Artin L–functions, Artin primitive roots Conjecture and applications

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An introduction to the Riemann zeta function
Exercises

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1. Check the formulae

$$\begin{array}{llll} \mu\star\mathbf{1}=\delta, & \mu\star j=\varphi, & |\mu|\star\kappa=\mathbf{1}, & \mu\star\kappa=\lambda,\\ |\mu|\star\mathbf{1}=2^\omega, & j^k\star\mathbf{1}=\sigma_k, & \mathbf{1}\star\mathbf{1}=\tau, & \mu\star\log=\Lambda. \end{array}$$

2. Check the formulae for the Dirichlet series $D(f,s)=\displaystyle\sum_{n\ge1}f(n)n^{-s}$.

$$\begin{array}{lll} D(j^k,s)=\zeta(s-k), & D(\kappa,s)=\zeta(2s), & D(\mu,s)=\dfrac{1}{\zeta(s)},\\ D(\tau,s)=\zeta(s)^2, & D(\sigma_k,s)=\zeta(s-k)\zeta(s), & D(\varphi,s)=\dfrac{\zeta(s-1)}{\zeta(s)},\\ D(|\mu|,s)=\dfrac{\zeta(s)}{\zeta(2s)}, & D(\lambda,s)=\dfrac{\zeta(2s)}{\zeta(s)}, & D(2^\omega,s)=\dfrac{\zeta(s)^2}{\zeta(2s)}\cdot \end{array}$$

3. Check that the multiplication by log is a derivation in the ring $\mathcal{A}$ of arithmetic functions:

$$\log\cdot(f\star g)=(\log\cdot f)\star g+f\star(\log\cdot g).$$

Check

$$D(\log,s)=-\zeta'(s),\qquad D(\Lambda,s)=-\frac{\zeta'(s)}{\zeta(s)}\cdot$$

3. Check the formula

$$\frac{\zeta'(s)}{\zeta(s)}=\frac{1}{2}\log\pi-\frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)}-\frac{1}{s}-\frac{1}{s-1}+(2s-1)\sum_{\rho\in Z_+}\frac{1}{(s-\rho)(s-(1-\rho))}\cdot$$

Deduce

$$\frac{\zeta'(1/2)}{\zeta(1/2)}=\frac{1}{2}\log(8\pi)+\frac{\gamma}{2}+\frac{\pi}{4}\cdot$$

`Hint:` one can use (without proof) the formula

$$\frac{\Gamma'(1/4)}{\Gamma(1/4)}=-3\log 2-\gamma-\frac{\pi}{2}\cdot$$