider

Let a, b be rational numbers, not integers. Then the number $\mathrm{B}(a,b)$ is transcendental.

Further results by Th. Schneider and S. Lang on abelian functions and algebraic groups.

Linear independence of transcendental numbers

A. Baker, J. Coates, D.W. Masser, G. Wüstholz,. . .




Values of Euler Beta and Gamma functions


J. Wolfart


G. Wüstholz

G. Wüstholz : any $\overline{\mathbb{Q}}$–linear relation among periods of an abelian variety arises from its endomorphisms.
(J. Wolfart and G. Wüstholz) : linear independence over the field of algebraic numbers of the values of the Euler Beta function at rational points (a, b).

Transcendence of values at algebraic points of hypergeometric functions with rational parameters.

Modular functions

$$P(q) = E_2(q) = 1 - 24 \sum_{n \geq 1} \frac{nq^n}{1 - q^n},$$

$$Q(q) = E_4(q) = 1 + 240 \sum_{n \geq 1} \frac{n^3 q^n}{1 - q^n},$$

$$R(q) = E_6(q) = 1 - 504 \sum_{n \geq 1} \frac{n^5 q^n}{1 - q^n}.$$


S. Ramanujan

$$\Delta = \frac{1}{1728}(Q^3 - R^2) \quad \text{and} \quad J = \frac{Q^3}{\Delta}.$$

Nesterenko's Theorem

1996, Yu. V. Nesterenko :
Let $q \in \mathbb{C}$ satisfy $0 < |q| < 1$. Then three at least of the numbers

$$q, P(q), Q(q), R(q)$$

are algebraically independent.


Y. Nesterenko

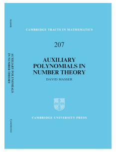
Corollary of Nesterenko's Theorem

The three numbers
π, e^π *and* $\Gamma(1/4)$
are algebraically independent.

Open problem :
Show that e *and* π *are algebraically independent.*

Transcendence of $\Gamma(i)$ following D.W. Masser

$$\Gamma(i)\overline{\Gamma(i)} = \Gamma(i)\Gamma(-i) = \frac{\Gamma(i)\Gamma(1-i)}{-i} = \frac{\pi}{-i\sin(i\pi)} = \frac{2\pi}{e^{\pi}-e^{-\pi}}.$$



D.W. Masser

D.W. Masser, Auxiliary Polynomials in Number Theory,
Cambridge Tracts in Mathematics **207** (2016),
Cambridge University Press.

Standard relations among Gamma values

(Translation) :

$$\Gamma(a+1) = a\Gamma(a)$$

(Reflection) :

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

(Multiplication) : for any positive number n,

$$\prod_{k=0}^{n-1} \Gamma\left(a+\frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)}\Gamma(na).$$

Conjecture of Rohrlich

Conjecture (D. Rohrlich)
Any multiplicative relation

$$\pi^{b/2}\prod_{a\in\mathbb{Q}}\Gamma(a)^{m_a}\in\overline{\mathbb{Q}}$$

with b and m_a in $\mathbb{Z}$ is in the ideal generated by the standard relations.

David Rohrlich

Conjecture of Rohrlich–Lang

D. Rohrlich

S. Lang

Conjecture (D. Rohrlich–S. Lang) *Any algebraic dependence relation among $(2\pi)^{-1/2}\Gamma(a)$ with $a\in\mathbb{Q}$ is in the ideal generated by the standard relations* (universal odd distribution).

Consequence of the conjecture of Rohrlich–Lang
(F. Adiceam) : the three numbers $\Gamma(1/5)$, $\Gamma(2/5)$ and $e^{\pi\sqrt{5}}$ are algebraically independent. *(Not yet know)*.

Gamma values

$$\frac{\Gamma(1/24)\Gamma(11/24)}{\Gamma(5/24)\Gamma(7/24)} = \sqrt{3}\sqrt{2+\sqrt{3}}.$$

Y. André

Yves André — Groupes de Galois motiviques et périodes.

Séminaire N. Bourbaki, Samedi 7 novembre 2015, 68ème année, 2015-2016, n° 1104.
http://www.bourbaki.ens.fr/TEXTES/1104.pdf

J. Ayoub : analogs for function fields of the periods conjectures of Grothendieck and Kontsevich–Zagier.

Riemann zeta function

L. Euler

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

B. Riemann

Euler : $s \in \mathbb{R}$.

Riemann : $s \in \mathbb{C}$.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Euler.html
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Riemann.html

Special values of the Riemann zeta function

Jacques Bernoulli
(1654–1705)

Leonhard Euler
(1707 – 1783)

$s \in \mathbb{Z}$: $\pi^{-2k}\zeta(2k) \in \mathbb{Q}$ *for* $k \geq 1$ (Bernoulli numbers).

Values of the Riemann zeta function at the positive integers

Even positive integers

$$\zeta(2n) = (-1)^{n-1}2^{2n-1}\frac{B_{2n}}{(2n)!}\pi^{2n} \quad (n \geq 1).$$

Odd positive integers : $\zeta(2n+1)$, $n \geq 1$?

Question : *for* $n \geq 1$, *is the number*

$$\frac{\zeta(2n+1)}{\pi^{2n+1}}$$

rational ?

Diophantine question

Determine all algebraic relations among the numbers

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7), \dots$$

Conjecture. *there is no algebraic relation : the numbers*

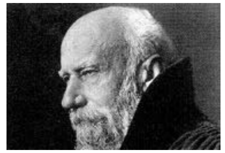
$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7), \dots$$

are algebraically independent.

As a consequence, one expects the numbers $\zeta(2n+1)$ and $\zeta(2n+1)/\pi^{2n+1}$ for $n \geq 1$ to be transcendental.

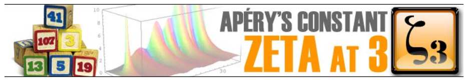
Values of ζ at the even positive integers

• F. Lindemann : π *is a transcendental number, hence* $\zeta(2k)$ *also for* $k \geq 1$.



F. Lindemann

Values of ζ at the odd positive integers



• Apéry (1978) : *The number*

$$\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3} = 1.202\,056\,903\,159\,594\,285\,399\,738\,161\,511\,\dots$$

is irrational.

• Rivoal (2000) + Ball, Zudilin, Fischler,... : *Infinitely many numbers among* $\zeta(2k+1)$ *are irrational + lower bound for the dimension of the* $\mathbb{Q}$*-space they span.*

Tanguy Rivoal

Let $\epsilon > 0$. *For any sufficiently large odd integer* a, *the dimension of the* $\mathbb{Q}$*–space spanned by the numbers* $1, \ \zeta(3), \ \zeta(5), \ \cdots, \zeta(a)$ *is at least*

$$\frac{1-\epsilon}{1+\log 2} \log a.$$

T. Rivoal

Wadim Zudilin

- *At least one of the four numbers* $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ *is irrational.*
- *There exists an odd number* j *in the interval* $[5, 69]$ *such that the three numbers* $1, \zeta(3), \zeta(j)$ *are* $\mathbb{Q}$*–linearly independent.*


W. Zudilin

References


S. Fischler

S. Fischler
Irrationalité de valeurs de zêta, (d'après Apéry, Rivoal, ...),
Sém. Nicolas Bourbaki, 2002-2003,
N° 910 (Novembre 2002).
`http//www.math.u-psud.fr/~fischler/publi.html`

C. Krattenthaler et T. Rivoal, *Hypergéométrie et fonction zêta de Riemann*, Mem. Amer. Math. Soc. **186** (2007), 93 p.
`http://www-fourier.ujf-grenoble.fr/~rivoal/articles.html`

Hurwitz zeta function

T. Rivoal (2006) : consider the Hurwitz zeta function

$$\zeta(s, z) = \sum_{k\geq 1} \frac{1}{(k+z)^s}.$$

Expand $\zeta(2, z)$ as a series in

$$\frac{z^2(z-1)^2\cdots(z-n+1)^2}{(z+1)^2\cdots(z+n)^2}.$$

The coefficients of the expansion belong to $\mathbb{Q} + \mathbb{Q}\zeta(3)$. This produces a new proof of Apéry's Theorem on the irrationality of $\zeta(3)$.
In the same way : new proof of the irrationality of $\log 2$ by expanding

$$\sum_{k\geq 1} \frac{(-1)^k}{k+z}.$$

On $\sum z^{2^n}$ and $\sum z^{n^2}$

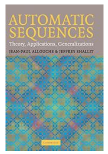
The name *Fredholm series* is often wrongly attributed to the power series

$$\sum_{n\geq 0} z^{2^n}$$

(see Allouche & Shallit, Notes on chapter 13). However Fredholm studied rather the theta series

$$\sum_{n\geq 0} z^{n^2}.$$


J-P. Allouche




J. Shallit

On $\sum z^{2^n}$

Let $\chi(z) = \sum_{n\geq 0} z^{2^n}$.

E. Catalan (1875), J. Sondow and W. Zudilin (2006) :

$$\gamma = \int_0^1 (\chi(x) - x) \frac{\mathrm{d}x}{x(1+x)}.$$


E. Catalan


J. Sondow


W. Zudilin

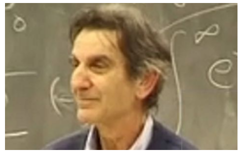
J. Sondow and W. Zudilin. Euler's constant, q-logarithms, and formulas of Ramanujan and Gosper. Ramanujan J. **12** (2006) 225–244.

On $\sum z^{2^n}$ and $\sum z^{n^2}$

Connection between the series $\sum z^{2^n}$, $\sum z^{n^2}$, theta functions, modular functions and paperfolding.

Michel Mendes-France and Ahmed Sebbar, Pliages de papier, fonctions thêta et méthode du cercle, Acta Math. **183** (1999), 101–139.

Ahmed Sebbar, Paperfolding and modular functions. in Exponential Analysis of Differential Equations and Related Topics (ed. Y. Takei). RIMS Kôkyûroku Bessatsu, B **52** (2014), 97–126.


M. Mendes–France


A. Sebbar

Mahler functions

A. J. Kempner (1916) proved the transcendence of the number

$$\chi(1/2) = \sum_{n\geq 0} 2^{-2^n}.$$

Kurt Mahler (1930, 1969) :
For $d \geq 2$, transcendence of the values at algebraic points of $\chi_d(z) = \sum_{n\geq 0} z^{d^n}$.


K. Mahler

`Tool`: The function χ_d satisfies the functional equation $\chi_d(z) = z + \chi_d(z^d)$ for $|z| < 1$.

Mahler functions : another example

The function

$$h(z) = \prod_{n\geq 0} (1 - z^{2^n})$$

satisfies the functional equation

$$h(z) = (1 - z^2) h(z^2).$$

The coefficients of its Taylor series at the origin

$$h(z) = \sum_{n\geq 0} (-1)^{t_n} z^n \qquad |z| < 1$$

are given by the Thue–Morse sequence $(t_n)_{n\geq 0} = (01101001100101 10\dots)$:

$$t_0 = 0, \quad t_{2n} = t_n, \quad t_{2n+1} = 1 - t_n.$$

Mahler theory

The proof of algebraic independence results for values of Mahler functions reduces to the proof of algebraic independence of the functions.

K. Mahler

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Conclusion

The transcendence theory of values of Siegel E–functions and of functions satisfying Mahler equations is strong, but a lot remains to be done in the other situations. One open problem is to prove the Hermite–Lindemann Theorem on the transcendence of $\log\alpha$ for nonzero algebraic number α by using the logarithmic function (i.e. the theory of G–functions) instead of the exponential function (i.e. the theory of E–functions).

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Irrationality and transcendence of values of special functions.

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