Hopf Algebras and Transcendental Numbers

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In dealing with multiple zeta values, the main diophantine challenge is to prove that known dependence relations among them suffice to deduce all algebraic relations. One tool which should be relevant is the structure of Hopf Algebras, which occurs in several disguises in this context. How to use it is not yet clear, but we point out that it already plays a role in transcendental number theory: Stéphane Fischler deduces interpolation lemmas from zero estimates by using a duality involving bicommutative (commutative and cocommutative) Hopf Algebras.

In the first section we state two transcendence results involving values of the exponential function; they are special cases of the linear subgroup Theorem which deals with commutative linear algebraic groups.

In the second section, following S. Fischler, we explain the connection between the data of the linear subgroup Theorem and bicommutative Hopf algebras of finite type.

In the third and last section we introduce non-bicommutative Hopf algebras related to multiple zeta values.

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1 Transcendence, exponential polynomials and commutative linear algebraic groups

We start with two examples of transcendence results; their proofs involve exponential polynomials and they occur as corollaries of a general result on commutative algebraic groups: the linear subgroup Theorem.

In this context, there is duality, which can be explained by means of the Fourier-Borel transform of exponential polynomials. This duality is revisited by S. Fischler from the view point of commutative linear algebraic groups, using Hopf algebras.

1.1 Transcendence results

Here is our first transcendence result ([B] Theorem 2.1).

Theorem 1.1 (Baker). *Let $\beta_0, \ldots, \beta_n$ be algebraic numbers and $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers. For $1 \leq i \leq n$, denote by $\log \alpha_i$ any complex logarithm of α_i. Assume*

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n = 0.$$

Then it holds that
1. $\beta_0 = 0$.
2. If $(\beta_1, \ldots, \beta_n) \neq (0, \ldots, 0)$, then the numbers $\log \alpha_1, \ldots, \log \alpha_n$ are $\mathbf{Q}$-linearly dependent.
3.If $(\log \alpha_1, \ldots, \log \alpha_n) \neq (0, \ldots, 0)$, then the numbers $\beta_1, \ldots, \beta_n$ are $\mathbf{Q}$-linearly dependent.

As is well known this result includes Hermite's result (1873) on the transcendence of e, Lindemann's result (1882) on the transcendence of π and more generally

Corollary 1.2 (Hermite-Lindemann). *If β is a non-zero algebraic number, then e^β is a transcendental number.*
Equivalently, if α is a non-zero algebraic number and if $\log \alpha$ is any non-zero logarithm of α, then $\log \alpha$ is a transcendental number.

This includes the transcendence of numbers like e, π, $e^{\sqrt{2}}$, $\log 2$.

Denote by $\overline{\mathbf{Q}}$ the field of all complex algebraic numbers, which is the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ and by $\mathcal{L} = \exp^{-1}(\overline{\mathbf{Q}}^\times)$ the $\mathbf{Q}$-vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \{\lambda \in \mathbf{C}\; ;\; e^\lambda \in \overline{\mathbf{Q}}^\times\} = \{\log \alpha\; ;\; \alpha \in \overline{\mathbf{Q}}^\times\}.$$

Hermite-Lindemann's Theorem asserts that $\mathcal{L}$ does not contain any non-zero algebraic number:

$$\mathcal{L} \cap \overline{\mathbf{Q}} = \{0\}.$$

Another corollary of Baker's Theorem 1.1 is the answer to Hilbert's seventh problem, given by A.O. Gel'fond and Th. Schneider in 1934:

Corollary 1.3 (Gel'fond-Schneider). *If β is an irrational algebraic number, α a non-zero algebraic number and $\log\alpha$ a non-zero logarithm of α, then the number*

$$\alpha^\beta = \exp(\beta \log \alpha)$$

is transcendental.

Gel'fond-Schneider's Theorem (Corollary 1.3) asserts that the quotient of two non-zero elements in $\mathcal{L}$ is either rational or else transcendental; Baker's Theorem 1.1 implies more generally that $\mathbf{Q}$-linearly independent elements of $\mathcal{L}$ are $\overline{\mathbf{Q}}$-linearly independent.

Theorem 1.1 also yields the transcendence of numbers like $e^{\sqrt{2}}2^{\sqrt{3}}$,

$$\int_0^1 \frac{dx}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right)$$

and more generally (under suitable assumptions – see [B] Theorems 2.2, 2.3 and 2.4) of numbers of the form

$$e^{\beta_0}\alpha_1^{\beta_1}\cdots\alpha_m^{\beta_m} \quad \text{and} \quad \beta_0 + \beta_1 \log\alpha_1 + \cdots + \beta_m \log\alpha_m$$

when the numbers α_i and β_j are algebraic.

It is to be remarked that Baker's Theorem does not include all known transcendence results related to the exponential function: here is an example ([W] p. 386).

Theorem 1.4 (Sharp six exponentials Theorem). *Let x_1, x_2 be two complex numbers which are $\mathbf{Q}$-linearly independent and y_1, y_2, y_3 three complex numbers which are also $\mathbf{Q}$-linearly independent. Further let β_{ij} $(i = 1, 2,\ j = 1, 2, 3)$ be six algebraic numbers. Assume*

$$e^{x_i y_j - \beta_{ij}} \in \overline{\mathbf{Q}} \quad \textit{for} \quad i = 1, 2,\ j = 1, 2, 3.$$

Then $x_i y_j = \beta_{ij}$ for $i = 1, 2,\ j = 1, 2, 3$.

The special case $\beta_{ij} = 0$ for all i, j is known as the six exponentials Theorem (due to Lang and Ramachandra in the 60's – see references in [W], § 1.3): *if x_1, x_2 are $\mathbf{Q}$-linearly independent and y_1, y_2, y_3 are also $\mathbf{Q}$-linearly independent, then at least one of the six numbers*

$$e^{x_i y_j} \quad (i = 1, 2,\ j = 1, 2, 3)$$

is transcendental.

The four exponentials Conjecture ([W] Conjecture 1.13) asserts that two values for y should suffice: *if x_1, x_2 are $\mathbf{Q}$-linearly independent and y_1, y_2 are also $\mathbf{Q}$-linearly independent, then at least one of the four numbers*

$$e^{x_i y_j} \quad (i = 1, 2,\ j = 1, 2)$$

is transcendental.

A sharper result is expected, which we call here the *sharp four exponentials Conjecture*: under the same assumptions as in the four exponentials Conjecture, *if β_{ij} $(i = 1, 2,\ j = 1, 2)$ are four algebraic numbers such that*

$$e^{x_i y_j - \beta_{ij}} \in \overline{\mathbf{Q}} \quad \textit{for} \quad i = 1, 2,\ j = 1, 2,$$

then one should have $x_i y_j = \beta_{ij}$ for $i = 1, 2,\ j = 1, 2$.

Conjecture 1.5 (Sharp five exponentials Conjecture). *If x_1, x_2 are $\mathbf{Q}$-linearly independent, if y_1, y_2 are $\mathbf{Q}$-linearly independent and if α, β_{11}, β_{12}, β_{21}, β_{22}, γ are six algebraic numbers such that*

$$e^{x_1 y_1 - \beta_{11}},\ e^{x_1 y_2 - \beta_{12}},\ e^{x_2 y_1 - \beta_{21}},\ e^{x_2 y_2 - \beta_{22}},\ e^{(\gamma x_2 / x_1) - \alpha}$$

are algebraic, then $x_i y_j = \beta_{ij}$ for $i = 1, 2,\ j = 1, 2$ and furthermore $\gamma x_2 = \alpha x_1$.

The case $\beta_{ij} = 0$ of Conjecture 1.5 is an easy consequence of the sharp six exponentials Theorem 1.4: this is the five exponentials Theorem ([W] p. 385): *If x_1, x_2 are $\mathbf{Q}$-linearly independent, if y_1, y_2 are $\mathbf{Q}$-linearly independent and if γ is a non-zero algebraic, then at least one of the five numbers*

$$e^{x_1 y_1},\ e^{x_1 y_2},\ e^{x_2 y_1},\ e^{x_2 y_2},\ e^{\gamma x_2 / x_1}$$

is transcendental.

Moreover, in the special case where the three numbers y_1, y_2 and γ/x_1 are $\mathbf{Q}$-linearly independent, the sharp five exponentials Conjecture 1.5 follows from the sharp six exponentials Theorem 1.4 by setting

$$y_3 = \gamma / x_1, \quad \beta_{13} = \gamma, \quad \beta_{23} = \alpha,$$

so that

$$x_1 y_3 - \beta_{13} = 0 \quad \text{and} \quad x_2 y_3 - \beta_{23} = (\gamma x_2 / x_1) - \alpha.$$

In the case where the three numbers y_1, y_2 and γ/x_1 are linearly dependent over $\mathbf{Q}$, the conjecture is open. A consequence of the sharp five exponentials Conjecture 1.5 is the transcendence of the number e^{π^2}: take

$$x_1 = y_1 = \gamma = 1,\ x_2 = y_2 = i\pi,\ \alpha = 0,\ \beta_{11} = 1,\ \beta_{ij} = 0 \quad \text{for} \quad (i, j) \neq (1, 1).$$

So far, we only know (W.D. Brownawell and the author) that at least one of the two statements holds:

- *e^{π^2} is transcendental.*
- *The two numbers e and π are algebraically independent.*

In the same way, setting

$$x_1 = y_1 = \gamma = 1,\ x_2 = y_2 = \lambda,\ \alpha = 0,\ \beta_{11} = 1,\ \beta_{ij} = 0 \quad \text{for} \quad (i,j) \neq (1,1),$$

we deduce from Conjecture 1.5 the transcendence of e^{λ^2} when λ is a non-zero logarithm of an algebraic number. Writing $\alpha = e^{\lambda}$ or $\lambda = \log \alpha$, we have

$$e^{\lambda^2} = \alpha^{\log \alpha}.$$

Only the following weaker statement is known: *at least one of the two numbers*

$$e^{\lambda^2} = \alpha^{\log \alpha},\ e^{\lambda^3} = \alpha^{(\log \alpha)^2}$$

is transcendental, which was proved initially by W.D. Brownawell and the author as a consequence of a result of algebraic independence; however it is also a consequence of the sharp six exponentials Theorem 1.4 with

$$x_1 = y_1 = 1,\ x_2 = y_2 = \lambda,\ y_3 = \lambda^2,\ \beta_{11} = 1,\ \beta_{ij} = 0 \quad \text{for} \quad (i,j) \neq (1,1).$$

The sharpest known result on this subject is the *strong six exponentials Theorem* due to D. Roy ([W] Corollary 11.16). Denote by $\widetilde{\mathcal{L}}$ the $\overline{\mathbf{Q}}$-vector space spanned by 1 and $\mathcal{L}$: hence $\widetilde{\mathcal{L}}$ is nothing else than the set of complex numbers of the form

$$\beta_0 + \sum_{i=1}^{n} \beta_i \log \alpha_i,$$

with $n \geq 0$, β_j algebraic numbers, α_i non-zero algebraic numbers and all values of their logarithm are considered. The strong six exponentials Theorem states that *if x_1, x_2 are $\overline{\mathbf{Q}}$-linearly independent and if y_1, y_2, y_3 are $\overline{\mathbf{Q}}$-linearly independent, then at least one of the six numbers*

$$x_i y_j \quad (i = 1,2,\ j = 1,2,3)$$

is not in $\widetilde{\mathcal{L}}$.

The strong four exponentials Conjecture ([W] Conjecture 11.17) claims that the same should hold with only two values y_1, y_2 in place of three.

1.2 Exponential polynomials

The proofs of both Theorems 1.1 and 1.4 involve exponential polynomials. Here are basic facts on them.

For the proof of Baker's Theorem 1.1, assume

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n.$$

In *Gel'fond–Baker's Method* (B_1), we consider the following $n+1$ functions

$$z_0,\ e^{z_1},\ \ldots,\ e^{z_{n-1}},\ e^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}}$$

of n variables $z_0, \dots, z_{n-1}$. At the points $\mathbf{Z}(1, \log\alpha_1, \dots, \log\alpha_{n-1}) \in \mathbf{C}^n$, all these functions take algebraic values. Moreover we also get algebraic numbers by taking derivatives with respect to the operators $\partial/\partial z_i$, $(0 \le i \le n-1)$.

Notice that there are n variables, $n+1$ functions, 1 point (together with its multiples) and n derivations (together with their compositions).

In *Generalized Schneider's Method* (B_2), we consider the $n+1$ functions: $z_0,\ z_1,\ \dots, z_{n-1}$ and

$$e^{z_0}\alpha_1^{z_1}\cdots\alpha_{n-1}^{z_{n-1}} = \exp\{z_0 + z_1\log\alpha_1 + \cdots + z_{n-1}\log\alpha_{n-1}\}$$

at the points: $\{0\} \times \mathbf{Z}^{n-1} + \mathbf{Z}(\beta_0, , \dots, \beta_{n-1}) \in \mathbf{C}^n$. Only one derivation yields algebraic numbers, namely $\partial/\partial z_0$.

In this alternative approach there are again n variables and $n+1$ functions, but a single derivation, while the points form a group of $\mathbf{Z}$-rank n.

1.3 Data for the proof of Theorem 1.4

Here are the main data for the proof of Theorem 1.4.

Assume $x_1, \dots, x_a$ are $\mathbf{Q}$-linearly independent, $y_1, \dots, y_b$ are $\mathbf{Q}$-linearly independent, β_{ij} are algebraic numbers and λ_{ij} are elements in $\mathcal{L}$ such that

$$\lambda_{ij} = x_i y_j - \beta_{ij} \quad \text{for} \quad i = 1, \dots, a,\ j = 1, \dots, b$$

with $ab > a + b$.

For Theorem 1.4 it would be sufficient to restrict to $a = 2$, $b = 3$, but it will be useful to introduce these two parameters a and b so that the situation becomes symmetric. As we shall see, we should assume $ab > a+b$, which means either $a \ge 2$ and $b \ge 3$ or else $a \ge 3$ and $b \ge 2$.

Consider the functions:

$$z_i,\ e^{(x_i/x_1)(z_{a+1}+z_1)-z_i} \quad (1 \le i \le a)$$

at the points of the subgroup in $\mathbf{C}$ spanned by

$$(\beta_{1j}, \dots, \beta_{aj}, \lambda_{1j}) \in \mathbf{C}^{a+1} \quad (1 \le j \le b).$$

These values are algebraic and the same holds for the values at the same points of the derivatives of these functions with respect to the differential operators $\partial/\partial z_i$ $(2 \le i \le a)$ and $\partial/\partial z_{a+1} - \partial/\partial z_1$.

Hence we are dealing with $2a$ functions in $a+1$ variables, b points (linearly independent) and a derivations.

1.4 Commutative linear algebraic groups

Theorems 1.1 and 1.4 are special cases of the linear subgroup Theorem. Consider a commutative linear algebraic group, say $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}$ (where $\mathbf{G}_a$ denotes the additive group and $\mathbf{G}_m$ the multiplicative group), over the field $\overline{\mathbf{Q}}$ of algebraic numbers. Its dimension is $d = d_0 + d_1$. Let $\mathcal{W} \subset T_e(G)$ be a $\mathbf{C}$-subspace which is rational over $\overline{\mathbf{Q}}$. Denote by ℓ_0 its dimension. Let $Y \subset T_e(G)$ be a finitely generated subgroup such that $\Gamma = \exp(Y)$ is contained in $G(\overline{\mathbf{Q}}) = \overline{\mathbf{Q}}^{d_0} \times (\overline{\mathbf{Q}}^{\times})^{d_1}$. Let ℓ_1 be the $\mathbf{Z}$-rank of Γ. Finally let $\mathcal{V} \subset T_e(G)$ be a $\mathbf{C}$-subspace containing both $\mathcal{W}$ and Y. Let n be the dimension of $\mathcal{V}$.

The conclusion of the linear subgroup Theorem below ([W] Theorem 11.5) is non-trivial only when

$$n(\ell_1 + d_1) < \ell_1 d_1 + \ell_0 d_1 + \ell_1 d_0. \tag{1.6}$$

For each connected algebraic subgroup G^* of G, defined over $\overline{\mathbf{Q}}$, we define

$$Y^* = Y \cap T_e(G^*), \quad \mathcal{V}^* = \mathcal{V} \cap T_e(G^*), \quad \mathcal{W}^* = \mathcal{W} \cap T_e(G^*)$$

and

$$d^* = \dim(G^*), \quad \ell_1^* = \operatorname{rank}_{\mathbf{Z}}(Y^*), \quad n^* = \dim_{\mathbf{C}}(\mathcal{V}^*), \quad \ell_0^* = \dim_{\mathbf{C}}(\mathcal{W}^*).$$

We may write $G^* = G_0^* \times G_1^*$ where G_0^* is an algebraic subgroup of G_0 and G_1^* is an algebraic subgroup of G_1. Define

$$d_0^* = \dim(G_0^*), \quad d_1^* = \dim(G_1^*),$$

so that $d^* = d_0^* + d_1^*$.

If we set

$$G_0' = \frac{G_0}{G_0^*}, \quad G_1' = \frac{G_1}{G_1^*}, \quad G' = \frac{G}{G^*} = G_0' \times G_1',$$

$$Y' = \frac{Y}{Y^*}, \quad \mathcal{V}' = \frac{\mathcal{V}}{\mathcal{V}^*}, \quad \mathcal{W}' = \frac{\mathcal{W}}{\mathcal{W}^*},$$

and

$$d_0' = \dim(G_0'), \quad d_1' = \dim(G_1'), \quad d' = \dim(G'),$$

$$\ell_1' = \operatorname{rank}_{\mathbf{Z}}(Y'), \quad n' = \dim_{\mathbf{C}}(\mathcal{V}'), \quad \ell_0' = \dim_{\mathbf{C}}(\mathcal{W}'),$$

then

$$d_0 = d_0^* + d_0', \quad d_1 = d_1^* + d_1', \quad d = d^* + d',$$

$$\ell_1 = \ell_1^* + \ell_1', \quad n = n^* + n', \quad \ell_0 = \ell_0^* + \ell_0'.$$

Theorem 1.7 (Linear subgroup Theorem).
(1) Assume $d > n$. Then there exists a connected algebraic subgroup G^* of G such that

$$d' > \ell_0' \quad \textit{and} \quad \frac{\ell_1' + d_1'}{d' - \ell_0'} \leq \frac{d_1}{d - n}.$$

(1') Assume $\ell_1 > 0$. Then there is a G^* for which

$$(d_1^*, \ell_1^*) \neq (0,0) \quad \textit{and} \quad \frac{d^* - \ell_0^*}{d_1^* + \ell_1^*} \leq \frac{n - \ell_0}{\ell_1}.$$

(2) Assume $d > n$ and $\ell_1 > 0$. Assume further that for any G^* for which $Y^* \neq \{0\}$, we have

$$\frac{n^* - \ell_0^*}{\ell_1^*} \geq \frac{n - \ell_0}{\ell_1}\cdot$$

Assume also that there is no G^* for which the three conditions $\ell_1' = 0$, $n' = \ell_0'$ and $d' > 0$ simultaneously hold. Then

$$d_1 > 0 \quad \textit{and} \quad \ell_1(d-n) \leq d_1(n-\ell_0).$$

(2') Assume $d > n$ and $\ell_1 > 0$. Assume further that for any G^* for which $d' > n'$, we have

$$\frac{d_1}{d-n} \leq \frac{d_1'}{d'-n'}\cdot$$

Assume also that there is no G^* for which the three conditions $d_1^* = 0$, $d^* = n^*$ and $d^* > 0$ simultaneously hold. Then

$$n > \ell_0 \quad \textit{and} \quad \ell_1(d-n) \leq d_1(n-\ell_0).$$

(3) Assume $\ell_1 > 0$. Then the family of G^* for which $\ell_1^* \neq 0$ and $(n^* - \ell_0^*)/\ell_1^*$ is minimal is not empty. Let G^* be such an element for which d^* is minimal. Then either $d^* = n^*$ or else

$$d_1^* > 0 \quad \textit{and} \quad \frac{n-\ell_0}{\ell_1} \geq \frac{n^*-\ell_0^*}{\ell_1^*} \geq \frac{d^*-n^*}{d_1^*}\cdot$$

(3') Assume $d > n$. Then the family of G^* for which $d' > n'$ and $d_1'/(d'-n')$ is minimal is not empty. Let G^* be such an element for which d' is minimal. Then either $\ell_1' = 0$ or else

$$n' > \ell_0' \quad \textit{and} \quad \frac{d_1}{d-n} \geq \frac{d_1'}{d'-n'} \geq \frac{\ell_1'}{n'-\ell_0'}\cdot$$

1.5 Fourier-Borel duality

Unifying the notation of §1.2 involving exponential polynomials, we let $d_0 + d_1$ be the number of functions, d_0 of which are linear and d_1 are exponential, ℓ_0 the number of derivations, ℓ_1 the number of points and n the number of variables.

	d_0	d_1	ℓ_0	ℓ_1	n
Baker B_1	1	n	n	1	n
Baker B_2	n	1	1	n	n
Sharp six exponentials	a	a	a	b	$a+1$

The inequality (1.6)

$$n(\ell_1 + d_1) < \ell_1 d_1 + \ell_0 d_1 + \ell_1 d_0.$$

is satisfied in the case of Baker's Theorem 1.1 since

$$n(\ell_1+d_1)=n^2+n, \qquad \ell_1 d_1+\ell_0 d_1+\ell_1 d_0=n^2+n+1.$$

For Theorem 1.4 the condition $a+b<ab$ is required:

$$n(\ell_1+d_1)=a^2+ab+a+b, \qquad \ell_1 d_1+\ell_0 d_1+\ell_1 d_0=a^2+2ab.$$

There is a duality in each two cases: it consists in permuting

$$(d_0,d_1,\ell_0,\ell_1)\longleftrightarrow(\ell_0,\ell_1,d_0,d_1)$$

For Baker's Theorem 1.1 it permutes methods B_1 and B_2. As pointed out to me by S. Fischler, for Theorem 1.4 it is not a mere permutation of a and b. Indeed the proof in § 1.3 involved the parameters

$$d_0=d_1=\ell_0=a, \quad \ell_1=b, \quad n=a+1,$$

henceforth the dual proof will involve the parameters

$$\ell_0=\ell_1=d_0=a, \quad d_1=b, \quad n=a+1.$$

In the dual proof there are $d_0+d_1=a+b$ functions, namely $z_1-z_{a+1},z_2,\ldots,z_a$ together with

$$e^{\beta_{1j}z_1+\cdots+\beta_{aj}z_a+\lambda_{1j}z_{a+1}} \quad (1\le j\le b),$$

the derivative operators are $\partial/\partial z_i$ $(1\le i\le a)$ and the points are $(0,\ldots,0,1)$ together with

$$(x_i/x_1,-\delta_{i2},\ldots,-\delta_{ia},x_i/x_1) \quad (2\le i\le a)$$

where δ_{ij} is Kronecker's symbol.

This duality rests on the analytic formula

$$\left(\frac{d}{dz}\right)^s \left(z^t e^{xz}\right)_{z=y}=\left(\frac{d}{dz}\right)^t \left(z^s e^{yz}\right)_{z=x}. \tag{1.8}$$

This formula (1.8) is related to the Fourier-Borel transform as follows. For s a non-negative integer and y a complex number, consider the analytic functional

$$\mathsf{L}_{sy}: f\longmapsto \left(\frac{d}{dz}\right)^s f(y).$$

Its Fourier-Borel transform is the analytic function $\mathsf{L}_{sy}(f_\zeta)$ of $\zeta\in\mathbf{C}$ which is the transform of the function $f_\zeta: z\mapsto e^{z\zeta}$:

$$f_\zeta(z)=e^{z\zeta}, \qquad \mathsf{L}_{sy}(f_\zeta)=\zeta^s e^{y\zeta}.$$

This yields (1.8) for $t=0$. The general case follows from

$$\mathsf{L}_{sy}(z^t f_\zeta)=\left(\frac{d}{d\zeta}\right)^t \mathsf{L}_{sy}(f_\zeta).$$

Formula (1.8) extends to the higher dimensional case (that is when $n > 1$). For $\underline{v} = (v_1, \ldots, v_n) \in \mathbf{C}^n$, set

$$D_{\underline{v}} = v_1 \frac{\partial}{\partial z_1} + \cdots + v_n \frac{\partial}{\partial z_n}\cdot$$

Let $\underline{w}_1, \ldots, \underline{w}_{\ell_0}$, $\underline{u}_1, \ldots, \underline{u}_{d_0}$, $\underline{\xi}$ and $\underline{\eta}$ in $\mathbf{C}^n$, $\underline{t} \in \mathbf{N}^{d_0}$ and $\underline{s} \in \mathbf{N}^{\ell_0}$. For $\underline{z} \in \mathbf{C}^n$, write

$$(\mathbf{u}\underline{z})^{\underline{t}} = (\underline{u}_1\underline{z})^{t_1} \cdots (\underline{u}_{d_0}\underline{z})^{t_{d_0}} \quad \text{and} \quad D_{\mathbf{w}}^{\underline{s}} = D_{\underline{w}_1}^{s_1} \cdots D_{\underline{w}_{\ell_0}}^{s_{\ell_0}}.$$

Then

$$D_{\mathbf{w}}^{\underline{s}}\bigl((\mathbf{u}\underline{z})^{\underline{t}} e^{\underline{\xi}\underline{z}}\bigr)\Big|_{\underline{z}=\underline{\eta}} = D_{\mathbf{u}}^{\underline{t}}\bigl((\mathbf{w}\underline{z})^{\underline{s}} e^{\underline{\eta}\underline{z}}\bigr)\Big|_{\underline{z}=\underline{\xi}}. \tag{1.9}$$

Example. In the proof of the sharp six exponentials Theorem 1.4 given in § 1.3 where $d_0 = d_1 = \ell_0 = a$, $\ell_1 = b$ and $n = a + 1$,

$$\underline{u}_i = (\delta_{i1}, \ldots, \delta_{ia}, 0) \quad (1 \le i \le a),$$

$\underline{w}_1 = (1, 0, \ldots, 0, -1)$ and

$$\underline{w}_i = (0, \delta_{i2}, \ldots, \delta_{ia}, 0) \quad (2 \le i \le a),$$

$\underline{\xi}$ is a linear combination of $\underline{\xi}_1, \ldots, \underline{\xi}_a$ with $\underline{\xi}_1 = (0, \ldots, 0, 1)$ and

$$\underline{\xi}_i = (x_i/x_1, -\delta_{i2}, \ldots, -\delta_{ia}, x_i/x_1) \quad (2 \le i \le a)$$

while $\underline{\eta}$ is a linear combination of $\underline{\eta}_1, \ldots, \underline{\eta}_b$ with

$$\underline{\eta}_j = (\beta_{1j}, \ldots, \beta_{aj}, \lambda_{1j}) \quad (1 \le j \le b).$$

Remark. The Fourier-Borel duality is not the same as the duality introduced by D. Roy in [Ro] which relates (1) and (1'), (2) and (2'), (3) and (3').

2 Bicommutative Hopf algebras

We consider commutative and cocommutative Hopf algebras (also called bicommutative Hopf algebras) over a field $\mathbf{k}$ of characteristic zero.

As the first example, the algebra of polynomials in one variable $H = \mathbf{k}[X]$ is endowed with a Hopf algebra structure with the coproduct Δ, the co-unit ϵ and the antipode S defined as the algebra morphisms for which

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \epsilon(X) = 0 \quad \text{and} \quad S(X) = -X.$$

If we identify $\mathbf{k}[X] \otimes \mathbf{k}[X]$ with $\mathbf{k}[T_1, T_2]$ by mapping $X \otimes 1$ to T_1 and $1 \otimes X$ to T_2, then

$$\Delta P(X) = P(T_1 + T_2), \quad \epsilon P(X) = P(0), \quad SP(X) = P(-X).$$

Since $\mathbf{G}_a(K) = \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}[X], K)$ and $\mathbf{k}[\mathbf{G}_a] = \mathbf{k}[X]$, it follows that $\mathbf{k}[\mathbf{G}_a]$ *is a bicommutative Hopf algebra of finite type.*

Our next example is the algebra of Laurent polynomials $H = \mathbf{k}[Y, Y^{-1}]$ which becomes a Hopf algebra with the coproduct Δ satisfying $\Delta(Y) = Y \otimes Y$, the co-unit ϵ for which $\epsilon(Y) = 1$ and the antipode S with $S(Y) = Y^{-1}$. The algebra isomorphism between $H \otimes H$ and $\mathbf{k}[T_1, T_1^{-1}, T_2, T_2^{-1}]$ defined by

$$Y \otimes 1 \mapsto T_1, \quad 1 \otimes Y \mapsto T_2$$

gives

$$\Delta P(Y) = P(T_1 T_2), \quad \epsilon P(Y) = P(1), \quad SP(Y) = P(Y^{-1}).$$

Since $\mathbf{G}_m(K) = \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}[Y, Y^{-1}], K)$, we have $\mathbf{k}[\mathbf{G}_m] = \mathbf{k}[Y, Y^{-1}]$ and again $\mathbf{k}[\mathbf{G}_m]$ is a *bicommutative Hopf algebra of finite type.*

Combining these two examples, one gets a whole family of Hopf algebras. Indeed let $d_0 \geq 0$ and $d_1 \geq 0$ be two integers with $d = d_0 + d_1 > 0$. The Hopf algebra

$$\mathbf{k}[X]^{\otimes d_0} \otimes \mathbf{k}[Y, Y^{-1}]^{\otimes d_1}$$

is isomorphic to

$$H = \mathbf{k}[X_1, \ldots, X_{d_0}, Y_1, Y_1^{-1}, \ldots, Y_{d_1}, Y_{d_1}^{-1}],$$

hence is isomorphic to $\mathbf{k}[G]$ with $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}$.

According to [A] Chap. 4 (p. 163), the category of $\mathbf{k}$-linear algebraic groups is anti-equivalent to the category of commutative $\mathbf{k}$-Hopf algebras of finite type. Hence *the category of commutative linear algebraic groups over* $\mathbf{k}$ *is anti-equivalent to the category of bicommutative Hopf algebras of finite type over* $\mathbf{k}$.

The commutative and connected linear algebraic groups over an algebraically closed fields are the groups $\mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}$ and the Hopf algebras $\mathbf{k}[\mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}]$ are the bicommutative Hopf algebras of finite type over $\mathbf{k}$ without zero divisors. In $\mathbf{k}[\mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}]$ the $\mathbf{k}$-vector space of primitive elements has dimension d_0, while the rank of group-like elements is d_1.

We exhausted the list of examples of bicommutative Hopf algebras without zero divisors and of finite type. However this is not the end of the story: let W be a $\mathbf{k}$-vector space of dimension ℓ_0. Then the symmetric algebra $\mathrm{Sym}(W)$ on W has a natural structure of bicommutative Hopf algebra of finite type [H3]. If $\partial_1, \ldots, \partial_{\ell_0}$ is a basis of W over $\mathbf{k}$, then $\mathrm{Sym}(W)$ is isomorphic to $\mathbf{k}[\partial_1, \ldots, \partial_{\ell_0}]$, hence to $\mathbf{k}[\mathbf{G}_a^{\ell_0}]$.

If Γ is a free $\mathbf{Z}$-module of finite type and rank ℓ_1, then the group algebra $\mathbf{k}\Gamma$ is a bicommutative Hopf algebra of finite type isomorphic to $\mathbf{k}[\mathbf{G}_m^{\ell_1}]$.

Therefore *the category of bicommutative Hopf algebras without zero divisors and of finite type over* $\mathbf{k}$ *is equivalent to the category of pairs* (W, Γ) *where* W *is a* $\mathbf{k}$*-vector space of finite dimension and* Γ *a free* $\mathbf{Z}$*-module of finite type.* In

$$H \simeq \mathrm{Sym}(W) \otimes \mathbf{k}\Gamma,$$

the space of primitive elements has dimension $\ell_0 = \dim W$, while the group-like elements have rank $\ell_1 = \mathrm{rank}\Gamma$.

We now take $\mathbf{k} = \overline{\mathbf{Q}}$. S. Fischler [F1] further introduces two more categories.

Let $\mathfrak{C}_1$ be the category whose objects are the triples (G, W, Γ) with $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}$ a commutative linear algebraic group over $\overline{\mathbf{Q}}$, $W \subset T_e(G)$ a subspace which is rational over $\overline{\mathbf{Q}}$ and $\Gamma \subset G(\overline{\mathbf{Q}})$ a torsion free finitely generated subgroup; moreover G is *minimal* for these properties: no algebraic subgroup G^* other than G itself satisfies $W \subset T_e(G^*)$ and $\Gamma \subset G^*(\overline{\mathbf{Q}})$.

We denote by ℓ_0 the dimension of W and by ℓ_1 the rank of Γ.

The morphisms $f : (G_1, W_1, \Gamma_1) \to (G_2, W_2, \Gamma_2)$ are given by a morphism $f : G_1 \to G_2$ of algebraic groups such that $f(\Gamma_1) \subset \Gamma_2$ such that the linear tangent map to f

$$df : T_e(G_1) \longrightarrow T_e(G_2)$$

satisfies $df(W_1) \subset W_2$.

The definition of the category $\mathfrak{C}_2$ requires the following additional data. Let H be an bicommutative Hopf algebra of finite type over $\overline{\mathbf{Q}}$ and without zero divisors. Denote by d_0 the dimension of the $\overline{\mathbf{Q}}$-vector space spanned by the primitive elements and by d_1 the rank of the group-like elements. Let H' be also a Hopf algebra, which is again bicommutative, without zero divisors and of finite type over $\overline{\mathbf{Q}}$. The dimension of the $\overline{\mathbf{Q}}$-vector space spanned by the primitive elements in H' is denoted by ℓ_0 while ℓ_1 is the rank of the group-like elements in H'. Let $\langle\cdot\rangle : H \times H' \longrightarrow \overline{\mathbf{Q}}$ be a bilinear map such that

$$\langle x, yy'\rangle = \langle \Delta x, y \otimes y'\rangle \quad \text{and} \quad \langle xx', y\rangle = \langle x \otimes x', \Delta y\rangle. \tag{2.1}$$

We used the notation

$$\langle \alpha \otimes \beta, \gamma \otimes \delta\rangle = \langle \alpha, \gamma\rangle\langle \beta, \delta\rangle.$$

The objects of the category $\mathfrak{C}_2$ are the triples $(H, H', \langle\cdot\rangle)$ given by two bicommutative Hopf algebras, without zero divisors and of finite type over $\overline{\mathbf{Q}}$, and a bilinear product satisfying (2.1). The morphisms are the pairs $(f, g) : (H_1, H_1', \langle\cdot\rangle_1) \to (H_2, H_2', \langle\cdot\rangle_2)$ where $f : H_1 \to H_2$ and $g : H_2' \to H_1'$ are Hopf algebra morphisms such that

$$\langle x_1, g(x_2')\rangle_1 = \langle f(x_1), x_2'\rangle_2.$$

One composes two morphisms $(f_1, g_1) : (H_1, H_1', \langle\cdot\rangle_1) \to (H_2, H_2', \langle\cdot\rangle_2)$ and $(f_2, g_2) : (H_2, H_2', \langle\cdot\rangle_2) \to (H_3, H_3', \langle\cdot\rangle_3)$ as

$$(f_2 \circ f_1, g_1 \circ g_2) : (H_1, H_1', \langle\cdot\rangle_1) \to (H_3, H_3', \langle\cdot\rangle_3).$$

Stéphane Fischler [F1] proves:

Theorem 2.2 (S. Fischler). *Both categories $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are equivalent. This equivalence preserves the parameters d_0, d_1, ℓ_0, ℓ_1.*

The category $\mathfrak{C}_2$ has a natural contravariant involution which consists in permuting H and H'. The corresponding involution in the category $\mathfrak{C}_1$ is the Fourier-Borel duality (1.9) we discussed above, which exchanges (d_0, d_1) and (ℓ_0, ℓ_1) in Theorem 1.7.

The main goal in [F1] is to establish new *interpolation lemmas*. Theorem 2.2 enables Fischler to obtain them by duality, starting from known *zero estimates*.

Roughly speaking, a zero estimate (see for instance [W] § 2.1) is a lower bound for the degree of a polynomial vanishing at a given finite set of points (multiplicities may be considered). An interpolation lemma provides a lower bound for an integer D with the following property: given a finite set of points (maybe with multiplicities), there is a polynomial of degree at most D taking given values at these points. In terms of matrices, the zero estimates states that a matrix, whose entries are the values of monomials at the given points, has maximal rank, if only there are enough monomials (hence the matrix is sufficiently rectangular), while the interpolation lemma states that such a matrix has maximal rank once there are enough points (again this means that the matrix is sufficiently rectangular, but in the other direction).

This method using a duality to deduce interpolation lemmas from zero estimates works only for *linear* commutative algebraic groups. Zero estimates are known more generally for commutative algebraic groups (hence for abelian and semi-abelian varieties), but duality does not extend to the non-linear case. Fischler [F2] uses other arguments to obtain interpolation lemmas for non-linear commutative algebraic groups.

3 Hopf algebras and multiple zeta values

Let $\mathfrak{S}$ denote the set of sequences $\underline{s} = (s_1, \ldots, s_k) \in \mathbf{N}^k$ with $k \geq 0$, $s_1 \geq 2$, $s_i \geq 1$ $(2 \leq i \leq k)$.

The *weight* $|\underline{s}|$ of $\underline{s}$ is $s_1 + \cdots + s_k$, while k is the *depth of* $\underline{s}$.

For $\underline{s} \in \mathfrak{S}$ set

$$\zeta(\underline{s}) = \sum_{n_1 > \cdots > n_k \geq 1} n_1^{-s_1} \cdots n_k^{-s_k}.$$

When $\underline{s}$ is the empty sequence (of weight and depth 0), we require $\zeta(\underline{s}) = 1$.

3.1 Goncharov's Conjecture

Denote by $\mathfrak{Z}$ the $\mathbf{Q}$-vector subspace of $\mathbf{C}$ spanned by the numbers

$$(2i\pi)^{-|s|}\zeta(\underline{s}) \quad (\underline{s} \in \mathfrak{S}).$$

As is well known (and as we shall see), for $\underline{s}$ and $\underline{s}'$ in $\mathfrak{S}$, the product $\zeta(\underline{s})\zeta(\underline{s}')$ is *in two ways* a linear combination with positive coefficients of numbers $\zeta(\underline{s}'')$.

Hence $\mathfrak{Z}$ is a $\mathbf{Q}$-sub-algebra of $\mathbf{C}$ with a double filtration by weight and depth.

For a graded Lie algebra $C_\bullet$ denote by $\mathfrak{U}C_\bullet$ its universal envelopping algebra and by

$$\mathfrak{U}C_\bullet^\vee = \bigoplus_{n\geq 0} (\mathfrak{U}C)_n^\vee$$

its graded dual, which is a commutative Hopf algebra.

Conjecture 3.1 (Goncharov [G]). *There exists a graded Lie algebra $C_\bullet$ and an isomorphism*

$$\mathfrak{Z} \simeq \mathfrak{U}C_\bullet^\vee$$

of bifiltered algebras, by the weight on the left and by the depth on the right.

Hopf algebras also occur in this theory in a non-conjectural way. They are used to describe the above mentioned quadratic relations expressing the product of two multiple zeta values as a linear combination of multiple zeta values.

3.2 The concatenation Hopf algebra

Let $X = \{x_0, x_1\}$ be an alphabet with two letters. The free monoid (of words) on X is

$$X^* = \{x_{\epsilon_1} \cdots x_{\epsilon_k}\ ;\ \epsilon_i \in \{0,1\},\ (1 \leq i \leq k),\ \ k \geq 0\}$$

whose product is concatenation, and its unity is the empty word e.

Let $\mathfrak{H}$ denote the free algebra $\overline{\mathbf{Q}}\langle X\rangle$ on X. An element $P \in \mathfrak{H}$ is written

$$P = \sum_{w \in X^*} \langle P|w\rangle w$$

with coefficients $\langle P|w\rangle \in \overline{\mathbf{Q}}$.

The *concatenation Hopf algebra* is $(\mathfrak{H}, \cdot, e, \Delta, \epsilon, S)$ where the coproduct is

$$\Delta P = P(x_0 \otimes 1 + 1 \otimes x_0, x_1 \otimes 1 + 1 \otimes x_1),$$

the co-unit $\epsilon(P) = \langle P \mid e\rangle$ and the antipode

$$S(x_1 \cdots x_n) = (-1)^n x_n \cdots x_1$$

for $n \geq 1$ and $x_1, \ldots, x_n$ in X.

It is a cocommutative, not commutative Hopf algebra.

3.3 The shuffle Hopf algebra

The shuffle product $ш : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$ is defined inductively by the conditions

$$u ш e = e ш u = u \quad \text{and} \quad xu ш yv = x(u ш yv) + y(xu ш v)$$

for x and y in X, u and v in X^*. It endows $\mathfrak{H}$ with a structure of commutative algebra $\mathfrak{H}_ш$.

According to [Re] Theorem 3.1, for $P \in \mathfrak{H}$,

$$\Delta P = \sum_{u,v \in X^*} (P | u ш v) u \otimes v. \tag{3.2}$$

The *shuffle Hopf algebra* is the commutative (not cocommutative) Hopf algebra $(\mathfrak{H}, ш, e, \Phi, \epsilon, S)$, with $\Phi : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}$ defined by

$$\langle \Phi(w) \mid u \otimes v \rangle = \langle uv \mid w \rangle.$$

Hence

$$\Phi(w) = \sum_{\substack{u,v \in X^* \\ uv=w}} u \otimes v.$$

From (3.2) it follows that the shuffle Hopf algebra is the graded dual of the concatenation Hopf algebra (see [Re] Chap. 1).

We need to consider subalgebras of $\mathfrak{H}$.

For $s \geq 1$ define $y_s = x_0^{s-1} x_1$. The subalgebra $\mathfrak{H}^1$ of $\mathfrak{H}$ spanned by $\{y_1, y_2, \ldots\}$ is free, and so is the subalgebra $\mathfrak{H}^0$ of $\mathfrak{H}^1$ spanned by $\{y_2, y_3, \ldots\}$. Also $\mathfrak{H}^1$ is the $\overline{\mathbf{Q}}$-vector space $\overline{\mathbf{Q}}e + \mathfrak{H}x_1$ spanned by $\{e\} \cup X^* x_1$, while $\mathfrak{H}^0$ is the $\overline{\mathbf{Q}}$-vector space $\overline{\mathbf{Q}}e + x_0 \mathfrak{H} x_1$ spanned by $\{e\} \cup x_0 X^* x_1$.

The shuffle ш makes $\mathfrak{H}^0$ and $\mathfrak{H}^1$ subalgebras of $\mathfrak{H}_ш$:

$$\mathfrak{H}^0_ш \subset \mathfrak{H}^1_ш \subset \mathfrak{H}_ш.$$

Define a mapping $\hat{\zeta} : x_0 X^* x_1 \to \mathbf{C}$ as follows. Each element w in $x_0 X^* x_1$ can be written in a unique way $y_{s_1} \cdots y_{s_k}$ with $\underline{s} = (s_1, \ldots, s_k) \in \mathfrak{S}$. The number of letters x_1 in w is the depth k of, while the total number of letters of w is the weight $s_1 + \cdots + s_k$ of $\underline{s}$. Define

$$\hat{\zeta}(y_{s_1} \cdots y_{s_k}) = \zeta(s_1, \ldots, s_k).$$

By $\overline{\mathbf{Q}}$-linearity one extends $\hat{\zeta}$ to a map from $\mathfrak{H}^0$ to $\mathbf{C}$ with $\hat{\zeta}(e) = 1$. Using the representation of $\hat{\zeta}$ as Chen iterated integrals, namely ([K], § XIX.11), for $w \in x_0 X^* x_1$,

$$\hat{\zeta}(x_{\epsilon_1} \cdots x_{\epsilon_p}) = \int_{1 > t_1 > \cdots > t_p > 0} \omega_{\epsilon_1}(t_1) \cdots \omega_{\epsilon_p}(t_p)$$

with $\epsilon_i \in \{0,1\}$ $(1 \leq i \leq p)$, $\epsilon_0 = 0$, $\epsilon_p = 1$,

$$\omega_0(t) = \frac{dt}{t} \quad \text{and} \quad \omega_1(t) = \frac{dt}{1-t},$$

one checks that $\hat{\zeta}$ *is a commutative algebra morphism of* $\mathfrak{H}^0_ш$ *into* $\mathbf{C}$.

The structure of the commutative algebra $\mathfrak{H}_ш$ is given by *Radford Theorem* [Re] Chap. 6. Consider the lexicographic order on X^* with $x_0 < x_1$. A *Lyndon word* is a word $w \in X^*$ such that, for each decomposition $w = uv$ with $u \neq e$

and $v \neq e$, the inequality $w < v$ holds. Examples of Lyndon words are x_0, x_1, $x_0 x_1^k$ $(k \geq 0)$, $x_0^\ell x_1$ $(\ell \geq 0)$, $x_0^2 x_1^2$. Denote by L the set of Lyndon words. Then the three shuffle algebras are (commutative) polynomial algebras

$$\mathfrak{H}_{ш} = K[\mathsf{L}]_{ш}, \quad \mathfrak{H}^1_{ш} = K\big[\mathsf{L} \setminus \{x_0\}\big]_{ш} \quad \text{and } \mathfrak{H}^0_{ш} = K\big[\mathsf{L} \setminus \{x_0, x_1\}\big]_{ш}.$$

Therefore

$$\mathfrak{H}_{ш} = \mathfrak{H}^1_{ш}[x_0]_{ш} = \mathfrak{H}^0_{ш}[x_0, x_1]_{ш} \quad \text{and } \mathfrak{H}^1_{ш} = \mathfrak{H}^0_{ш}[x_1]_{ш}. \tag{3.3}$$

3.4 Harmonic algebra

There is another product being shuffle-like law on $\mathfrak{H}$, called *harmonic product* by M. Hoffman ([H1], [H2]) and *stuffle* by other authors [BBBL], denoted with a star, which also gives rise to subalgebras

$$\mathfrak{H}^0_\star \subset \mathfrak{H}^1_\star \subset \mathfrak{H}_\star.$$

It is defined as follows. First on X^*, the map $\star : X^* \times X^* \to \mathfrak{H}$ is defined by induction, starting with

$$x_0^n \star w = w \star x_0^n = w x_0^n$$

for any $w \in X^*$ and any $n \geq 0$ (for $n = 0$ it means $e \star w = w \star e = w$ for all $w \in X^*$) and then

$$y_s u \star y_t v = y_s(u \star y_t v) + y_t(y_s u \star v) + y_{s+t}(u \star v)$$

for u and v in X^*, s and t positive integers.

The harmonic product is an efficient way of writing the quadratic relations among multiple zeta values arising from the expression of $\zeta(\underline{s})$ as series: *$\hat{\zeta}$ is a commutative algebra morphism of $\mathfrak{H}^0_\star$ into $\mathbf{C}$.*

Hoffman [H1] gives the structure of the quasi-harmonic algebra $\mathfrak{H}_\star$ as well as of its subalgebras $\mathfrak{H}^1_\star$ and $\mathfrak{H}^0_\star$: they are again polynomial algebras on Lyndon words:

$$\mathfrak{H}_\star = K[\mathsf{L}]_\star, \quad \mathfrak{H}^0_\star = K\big[\mathsf{L} \setminus \{x_0\}\big]_\star \quad \text{and} \quad \mathfrak{H}^1_\star = K\big[\mathsf{L} \setminus \{x_0, x_1\}\big]_\star.$$

Hence

$$\mathfrak{H}_\star = \mathfrak{H}^1_\star[x_0]_\star = \mathfrak{H}^0_\star[x_0, x_1]_\star \quad \text{and} \quad \mathfrak{H}^1_\star = \mathfrak{H}^0_\star[x_1]_\star. \tag{3.4}$$

The *quasi-shuffle* Hopf algebra is the commutative algebra $\mathfrak{H}^1_\star$ with the coproduct Δ defined by the conditions

$$\Delta(y_i) = y_i \otimes e + e \otimes y_i$$

for $i \geq 1$, the co-unit

$$\epsilon(P) = \langle P \mid e \rangle$$

and the antipode

$$S(y_{s_1} \cdots y_{s_k}) = (-1)^k y_{s_k} \cdots y_{s_1}.$$

This quasi-shuffle Hopf algebra is isomorphic to the Hopf algebra of non-commutative symmetric series, whose graded dual is the Hopf algebra of quasi-symmetric series (see [H2] and [H3]) .

3.5 Regularized double shuffle relations

As we have seen the map $\hat\zeta$ is a commutative algebra morphism of $\mathfrak{H}^0_{ш}$ into $\mathbf{C}$ and also of $\mathfrak{H}^0_\star$ into $\mathbf{C}$. Hence the kernel of $\hat\zeta$ in $\mathfrak{H}^0$ is an ideal for the two algebra structures ш and $\star$. A fundamental question (cf. Goncharov's Conjecture 3.1) is to describe this kernel.

The relations

$$\hat\zeta(u ш v) = \hat\zeta(u)\hat\zeta(v) \quad \text{and} \quad \hat\zeta(u \star v) = \hat\zeta(u)\hat\zeta(v) \quad \text{for } u \text{ and } v \text{ in } \mathfrak{H}^0$$

show that, for any u and v in $\mathfrak{H}^0$, $u ш v - u \star v$ belong to the kernel of $\hat\zeta$. The equations

$$\hat\zeta(u ш v - u \star v) = 0 \quad \text{for } u \text{ and } v \text{ in } \mathfrak{H}^0 \tag{3.5}$$

are called *the standard linear relations among multiple zeta values.*

Other elements belong to the kernel of $\hat\zeta$: *Hoffman's relations* (see for instance [Z]) are

$$\hat\zeta(x_1 ш v - x_1 \star v) = 0 \quad \text{for } v \text{ in } \mathfrak{H}^0. \tag{3.6}$$

Notice that $x_1 ш v - x_1 \star v \in \mathfrak{H}^0$ for $v \in \mathfrak{H}^0$. The simplest example

$$x_1 ш y_2 - x_1 \star y_2 \in \ker\hat\zeta$$

yields the relation $\zeta(2,1) = \zeta(3)$ known by Euler.

It was conjectured in [MJOP] that the elements

$$u ш v - u \star v \quad \text{and} \quad x_1 ш v - x_1 \star v,$$

when u and v range over the set $\mathfrak{H}^0$, span the $\mathbf{Q}$ vector space $\ker\hat\zeta$. This conjecture is not yet disproved, but there is a little doubt about it for the following reason.

From (3.3) and (3.4) it follows that there are two uniquely determined algebra morphisms

$$\hat Z_ш : \mathfrak{H}^1_ш \longrightarrow \mathbf{R}[T] \quad \text{and} \quad \hat Z_\star : \mathfrak{H}^1_\star \longrightarrow \mathbf{R}[T]$$

which extend $\hat\zeta$ and map x_1 to T. According to [C], the next result is due to Boutet de Monvel and Zagier (see also [I-K]).

Proposition 3.7 *There is a* $\mathbf{R}$*-linear isomorphism* $\varrho : \mathbf{R}[T] \to \mathbf{R}[T]$ *which makes the following diagram commutative:*

$$\begin{array}{ccc} \mathfrak{H}^1 & \xrightarrow{\hat Z_\star} & \mathbf{R}[T] \\ \Big\| & & \Big\downarrow \varrho \\ \mathfrak{H}^1 & \xrightarrow{\hat Z_ш} & \mathbf{R}[T] \end{array}$$

An explicit formula for ϱ *is given by means of the generating series*

$$\sum_{\ell \ge 0} \varrho(T^\ell)\frac{t^\ell}{\ell!} = \exp\left(Tt + \sum_{n=2}^\infty (-1)^n \frac{\zeta(n)}{n} t^n\right). \tag{3.8}$$

It is instructive to compare the right hand side of (3.8) with the formula giving the expansion of the logarithm of Euler Gamma function:

$$\Gamma(1+t) = \exp\left(-\gamma t + \sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n} t^n\right).$$

Accordingly, ϱ may be viewed as the differential operator of infinite order

$$\exp\left(\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n}\left(\frac{\partial}{\partial T}\right)^n\right)$$

(just consider the image of e^{tT}).

In [I-K] Ihara and Kaneko propose a regularization of the divergent multiple zeta values as follows.

Recall that $\mathfrak{H}_{ш} = \mathfrak{H}^0[x_0, x_1]_{ш}$. Denote by $\mathrm{reg}_{ш}$ the $\mathbf{Q}$-linear map $\mathfrak{H} \to \mathfrak{H}^0$ which maps $w \in \mathfrak{H}$ to its constant term in its expansion as a polynomial in x_0, x_1 in the shuffle algebra $\mathfrak{H}^0[x_0, x_1]_{ш}$. Then $\mathrm{reg}_{ш}$ is an algebra morphism $\mathfrak{H}_{ш} \to \mathfrak{H}^0_{ш}$. Clearly for $w \in \mathfrak{H}^0$ we have

$$\mathrm{reg}_{ш}(w) = w.$$

Theorem 3.9 (Ihara, Kaneko). *Let w be any word in X^*. Write $w = x_1^m w_0 x_0^n$ with $w_0 \in \mathfrak{H}^0$, $m \geq 0$ and $n \geq 0$. Then*

$$\mathrm{reg}_{ш}(w) = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j} x_1^i ш (x_1^{m-i} w_0 x_0^{n-j}) ш x_0^j.$$

Special cases are:

$$\mathrm{reg}_{ш}(x_1^m) = \mathrm{reg}_{ш}(x_0^n) = 0 \quad \text{for} \quad m \geq 1 \quad \text{and} \quad n \geq 1.$$

$$\mathrm{reg}_{ш}(x_1^m x_0^n) = (-1)^{m+n-1} x_0^n x_1^m \quad \text{for} \quad m \geq 1 \quad \text{and} \quad n \geq 1.$$

$$\mathrm{reg}_{ш}(x_1^m x_0 u) = (-1)^m x_0 (x_1^m ш u) \quad \text{for} \quad m \geq 0 \quad \text{and} \quad u \in X^* x_1.$$

$$\mathrm{reg}_{ш}(u x_1 x_0^n) = (-1)^n (u ш x_0^n) x_1 \quad \text{for} \quad n \geq 0 \quad \text{and} \quad u \in x_0 X^*.$$

Moreover there is an explicit expression for w as a polynomial in x_0 and x_1 in the algebra $\mathfrak{H}^0[x_0, x_1]_{ш}$:

$$w = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathrm{reg}_{ш}(x_1^{m-i} w_0 x_0^{n-j}) ш x_1^i ш x_0^j.$$

The *regularized double shuffle relations* of Ihara and Kaneko in [I-K] produce a number of linear relations among multiple zeta values:

Theorem 3.10 (Ihara, Kaneko). *For $w \in \mathfrak{H}^1$ and $w_0 \in \mathfrak{H}^0$,*

$$\mathrm{reg}_{ш}(w ш w_0 - w \star w_0) \in \ker \hat{\zeta}. \tag{3.11}$$

Special cases of (3.11) – for which no regularization is required – are the standard relations (3.5) which correspond to $w \in \mathfrak{H}^0$ and Hoffman's relations (3.6) which correspond to $w = x_1$.

An example of u and v in $\mathfrak{H}^1$ for which $u ш v - u \star v \in \mathfrak{H}^0$ but $\hat{\zeta}(u ш v - u \star v) \neq 0$ is $u = v = x_1$.

3.6 The main diophantine Conjecture

The main diophantine Conjecture arose after the works of several mathematicians, including D. Zagier, A.B. Goncharov, M. Kontsevich, M. Hoffman, M. Petitot and Hoang Ngoc Minh, K. Ihara and M. Kaneko, J. Écalle, P. Cartier (see [C]).

Conjecture 3.12 *The kernel of $\hat{\zeta}$ is spanned by the elements*

$$\mathrm{reg}_{ш}(w ш w_0 - w \star w_0)$$

where w ranges over $\mathfrak{H}^1$ and w_0 over $\mathfrak{H}^0$.

Conjecture 3.12 means that the ideal of algebraic relations among multiple zeta values is generated by the double shuffle relations of Ihara and Kaneko in Theorem 3.10.

More precisely, we introduce independent variables Z_u, where u ranges over the set X^*x_1. For $v = \sum_u c_u u$ in $\mathfrak{H}^1$, we set

$$Z_v = \sum_u c_u Z_u$$

where $Z_e = 1$. In particular, for u_1 and u_2 in $x_0X^*x_1$, $Z_{u_1 ш u_2}$ and $Z_{u_1 \star u_2}$ are linear forms in Z_u, $u \in x_0X^*x_1$. Also, for $v \in x_0\mathfrak{H}x_1$, $Z_{x_1 ш v - x_1 \star v}$ is a linear form in Z_u, $u \in x_0X^*x_1$.

Denote by $\mathfrak{R}$ the ring of polynomials with coefficients in $\overline{\mathbf{Q}}$ in the variables Z_u, where u ranges over the set of words in $x_0X^*x_1$ which start with x_0 and end with x_1. Further, denote by $\mathfrak{I}$ the ideal of $\mathfrak{R}$ consisting of all polynomials which vanish under the specialization map $\mathfrak{R} \to \mathbf{R}$ which is the $\overline{\mathbf{Q}}$-algebra morphism defined by

$$Z_u \mapsto \hat{\zeta}(u) \quad (u \in x_0X^*x_1).$$

The $\overline{\mathbf{Q}}$-sub-algebra in $\mathbf{C}$ of multiple zeta values (up to the normalization with powers of $2\pi i$, this is the algebra $\mathfrak{Z}$ of Goncharov's Conjecture 3.1) is isomorphic to the quotient $\mathfrak{R}/\mathfrak{I}$.

Let $\mathfrak{J}$ be the ideal of $\mathfrak{R}$ generated by the polynomials

$$Z_u Z_v - Z_{u ш v} \quad \text{and} \quad Z_r \quad \text{with} \quad r = \mathrm{reg}_{ш}(w ш w_0 - w \star w_0),$$

where u, v, w_0 range over $\mathfrak{H}^0$ and w over $\mathfrak{H}^1$.

Theorem 3.10 can be written $\mathfrak{J} \subset \mathfrak{I}$ and Conjecture 3.12 means $\mathfrak{J} = \mathfrak{I}$.

The ideal of $\mathfrak{R}$ associated to the above mentioned conjecture of [MJOP] (see § 3.5) is the ideal, contained in $\mathfrak{J}$, generated by the polynomials

$$Z_u Z_v - Z_{u ш v}, \quad Z_u Z_v - Z_{u \star v} \quad \text{and} \quad Z_{x_1 ш u - x_1 \star u},$$

where u and v range over the set of elements in $x_0 X^* x_1$.

The structure of the quotient of $\mathfrak{R}/\mathfrak{J}$ is being studied by Jean Écalle [E].

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