Artin $L$–functions, Artin primitive roots Conjecture and applications

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# The Riemann zeta function

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# Content

# The fundamental Theorem of arithmetic

$$n = \prod_p p^{v_p(n)}$$

$$\omega(n) = \sum_{p|n} 1, \quad \Omega(n) = \sum_{p|n} v_p(n).$$

Divisor function

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# Arithmetic functions $f : \mathbb{N} \to \mathbb{C}$

*Multiplicative function* : $f(ab) = f(a)f(b)$ if $\gcd(a, b) = 1$.

Example : for $k \in \mathbb{C}$, $\sigma_k(n) = \sum_{d|n} d^k$.

*Completely multiplicative function* : $f(ab) = f(a)f(b)$ for all $a$, $b$.

Example : for $k \in \mathbb{C}$, $j^k = p_k : n \mapsto n^k$.

A multiplicative function is characterised by its values at prime powers :

$$f(n) = \prod_p f(p^{v_p(n)}).$$

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# Dirichlet series

$$D(f,s)=\sum_{n\geq 1} f(n)n^{-s}.$$

Main example : Euler (1736), Riemann (1859)

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}.$$

Euler : $\zeta(2)=\pi^2/6$, $\zeta(4)=\pi^4/90$, $\zeta(2n)\pi^{-2n}\in\mathbb{Q}$ $(n\geq 0)$,
$\zeta(0)=-1/2$, $\zeta(-1)=-1/12$,
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# Product of Dirichlet series

$$D(f,s)D(g,s) = D(f \star g, s).$$

Convolution :

$$f \star g(n) = \sum_{ab=n} f(a)g(b).$$

# The ring $\mathcal{A}$ of arithmetic functions

The ring of arithmetic functions is a commutative factorial ring $\mathcal{A}$, the unity of which is the Kronecker delta function $\delta$ with

$$\delta(n) = \begin{cases} 1 & \text{for } n = 1, \\ 0 & \text{for } n > 1. \end{cases}$$

The units are the arithmetic functions $f$ such that $f(1) \neq 0$.
`Exercise:` check that the inverse $g$ of a unit $f$ is defined recursively by

$$g(1) = f(1)^{-1}, \quad g(n) = -f(1)^{-1} \sum_{d|n, d<n} f(n/d)g(d) \quad (n > 1).$$

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Iwaniec Chap. I

# Examples of arithmetic functions

The constant function $\mathbf{1} = j^0 : n \mapsto 1$.

The Kronecker delta function $\delta$.

The Möbius function : $\mu$.

Euler totient function : $\varphi$.

The function $j : j(n) = n$ for all $n \geq 1$.

The characteristic function of the squares : $\kappa$.

The lambda function : $\lambda = (-1)^\Omega$.

The von Mangoldt function :

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ with } p \text{ prime and } m \geq 1, \\ 0 & \text{if } n \text{ is not a prime power.} \end{cases}$$

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# Exercises

$$\mu \star \mathbf{1} = \delta, \qquad \mu \star j = \varphi,$$

$$|\mu| \star \kappa = \mathbf{1}, \qquad \mu \star \kappa = \lambda,$$

$$|\mu| \star \mathbf{1} = 2^{\omega}, \qquad j^k \star \mathbf{1} = \sigma_k,$$

$$\mathbf{1} \star \mathbf{1} = \tau, \qquad \mu \star \log = \Lambda.$$

Möbius inversion formula :

$$g = \mathbf{1} \star f \quad \Longleftrightarrow \quad f = \mu \star g.$$

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$$D(\mathbf{1}, s) = \zeta(s) \qquad D(j^k, s) = \zeta(s-k)$$

$$D(\kappa, s) = \zeta(2s) \qquad D(\mu, s) = \frac{1}{\zeta(s)}$$

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More exercises : Patterson p.10 exercise 1.3

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# The Prime Number Theorem.

For $x > 0$ let $\pi(x)$ be the number of prime numbers $\leq x$ :

$$\pi(x) = \sum_{p \leq x} 1.$$

$\pi(10) = 4$, $\pi(100) = 25$, $\pi(1000) = 168$, $\pi(10\,000) = 1229$.

Conjecture of Gauss (1792) and Adrien-Marie Legendre (1798), proved by Hadamard and de la Vallée Poussin (1896) : **PNT**. For $x \to \infty$,

$$\pi(x) \sim \frac{x}{\log x}.$$

This is equivalent to $p_n \sim n \log n$.

Better approximation of $\pi(x)$ : *integral logarithm*

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

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# Distribution of prime numbers : elementary results.

$$\theta(x) = \sum_{p \le x} \log p, \qquad \psi(x) = \sum_{p^m \le x} \log p = \sum_{n \le x} \Lambda(n)$$

$$M(x) = \sum_{n \le x} \mu(n).$$

**Theorem** (elementary PNT) (P.L. Tchebyshev) : 1860's — *cf.* course of Francesco Pappalardi

$$\pi(x) \asymp \frac{x}{\log x}, \qquad \theta(x) \asymp x, \qquad \psi(x) \asymp x \quad \text{for } x \to \infty.$$

The Prime Number Theorem $\pi(x) \sim x/\log x$ is equivalent $\theta(x) \sim x$ and to $\psi(x) \sim x$ and to $M(x) = o(x)$ as $x \to \infty$.

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# Elementary results on prime numbers.

For $n \geq 1$,

$$2^n \leq \binom{2n}{n} \leq 4^n.$$

For $n \geq 1$ and $p$ prime,

$$v_p(n!) = \sum_{m \geq 1} \left\lfloor \frac{n}{p^m} \right\rfloor.$$

For $n \geq 1$,

$$\theta(2n) - \theta(n) \leq 4 \log n.$$

Consequence :

$$\theta(x) \leq C_1 x, \qquad \pi(x) \geq C_2 \frac{x}{\log x}.$$

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$$\theta(2n) - \theta(n) \leq 4 \log n.$$

Consequence :

$$\theta(x) \leq C_1 x, \qquad \pi(x) \geq C_2 \frac{x}{\log x}.$$

# Elementary results on prime numbers.

For $n \geq 1$,

$$2^n \leq \binom{2n}{n} \leq 4^n.$$

For $n \geq 1$ and $p$ prime,

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For $2 \le y < x$,

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# Exercises

Deduce

$$\sum_{p \le x} \frac{\log p}{p} = \log x + O(1)$$

and

$$\sum_{p \le x} \frac{1}{p} = \log \log x + C + O(1/\log x)$$

# von Mangoldt function

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n\geq 1} \frac{\Lambda(n)}{n^s}$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ with } p \text{ prime and } m \geq 1, \\ 0 & \text{if } n \text{ is not a prime power.} \end{cases}$$

# Exercise

Check

$$\lim_{s\to 1}\left(\zeta(s)-\frac{1}{s-1}\right)=\gamma$$

where $\gamma$ is *Euler constant* :

$$\gamma=\lim_{N\to\infty}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}-\log N.$$

`Hint:`

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}=1+\int_1^N([t]-t)\frac{dt}{t^2}$$

# Abel summation.

$$A_0 = 0, \quad A_n = \sum_{m=1}^{n} a_m, \quad a_n = A_n - A_{n-1} \quad (n \geq 1)$$

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N-1} A_n (b_n - b_{n+1}) + A_N b_N.$$

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# Abel's Partial Summation Formula PSF.

Let $a_n$ be a sequence of complex numbers and $A(x) = \sum_{n \leq x} a_n$. Let $f$ be a function of class $\mathcal{C}^1$. Then

$$\sum_{y<n\leq x} a_n f(n) = A(x) f(x) - \int_y^x A(t) f'(t) dt.$$

Hindry p. 129–131

# Abel summation : an example.

Take $a_n = 1$ for all $n$, so that $A(t) = \lfloor t \rfloor$. Then

$$\sum_{n=M+1}^{N} f(n) = \int_M^N f(t)dt + \int_M^N (t - \lfloor t \rfloor) f'(t) dt$$

Take $f(t) = 1/t$. Then

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + O(1/N)$$

where $\gamma$ is *Euler constant*

$$\gamma = 1 - \int_1^\infty \{t\} \frac{dt}{t^2} \qquad \text{where} \qquad \{t\} = t - \lfloor t \rfloor.$$

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# Abel summation for $\zeta$.

Take $a_n = n^{-s}$ for all $n$; assume $\mathrm{Re}(s) > 1$. An approximation of the series

$$\sum_{n \geq 1} n^{-s}$$

is

$$\int_1^\infty t^{-s} dt = \frac{1}{s-1}.$$

It is a fact that $(s-1)\zeta(s) \to 1$ as $s \to 1_+$.

We will see that if we remove this singularity of $\zeta$ at $s = 1$, the difference

$$\zeta(s) - \frac{1}{s-1}$$

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# Analytic continuation of $\zeta(s)$ to $\mathrm{Re}(s) > 0$

$$(1-2^{1-s})\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots + (2n-1)^{-s} - (2n)^{-s} + \cdots$$

$$\log(1-u) = -\frac{u}{1} - \frac{u^2}{2} - \frac{u^3}{3} - \cdots - \frac{u^n}{n} - \cdots$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2.$$

$$\lim_{s \to 1_+} (1 - 2^{1-s})\zeta(s) = \log 2$$

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Hardy and Wright : $\lim_{s\to 1} \sum a_n n^{-s} = \sum a_n n^{-1}$ if the series in RHS converges.

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# Analytic continuation of $\zeta(s)$ (continued)

The function $\zeta(s) - 1/(s-1)$ extends to an analytic function in $\mathrm{Re}(s) > 0$.

$$\frac{1}{n^s} = s\int_n^\infty t^{-s-1}dt, \qquad \sum_{n=1}^{t} 1 = [t].$$

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Hindry p. 139

# Abscissa of convergence

Let $f(s) = \sum_{n\geq 1} a_n n^{-s}$ be a Dirichlet series converging at $s_0$. Then for any $C \geq 1$ it converges uniformly on the sector

$$\left\{ s \in \mathbb{C} \;\middle|\; \mathrm{Re}(s - s_0) \geq 0,\ |s - s_0| \leq C\mathrm{Re}(s - s_0) \right\}$$

A Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ has an absissa of convergence $\sigma_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$, such that the series converges for $\mathrm{Re}(s) > \sigma_0$ and diverges for $\mathrm{Re}(s) < \sigma_0$.

The sum of the series is analytic in the half plane $\mathrm{Re}(s) > \sigma_0$ and its derivatives are given by

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# Absolute abscissa of convergence

The absolute abscissa of convergence $\sigma_a$ is defined in the same way.

One can prove

$$\sigma_0 \leq \sigma_a \leq \sigma_0 + 1.$$

For the Riemann zeta function, $\sigma_0 = \sigma_a = 1$.

For the Dirichlet series

$$(1 - 2^{1-s})\zeta(s) = \sum_{n \geq 1} (-1)^{n-1} n^{-s},$$

we have $\sigma_0 = 1$ and $\sigma_a = 0$.

**Riemann Hypothesis** : the abscissa of convergence of the Dirichlet series $D(\mu, s) = \dfrac{1}{\zeta(s)}$ is $1/2$.

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# Infinite products

Definition of the convergence of a product.

Remark. If $z_n$ is any sequence of complex numbers and if one at least of the $z_n$ vanishes, then the sequence

$$(z_0 z_1 z_2 \cdots z_n)_{n \geq 0}$$

converges to $0$.

Remark.

$$\lim_{N \to \infty} \prod_{n=1}^{N} \left(1 - \frac{1}{n+1}\right) = 0.$$

Hindry p. 135

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# Convergent infinite products

Definitions. An infinite product $\prod_{n\geq 0}(1+u_n)$ is *convergent* (resp. *absolutely convergent*) if there exists $n_0 \geq 0$ such that $|u_n| < 1$ for $n \geq n_0$ and the series $\sum_{n\geq n_0} \log(1+u_n)$ is convergent (resp. absolutely convergent).

An infinite product of functions $\prod_{n\geq 0}(1+u_n(x))$ is *uniformly convergent* on a compact $K$ if there exists $n_0 \geq 0$ such that $|u_n(z)| < 1$ for $n \geq n_0$ and all $z \in K$ and such that the series $\sum_{n\geq n_0} \log(1+u_n(z))$ is uniformly convergent on $K$.

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# Convergent infinite products

A product $\prod_{n\geq 0}(1+u_n)$ is absolutely convergent if and only if the series $\sum_{n\geq 0}|u_n|$ is convergent

A convergent infinite product is zero if and only if one of the factors is zero.

If the series $\sum_{n\geq 0}\log(1+u_n(z))$ is uniformly convergent on every compact subset of an open set $U$, then the infinite product $\prod_{n\geq 0}(1+u_n(z))$ is holomorphic on $U$.

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# Euler Product

The infinite product $\prod_p (1 - p^{-s})$ is uniformly convergent on any compact included in the half plane $\mathrm{Re}(s) > 1$. It defines an analytic function in this half plane and we have

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

For $X > 0$, let $\mathcal{N}(X)$ be the set of positive integers, the prime factors of which are $\leq X$. Then

$$\prod_{p \leq X} \frac{1}{1 - p^{-s}} = \prod_{p \leq X} \sum_{m \geq 0} \frac{1}{p^{ms}} = \sum_{n \in \mathcal{N}(X)} \frac{1}{n^s}.$$

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# Euler Product

The infinite product $\prod_p (1 - p^{-s})$ is uniformly convergent on any compact included in the half plane $\mathrm{Re}(s) > 1$. It defines an analytic function in this half plane and we have

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

For $X > 0$, let $\mathcal{N}(X)$ be the set of positive integers, the prime factors of which are $\leq X$. Then

$$\prod_{p \leq X} \frac{1}{1 - p^{-s}} = \prod_{p \leq X} \sum_{m \geq 0} \frac{1}{p^{ms}} = \sum_{n \in \mathcal{N}(X)} \frac{1}{n^s}.$$

Hindry p. 137

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# $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$

For $\mathrm{Re}(s) = \sigma > 1$,

$$\left|\zeta(s) - \prod_{p \le X} \frac{1}{1 - p^{-s}}\right| = \left|\sum_{n \notin \mathcal{N}(X)} \frac{1}{n^s}\right| \le \sum_{n > X} \left|\frac{1}{n^s}\right| = \sum_{n > X} \frac{1}{n^\sigma}.$$

$$\log(1 - u) = -\sum_{m \ge 1} \frac{u^m}{m}.$$

$$\log(1 - p^{-s}) = -\sum_{m \ge 1} \frac{p^{-ms}}{m}$$

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## $\log \zeta(s)$ for $\mathrm{Re}(s) > 1$

$$\log \zeta(s) = \sum_p \sum_{m \geq 1} \frac{1}{m p^{ms}}.$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \sum_{m \geq 1} \frac{\log p}{p^{ms}}.$$

Recall the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ with } p \text{ prime and } m \geq 1, \\ 0 & \text{if } n \text{ is not a prime power.} \end{cases}$$

and the formula

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# Euler products for multiplicative functions

**Proposition** : an arithmetic function $f$ is multiplicative (resp. completely multiplicative) if and only if

$$D(f,s) = \prod_p \left( \sum_{\nu=0}^{\infty} f(p^\nu) p^{-\nu s} \right),$$

(resp.

$$D(f,s) = \prod_p \left( \sum_{\nu=0}^{\infty} (f(p)p^{-s})^\nu \right) = \prod_p \frac{1}{1 - f(p)p^{-s}}. )$$

# The Prime Number Theorem

**Theorem**. The integral

$$\int_1^\infty (\theta(t) - t)\frac{dt}{t^2}$$

is convergent.

**Corollary :** $\theta(x) \sim x$ as $x \to \infty$.

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# Laplace Transform

Let $h(t)$ be a bounded piecewise continuous function. Then the integral

$$F(s) = \int_0^\infty h(u)e^{-su}du$$

is convergent and defines a holomorphic function on the half plane $\mathrm{Re}(s) > 0$.

If this function $F$ can be analytically continued to a holomorphic function on an open set containing the closed half plane $\mathrm{Re}(s) \geq 0$, the the integral converges for $s = 0$ and

$$F(0) = \int_0^\infty h(u)du.$$

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# Consequence

Take $h(u) = \theta(e^u)e^{-u} - 1$. Then the function

$$F(s) = \int_0^\infty h(u)e^{-su}du = \int_1^\infty \frac{\theta(t) - t}{t^{s+2}}dt$$

can be written as the sum of

$$-\frac{\zeta'(s+1)}{(s+1)\zeta(s+1)} - \frac{1}{s}$$

and a holomorphic function on $\mathrm{Re}(s) \geq -1/2$.
The key point in the proof of the PNT is the following result of Hadamard and de la Vallée Poussin
**Theorem**. The function $\zeta(s)$ has no zero on the line $\mathrm{Re}(s) = 1$.

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# No zero of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$

Trigonometric formula :

$$4\cos x + cos(2x) + 3 = 2(1 + \cos x)^2 \geq 0.$$

Consequence :

$$|\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)\zeta(\sigma)^3 \geq 1$$

for $\sigma \geq 1$ and $t \in \mathbb{R}$.

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# Euler Gamma function

The integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

defines an analytic function on the half plane $\mathrm{Re}(s) > 0$ which satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s).$$

The Gamma function can be analytically continued to a meromorphic function in the complex plane $\mathbb{C}$ with a simple pole at any integer $\leq 0$. The residue at $s = -k$ $(k \geq 0)$ is $(-1)^k/k!$.

# Euler Gamma function

$$\Gamma(s) = \left[\frac{1}{s}e^{-s}t^s\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-t}t^s dt = \frac{1}{s}\Gamma(s+1).$$

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n)}.$$

Remark. From $\Gamma(1) = 1$ we deduce $\Gamma(n+1) = n!$ for $n \geq 0$.

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## $\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma(1/2) = \int_0^\infty e^{-t} t^{-1/2} dt = 2 \int_0^\infty e^{-x^2} dx.$$

Hence

$$\begin{aligned} \frac{1}{4}\Gamma(1/2)^2 &= \int_0^\infty \int_0^\infty e^{-x^2-y^2} dxdy \\ &= \int_0^\infty \int_0^{\pi/2} e^{-r^2} rdrd\theta \\ &= \left[-\frac{1}{2}e^{-r^2}\right]_0^\infty \frac{\pi}{2} \\ &= \frac{\pi}{4}. \end{aligned}$$

# Euler Gamma function

Exercises:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\Gamma\left(\frac{1}{2}\right)\Gamma(s).$$

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^s n!}{s(s+1)\cdots(s+n)}.$$

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right) e^{-s/n}.$$

# Analytic continuation of $\zeta(s)$ (continued)

For $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{e^t - 1} t^s \frac{dt}{t},$$

$$\frac{1}{e^t - 1} = \sum_{n \geq 1} e^{-nt},$$

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Colmez p. 40

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# Analytic continuation

**Lemma.** Let $f \in \mathcal{C}^\infty(\mathbb{R}_{>0})$ be a fast decreasing function at infinity. Then the function defined for $\mathrm{Re}(s) > 0$ by the integral

$$L(f, s) = \frac{1}{\Gamma(s)} \int_0^\infty f(t) t^s \frac{dt}{t}$$

has an analytic continuation to $\mathbb{C}$.

Special values :

Under the assumptions of the lemma, for $n \geq 0$ we have

$$L(f, -n) = (-1)^n f^{(n)}(0).$$

Colmez p. 41

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# Bernoulli numbers

The function

$$f_0(t) = \frac{t}{e^t - 1}.$$

satisfies the hypotheses of the Lemma. Define $(B_n)_{n\geq 0}$ by

$$f_0(t) = \sum_{n\geq 0} B_n \frac{t^n}{n!}.$$

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42},$$

$$B_3 = B_5 = B_7 = \cdots = 0.$$

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http://www.bernoulli.org/

| $n$ | 0 | 1 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $B_n$ | 1 | $-\frac{1}{2}$ | $\frac{1}{6}$ | $-\frac{1}{30}$ | $\frac{1}{42}$ | $-\frac{1}{30}$ | $\frac{5}{66}$ | $-\frac{691}{2730}$ | $\frac{7}{6}$ | $-\frac{3617}{510}$ | $\frac{43867}{798}$ | $-\frac{174611}{330}$ |
| $\frac{B_n}{n}$ | | | $\frac{1}{12}$ | $-\frac{1}{120}$ | $\frac{1}{252}$ | $-\frac{1}{240}$ | $\frac{1}{132}$ | $-\frac{691}{32760}$ | $\frac{1}{12}$ | $-\frac{3617}{8160}$ | $\frac{43867}{14364}$ | $-\frac{174611}{6600}$ |

## $\zeta(s)$ as an integral

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$$

Poles :

For $\Gamma(s)$ : $s = 0, -1, -2, \ldots$ residue $(-1)^n/n!$ at $s = -n$.

For $\int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$ : $s = 1, 0, -1, -2, \ldots$ residue $B_{n+1}/(n+1)!$ at $s = -n$.

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# Values at negative integers

The Riemann zeta function $\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$ which is analytic in $\mathbb{C} \setminus \{1\}$, with a simple pole at $s = 1$ with residue $1$.

For $n$ a positive integer,

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}.$$

In particular $\zeta(-n) \in \mathbb{Q}$ for $n \geq 0$ and $\zeta(-2n) = 0$ for $n \geq 1$.

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12}, \quad \zeta(-3) = \frac{1}{120},$$

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# Values at positive even integers

**Theorem**. Let $n \geq 1$ be a positive integer. Then

$$\zeta(2n) = -\frac{1}{2} B_{2n} \frac{(2i\pi)^{2n}}{(2n)!}.$$

In particular $\zeta(2n)/\pi^{2n}$ is a rational number.

Examples :
$\zeta(2) = \pi^2/6$ (The Basel problem).
$\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450$.
The Basel problem, first posed by Pietro Mengoli in 1644, was solved by Leonhard Euler in 1735, when he was 28 only.

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# “Proof” of $\zeta(2) = \pi^2/6$, following Euler

The sum of the inverses of the roots of a polynomial $f$ with $f(0) = 1$ is $-f'(0)$ : for

$$1 + a_1 z + a_2 z^2 + \cdots + a_n z^n = (1 - \alpha_1 z) \cdots (1 - \alpha_n z)$$

we have $\alpha_1 + \cdots + \alpha_n = -a_1$.

Write

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Set $z = x^2$. The zeroes of the function

$$\frac{\sin \sqrt{z}}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \cdots$$

are $\pi^2, 4\pi^2, 9\pi^2, \ldots$ *hence* the sum of the inverses of these numbers is

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Set $z = x^2$. The zeroes of the function

$$\frac{\sin \sqrt{z}}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \cdots$$

are $\pi^2$, $4\pi^2$, $9\pi^2$, ... *hence* the sum of the inverses of these numbers is

$$\sum_{n \geq 1} \frac{1}{n^2 \pi^2} = \frac{1}{6}.$$

# "Proof" of $\zeta(2) = \pi^2/6$, following Euler

The sum of the inverses of the roots of a polynomial $f$ with $f(0) = 1$ is $-f'(0)$ : for

$$1 + a_1 z + a_2 z^2 + \cdots + a_n z^n = (1 - \alpha_1 z) \cdots (1 - \alpha_n z)$$

we have $\alpha_1 + \cdots + \alpha_n = -a_1$.
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# Remark

Let $\lambda \in \mathbb{C}$. The functions

$$f(z) = 1 + a_1 z + a_2 z^2 + \cdots$$

and

$$e^{\lambda z} f(z) = 1 + (a_1 + \lambda) z + \cdots$$

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The sum $\sum_i \alpha_i$ cannot be at the same time $-a_1$ and $-a_1 - \lambda$.

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# Completing Euler's proof

$$\frac{\sin x}{x} = \prod_{n\geq 1}\left(1 - \frac{x^2}{n^2\pi^2}\right).$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \cdots \implies \sum_{n\geq 1}\frac{1}{n^2\pi^2} = \frac{1}{6}.$$

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Evaluating $\zeta(2)$. Fourteen proofs compiled by Robin Chapman.

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# Another proof (Calabi)

P. Cartier. – *Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents.* Sém. Bourbaki no. 885 Astérisque **282** (2002), 137-173.

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$$\frac{1}{1-x^2y^2} = \sum_{n\geq 0} x^{2n}y^{2n}.$$

$$\int_0^1 x^{2n}dx = \frac{1}{2n+1}.$$

$$\int_0^1\int_0^1 \frac{dxdy}{1-x^2y^2} = \sum_{n\geq 0}\frac{1}{(2n+1)^2}.$$

$$x = \frac{\sin u}{\cos v}, \qquad y = \frac{\sin v}{\cos u},$$

$$\int_0^1\int_0^1 \frac{dxdy}{1-x^2y^2} = \int_{0\leq u\leq \pi/2,\ 0\leq v\leq \pi/2,\ u+v\leq\pi/2} dudv = \frac{\pi^2}{8}.$$

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# Completing Calabi's proof of $\zeta(2) = \pi^2/6$

From
$$\sum_{n\geq 0} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

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$$\sum_{n\geq 1} \frac{1}{n^2} = \sum_{n\geq 1} \frac{1}{(2n)^2} + \sum_{n\geq 0} \frac{1}{(2n+1)^2}.$$

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# The function $\pi\mathrm{cotg}(\pi z)$

Proposition : for $z \in \mathbb{C} \setminus \mathbb{Z}$,

$$\pi\mathrm{cotg}\pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} + \frac{1}{z-n}\right).$$

$$\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2} = -2\sum_{n=1}^{\infty}\zeta(2n)z^{2n-1}.$$

$$\pi\mathrm{cotg}\pi z = i\pi\frac{e^{2i\pi z}+1}{e^{2i\pi z}-1} = i\pi + \frac{2i\pi}{e^{2i\pi z}-1} = i\pi + \frac{1}{z}\sum_{n=0}^{\infty}B_n\frac{(2i\pi z)^n}{n!}.$$

Colmez p. 42

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# Functional equation

An entire function (analytic in $\mathbb{C}$) is defined by

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

$$\xi(0) = \xi(1) = 1.$$

**Theorem** (Riemann) :

$$\xi(s) = \xi(1-s).$$

# Non trivial zeroes

Denote by $Z$ the multiset of zeroes (counting multiplicities) of $\zeta(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ and by $Z_+$ the multiset of zeroes (counting multiplicities) of $\zeta(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ with positive imaginary part.
Then

$$Z_+ = Z \cup \{1 - \rho \mid \rho \in Z\}.$$

# Hadamard product expansion

Explicit formula :

$$s(1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s) = -e^{Bs}\prod_{\rho\in Z}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$$

with

$$B = -\frac{1}{2}\sum_{\rho\in Z}\frac{1}{\rho(1-\rho)} = -\frac{\gamma}{2} - 1 + \frac{1}{2}\log(4\pi) = -0.023095\ldots$$

We can write

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Patterson p.33-34, Iwaniec Chap. 8

# Explicit formula for the logarithmic derivative

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} - \frac{1}{s} - \frac{1}{s-1} + \sum_{\rho\in Z}\frac{1}{s-\rho}.$$

# Poisson formula

For $f \in L^1(\mathbb{R})$ let $\hat{f}$ be its Fourier transform :

$$\widehat{f}(y) = \int_{-\infty}^{+\infty} f(x) e^{2i\pi xy} dx.$$

Assume that the function $x \mapsto \sum_{n\in\mathbb{Z}} f(x+n)$ is continuous with bounded variation on $[0,1]$ ; then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m).$$

Recall van der Corput Formula (first course by Francesco Pappalardi).

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Corollary. The *theta series*

$$\theta(u) = \sum_{n\in\mathbb{Z}} e^{-\pi u n^2}$$

satisfies the functional equation , for $u > 0$ :

$$\theta(1/u) = \sqrt{u}\theta(u).$$

For $\mathrm{Re}(s) > 1$,

$$\xi(s) = s(s-1)\int_0^\infty \frac{(\theta(u)-1)u^{s/2}}{2u}du.$$

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Colmez p. 51

# The Riemann Memoir (1859).

**On the number of primes less than a given magnitude** (9p.)

- The function $\zeta(s)$ defined by the Dirichlet series $\sum_{n\geq 1} n^{-s}$ has an analytic continuation to the whole complex plane where it is holomorphic except a simple pole at $s = 1$ with residue $1$.
- The following functional equation holds

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

- The Riemann zeta function $\zeta(s)$ has simple zeroes at $s = -2, -4, , -6, \ldots$ which are called the trivial zeroes, and infinitely many non-trivial zeroes in the critical strip of the form $\rho = \beta + i\gamma$ with $0 \leq \beta \leq 1$ and $\gamma \in \mathbb{R}$.

http://www.claymath.org/sites/default/files/ezeta.pdf

# The Riemann Memoir (1859) (continued).

- The following product formula holds

$$s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = \prod_{\rho}\left(1 - \frac{s}{\rho}\right)$$

- The following prime number formula holds

$$\psi^{\flat}(x) = \sum_{n\leq x}{}^{\flat}\Lambda(n) = x - \sum_{\rho}\frac{x^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

- The Riemann Hypothesis. Every non-trivial zero of $\zeta(s)$ is on the critical line $\mathrm{Re}(s) = 1/2$ :

$$\rho = \frac{1}{2} + i\gamma.$$

# The Riemann Hypothesis.

The complex zeroes of the Riemann zeta function $\zeta(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ lie on the critical line $\mathrm{Re}(s) = 1/2$ :

$$s \in \mathbb{C},\ 0 < \mathrm{Re}(s) < 1 \text{ and } \zeta(s) = 0 \quad \Longrightarrow \quad \mathrm{Re}(s) = 1/2.$$

Equivalent statement:

$$\pi(x) = \mathrm{Li}(x) + O(x^{1/2} \log x)$$

as $x \to \infty$.

Asymptotic expansion :

$$\mathrm{Li}(x) \simeq \frac{x}{\log x} \sum_{n \geq 0} \frac{n!}{(\log x)^n} \simeq \frac{x}{\log x} + \frac{x}{(\log x)^2} + \cdots$$

Riemann hypothesis for the Möbius function :

$$M(x) = O(x^{1/2+\epsilon})$$

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# Notes by Riemann

Non–trivial zeroes :

$$\gamma_1 = 14.134725\ldots$$
$$\gamma_2 = 21.022039\ldots$$
$$\gamma_3 = 25.01085\ldots$$
$$\gamma_4 = 30.42487\ldots$$

http://oeis.org/wiki/Riemann_zeta_function

**Table of nontrivial zeros**[5]

| $n$ | Imaginary part (base 10) of $n^{\text{th}}$ nontrivial zero (above the real axis) | OEIS |
|---|---|---|
| **1** | 14.134725141734693790457251983562470270784257115699243175685567460149... | A058303 |
| **2** | 21.022039638771554992628479593896902777334340524902781754629520403587... | A065434 |
| **3** | 25.010857580145688763213790992562821818659549672557996672496542006745... | A065452 |
| **4** | 30.424876125859513210311897530584091320181560023715440180962146036993... | A065453 |
| **5** | 32.935061587739189690662368964074903488812715603517039009280003440784... | A192492 |
| **6** | 37.586178158825671257217763480705332821405597350830793218333001113622... | |
| **7** | 40.918719012147495187398126914633254395726165962777279536161303667253... | |
| **8** | 43.327073280914999519496122165406805782645668371836871446878893685521... | |
| **9** | 48.005150881167159727942472749427516041686844001144425117775312519814... | |
| **10** | 49.773832477672302181916784678563724057723178299676662100781955750433... | |

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# Non trivial zeroes of $\zeta(s)$

Hardy (1914) : infinitely many non-trivial zeroes of $\zeta(s)$ are on the critical line.

Norman Levinson (1974) : $\geq 1/3$ of the non-trivial zeroes of $\zeta(s)$ are on the critical line.

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# The asymptotic formula for $N(T)$

Let $N(T)$ be the number of zeroes $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle

$$0 < \beta < 1, \qquad 0 < \gamma \leq T.$$

Then

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$$

for $T \geq 2$.

Iwaniec Chap. 9

# Zero free region for $\zeta(s)$

De la Vallée Poussin (1896) :

$$\sigma > 1 - \frac{c}{\log(2 + |t|)}$$

for an absolute constant $c > 0$.

$$\psi(x) = x + O\big(x \exp(-c\sqrt{\log x})\big).$$

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$$\sigma > 1 - c(\log t)^{-2/3}, \qquad t \geq 2$$

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# Diophantine problem

**Conjecture**. The numbers

$$\pi, \zeta(3), \zeta(5), \dots, \zeta(2n+1), \dots$$

are algebraically independent.

Apéry (1978) : $\zeta(3) \notin \mathbb{Q}$

Rivoal (2000) : infinitely many $\zeta(2n+1)$ are irrational ; the numbers $\zeta(2n+1)$ span a $\mathbb{Q}$–vector space of infinite dimension.

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# Multizeta values (MZV)

Euler

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

for $s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$.

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# $\zeta(s)$ is a period

For $s$ integer $\geq 2$,

$$\zeta(s) = \int_{1>t_1>t_2\cdots>t_s>0} \frac{dt_1}{t_1} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s}.$$

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$$\int_{t_1>t_2\cdots>t_s>0} \frac{dt_2}{t_2} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s} = \sum_{n\geq 1} \frac{t_1^{n-1}}{n^{s-1}}.$$

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# MZV are periods

$$\zeta(2,1) = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} \cdot \frac{dt_3}{1-t_3}.$$

Proof.
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$$\int_0^{t_2} \frac{dt_3}{1-t_3} = \sum_{n\geq 1} \frac{t_2^{n-1}}{n}, \quad \text{next} \quad \int_0^{t_1} \frac{t_2^{n-1}dt_2}{t_2-1} = \sum_{m>n} \frac{t_1^m}{m},$$

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# Linear relations among MZV

As a consequence, multiple zeta values satisfy a lot of independent linear relations with integer coefficients.

**Example**

Product of series :

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$$

Product of integrals :

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1)$$

Hence

$$\zeta(4) = 4\zeta(3,1).$$

# Conjecture Rohrlich–Lang

Any algebraic dependence relation among the numbers $(2\pi)^{-1/2}\Gamma(a)$ with $a \in \mathbb{Q}$ lies in the ideal generated by the standard relations :

(1) Translation :

$$\Gamma(a+1) = a\Gamma(a),$$

(2) Reflexion :

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}.$$

(3) Multiplication : for any positive integer $n$, we have

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)} \Gamma(na).$$

(Universal odd distribution).

Artin $L$–functions, Artin primitive roots Conjecture and applications

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# The Riemann zeta function

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