Artin L–functions, Artin primitive roots Conjecture and applications

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An introduction to the Riemann zeta function

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Content of this file (29 pages).

- p.2–3: a list of references.
- p. 4–22: from the book *Arithmetics* by M. Hindry: p. 127–131, 134–139, 148–155.
- p. 23–29: from *Éléments d'analyse et d'algèbre (et de théorie des nombres)* by P. Colmez : p. 40–43 and 51–53.

Further main references (not included in this file):

- *Lectures on the Riemann zeta function* by H. Iwaniec, Chap. 1–12, p. 3–43.
- *An introduction to the theory of the Riemann zeta-function* by S. J. Patterson, Chap. 1–11, p. 1–66.
- *An introduction to zeta functions* by P. Cartier, p 1–63.

References

LMFDB is the database of L-functions, modular forms, and related objects. It is an extensive database of mathematical objects arising in Number Theory.

`http://www.lmfdb.org/`

A basic course in number theory, containing the basic elements of analytic number theory, is

- M. Hindry, *Arithmetics*, Universitext, Springer, London, 2011. Translated from the 2008 French original.

Perhaps the simplest and more direct introduction to the Riemann zeta function is contained in the recent small (and a bit synthetic) book :

- H. Iwaniec, *Lectures on the Riemann zeta function*, vol. 62 of University Lecture Series, American Mathematical Society, Providence, RI, 2014.

Chapters 5–7 (p.21–26) present the theory from the beginning to the functional equation; chapters 8–11 (p.27–42) contain the rest of the basic theory, ending with the proof of PNT.

A discussion of the special values of the Riemann zeta function is included in

- P. Cartier, *An introduction to zeta functions*, in From number theory to physics (Les Houches, 1989), Springer, Berlin, 1992, pp. 1–63.

One of the most classical reference is

- G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford University Press, Oxford, sixth ed., 2008. Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles.

Other classical books, all containing the basic theory of $\zeta(s)$, are:

- A. E. Ingham, *The distribution of prime numbers*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1990. Reprint of the 1932 original, With a foreword by R. C. Vaughan.

- E. C. Titchmarsh, *The theory of the Riemann zeta-function*, The Clarendon Press, Oxford University Press, New York, second ed., 1986. Edited and with a preface by D. R. Heath-Brown.
- T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York-Heidelberg, 1976. Undergraduate Texts in Mathematics.
- A. A. Karatsuba, *Basic analytic number theory*, Springer-Verlag, Berlin, 1993. Translated from the second (1983) Russian edition and with a preface by Melvyn B. Nathanson.
- S. J. Patterson, *An introduction to the theory of the Riemann zeta-function*, vol. 14 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1988.
- H. Davenport, *Multiplicative number theory*, vol. 74 of Graduate Texts in Mathematics, Springer-Verlag, New York, third ed., 2000. Revised and with a preface by Hugh L. Montgomery.
- H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory. I. Classical theory*, vol. 97 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2007.
- G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, vol. 163 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, third ed., 2015. Translated from the 2008 French edition by Patrick D. F. Ion.
- P. Colmez, *Éléments d'analyse et d'algèbre (et de théorie des nombres)*, les éditions de l'école polytechnique, 2011, 678 pages.

1.6. Theorem. *As x tends to infinity, we have the following asymptotic behavior:*

$$\pi(x;a,b) := \mathrm{card}\{p \text{ prime}\ ,\ p \leqslant x,\ p \equiv a \bmod b\} \sim \frac{x}{\phi(b)\log x}\cdot \tag{4.4}$$

In this section, we will expand on some so-called "elementary" methods (which in this context means that they do not involve complex variables) which can be used to prove the previous assertions, except for Dirichlet's theorem on arithmetic progressions and the prime number theorem. They will however allow us to prove a partial version: there exist two constants, $c_1, c_2 > 0$, such that $c_1 x/\log x \leqslant \pi(x) \leqslant c_2 x/\log x$.

1.7. Lemma. *The following estimate holds:* $n\log 2 \leqslant \log\binom{2n}{n} \leqslant n\log 4$.

Proof. From the binomial theorem, we know that $\binom{2n}{n} \leqslant \sum_{k=0}^{2n}\binom{2n}{k} = (1+1)^{2n} = 4^n$. Next, we have the following lower bound: $\displaystyle\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{2n(2n-1)\cdots(n+1)}{n(n-1)\cdots 1} \geqslant 2^n$. □

1.8. Lemma. *The following formula holds:* $\mathrm{ord}_p(n!) = \sum_{m\geqslant 1}\left\lfloor \frac{n}{p^m}\right\rfloor$*; furthermore, the sum can be restricted to* $m \leqslant \log n/\log p$.

Proof. Write $n! = 1\cdot 2\cdot 3\cdots n = \prod_{k=1}^{n} k$. The number of integers $\leqslant n$ which are divisible by p is $\lfloor n/p\rfloor$, and the number of integers $\leqslant n$ divisible by p^2 is $\lfloor n/p^2\rfloor$, etc. Thus $\mathrm{ord}_p(n!)$ is the sum of the $\lfloor n/p^m\rfloor$. Finally, $p^m \leqslant n$ is equivalent to $m \leqslant \log n/\log p$, hence the first statement is proved. □

We can therefore write

$$\log\binom{2n}{n} = \sum_{p\leqslant 2n}\mathrm{ord}_p\binom{2n}{n}\log p = \sum_{p\leqslant 2n}\left(\sum_{m\geqslant 1}\left\lfloor\frac{2n}{p^m}\right\rfloor - 2\left\lfloor\frac{n}{p^m}\right\rfloor\right)\log p. \tag{4.5}$$

To find a lower bound, we only keep the terms that satisfy $n < p \leqslant 2n$. In fact, such a p clearly divides $\binom{2n}{n} = (2n)!/(n!)^2$, and thus we obtain

$$n\log 4 \geqslant \log\binom{2n}{n} \geqslant \sum_{n<p\leqslant 2n}\log p = \theta(2n) - \theta(n).$$

From this, we obtain an upper bound of the form $\theta(x) \leqslant Cx$. This is true because

$$\theta(2^m) = \sum_{k=0}^{m-1} \theta(2^{k+1}) - \theta(2^k) \leqslant \sum_{k=0}^{m-1} 2^k \log 4 = (2^m - 1)\log 4.$$

Therefore, if $2^m \leqslant x < 2^{m+1}$, then

$$\theta(x) \leqslant \theta(2^{m+1}) \leqslant 2^{m+1} \log 4 \leqslant (2\log 4)x. \tag{4.6}$$

To obtain an upper bound, we could notice that $\lfloor 2u \rfloor - 2\lfloor u \rfloor$ always equals 0 or 1 and equals 0 whenever $u < 1/2$. Thus

$$n \log 2 \leqslant \log \binom{2n}{n} = \sum_{p \leqslant 2n} \left(\sum_{m \geqslant 1} \left\lfloor \frac{2n}{p^m} \right\rfloor - 2 \left\lfloor \frac{n}{p^m} \right\rfloor \right) \log p$$

$$\leqslant \sum_{p \leqslant 2n} \left(\frac{\log(2n)}{\log p} \right) \log p = \log(2n)\pi(2n).$$

From this, we have a lower bound of the form $\pi(x) \geqslant Cx/\log x$. This is true because if $2n \leqslant x < 2(n+1)$, then

$$\pi(x) \geqslant \pi(2n) \geqslant \frac{n \log 2}{\log(2n)} \geqslant \left(\frac{x}{2} - 1\right) \frac{\log 2}{\log x}. \tag{4.7}$$

Furthermore, we can easily see that

$$\theta(x) = \sum_{p \leqslant x} \log p \leqslant \log x \sum_{p \leqslant x} 1 = \pi(x) \log x. \tag{4.8}$$

Next, notice that for $2 \leqslant y < x$,

$$\pi(x) - \pi(y) = \sum_{y < p \leqslant x} 1 \leqslant \frac{1}{\log y} \sum_{y < p \leqslant x} \log p = \frac{1}{\log y} \left(\theta(x) - \theta(y)\right).$$

It follows that

$$\pi(x) \leqslant \frac{\theta(x)}{\log y} + \pi(y) \leqslant \frac{\theta(x)}{\log y} + y.$$

By choosing $y = x/(\log x)^2$ and by recalling the previous inequality (4.8), we have

$$\frac{\theta(x)}{\log x} \leqslant \pi(x) \leqslant \frac{\theta(x)}{\log x + 2\log\log x} + \frac{x}{(\log x)^2}. \tag{4.9}$$

To summarize, it is easy to see from inequalities (4.6), (4.7), (4.8) and (4.9) that $(\theta(x) \sim x)$ is equivalent to $(\pi(x) \sim x/\log x)$ and that

$$C_1 x \leqslant \theta(x) \leqslant C_2 x \qquad \text{and} \qquad C_3 x/\log x \leqslant \pi(x) \leqslant C_4/\log x. \tag{4.10}$$

Furthermore, the following comparison of the function $\theta(x)$ to the function $\psi(x)$ is not difficult to see:

$$\theta(x) \leqslant \psi(x) := \sum_{p^m \leqslant x} \log p = \theta(x) + \theta(\sqrt{x}) + \theta(\sqrt[3]{x}) + \dots$$
$$\leqslant \theta(x) + \frac{\log(x)}{\log 2}\theta(\sqrt{x}) \leqslant \theta(x) + C \log x \sqrt{x}.$$

Finally, if we denote by p_n the nth prime number, we have $\pi(p_n) = n$ by definition. The prime number theorem therefore implies that $n \sim p_n/\log(p_n)$ and that $p_n \sim n \log n$. We can check that the latter statement is in fact equivalent to the prime number theorem.

1.9. Lemma. (Abel's formula) *Let* $A(x) := \sum_{n \leqslant x} a_n$ *and* f *be a function of class* $\mathscr{C}^1$. *Then,*

$$\sum_{y<n\leqslant x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt. \tag{4.11}$$

Proof. We first point out that $\int_n^{n+1} A(t)f'(t)dt = A(n)\int_n^{n+1} f'(t)dt = A(n)\left(f(n+1) - f(n)\right)$. Therefore, setting $N = \lfloor x \rfloor$ and $M = \lfloor y \rfloor$ yields

$$\int_M^N A(t)f'(t)dt = \sum_{n=M}^{N-1} \int_n^{n+1} A(t)f'(t)dt = \sum_{n=M}^{N-1} A(n+1)\left(f(n) - f(n)\right)$$
$$= \sum_{n=M+1}^{N} f(n)(A(n-1) - A(n)) + f(N)A(N) - A(M)f(M)$$
$$= -\sum_{n=M+1}^{N} f(n)a_n + f(N)A(N) - A(M)f(M).$$

This proves the formula when x and y are integers. For the general formula, observe that

$$\int_{\lfloor x \rfloor}^x A(t)f'(t)dt = A(\lfloor x \rfloor)\left(f(x) - f(\lfloor x \rfloor)\right) = A(x)f(x) - A(\lfloor x \rfloor)f(\lfloor x \rfloor). \square$$

Applications. 1) The formula gives a fairly precise comparison between the "sum" and the "integral" (see Exercise 4-6.10 for some refinements). To be more precise, if we take $a_n = 1$ and integrate by parts, we have:

$$\sum_{n=M+1}^{N} f(n) = \int_M^N f(t)dt + \int_M^N (t - \lfloor t \rfloor)f'(t)dt. \tag{4.12}$$

If we choose $f(t) = 1/t$, we obtain

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n} &= 1+\int_1^N\frac{dt}{t}-\int_1^N(t-\lfloor t\rfloor)\frac{dt}{t^2}\\ &= \log N+\left(1-\int_1^\infty(t-\lfloor t\rfloor)\frac{dt}{t^2}\right)+\int_N^\infty(t-\lfloor t\rfloor)\frac{dt}{t^2}\\ &= \log N+\gamma+O\left(\frac{1}{N}\right),\end{aligned}$$

where $\gamma := 1-\int_1^\infty(t-\lfloor t\rfloor)\frac{dt}{t^2}$ is *Euler's constant.*

2) Take $a_n=1$, so $A(t)=\lfloor t\rfloor$, $y=1$ and $f(t)=\log t$. We therefore have

$$\begin{aligned}\log(\lfloor x\rfloor!) &= \lfloor x\rfloor\log(x)-\int_1^x\frac{\lfloor t\rfloor dt}{t}\\ &= x\log x-\int_1^x dt+(\lfloor x\rfloor-x)\log x-\int_1^x\frac{\lfloor t\rfloor-t}{t}dt\\ &= x\log x-x+O(\log x).\end{aligned}$$

We should point out that Stirling's formula gives a slightly more precise statement, namely $n!\sim n^ne^{-n}\sqrt{2\pi n}$, and hence $\log(n!)=n\log n-n+\frac{1}{2}\log n+\frac{1}{2}\log(2\pi)+\epsilon(n)$ where $\lim_{n\to\infty}\epsilon(n)=0$.

Furthermore, we see that

$$\begin{aligned}\log(\lfloor x\rfloor!) &= \sum_{p\leqslant x}\mathrm{ord}_p(\lfloor x\rfloor!)\log p\\ &= \sum_{p\leqslant x}\sum_{m\geqslant 1}\left\lfloor\frac{x}{p^m}\right\rfloor\log p\\ &= x\sum_{p\leqslant x}\frac{\log p}{p}+\sum_{p\leqslant x}\log p\left(\left\lfloor\frac{x}{p}\right\rfloor-\frac{x}{p}\right)+\sum_{p\leqslant x}\sum_{m\geqslant 2}\left\lfloor\frac{x}{p^m}\right\rfloor\log p\\ &= x\sum_{p\leqslant x}\frac{\log p}{p}+O(x),\end{aligned}$$

where the last estimate comes from the upper bound $\theta(x)=\sum_{p\leqslant x}\log p=O(x)$, from (4.6) and the estimate

$$\sum_{p\leqslant x}\sum_{m\geqslant 2}\left\lfloor\frac{x}{p^m}\right\rfloor\log p\leqslant x\sum_{p\leqslant x}\sum_{m\geqslant 2}\frac{\log p}{p^m}=x\sum_{p\leqslant x}\frac{\log p}{p(p-1)}=O(x).$$

From this, we can deduce the first formula in Proposition 4-1.2,

$$\sum_{p\leqslant x}\frac{\log p}{p}=\log x+O(1). \tag{4.13}$$

To get the second, we apply Abel's formula (Lemma 4-1.9) letting $f(t) = 1/\log t$ and $a_n = \log p/p$ if $n = p$ is prime and $a_n = 0$ otherwise. By setting $A(x) = \sum_{p\leqslant x} \dfrac{\log p}{p}$, we have that

$$\begin{aligned}\sum_{p\leqslant x} \frac{1}{p} &= \sum_{n\leqslant x} a_n f(n)\\ &= \frac{A(x)}{\log x} + \int_2^x \frac{A(t)dt}{t(\log t)^2}\\ &= 1 + O(1/\log x) + \int_2^x \frac{dt}{t\log t} + \int_2^x \frac{(A(t)-\log t)}{t(\log t)^2}dt\\ &= \log\log x - \log\log 2 + 1 + \int_2^\infty \frac{(A(t)-\log t)}{t(\log t)^2}dt + O(1/\log x).\end{aligned}$$

2. Holomorphic Functions (Summary/Reminders)

This section, without proofs, is a summary of some of the fundamental properties of functions of a complex variable that we will be using. It could be helpful to use [74] as a reference.

Concerning series, we will use the product rule for calculating the product of two *absolutely* convergent series:

$$\left(\sum_{n=0}^\infty a_n\right)\left(\sum_{n=0}^\infty b_n\right) = \sum_{n=0}^\infty \left(\sum_{k=0}^n a_k b_{n-k}\right),$$

as well as rearrangement of the order of summation in a series with *positive* terms $a_{m,n}$:

$$\sum_{n=0}^\infty \left(\sum_{m=0}^\infty a_{m,n}\right) = \sum_{m=0}^\infty \left(\sum_{n=0}^\infty a_{m,n}\right).$$

A power series $S(z) = \sum_{n=0}^\infty a_n z^n$ is said to have a radius of convergence $R \geqslant 0$ (possibly $R = 0$ or $R = +\infty$) if the series converges for all $|z| < R$ and diverges for all $|z| > R$; furthermore, the convergence is absolute in the interior of the disc of convergence and the function is of class $\mathscr{C}^\infty$ with $S^{(k)}(z) = \sum_{n=k}^\infty n(n-1)\cdots(n-k+1)a_n z^{n-k}$. In fact, the function S can be expanded as a power series around every point $z_0 \in D(0,R)$, in other words, for every $z \in D(z_0,r) \subset D(0,R)$, we have $S(z) = \sum_{n=0}^\infty b_n(z-z_0)^n$ (with $b_n = S^{(n)}(z_0)/n!$). Such a function only has a finite number of zeros in every closed disc (or compact set) which is contained in $D(0,R)$. We define the multiplicity of a zero, z_0, as the integer k such that $S(z) =$

The series converges normally at every point in the open disk $D(1,1) = \{z \in \mathbf{C} \mid |1-z| < 1\}$, and the convergence is uniform (and also normal) on every closed disk with center 1 and radius $r < 1$. Therefore, $F(z)$ is holomorphic on $D(1,1)$. If z is a real number in the interval $]0,2[$, we can see that $F(z) = \log z$ (ordinary logarithm), and in particular,

$$\exp(F(z)) = z.$$

The previous formula indicates that the two functions, the identity and $\exp \circ F$, which are analytic on the disk $D(1,1)$, coincide on the segment $]0,2[$ and hence on the whole disk. Thus F defines a complex logarithm on the disk $|z-1| < 1$.

2.8. Definition. Let $f(z)$ be a holomorphic function on U. We say that $F(z)$ is a *branch of the logarithm* of f on U (and we write, with a slight abuse of notation, $F(z) = \log f(z)$) if $F(z)$ is holomorphic and if $\exp(F(z)) = f(z)$.

2.9. Remark. If $F(z)$ is a branch of the logarithm of f, then f is never 0 on U, we have $|\exp(F(z))| = \exp(\operatorname{Re} F(z)) = |f(z)|$, and hence

$$\operatorname{Re} \log f(z) = \log |f(z)|.$$

Likewise, $f'(z)/f(z) = F'(z) \exp(F(z)) / \exp(F(z)) = F'(z)$, and also

$$\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)} \cdot$$

Finally, if F_1 and F_2 are two logarithms, then $F_2(z) = F_1(z) + 2k\pi i$ on any connected set U.

This remark suggests that we should construct the logarithm of $f(z)$ as an antiderivative $f'(z)/f(z)$, with the condition that f is not zero. We have seen that this is possible if U is simply connected.

2.10. Proposition. *Let U be a simply connected open subset of the complex plane and $f(s)$ a holomorphic function without any zeros in U. Then there exists a holomorphic branch $F(s) = \log f(s)$ on U. Two such branches differ by an integer multiple of $2\pi i$.*

We will finish this summary by explaining the notion of an infinite product. The first idea consists of saying that a product is convergent if $\lim_N \prod_{n=0}^{N} p_n$ exists. This could be confusing because it is not true that such a product is zero if and only if one of the factors is zero. For example, it can be easily checked that

$$\lim_{N\to\infty}\prod_{n=1}^{N}\Big(1-\frac{1}{n+1}\Big)=0.$$

We can overcome this inconvenience by defining infinite products a little differently. Observe first that a necessary condition for the convergence of a non-zero product $\prod_n p_n$ is to have $\lim_n p_n = 1$; it therefore does not hurt to assume that $p_n = 1 + u_n$ where u_n tends to zero. In particular, $\log(1+u_n) = \sum_{k=1}^{\infty}(-1)^k u_n^k/k$ is well-defined when $|u_n| < 1$ and hence when $n \geqslant n_0$, which justifies the following definition.

2.11. Definition. A product $\prod_{n=0}^{\infty}(1+u_n)$ is *convergent* (resp. *absolutely convergent*) if there exists n_0 such that $|u_n| < 1$ for all $n \geqslant n_0$ and the series $\sum_{n=n_0}^{\infty}\log(1+u_n)$ is convergent (resp. absolutely convergent). A product of functions $\prod_{n=0}^{\infty}(1+u_n(z))$ is *uniformly convergent* on K if there exists n_0 such that $|u_n(z)| < 1$ for all $n \geqslant n_0$ and $z \in K$ and the series $\sum_{n=n_0}^{\infty}\log(1+u_n(z))$ is uniformly convergent (on K).

2.12. Lemma. *A product $P := \prod_{n=0}^{\infty}(1+u_n)$ is absolutely convergent if and only if the series $\sum_{n=0}^{\infty}|u_n|$ is convergent. If P is convergent, it is zero if and only if one of the factors $1+u_n$ is zero.*

2.13. Proposition. *Let $(u_n(z))$ be a sequence of holomorphic functions on an open set U such that the series $\sum_n \log(1+u_n(z))$ converges uniformly on every compact subset of U.*

i) Then the function defined by the infinite product

$$P(z) := \prod_{n=0}^{\infty}(1+u_n(z))$$

is holomorphic on U.

ii) For every $z_0 \in U$, only a finite number of $p_n(z) := 1+u_n(z)$ are zero at z_0, and hence

$$\operatorname{ord}_{z_0} P(z) = \sum_{n=0}^{\infty}\operatorname{ord}_{z_0} p_n(z).$$

3. Dirichlet Series and the Function $\zeta(s)$

We call a *Dirichlet series* a series of the form $F(s) = \sum_{n=1}^{\infty}\dfrac{a_n}{n^s}\cdot$ We will now state its first important property.

3.1. Proposition. *Let* $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *be a Dirichlet series that we will assume to be convergent at* s_0*. Then it converges uniformly on the sets* $E_{C,s_0} = \{s \in \mathbf{C} \mid \operatorname{Re}(s - s_0) \geqslant 0, |s - s_0| \leqslant C \operatorname{Re}(s - s_0)\}$*.*

Proof. For $M \geqslant 1$, set $A_M(x) := \sum_{M<n\leqslant x} a_n n^{-s_0}$. By the hypothesis, we then have $|A_M(x)| \leqslant \epsilon(M)$ where $\epsilon(M)$ tends to zero as M tends to infinity. Abel's formula gives

$$\sum_{M<n\leqslant N} a_n n^{-s} = \sum_{M<n\leqslant N} a_n n^{-s_0} n^{-(s-s_0)}$$
$$= A_M(N) N^{-(s-s_0)} + (s - s_0) \int_M^N A_M(t) t^{-(s-s_0+1)} dt.$$

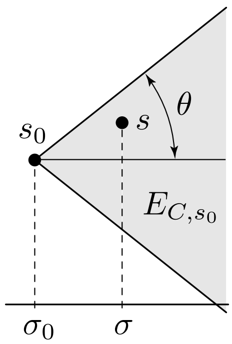
We can find an upper bound for the integral as follows:

$$\left| \int_M^N A_M(t) t^{-(s-s_0+1)} dt \right| \leqslant \epsilon(M) \int_M^N t^{-(\sigma-\sigma_0+1)} dt$$
$$= \epsilon(M) \frac{M^{-(\sigma-\sigma_0)} - N^{-(\sigma-\sigma_0)}}{(\sigma - \sigma_0)} \cdot$$

By restricting to an angular sector E_{C,s_0} bounded by $\sigma - \sigma_0 = \operatorname{Re}(s) - \operatorname{Re}(s_0) \geqslant 0$ and $|s - s_0| = C(\sigma - \sigma_0)$, we obtain

$$\left| \sum_{M<n\leqslant N} a_n n^{-s} \right| \leqslant \epsilon(M)(1 + C),$$

which suffices to show the uniform convergence on this sector (cf. the figure below). □



the domain $|s - s_0| \leqslant \dfrac{\sigma - \sigma_0}{\cos\theta}$

The following corollary is a result of the general theorems recalled in the previous section (in particular, Proposition 4-2.6).

3.2. Corollary. *Every Dirichlet series* $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *has an abscissa of convergence, say* σ_0*, such that the series converges for* $\operatorname{Re}(s) > \sigma_0$ *and* *diverges for* $\mathrm{Re}(s) < \sigma_0$. *Furthermore, the function* F *defined by the series is holomorphic in the half-plane of convergence* $\mathrm{Re}(s) > \sigma_0$, *and its derivatives are given by* $F^{(k)}(s) = \sum_{n=1}^{\infty} a_n(-\log n)^k n^{-s}$.

Proof. It suffices to let $\sigma_0 = \inf\{\sigma \in \mathbf{R} \mid \text{ the series converges at } \sigma\}$, then to observe that every compact set in the (open) half-plane of convergence is contained in a sector, as above, where the convergence is uniform. □

3.3. Remarks. 1) If a Dirichlet series converges at $s_0 = \sigma_0 + it_0$ to the number S, then the proof of Proposition 4-3.1 (above) shows that $S = \lim_{\epsilon \to 0^+} F(\sigma_0 + \epsilon + it_0)$.

2) Set $A(t) := \sum_{n \leqslant t} a_n$. The previous proof allows us to establish the formula

$$\sum_{n=1}^{\infty} a_n n^{-s} = s \int_1^{\infty} \left(\sum_{n \leqslant t} a_n \right) t^{-s-1} dt = s \int_1^{\infty} A(t) t^{-s-1} dt. \tag{4.14}$$

In particular, if $A(t) = \sum_{n \leqslant t} a_n$ is bounded, then the series converges whenever $\mathrm{Re}(s) > 0$.

The most famous Dirichlet series is the Riemann *zeta function*, defined by the series $\sum_{n=1}^{\infty} n^{-s}$. It is well-known, at least for real values and the general case follows from the real case, that the abscissa of convergence is $+1$.

3.4. Theorem. (Euler Product) *If* $\mathrm{Re}(s) > 1$, *then the following formula holds*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{4.15}$$

Proof. Notice that $\sum_p |p^{-s}| = \sum_p p^{-\sigma} \leqslant \sum_n n^{-\sigma}$, hence the product is absolutely convergent. If $\mathrm{Re}(s) > 0$, the convergence of geometric series allows us to write $(1 - p^{-s})^{-1} = \sum_{m=0}^{\infty} p^{-ms}$. By taking the product over the prime numbers $p_1, \ldots, p_r$ which are $\leqslant T$, we obtain

$$\begin{aligned} \prod_{p \leqslant T} (1 - p^{-s})^{-1} &= \prod_{p \leqslant T} \left(\sum_{m=0}^{\infty} p^{-ms} \right) \\ &= \sum_{\substack{m_1, \ldots, m_r \geqslant 1 \\ p_1, \ldots, p_r \leqslant T}} (p_1^{m_1} \cdots p_r^{m_r})^{-s} \\ &= \sum_{n \in \mathcal{N}(T)} n^{-s}, \end{aligned}$$

where we denote by $\mathscr{N}(T)$ the set of integers all of whose prime factors are $\leqslant T$. Thus whenever $\mathrm{Re}(s) > 1$, we have

$$\left|\sum_{n=1}^{\infty} \frac{1}{n^s} - \prod_{p\leqslant T}\left(1-\frac{1}{p^s}\right)^{-1}\right| = \left|\sum_{n\notin\mathscr{N}(T)} \frac{1}{n^s}\right| \leqslant \sum_{n>T}\left|\frac{1}{n^s}\right| = \sum_{n>T}\frac{1}{n^\sigma}\cdot$$

The last sum is the tail-end of a real convergent series (when $\sigma := \mathrm{Re}(s) > 1$) and thus tends to zero, which proves both the convergence of the product and Euler's formula. □

3.5. Corollary. *The function* $\zeta(s)$ *does not have any zeros in the open half-plane* $\mathrm{Re}(s) > 1$*. A holomorphic branch of* $\log\zeta(s)$ *for* $\mathrm{Re}(s) > 1$ *can be constructed by setting*

$$\log\zeta(s) = \sum_p \sum_{m\geqslant 1} \frac{p^{-ms}}{m}\cdot \tag{4.16}$$

Furthermore, if we define the von Mangoldt function *by*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, \\ 0 & \text{if not,}\end{cases}$$

then

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{m\geqslant 1} \frac{\log p}{p^{ms}} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\cdot \tag{4.17}$$

Proof. We know $1-p^{-s}\neq 0$ and that the product is convergent. The first assertion is therefore obvious. The second formula can be deduced from Euler's formula by taking the series expansion of the logarithm (valid for $|x|<1$) and summing:

$$\log\left((1-x)^{-1}\right) = \sum_{m=1}^{\infty}\frac{x^m}{m}\cdot$$

The second formula is therefore gotten by differentiating the first. □

Interlude (I). These formulas can be generalized by replacing $\mathbf{Z}$ and $\mathbf{Q}$ by $\mathscr{O}_K$ and K, and the uniqueness of the decomposition into prime factors by the uniqueness of the decomposition into prime ideals (Theorem 3-4.18). We denote by $\mathscr{I}_K$ the set of non-zero ideals and $\mathscr{P}_K$ the set of non-zero (maximal) prime ideals in $\mathscr{O}_K$. Now we can introduce the Dedekind zeta function and prove that

$$\zeta_K(s) := \sum_{I\in\mathscr{I}_K} \mathrm{N}(I)^{-s} = \prod_{\mathfrak{p}\in\mathscr{P}_K}\left(1-\mathrm{N}(\mathfrak{p})^{-s}\right)^{-1} \text{ for } \mathrm{Re}(s) > 1.$$

3.6. Proposition. *The function* $\zeta(s)$ *can be analytically continued to a meromorphic function to the half-plane* $\mathrm{Re}(s) > 0$*, with a unique pole at* $s = 1$ *with residue equal to* $+1$.

Proof. The statement says that $\zeta(s) - 1/(s-1)$, originally defined for $\mathrm{Re}(s) > 1$, has a holomorphic continuation to the half-plane $\mathrm{Re}(s) > 0$. To prove this, we can write $\zeta(s)$, using the expression $\lfloor t \rfloor = \sum_{n \leqslant t} 1$, as

$$\begin{aligned}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} &= s \int_1^{\infty} \lfloor t \rfloor t^{-s-1} dt \\ &= s \int_1^{\infty} t^{-s} dt + s \int_1^{\infty} (\lfloor t \rfloor - t)\, t^{-s-1} dt \\ &= \frac{1}{s-1} + 1 + s \int_1^{\infty} (\lfloor t \rfloor - t)\, t^{-s-1} dt.\end{aligned}$$

We know that $|\lfloor t \rfloor - t| \leqslant 1$. The last integral is hence convergent and defines a holomorphic function for $\mathrm{Re}(s) > 0$. □

3.7. Remark. Actually, the function $\zeta(s) - 1/(s-1)$ can be extended to the whole complex plane, and moreover, $\zeta(s)$ satisfies a functional equation (see Theorem 4-5.6 further down).

4. Characters and Dirichlet's Theorem

4.1. Definition. If G is a finite abelian group, a homomorphism from G to $\mathbf{C}^*$ is called a *character*. The set of characters of G forms a group denoted by $\hat{G}$.

4.2. Proposition. *The group* $\hat{G}$ *is isomorphic to the group* G *(but not canonically).*

Proof. If $G = \mathbf{Z}/n\mathbf{Z}$, a character satisfies $\chi(1) \in \mu_n$ (where μ_n denotes as usual the group of nth roots of unity), and the map $\chi \mapsto \chi(1)$ provides an isomorphism between $\hat{G}$ and μ_n. The latter is isomorphic to $\mathbf{Z}/n\mathbf{Z}$ and hence to G. We will now show that $\widehat{G_1 \times G_2} \cong \hat{G}_1 \times \hat{G}_2$; to see why this is true, a character χ of $G_1 \times G_2$ can be written as $\chi(g_1, g_2) = \chi(g_1, e_2)\chi(e_1, g_2)$, and, by setting $\chi_1 = \chi(\cdot\,, e_2)$ and $\chi_2 = \chi(e_1, \cdot)$, we obtain an isomorphism $\chi \mapsto (\chi_1, \chi_2)$ from $\widehat{G_1 \times G_2}$ to $\hat{G}_1 \times \hat{G}_2$. The general case is now easy: we have $G \cong \mathbf{Z}/n_1\mathbf{Z} \times \cdots \times \mathbf{Z}/n_r\mathbf{Z}$, hence

$$\begin{aligned}\hat{G} &\cong (\mathbf{Z}/n_1\mathbf{Z} \times \cdots \times \mathbf{Z}/n_r\mathbf{Z})\widehat{\ } \\ &\cong \widehat{\mathbf{Z}/n_1\mathbf{Z}} \times \cdots \times \widehat{\mathbf{Z}/n_r\mathbf{Z}} \cong \mathbf{Z}/n_1\mathbf{Z} \times \cdots \times \mathbf{Z}/n_r\mathbf{Z} \cong G.\end{aligned}$$

□

Dedekind zeta function of the field $\mathbf{Q}(\exp(2\pi i/m))$, which explains in a more conceptual manner why the coefficients are positive.

5. The Prime Number Theorem

We will prove the following form of the prime number theorem.

5.1. Theorem. *The integral* $\int_1^\infty (\theta(t)-t)t^{-2}dt$ *is convergent.*

We will show that the convergence of this integral implies $\theta(x) \sim x$, and hence $\pi(x) \sim x/\log x$. To see this, suppose that $\limsup \theta(x)x^{-1} > 1$, then there exists $\epsilon > 0$ and x_n tending to infinity such that $\theta(x_n)x_n^{-1} \geqslant 1+\epsilon$. For $t \in [x_n, (1+\epsilon/2)x_n]$, we therefore have

$$\frac{\theta(t)-t}{t^2} \geqslant \frac{\theta(x_n)-(1+\epsilon/2)x_n}{t^2} \geqslant \frac{\epsilon x_n/2}{t^2} \geqslant \frac{\epsilon x_n}{2(1+\epsilon/2)^2 x_n^2},$$

and consequently

$$\int_{x_n}^{(1+\epsilon/2)x_n} \frac{\theta(t)-t}{t^2}dt \geqslant \frac{\epsilon^2}{4(1+\epsilon/2)^2},$$

which contradicts the convergence of the integral. We can therefore conclude that $\limsup \theta(x)x^{-1} \leqslant 1$. A symmetric argument shows that $\liminf \theta(x)x^{-1} \geqslant 1$. It follows that $\lim \theta(x)x^{-1} = 1$.

To prove the theorem, we will use the following result from complex analysis (due to Newman, see [55, 81]) concerning the Laplace transform.

5.2. Theorem. ("The analytic theorem") *Let $h(t)$ be a bounded, piecewise continuous function. Then the integral*

$$F(s) = \int_0^{+\infty} h(u)e^{-su}du$$

is convergent and defines a holomorphic function on the half-plane $\mathrm{Re}(s) > 0$. *Suppose that this function can be analytically continued to a holomorphic function on the* closed *half-plane* $\mathrm{Re}(s) \geqslant 0$. *Then the integral converges for $s = 0$ and*

$$F(0) = \int_0^{+\infty} h(u)du.$$

Let us provisionally admit that this result is true and see how to apply it to the function

$$\begin{aligned} F(s) &= \int_1^{+\infty} \frac{\theta(t)-t}{t^{s+2}}\,dt \\ &= \int_0^{+\infty} \frac{\theta(e^u)-e^u}{e^{u(s+2)}}\,e^u du = \int_0^{+\infty} \left[\theta(e^u)e^{-u}-1\right] e^{-us}du. \end{aligned}$$

The function $h(u) := \theta(e^u)e^{-u} - 1$ is indeed bounded and piecewise continuous. If the analytic continuation hypothesis is satisfied, then we know that $F(0) = \int_0^{+\infty}(\theta(e^u)e^{-u}-1)du = \int_1^{+\infty} \frac{\theta(t)-t}{t^2}dt$ is indeed convergent.

We could transform the integral which defines $F(s)$ (for $\mathrm{Re}(s) > 0$) as follows:

$$\begin{aligned} F(s) &= \int_1^{+\infty} \frac{\theta(t)-t}{t^{s+2}}\,dt \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \theta(t)t^{-s-2}dt - \int_1^{+\infty} t^{-s-1}dt \\ &= \sum_{n=1}^{\infty} \theta(n)\frac{n^{-s-1}-(n+1)^{-s-1}}{s+1} - \frac{1}{s} \\ &= \frac{1}{s+1}\sum_{n=1}^{\infty} n^{-s-1}\left(\theta(n)-\theta(n-1)\right) - \frac{1}{s} \\ &= \frac{1}{s+1}\sum_p p^{-s-1}\log(p) - \frac{1}{s}\cdot \end{aligned}$$

We have also seen that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m\geqslant 1} \log(p)p^{-ms} = \sum_p \log(p)p^{-s} + \sum_{p,m\geqslant 2} \log(p)p^{-ms}.$$

The second term in the last expression is a convergent series and hence holomorphic for $\mathrm{Re}(s) > 1/2$. From this, we can deduce that $\sum_p \log(p)p^{-s} = -\frac{\zeta'(s)}{\zeta(s)} +$ a holomorphic function on $\mathrm{Re}(s) > 1/2$ and finally that

$$F(s) = -\frac{\zeta'(s+1)}{(s+1)\zeta(s+1)} - \frac{1}{s} + \text{ a holomorphic function on } \mathrm{Re}(s) > -1/2.$$

The key point in the proof is therefore the following result.

5.3. Theorem. (Hadamard, de la Vallée-Poussin) *The function* $\zeta(s)$ *does not have any zeros on the line* $\mathrm{Re}(s) = 1$.

Proof. We start with the formula

$$4\cos(x) + \cos(2x) + 3 = 2(1 + \cos(x))^2 \geqslant 0.$$

Recall that

$$\log \zeta(\sigma + it) = \sum_{p,m} \frac{p^{-m\sigma - mit}}{m}, \quad \text{and}$$

$$\log |\zeta(\sigma + it)| = \sum_{p,m} \frac{p^{-m\sigma}}{m} \cos(mt \log p).$$

This implies that

$$\log \left(|\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \zeta(\sigma)^3 \right) = \sum_{p,m \geqslant 1} \frac{p^{-m\sigma}}{m} (4\cos(mt\log p) + \cos(2mt\log p) + 3) \geqslant 0.$$

We can conclude from this, assuming $\sigma > 1$, that

$$|\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \zeta(\sigma)^3 \geqslant 1. \tag{4.22}$$

Now, if $\zeta(s)$ had a zero of order k at $1 + it$ and with order ℓ at $1 + 2it$, then $|\zeta(\sigma + it)| \sim a(\sigma - 1)^k$, $|\zeta(\sigma + 2it)| \sim b(\sigma - 1)^\ell$ and $\zeta(\sigma) \sim (\sigma - 1)^{-1}$ (where σ tends to 1 from above). The left-hand side of inequality (4.22) is therefore (asymptotically) equivalent to $c(\sigma - 1)^{4k+\ell-3}$, which implies that $4k + \ell - 3 \leqslant 0$, and hence $k = 0$. □

5.4. Corollary. *The function defined on* $\mathrm{Re}(s) > 1$ *by*

$$G(s) := -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1}$$

extends to a holomorphic function on $\mathrm{Re}(s) \geqslant 1$.

Proof. The previous theorem shows that the function $\zeta'(s)/\zeta(s)$ is holomorphic on the line $\mathrm{Re}(s) = 1$, except for $s = 1$. Consequently, the function $G(s)$ also is. To study $G(s)$ in a neighborhood of $s = 1$, we use the fact that $\zeta(s)$ has a simple pole at $s = 1$, and consequently $\zeta'(s)/\zeta(s) = -1/(s-1) + g(s)$, where $g(s)$ is holomorphic in a neighborhood of 1. Thus $G(s)$ is indeed holomorphic in a neighborhood of 1 and hence on the line $\mathrm{Re}(s) = 1$. □

Appendix. Proof of the "analytic theorem"

Recall the statement of the analytic result used in the proof of the prime number theorem.

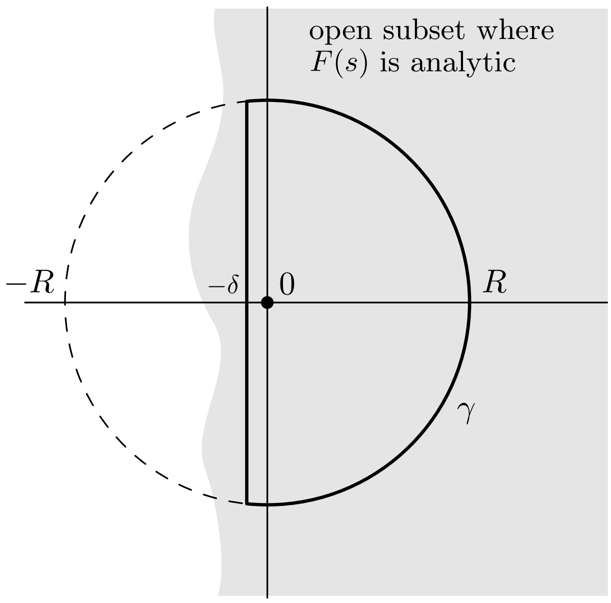
5.5. Theorem. *If $h(t)$ is a bounded piecewise continuous function, then the integral (the Laplace transform of h)*

$$F(s) = \int_0^{+\infty} h(u)e^{-su}du$$

is convergent and defines a function which is holomorphic on the half-plane $\mathrm{Re}(s) > 0$. *Suppose that this function can be analytically continued to a holomorphic function on the* closed *half-plane* $\mathrm{Re}(s) \geqslant 0$. *Then the integral for $s = 0$ converges and*

$$F(0) = \int_0^{+\infty} h(u)du.$$

Proof. The first part is analogous to the theorem of convergence for Dirichlet series (see Exercise 4-6.2). We will therefore prove the second statement. For a (large) real number T, let $F_T(s) := \int_0^T h(t)e^{-st}dt$; these are functions which are holomorphic for all $s \in \mathbf{C}$. We now need to show that $\lim_{T\to\infty} F_T(0)$ exists and equals $F(0)$. To do this, we consider for some large R the contour $\gamma = \gamma(R, \delta)$ which bounds the region $S := \{s \in \mathbf{C} \mid \mathrm{Re}(z) \geqslant -\delta \text{ and } |s| \leqslant R\}$. Once we have fixed R, we can choose $\delta > 0$ sufficiently small so that $F(s)$ is analytic on this region.



The trick lies in introducing the function

$$G_T(s) := (F(s) - F_T(s))\, e^{sT}\left(1 + \frac{s^2}{R^2}\right),$$

so that $G_T(0) = F(0) - F_T(0)$. Therefore, everything comes back to proving that $\lim_{T\to\infty} G_T(0) = 0$. To do this, we will use the residue theorem a first time, noticing that

$$G_T(0) = F(0) - F_T(0) = \frac{1}{2\pi i}\int_\gamma (F(s) - F_T(s))\, e^{sT}\left(1 + \frac{s^2}{R^2}\right)\frac{ds}{s}\cdot$$

To find an upper bound on this integral, we cut the contour into two pieces: γ_1, which is the piece of γ which lives in the half-plane $\operatorname{Re}(s) > 0$, and γ_2, which lives in the half-plane $\operatorname{Re}(s) < 0$. We then carry out the following computation.

Let s be a number such that $|s| = R$ or $s = Re^{i\theta}$. Then we have

$$\left|e^{sT}\left(1 + \frac{s^2}{R^2}\right)\frac{1}{s}\right| = e^{\operatorname{Re}(s)T}\left|e^{-i\theta} + e^{i\theta}\right|\frac{1}{R} = e^{\operatorname{Re}(s)T}\,\frac{2\operatorname{Re}(s)}{R^2}\cdot$$

We also have the upper bound

$$|F(s) - F_T(s)| = \left|\int_T^\infty h(t)e^{-st}dt\right| \leqslant M\int_T^\infty \left|e^{-st}\right| dt = \frac{Me^{-\operatorname{Re}(s)T}}{\operatorname{Re}(s)}\cdot$$

This gives us

$$\left|\frac{1}{2\pi i}\int_{\gamma_1} (F(s) - F_T(s))\, e^{sT}\left(1 + \frac{s^2}{R^2}\right)\frac{ds}{s}\right| \leqslant \frac{M}{R}\cdot$$

Thus assuming that R is very large, this part of the integral will be arbitrarily small. Now, cut the integral over γ_2 into two pieces, I_1 and I_2, where

$$I_1 := \frac{1}{2\pi i}\int_{\gamma_2} F(s)e^{sT}\left(1 + \frac{s^2}{R^2}\right)\frac{ds}{s},$$
$$I_2 := \frac{1}{2\pi i}\int_{\gamma_2} F_T(s)e^{sT}\left(1 + \frac{s^2}{R^2}\right)\frac{ds}{s}\cdot$$

To find an upper bound on I_2, observe first that $F_T(s)$ is entire. The residue theorem (or actually the Cauchy formula in this case) allows us to then replace the contour γ_2 by the arc of a circle of radius R which lives in the half-plane $\operatorname{Re}(s) < 0$ and, by using the same upper bounds, to conclude that $|I_2| \leqslant M/R$. To find an upper bound of I_1, simply notice that the function $F(s)e^{sT}\left(1 + \frac{s^2}{R^2}\right)\frac{1}{s}$ converges to 0 when T tends to $+\infty$ and converges uniformly on every compact set contained in $\operatorname{Re}(s) < 0$. Consequently,

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma_2} F(s)e^{sT}\left(1 + \frac{s^2}{R^2}\right)\frac{ds}{s} = 0.$$

By putting the three upper bounds together, we see that

$$|F_T(0) - F(0)| \leqslant \frac{2M}{R} + \epsilon(T)$$

where $\epsilon(T)$ tends to zero (in a way dependent on R). We needed to show that $\lim F_T(0) = F(0)$, which is now accomplished. □

Supplement. Analytic continuation and the functional equation

We will now outline the main steps of the proof of the following theorem due to Riemann.

5.6. Theorem. (The functional equation of the Riemann zeta function) *The function $\zeta(s) - 1/(s-1)$ can be analytically continued to the whole complex plane. Furthermore, the function $\zeta(s)$ satisfies the functional equation given by*

$$\xi(s) = \xi(1-s), \tag{4.23}$$

where $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

As a preliminary, we will recall the construction of the function $\Gamma(s)$ and the Poisson formula which gives the functional equation for the theta series.

5.7. Lemma. *The integral $\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt$ defines a holomorphic function for $\mathrm{Re}(s) > 0$, which satisfies the functional equation $\Gamma(s+1) = s\Gamma(s)$. It can be continued to all of $\mathbf{C}$ as a meromorphic function with simple poles at $0, -1, -2, -3, \ldots$.*

Proof. Showing that the integral is convergent does not pose any problems. The functional equation can be obtained by integrating by parts. The functional equation also allows us to analytically continue by induction from $\mathrm{Re}(s) > -n$ to $\mathrm{Re}(s) > -n-1$ by using the fact that $\Gamma(s) = s^{-1}\Gamma(s+1)$. Finally, the expression

$$\Gamma(s) = \frac{1}{s(s+1)\cdots(s+n)}\Gamma(s+n+1)$$

makes it clear where the poles are. □

We can also prove that for all s, $\Gamma(s) \neq 0$ (see Exercise 4-6.19).

5.8. Lemma. (Poisson formula) *Let $f(x)$ be an integrable function over $\mathbf{R}$ (i.e., in $L^1(\mathbf{R})$). We define its Fourier transform by*

$$\hat{f}(y) := \int_{-\infty}^{+\infty} f(x)\exp(2\pi ixy)dx$$

and assume that the function $\sum_{n\in\mathbf{Z}} f(x+n)$ *is of bounded variation on* $[0,1]$ *and continuous. Then the following formula holds:*

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{m\in\mathbf{Z}} \hat{f}(m). \tag{4.24}$$

Proof. We introduce the function $G(x) := \sum_{n\in\mathbf{Z}} f(x+n)$ (the hypotheses guarantee the existence and continuity of such a function), which is clearly a periodic function. Dirichlet's theorem on Fourier series allows us to write its Fourier expansion as

$$G(x) = \sum_{m\in\mathbf{Z}} \hat{G}(m)\exp(2\pi i m x),$$

where the Fourier coefficients can be calculated as follows:

$$\begin{aligned}\hat{G}(m) := \int_0^1 G(t)\exp(-2\pi i m t)dt &= \sum_{n\in\mathbf{Z}} \int_0^1 f(t+n)\exp(-2\pi i m t)dt \\ = \int_{-\infty}^{+\infty} f(x)\exp(-2\pi i x m)dx &= \hat{f}(-m).\end{aligned}$$

This gives

$$\sum_{n\in\mathbf{Z}} f(x+n) = \sum_{m\in\mathbf{Z}} \hat{f}(m)\exp(-2\pi i m x).$$

The Poisson formula follows from that by taking $x = 0$. □

This formula is most often applied to a function f which is continuously differentiable and fast decreasing (i.e., $f(x) = O(|x|^{-M})$ for all M), and therefore the function G is itself continuously differentiable. This is the case when applying the formula to the following "theta" function.

5.9. Corollary. *The function*[2] $\theta(u) := \sum_{n\in\mathbf{Z}} \exp\left(-\pi u n^2\right)$ *satisfies the functional equation for all* $u \in \mathbf{R}_+^*$ *given by:*

$$\theta(1/u) = \sqrt{u}\,\theta(u). \tag{4.25}$$

Proof. It suffices to apply the Poisson formula to the function $f(x) = \exp(-\pi u x^2)$ and to verify that $\hat{f}(y) = \exp(-\pi y^2/u)/\sqrt{u}$. □

Proof. (of Theorem 4-5.6) We start with the following computation (where

[2] We hope that the context will allow the reader to distinguish this function from the Tchebychev function $\theta(x) = \sum_{p\leqslant x} \log p$.

we introduce $t = \pi n^2 u$) which is valid for $\mathrm{Re}(s) > 1$.

$$\begin{aligned}\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) &= \sum_{n\geqslant 1}\int_0^\infty e^{-t}t^{s/2}\pi^{-s/2}n^{-s}\,\frac{dt}{t}\\ &= \int_0^\infty \left\{\sum_{n\geqslant 1}\exp(-\pi u n^2)\right\} u^{s/2}\,\frac{du}{u}\\ &= \int_0^\infty \tilde{\theta}(u)\,\frac{u^{s/2}du}{u}\end{aligned}$$

where

$$\tilde{\theta}(u) := \sum_{n\geqslant 1}\exp\left(-\pi u n^2\right) = \frac{\theta(u)-1}{2}\cdot$$

Let us point out that $\tilde{\theta}(u) = O(\exp(-\pi u))$ when u tends to infinity and that the functional equation of the function θ can be translated into

$$\tilde{\theta}\left(\frac{1}{u}\right) = \sqrt{u}\,\tilde{\theta}(u) + \frac{1}{2}\left(\sqrt{u}-1\right). \tag{4.26}$$

By using the simple computation $\int_1^\infty t^{-s} = 1/(s-1)$ and the functional equation of the theta function (4.25), we obtain

$$\begin{aligned}\xi(s) &= \int_0^1 \tilde{\theta}(u)\,\frac{u^{s/2}du}{u} + \int_1^\infty \tilde{\theta}(u)\,\frac{u^{s/2}du}{u}\\ &= \int_1^\infty \tilde{\theta}(1/u)\,\frac{u^{-s/2}du}{u} + \int_1^\infty \tilde{\theta}(u)\,\frac{u^{s/2}du}{u}\\ &= \int_1^\infty \left\{\sqrt{u}\tilde{\theta}(u) + \frac{1}{2}\left(\sqrt{u}-1\right)\right\}\frac{u^{-s/2}du}{u} + \int_1^\infty \tilde{\theta}(u)\,\frac{u^{s/2}du}{u}\\ &= \int_1^\infty \tilde{\theta}(u)\left\{u^{\frac{s}{2}} + u^{\frac{1-s}{2}}\right\}\frac{du}{u} + \frac{1}{s-1} - \frac{1}{s}\cdot\end{aligned}$$

We have a priori obtained the desired expression only for $\mathrm{Re}(s) > 1$, but we can easily see that the integral defines an entire function since $\tilde{\theta}(u) = O(\exp(-\pi u))$ and since it is symmetric under the transformation $s \mapsto 1-s$. □

Supplement without proofs

1) To establish the prime number theorem, we could, in the place of the "analytic theorem", use Ikehara's theorem [40] (sometimes called the Ikehara-Wiener theorem), which is more powerful but also more tricky to prove. We will settle with stating the theorem. Its extension to the case of a multiple pole was proven by Delange [25].

seulement si son terme général tend vers 0), nous avons consacré un chapitre à leur construction (la construction du corps $\mathbf{C}_p$ est un peu plus longue que celle de $\mathbf{C}$, mais pas beaucoup) et un chapitre à l'analyse sur $\mathbf{Z}_p$. Le dernier chapitre est, quant à lui, consacré à la construction de la fonction zêta p-adique. Le second chapitre contient beaucoup plus de choses que ce qui est nécessaire pour cette construction (les mesures suffiraient, mais les distributions deviennent indispensables pour construire des fonction L p-adiques plus générales, par exemple celles attachées aux formes modulaires). Pour d'autres points de vue sur les nombres p-adiques, le lecteur pourra consulter les livres *p-adic numbers, p-adic analysis, and zeta-functions* et *p-adic analysis : a short course on recent work* de Koblitz ou *An introduction to G-functions* de Dwork, Gerotto et Sullivan. L'énoncé précis du théorème de Mazur et Wiles n'est pas donné dans le texte, pas plus que l'application au théorème de Kummer et nous renvoyons le lecteur intéressé aux livres *Introduction to cyclotomic fields* de Washington ou *Cyclotomic fields I and II* de Lang (pour le théorème de Kummer, on peut aussi lire la démonstration dans l'ouvrage de Borevich et Chafarevitch déjà cité).

CHAPITRE I
LA FONCTION ZÊTA DE RIEMANN

I.1. Prolongement analytique et valeurs aux entiers négatifs

Soit $\zeta(s) = \sum_{n=1}^{+\infty} n^{-s} = \prod_p (1-p^{-s})^{-1}$ la fonction zêta de Riemann. Soit $\Gamma(s) = \int_{t=0}^{+\infty} e^{-t} t^s \frac{dt}{t}$ la fonction Γ d'Euler. Cette fonction est holomorphe pour $\mathrm{Re}(s) > 0$ et satisfait l'équation fonctionnelle $\Gamma(s+1) = s\Gamma(s)$, ce qui permet de la prolonger en une fonction méromorphe sur $\mathbf{C}$ tout entier.

Lemme I.1.1. *Si* $\mathrm{Re}(s) > 1$*, alors*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{1}{e^t - 1} t^s \frac{dt}{t}.$$

Démonstration. Il suffit d'écrire $\frac{1}{e^t-1}$ sous la forme $\sum_{n=1}^{+\infty} e^{-nt}$ et d'utiliser la formule $\int_0^{+\infty} e^{-nt} t^s \frac{dt}{t} = \frac{\Gamma(s)}{n^s}$.

Proposition I.1.2. *Si f est une fonction $\mathscr{C}^\infty$ sur $\mathbf{R}_+$ à décroissance rapide à l'infini, alors la fonction*

$$\mathrm{L}(f,s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f(t) t^s \frac{dt}{t}$$

définie pour $\mathrm{Re}(s) > 0$ *admet un prolongement holomorphe à* $\mathbf{C}$ *tout entier et, si* $n \in \mathbf{N}$, *alors* $\mathrm{L}(f,-n) = (-1)^n f^{(n)}(0)$.

Démonstration. Soit φ une fonction $\mathscr{C}^\infty$ sur $\mathbf{R}_+$, valant 1 sur $[0,1]$ et 0 sur $[2,+\infty[$. On peut écrire f sous la forme $\varphi f + (1-\varphi)f$ et $\mathrm{L}(f,s)$ sous la forme $\mathrm{L}(\varphi f, s) + \mathrm{L}((1-\varphi)f, s)$ et comme $(1-\varphi)f$ est nulle dans un voisinage de 0 et à décroissance rapide à l'infini, l'intégrale $\displaystyle\int_0^{+\infty} f(t)t^s \frac{dt}{t}$ définit une fonction holomorphe sur $\mathbf{C}$ tout entier. Comme de plus, $1/\Gamma(s)$ s'annule aux entiers négatifs, on a $\mathrm{L}((1-\varphi)f, -n) = 0$ si $n \in \mathbf{N}$. On voit donc que, quitte à remplacer f par $(1-\varphi)f$, on peut supposer f à support compact. Une intégration par partie nous fournit alors la formule $\mathrm{L}(f,s) = -\mathrm{L}(f', s+1)$ si $\mathrm{Re}(s) > 1$, ce qui permet de prolonger $\mathrm{L}(f,s)$ en une fonction holomorphe sur $\mathbf{C}$ tout entier. D'autre part, on a

$$\mathrm{L}(f,-n) = (-1)^{n+1}\mathrm{L}(f^{(n+1)},1) = (-1)^{n+1}\int_0^{+\infty} f^{(n+1)}(t)dt = (-1)^n f^{(n)}(0),$$

ce qui termine la démonstration.

On peut en particulier appliquer cette proposition à $f_0(t) = \dfrac{t}{e^t - 1}$. Soit $\sum_{n=0}^{+\infty} \mathrm{B}_n t^n/n!$ le développement de Taylor de f_0 en 0. Les B_n sont des nombres rationnels appelés nombres de Bernoulli et qu'on retrouve dans toutes les branches des mathématiques. On a en particulier

$$\mathrm{B}_0 = 1, \quad \mathrm{B}_1 = -\frac{1}{2}, \quad \mathrm{B}_2 = \frac{1}{6}, \quad \mathrm{B}_4 = \frac{-1}{30}, \quad \dots, \quad \mathrm{B}_{12} = \frac{-691}{2730},$$

et comme $f_0(t) - f_0(-t) = -t$, la fonction f_0 est presque paire et $\mathrm{B}_{2k+1} = 0$ si $k \geqslant 1$. Un test presque infaillible pour savoir si une suite de nombres a un rapport avec les nombres de Bernoulli est de regarder si 691 apparaît dans les premiers termes de cette suite.

Théorème I.1.3

(i) *La fonction ζ a un prolongement méromorphe à $\mathbf{C}$ tout entier, holomorphe en dehors d'un pôle simple en $s = 1$ de résidu 1.*

(ii) *Si $n \in \mathbf{Q}$, alors $\zeta(-n) = (-1)^n \dfrac{\mathrm{B}_{n+1}}{n+1}$; en particulier, $\zeta(-n) \in \mathbf{Q}$.*

Démonstration. On a $\zeta(s) = \frac{1}{s-1}\mathrm{L}(f_0, s-1)$ comme on le constate en utilisant la formule $\Gamma(s) = (s-1)\Gamma(s-1)$; on en déduit le résultat.

I.2. Valeurs aux entiers positifs pairs

Proposition I.2.1. *Si* $z \in \mathbf{C} - \mathbf{Z}$, *alors*

$$\frac{1}{z} + \sum_{n=1}^{+\infty} \Big(\frac{1}{z+n} + \frac{1}{z-n} \Big) = \pi \mathrm{cotg}\, \pi z.$$

Démonstration. Notons $\mathrm{F}(z)$ la fonction

$$\mathrm{F}(z) = \frac{1}{z} + \sum_{n=1}^{+\infty} \Big(\frac{1}{z+n} + \frac{1}{z-n} \Big) = \frac{1}{z} + \sum_{n=1}^{+\infty} \frac{2z}{z^2 - n^2}.$$

La convergence absolue de la série dans le second membre montre que $\mathrm{F}(z)$ est une fonction méromorphe sur $\mathbf{C}$, holomorphe sur $\mathbf{C} - \mathbf{Z}$ avec des pôles simples de résidu 1 en les entiers, que F est impaire et périodique de période 1. La fonction $\mathrm{G}(z) = \mathrm{F}(z) - \pi \mathrm{cotg}\, \pi z$ est donc holomorphe sur $\mathbf{C}$, impaire et périodique de période 1.

Si $-1/2 \leqslant x \leqslant 1/2$, on a $|z^2 - n^2| \geqslant y^2 + n^2 - 1/4$ et $|z| \leqslant y + 1/2$. On obtient donc

$$\begin{aligned} \Big| \sum_{n=1}^{+\infty} \frac{2z}{z^2 - n^2} \Big| \leqslant \sum_{n=1}^{+\infty} \Big| \frac{2z}{z^2 - n^2} \Big| &\leqslant \sum_{n=1}^{+\infty} \frac{2y+1}{y^2 - \frac{1}{4} + n^2} \\ &\leqslant \int_0^{+\infty} \frac{2y+1}{y^2 - \frac{1}{4} + x^2} dx = \frac{\pi}{2} \cdot \frac{2y+1}{\sqrt{y^2 - \frac{1}{4}}}. \end{aligned}$$

Comme la fonction $\pi \mathrm{cotg}\, \pi z$ est bornée sur $|\mathrm{Im}(z)| \geqslant 1$, il existe $c > 0$ tel que $|\mathrm{G}(z)| \leqslant c$ si $z = x + iy$ et $-1/2 \leqslant x \leqslant 1/2$, $y \geqslant 1$. De plus, G étant holomorphe, il existe $c' > 0$ tel que $|\mathrm{G}(z)| \leqslant c'$ si $z = x + iy$ et $-1/2 \leqslant x \leqslant 1/2$, $0 \leqslant y \leqslant 1$ et G étant impaire et périodique de période 1, on a alors $|\mathrm{G}(z)| \leqslant \sup(c, c')$ quel que soit $z \in \mathbf{C}$. La fonction G est donc bornée sur $\mathbf{C}$ tout entier, donc constante et nulle car impaire. Ceci permet de conclure.

Maintenant, on a d'une part

$$\frac{1}{z} + \sum_{n=1}^{+\infty} \frac{2z}{z^2 - n^2} = \frac{1}{z} - \sum_{n=1}^{+\infty} 2z \sum_{k=0}^{+\infty} \frac{z^{2k}}{n^{2k+2}} = \frac{1}{z} - 2 \sum_{k=1}^{+\infty} \zeta(2k) z^{2k-1},$$

et d'autre part,

$$\pi \mathrm{cotg}\, \pi z = i\pi \frac{e^{2i\pi z} + 1}{e^{2i\pi z} - 1} = i\pi + \frac{2i\pi}{e^{2i\pi z} - 1} = i\pi + \frac{1}{z} \sum_{n=0}^{+\infty} \mathrm{B}_n \frac{(2i\pi z)^n}{n!}.$$

On en déduit le résultat suivant.

Théorème I.2.2. *Si k est un entier $\geqslant 1$, alors*

$$\zeta(2k) = -\frac{1}{2}\mathrm{B}_{2k}\frac{(2i\pi)^{2k}}{(2k)!}.$$

En particulier, $\pi^{-2k}\zeta(2k)$ est un nombre rationnel.

I.3. Polylogarithmes et valeurs aux entiers positifs de la fonction zêta

1. Polylogarithmes

Si k est un entier $\geqslant 1$, on note $\mathrm{Li}_k(z)$ la fonction définie, pour $|z| < 1$, par la formule

$$\mathrm{Li}_k(z) = \sum_{n=1}^{+\infty}\frac{z^n}{n^k}.$$

Ces fonctions Li_k, appelées *polylogarithmes* (Li_2 est le *dilogarithme*, Li_3 le *trilogarithme*...), ont été introduites par Leibnitz, mais n'ont commencé à jouer un rôle important que depuis une vingtaine d'années ; on s'est aperçu récemment qu'elles apparaissaient naturellement dans de multiples questions (volumes de variétés hyperboliques, valeurs aux entiers des fonctions zêtas, cohomologie de $\mathbf{GL}_n(\mathbf{C})$...).

On a $\mathrm{Li}_1(z) = -\log(1-z)$, et $\frac{d}{dz}\mathrm{Li}_k(z) = z^{-1}\mathrm{Li}_{k-1}(z)$, ce qui permet de prolonger analytiquement $\mathrm{Li}_k(z)$, par récurrence sur k, en une fonction holomorphe multivaluée sur $\mathbf{C} - \{0, 1\}$. (On est obligé de supprimer 0 bien que la série $\sum_{n=1}^{+\infty} z^n/n^k$ n'ait pas de singularité en 0 car, après un tour autour de 1, la fonction $\mathrm{Li}_1(z) = -\log(1-z)$ augmente de $2i\pi$ et donc vaut $2i\pi$ en 0, ce qui fait que $\mathrm{Li}_2(z) = \int z^{-1}\mathrm{Li}_1(z)$ a une singularité logarithmique en 0.) Pour prolonger analytiquement $\log z$ et les $\mathrm{Li}_k(z)$, $k \geqslant 1$, il faut choisir un chemin γ_z dans $\mathbf{C} - \{0, 1\}$ reliant un point fixe (nous prendrons 1/2 comme point de base de tous nos chemins) à z et poser

$$\log z = -\log 2 + \int_{\gamma_z}\frac{dt}{t},$$

$$\mathrm{Li}_1(z) = \log 2 + \int_{\gamma_z}\frac{dt}{1-t}$$

et

$$\mathrm{Li}_k(z) = \sum_{n=1}^{+\infty}\frac{2^{-n}}{n^k} + \int_{\gamma_z}\mathrm{Li}_{k-1}(t)\frac{dt}{t}.$$

Cette formule étant valable quel que soit φ, la formule d'inversion de Fourier $\widehat{\widehat{\varphi}}(0) = \varphi(0)$ nous permet de d'obtenir l'égalité

$$c_n = \zeta(n) c_{n-1}$$

qui permet de conclure.

Remarque I.3.9. En voyant la formule du théorème, on se dit qu'il doit exister une « décomposition » de $\mathbf{SL}_n$ en morceaux reflétant la factorisation naturelle du volume de $\mathbf{SL}_n(\mathbf{R})/\mathbf{SL}_n(\mathbf{Z})$. C'est cette vague intuition qui mène à la K-théorie et aux régulateurs de Borel (*cf.* démonstration du théorème I.3.7).

I.4. Équation fonctionnelle de la fonction zêta

Soit $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Théorème I.4.1. *La fonction $\xi(s)$ admet un prolongement méromorphe à $\mathbf{C}$ tout entier, holomorphe en dehors de pôles simples de résidus respectifs -1 et 1 en $s = 0$ et $s = 1$, et vérifie l'équation fonctionnelle*

$$\xi(s) = \xi(1 - s).$$

Démonstration. Il y a un nombre assez conséquent de méthodes pour arriver au résultat. Nous en donnerons deux dans ce n$^{\text{o}}$ et une autre au § III.4 (*cf.* th. III.4.5).

1. Première méthode : la fonction thêta

Si $t > 0$, on a

$$\int_{-\infty}^{+\infty} e^{-\pi t x^2} e^{-2i\pi xy} dx = e^{-\pi t^{-1} y^2} \int_{-\infty}^{+\infty} e^{-\pi t (x + i t^{-1} y)^2} dx = t^{-1/2} e^{-\pi t^{-1} y^2}.$$

(C'est classique ; faire le changement de variables $u = \sqrt{t}(x + it^{-1}y)$, utiliser le théorème des résidus pour revenir sur la droite réelle et la formule $\int_{-\infty}^{+\infty} e^{-\pi u^2} du = 1$ pour conclure.)

Si $t > 0$, soit $\theta(t) = \sum_{n\in\mathbf{Z}} e^{-\pi t n^2}$. La formule de Poisson

$$\sum_{n\in\mathbf{Z}} \varphi(n) = \sum_{n\in\mathbf{Z}} \int_{-\infty}^{+\infty} \varphi(x) e^{-2i\pi n x} dx,$$

utilisée pour $\varphi(x) = e^{-\pi t x^2}$, nous fournit l'équation fonctionnelle

$$\theta(t) = t^{-1/2}\theta(t^{-1}).$$

On a alors

$$\begin{aligned}\xi(s) &= \frac{1}{2}\int_0^{+\infty}(\theta(t)-1)t^{s/2}\frac{dt}{t}\\ &= \frac{1}{2}\int_0^{1}(t^{-1/2}\theta(t^{-1})-1)t^{s/2}\frac{dt}{t}+\frac{1}{2}\int_1^{+\infty}(\theta(t)-1)t^{s/2}\frac{dt}{t}\end{aligned}$$

et on peut changer t en t^{-1} dans la première intégrale pour obtenir

$$\xi(s) = -\frac{1}{s}+\frac{1}{s-1}+\frac{1}{2}\int_1^{+\infty}(\theta(t)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t}.$$

On en déduit le résultat car $\theta(t)-1$ est à décroissance rapide à l'infini, ce qui implique que l'intégrale est une fonction holomorphe de s sur $\mathbf{C}$ tout entier, et le membre de droite est évidemment invariant par $s \mapsto 1-s$.

2. Deuxième méthode : intégrale sur un contour

Si $c > 0$, soit C_c le contour obtenu en suivant la droite réelle de $+\infty$ à $c\pi$, puis en parcourant le carré de sommets $c\pi(\pm 1 \pm i)$ dans le sens trigonométrique, et en retournant en $+\infty$ le long de l'axe réel. Soit

$$\mathrm{F}_c(s) = \int_{\mathrm{C}_c}\frac{1}{e^z-1}(-z)^s\frac{dz}{z},$$

où $(-z)^s = \exp(s\log(-z))$ et la branche du logarithme choisie est celle dont la partie imaginaire est comprise entre $-\pi$ et π ; en particulier, on a $(-z)^s = e^{-i\pi s}z^s$ de $+\infty$ à $c\pi$ et $(-z)^s = e^{i\pi s}z^s$ de $c\pi$ à $+\infty$ (après avoir parcouru le carré). Comme $\frac{1}{e^z-1}$ est à décroissance rapide à l'infini, la fonction $\mathrm{F}_c(s)$ est holomorphe sur $\mathbf{C}$ pour tout c qui n'est pas un entier pair (pour éviter les pôles de $\frac{1}{e^z-1}$). D'autre part, le théorème des résidus montre que $\mathrm{F}_c(s)$ ne dépend pas de c si c reste dans un intervalle du type $]2\mathrm{N}, 2\mathrm{N}+2[$, avec $\mathrm{N} \in \mathbf{N}$. En particulier, on a $\mathrm{F}_1(s) = \mathrm{F}_c(s)$ quel que soit $c \in]0, 2[$. Si $\mathrm{Re}(s) > 1$, quand c tend vers 0, l'intégrale sur le carré de sommets $c\pi(\pm 1 \pm i)$ tend vers 0, et on obtient, en passant à la limite

$$\mathrm{F}_1(s) = e^{-i\pi s}\int_{+\infty}^{0}\frac{1}{e^z-1}z^s\frac{dz}{z}+e^{i\pi s}\int_0^{+\infty}\frac{1}{e^z-1}z^s\frac{dz}{z} = 2i\cdot\sin\pi s\cdot\Gamma(s)\cdot\zeta(s).$$

Maintenant, quand N tend vers $+\infty$, la fonction $\mathrm{F}_{2\mathrm{N}+1}(s)$ tend vers 0 quand $\mathrm{Re}(s) < 0$ car $\frac{1}{e^z-1}$ est majorée, indépendamment de N, sur $\mathrm{C}_{2\mathrm{N}+1}$. La différence entre $\mathrm{F}_1(s)$ et $\mathrm{F}_{2\mathrm{N}+1}(s)$ peut se calculer grâce au théorème des résidus. La fonction $\frac{1}{e^z-1}(-z)^{s-1}$ a des pôles en $z = \pm 2i\pi, \pm 4i\pi, \dots, \pm 2\mathrm{N}i\pi$ dans le contour délimité par la différence entre $\mathrm{C}_{2\mathrm{N}+1}$ et C_1. Si $k \in \{1, \dots, \mathrm{N}\}$, le résidu de $\frac{1}{e^z-1}(-z)^{s-1}$ est $(2k\pi)^{s-1}e^{-i\pi(s-1)/2}$ en $2ik\pi$ et $(2k\pi)^{s-1}e^{i\pi(s-1)/2}$

en $-2ik\pi$, ce qui nous donne

$$\frac{1}{2i\pi}(\mathrm{F}_{2\mathrm{N}+1}(s) - \mathrm{F}_1(s)) = 2\cos\pi\frac{s-1}{2}\cdot\sum_{k=1}^{\mathrm{N}}(2k\pi)^{s-1}.$$

En passant à la limite, on obtient donc $\mathrm{F}_1(s) = 4i\pi\cdot\cos(\pi\frac{s-1}{2})\cdot(2\pi)^{s-1}\cdot\zeta(1-s)$, si $\mathrm{Re}(s) < 0$. On en déduit l'équation fonctionnelle

$$\sin\pi s\cdot\Gamma(s)\cdot\zeta(s) = \cos\pi\frac{s-1}{2}\cdot(2\pi)^s\cdot\zeta(1-s).$$

Pour passer de cette équation fonctionnelle à celle de ξ, il faut utiliser les formules classiques

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s},\quad \Gamma\big(\frac{s}{2}\big)\Gamma\big(\frac{s+1}{2}\big) = 2^{1-s}\Gamma\big(\frac{1}{2}\big)\Gamma(s),\quad \text{et}\quad \Gamma\big(\frac{1}{2}\big) = \sqrt{\pi}.$$

I.5. Fonctions L de Dirichlet

1. Caractères de Dirichlet et sommes de Gauss

Si d est un entier, on appelle caractère de Dirichlet modulo d un morphisme de groupes de $(\mathbf{Z}/d\mathbf{Z})^*$ dans $\mathbf{C}^*$. L'image d'un caractère de Dirichlet est bien évidemment incluse dans le groupe des racines de l'unité.

Si d' est un diviseur de d et χ est un caractère de Dirichlet modulo d', on peut aussi voir χ comme un caractère de Dirichlet modulo d en composant χ avec la projection $(\mathbf{Z}/d\mathbf{Z})^* \to (\mathbf{Z}/d'\mathbf{Z})^*$. On dit que χ est de conducteur d si on ne peut pas trouver de diviseur d' de d distinct de d, tel que χ provienne d'un caractère modulo d'. De manière équivalente, χ est de conducteur d si quel que soit d' diviseur de d distinct de d, la restriction de χ au noyau de la projection $(\mathbf{Z}/d\mathbf{Z})^* \to (\mathbf{Z}/d'\mathbf{Z})^*$ n'est pas triviale.

Si χ est un caractère de Dirichlet modulo d, on note χ^{-1} le caractère de Dirichlet modulo d défini par $\chi^{-1}(n) = (\chi(n))^{-1}$ si $n \in (\mathbf{Z}/d\mathbf{Z})^*$.

Si χ est un caractère de Dirichlet modulo d, on considère aussi souvent χ comme une fonction périodique sur $\mathbf{Z}$ de période d en composant χ avec la projection naturelle de $\mathbf{Z}$ sur $\mathbf{Z}/d\mathbf{Z}$ et en étendant χ par 0 sur les entiers non premiers à d. On a donc $\chi^{-1}(n) = (\chi(n))^{-1}$ si $(n,d) = 1$, mais $\chi^{-1}(n) = 0$ si $(n,d) \neq 1$.

Si d est un entier, χ un caractère de Dirichlet de conducteur d et si $n \in \mathbf{Z}$, on définit la somme de Gauss tordue $\mathrm{G}(\chi,n)$ par la formule

$$\mathrm{G}(\chi,n) = \sum_{a \bmod d}\chi(a)e^{2i\pi na/d}$$

et on pose $\mathrm{G}(\chi) = \mathrm{G}(\chi,1)$.