

# Counting Integer Flows in Networks.

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## Introduction

A *network* is a graph with directed edges, with multiple copies of the edges allowed, and where each node  $v$  has an integer value specified, the so called *excess* of  $v$ , and each arc has an assigned positive integer value called its *capacity*. A *feasible flow* is an assignment of real values to the arcs of the network so that for any node  $v$  the difference between the sum of values in outgoing arcs minus the sum of values in incoming arcs equals the prescribed excess of the node  $v$  and the capacities of the arcs are not surpassed. In this paper we study the problem of effectively counting the number of different *integral feasible flows* in a network. It is well-known that this problem is  $\#P$ -hard in the computational category of counting problems [13] because the problem of counting perfect matchings in bipartite graphs reduces to it. Despite this bad complexity, concrete applications abound in graph theory [14], representation theory [15], and statistics [12] and thus finding good methods for attacking concrete examples is of importance. Our goal is to show that using the algebraic-analytic structure of the problem allows us to count flows in complicated instances very fast, surpassing traditional exhaustive enumeration. Continuing the work started in [2] we present effective counting algorithms from which one can in fact derive counting formulas when the excess function has parameters.

The set of all *feasible flows* with given excess vector  $b$  and capacity vector  $c$  is a convex polytope, the well-known *flow polytope*, which is defined by the constraints  $\Phi_G x = b$ ,  $0 \leq x \leq c$ , where  $\Phi_G$  denotes the node-arc incidence matrix of  $G$  (a *network matrix*). The incidence matrix  $\Phi_G$  has one column per arc and one row per node. Each column of  $\Phi_G$  has as many entries as nodes. For an arc going from  $i$  to  $j$ , its corresponding column has zeros everywhere except at the  $i$ -th and  $j$ -th entries. The  $j$ -th entry, the *head* of the arrow, receives a  $-1$  and the  $i$ -th entry, *tail* of the arrow, a  $1$ . A famous

instance is the *max-flow min-cut* problem [19]. This is the case when  $b$  has first entry  $v$ , last entry  $-v$  and 0 elsewhere. In part (B) of Figure 1 we list all possible flows with  $v = 11$ , the maximal possible from the network information specified in part (A).

For us, an important feature of the network incidence matrix  $\Phi_G$  is that it is unimodular. We say that the system  $\Phi_G$  is *unimodular*, if the columns of  $\Phi_G$  span a lattice, denoted by  $\mathbb{Z}\Phi_G$  and, whenever  $a$  is in this lattice  $\mathbb{Z}\Phi_G$ , the polytope  $P(\Phi_G, a) = \{x \mid x \geq 0 : \Phi_G x = a\}$  has vertices with integral coordinates. Even more strongly, network matrices are in fact totally unimodular matrices [19], which means that the lattice generated by their columns is the standard integral lattice  $\mathbb{Z}^n$ . Note that the integral feasible flows are precisely the integer lattice points inside the flow polytope.

Here is an example: The node-arc incidence matrix for the graph  $G_1$  in Figure 2 is defined by:

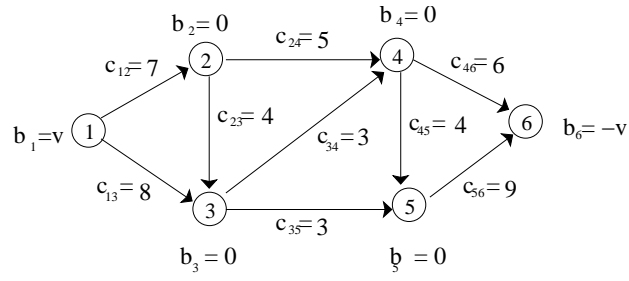
$$\Phi_{G_1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix}.$$

The equation  $\Phi_{G_1} x = b$  reads as the series of equations  $x_1 - x_2 = b_1$ ,  $x_2 - x_3 + x_4 = b_2$ ,  $-x_1 + x_3 - x_4 = b_3$ . These 3 equations express the fact that, at each node  $v \in \{1, 2, 3\}$ , the difference between the sum of values in outgoing arcs minus the sum of values in incoming arcs equals the prescribed excess  $b_i$  of the node  $v$ . Feasible flows are restricted furthermore by the conditions  $0 \leq x_i \leq c_i$ .

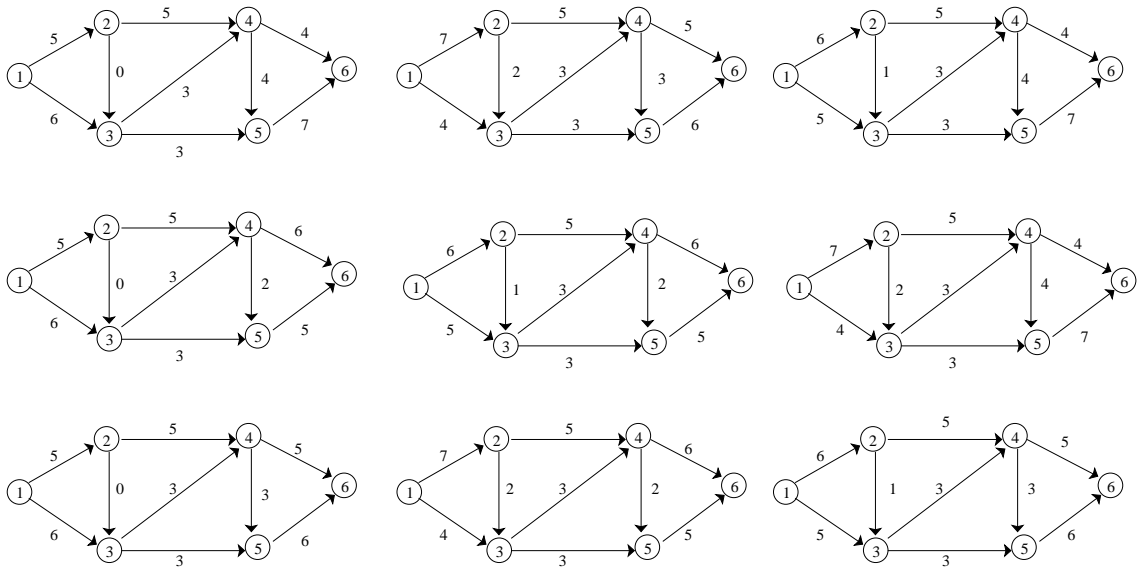
The algorithm and formulas deduced here are based on the notion of *total residue* (see Section 1), the main concept involved being the study of rational functions with poles on an arrangement of hyperplanes. The enumeration theory we present was extended to arbitrary rational polyhedra in [23]. The particular description we do here is valid for all *unimodular matrices* (again, remember that a matrix  $A$  is unimodular if  $A$  has integral coefficients and the polytope  $\mathcal{P} = \{x \in \mathbb{R}_+^m \mid Ax = b\}$ , has only integral vertices whenever  $b$  is in the lattice spanned by the columns of  $A$ ).

The following lemma implies that it is enough to describe our counting formulas and techniques for networks without restricted capacities on the arcs and that have no directed cycles; these are called *acyclic uncapacitated networks*:

**Lemma 1** *Given a network  $G$  with  $n$  nodes and  $m$  arcs, with capacity  $c$  and excess function  $b$ , there is an acyclic uncapacitated network  $\widehat{G}$  with  $n + m$  nodes,  $2m$  arcs, and excess function  $\widehat{b}$  (a linear combination of  $b, c$ ) such*



(A)



(B)

Figure 1: counting all maximum flows (part B) of an specific network (part A)

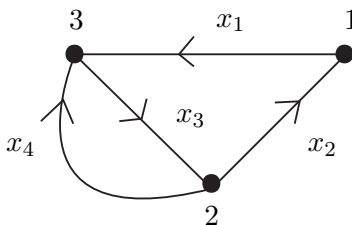
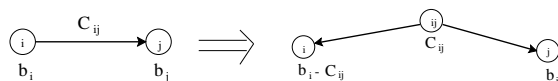


Figure 2: Network  $G_1$  with nodes 1, 2, 3, edges  $x_1, x_2, x_3, x_4$ , excess function  $b_1 = 3, b_2 = -2, b_3 = -1$  and capacity function  $c_{x_1} = 1, c_{x_2} = 1, c_{x_3} = 2, c_{x_4} = 1$ .

that the integral flows in both networks are in bijection. The network  $\widehat{G}$  is obtained from  $G$  by replacing each arc by two new arcs as illustrated in the figure below.



**Proof:** For the network  $G$  with capacity  $c$ , the flows are the solutions of  $\Phi_G x = b, \quad 0 \leq x \leq c$  (\*). There is a clear bijection (a projection) between the solutions of system (\*) and the solutions of

$$\begin{bmatrix} \Phi_G & 0 \\ I & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}, \quad x, y \geq 0.$$

The new enlarged matrix is denoted  $\widehat{\Phi}_G$  and called the *extended network matrix*. To the network  $G$ , with its set of nodes  $V$  and its set of arcs  $E$ , we have associated the new network  $\widehat{G}$ . The set of nodes of  $\widehat{G}$  is the disjoint union of the two sets  $V$  and  $E$  and the network  $\widehat{G}$  is obtained from  $G$  by replacing each arc by two new arcs as illustrated in the figure above: that is to each  $f \in E$  is associated  $f_1 = [f, j]$  and  $f_2 = [f, i]$  where  $i$  is the tail of  $f$  and  $j$  is the head of  $f$ . Both arrows  $f_1$  and  $f_2$  are oriented with their common tail  $\{f\}$  belonging to the set  $E$  and their heads  $\{i\}$  and  $\{j\}$  in the set  $V$ . Thus  $\widehat{G}$  is a directed graph, with  $n + m$  nodes and  $2m$  arcs. If  $b \in \mathbb{R}^n$  is the excess vector and  $c \in \mathbb{R}^m$  is the capacity vector of the network  $G$ , we define a new excess vector  $\hat{b} \in \mathbb{R}^n \oplus \mathbb{R}^m$ . The projection of  $\hat{b}$  on  $\mathbb{R}^n$  has coordinates  $\hat{b}_i = b_i - \sum_{f \in E | \text{tail}(f)=i} \text{capacity}(f)$ . The projection of  $\hat{b}$  of  $\mathbb{R}^m$

is the capacity vector  $c$ . Let  $T_G$  be the matrix with one column per arc and one row per node defined as follows. The column corresponding to an arc has just *one* non zero entry: the *tail* of the arrow receives a 1. Then  $\Phi_G - T_G$  is the matrix with one column per arc and just the *head* of the arrow receives a  $-1$ . All other entries are 0. Thus

$$\begin{bmatrix} I & -T_G \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi_G & 0 \\ I & I \end{bmatrix} = \begin{bmatrix} \Phi_G - T_G & -T_G \\ I & I \end{bmatrix}$$

is equal by definition to the matrix  $\Phi_{\hat{G}}$ , the  $m$  first columns corresponding to new arrows  $f_1$ , and the last columns corresponding to new arrows  $f_2$ . Solutions of

$$\begin{bmatrix} \Phi_G & 0 \\ I & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$$

are solutions of the equation

$$\Phi_{\hat{G}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & -T_G \\ 0 & I \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \hat{b}.$$

Thus we obtain a bijection between feasible flows of the network  $G$  with feasible flows of the uncapacitated network  $\hat{G}$ . The correspondence assigns to the arc  $f_1$  the value  $x_f$  and to the arc  $f_2$  the value  $y_f = c_f - x_f$ .

**Example 2** Consider the network  $G_1$  of Figure 2. Using the transformation of the previous lemma we would pass from the capacitated network to the uncapacitated network  $G_2$  illustrated in Figure 3 and the excesses of its nodes are in the caption.

Because of Lemma 1 and due to interesting applications in representation theory, it makes sense to focus our efforts on the special case of uncapacitated acyclic graphs, and we do so on Section 2. A particular case is what representation theorists would call the *Kostant partition functions* associated to the complete graph  $K_n$  with  $n$  nodes. There are many ways to induce an acyclic orientation to the complete graph, here we take the following convention of orientation: whenever there is an edge of the graph  $G$  between  $i$  and  $j$ , with  $i < j$ , then we direct the arrow from  $i$  to  $j$ .

**Example 3** Consider the complete graph  $G$  on vertices  $1, 2, 3, 4$ . In this case, each vertex is joined to all the others and the incidence matrix of the network is

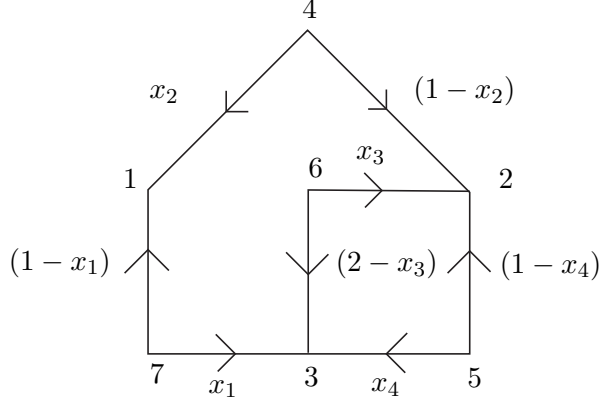


Figure 3: Network  $G_2$  with excess  $\{2, -4, -3, 1, 1, 2, 1\}$  and no capacities resulting from capacitated network in Figure 2.

$$\Phi_G = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}.$$

Another example of flow polytope is the *Pitman-Stanley polytope* [17] that is constructed starting from a multiple edge graph:

**Example 4** Consider the graph with vertices  $(1, \dots, n)$  and edges from  $\{i, i+1\}$  and  $\{i, n\}$  and the last edge  $\{n-1, n\}$  of multiplicity two. In the case  $n = 3$  then

$$\Phi_G = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix}.$$

Also, within the class of flow polytopes, we will be investigating the famous *transportation polytopes* [19]. These polytopes are usually described in terms of  $m$  by  $n$  real matrices (denoted here by  $M_{m,n}(\mathbb{R})$ ): Fix  $c = (c_1, \dots, c_n) \in \mathbb{R}_+^n$  and  $d = (d_1, \dots, d_m) \in \mathbb{R}_+^m$  such that  $\sum_{i=1}^m d_i = \sum_{i=1}^n c_i$  and define  $T_{m,n}(d, c)$  as the set

$$\left\{ X = \{x_{ij}\} \in M_{m,n}(\mathbb{R}) \ ; \ \begin{array}{ll} x_{ij} \geq 0, 1 \leq i \leq m, 1 \leq j \leq n & \\ \sum_k x_{ik} = d_i & 1 \leq i \leq m \\ \sum_k x_{kj} = c_j & 1 \leq j \leq n \end{array} \right\}.$$

Then  $T_{m,n}(d, c)$  is a polytope called the *transportation polytope* associated for the vectors  $d, c$ . We can easily see that this is another flow polytope over a complete bipartite network  $K_{m,n}$  (see Figure 4 where the first  $m$  nodes receive excess values  $(d_1, \dots, d_m)$  and the  $n$  nodes in the second block receive the excess values  $(-c_1, -c_2, \dots, -c_n)$ . The arcs are oriented from the first block to the second. In the family of transportation polytopes there is a distinguished member, the *Birkhoff polytope* that has been extensively studied (see for instance the references in the recent paper [4]).

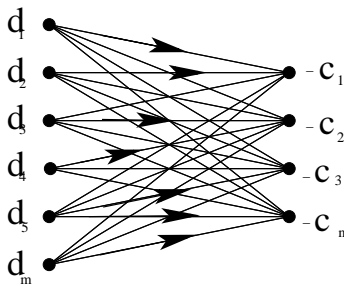


Figure 4: The transportation polytopes are network polytopes of complete bipartite graphs

It is well-known that the counting formulas of integer flows in a network come in piecewise polynomial functions (see [8, 22]). It is therefore of interest to understand the regions of validity of each polynomial formula, the so called *chambers*. We dedicate in Section 3 some effort to understand the structure of the chambers and how to determine the number of chambers. The question of how many chambers are possible was first raised in [15]. The combinatorial investigations of the chambers for the partition functions was initiated by [1]. See also [10].

## 1 Formulas for the volume and the number of integral points of flow polytopes.

In this section, we outline the principles used in the algorithms we implemented for counting integer flows. The method is valid for general convex polytopes [2, 23], thus we describe things in a general setting when possible. In Section 2, we will use particular properties of flow polytopes associated with graphs to calculate the counting formulas.

Let  $\Phi$  be an integral  $r$  by  $N$  matrix with columns vectors  $\phi_1, \dots, \phi_N$ . Let

$b$  be an  $r$ -dimensional column vector and  $\mathcal{P} = \{x \in \mathbb{R}_+^N \mid \Phi x = b\}$ , the rational convex polytope associated to  $\Phi$  and  $b$ . We assume that  $b$  is in the cone  $C(\Phi)$  spanned by the non-negative linear combinations of columns  $\phi_1, \phi_2, \dots, \phi_N$  of  $\Phi$ . Without loss of generality we may assume that  $\text{rank}(\Phi) = r$ . If this is not the case, take the subspace of  $\mathbb{R}^r$  generated by the columns of our matrix and rewrite the polytope in term of an appropriate rank  $k$  matrix of dimension  $k$  by  $N$ . For example, for the network polytopes the matrices are not of full rank but deleting one of the rows turns them into one.

In what follows we assume that  $\text{kernel}(\Phi) \cap \mathbb{R}_+^N = \{0\}$ . Then 0 is not in the convex hull of the vectors  $\phi_k$  and the cone  $C(\Phi)$  is an acute cone in  $\mathbb{R}^r$ . For  $a \in \mathbb{R}^r$  we denote by

$$P(\Phi, a) = \{(x_1, x_2, \dots, x_N) \in \mathbb{R}_+^N \mid \sum_{j=1}^N x_j \phi_j = a\}.$$

It is obvious that  $P(\Phi, a)$  is a convex polytope determined by the matrix  $\Phi$ . Define

$$v(\Phi, a) = \text{volume}(P(\Phi, a)).$$

If  $\Phi$  spans a lattice in  $\mathbb{R}^r$  and  $a$  belongs to this lattice, then define

$$k(\Phi, a) = |P(\Phi, a) \cap \mathbb{Z}^N|.$$

Thus  $k(\Phi, a)$  is the number of solutions  $(x_1, x_2, \dots, x_N)$ , in non-negative integers  $x_j$ , of the equation  $\sum_{j=1}^N x_j \phi_j = a$ . The function  $k(\Phi, a)$  is called the *vector partition function* associated to  $\Phi$ . The name partition comes from the fact that if  $\Phi = [e_1, e_1, \dots, e_1]$  is the sequence of  $N$  times the standard basis vector of  $\mathbb{R}$ , then  $P_\Phi(ae_1) \cap \mathbb{Z}^N$  is the set of solutions of the equation  $a = x_1 + x_2 + \dots + x_N$ , that is the partition of the integer  $a$  in  $N$  integers. In particular, the function  $a \rightarrow k(\Phi, a)$  depends strongly of the multiplicities in the system  $\Phi$ . The basic starting observation is

**Theorem 5** *Let  $z \in \mathbb{R}^r$  denote a vector in the dual cone to  $C(\Phi)$ . Then,*

$$\int_{C(\Phi)} v(\Phi, a) e^{-\langle a, z \rangle} da = \frac{1}{\prod_{\phi \in \Phi} \langle \phi, z \rangle},$$

$$\sum_{a \in C(\Phi) \cap \mathbb{Z}^r} k(\Phi, a) e^{-\langle a, z \rangle} = \frac{1}{\prod_{\phi \in \Phi} 1 - e^{-\langle \phi, z \rangle}}.$$

The goal is to compute the inverses of these two equations. The point is that one can write efficient formulas for the inversion of Laplace transforms in terms of residues. In the sequel, we will write indifferently  $\langle \phi, z \rangle$  or  $\phi(z)$ .



Let  $\Delta^+$  the set  $\{\Phi\}$ , this means the elements of  $\Phi$  are present *without multiplicities*. We define  $\Delta = \Delta^+ \cup -\Delta^+$ . A subset  $\sigma$  of  $\Delta^+$  is called a *basic* subset if  $\{\sigma\}$  form a vector space basis of  $\mathbb{R}^r$ . The *chamber complex* is the polyhedral subdivision of the cone  $C(\Delta^+)$  which is defined as the common refinement of the simplicial cones  $C(\sigma)$  running over all possible basic subsets of  $\Delta^+$ . The pieces of this subdivision are called *chambers*. We will discuss the chambers in detail, specially how to compute the chambers, in Section 3. The important fact to remember is that for each chamber there is a quasipolynomial formula for  $k(\Phi, a)$  and we explain now how to derive the formula on a given chamber.

Each  $\phi \in \Delta$  determines a linear form on  $\mathbb{C}^r$  and a complex hyperplane  $\{z \in \mathbb{C}^r | \phi(z) = 0\}$  in  $\mathbb{C}^r$ . Consider the hyperplane arrangement

$$\mathcal{H}_{\mathbb{C}} = \bigcup_{\phi \in \Delta} \{z \in \mathbb{C}^r | \phi(z) = 0\}$$

and let  $R_{\Delta}$  denote the space of rational functions of  $z \in \mathbb{C}^r$  with poles on  $\mathcal{H}_{\mathbb{C}}$ . A function in  $R_{\Delta}$  can be written  $P(z) / \prod_{\phi \in \Delta} \phi(z)^{n_{\phi}}$  where  $P$  is a polynomial function on  $r$  complex variables and  $n_{\phi}$  are non negative integers. A subset  $\sigma$  of  $\Delta$  is called a **basic subset of  $\Delta$** , if the elements  $\phi \in \sigma$  form a vector space basis for  $\mathbb{R}^r$ . For such  $\sigma$ , set

$$f_{\sigma}(z) := \frac{1}{\prod_{\phi \in \sigma} \phi(z)}.$$

After a linear change of coordinates, the function  $f_{\sigma}$  is simply  $\frac{1}{z_1 z_2 \dots z_r}$  and we denote by  $S_{\Delta}$  the subspace of  $R_{\Delta}$  spanned by such “simple” elements  $f_{\sigma}$ . Elements  $f_{\sigma}$  are, in general, *not* linearly independent, as we see in the example below.

**Example 6** Let  $\Delta^+$  be the set  $\Delta^+ = \{e_1, e_2, (e_1 - e_2)\}$ . Then we have the linear relation

$$\frac{1}{xy} = \frac{1}{y(x-y)} - \frac{1}{x(x-y)}$$

between elements  $f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3}$  with  $\sigma_1 = \{e_1, e_2\}$ ,  $\sigma_2 = \{e_1, (e_1 - e_2)\}$  and  $\sigma_3 = \{e_2, (e_1 - e_2)\}$  basic subsets of  $\Delta^+$ .

Partial differentiation  $\partial_i$  preserves the space  $R_{\Delta}$ . The key result we need is that there is a well-defined decomposition of  $R_{\Delta}$  under the action of partial differentiations, a free module part generated by the basic rational functions  $f_{\sigma}$ , and a torsion module part, which is unnecessary for calculations and can be neglected.

**Theorem 7 (Brion-Vergne [9])** *The vector space  $S_\Delta$  is contained in the homogeneous component of degree  $-r$  of  $R_\Delta$  and we have the direct sum decomposition*

$$R_\Delta = S_\Delta \oplus \left( \sum_{i=1}^r \partial_i R_\Delta \right).$$

We call the projection map

$$Tres_\Delta : R_\Delta \rightarrow S_\Delta$$

according to this decomposition the **total residue map**.

The projection  $Tres_\Delta(f)$  of a function  $f$  with poles on the union of hyperplanes  $\mathcal{H}_\mathbb{C}$  depends only of the smallest hyperplane arrangement  $\mathcal{H}'_\mathbb{C}$  containing the poles of  $f$ . Therefore we just denote by  $Tres(f)$  the residue of a rational function  $f$  with denominator product of linear forms.

**Example 8** *Observe that if we work in  $\mathbb{R}^1$  and  $\Delta = \{\pm e_1\}$ , then  $R_\Delta$  is the space of Laurent series*

$$L = \{f(z) = \sum_{k \geq -q} a_k z^k\}.$$

The total residue of a function  $f(z) \in L$  is the function  $\frac{a_{-1}}{z}$ . The usual residue, denoted  $Res_{z=0} f$ , is the constant  $a_{-1}$ .

We denote by  $\hat{R}_\Delta$  the obvious extension of  $R_\Delta$ , when we replace the space of polynomial functions on  $r$  variables by the space of formal power series on  $r$  variables. Let  $F : \mathbb{C}^r \rightarrow \mathbb{C}^r$  be an analytic map, such that  $F(0) = 0$  and preserving each hyperplane  $\phi = 0$ . If  $f \in \hat{R}_\Delta$ , the function  $(F^* f)(z) = f(F(z))$  is again in  $\hat{R}_\Delta$ . Let  $Jac(F)$  be the Jacobian of the map  $F$ . The function  $Jac(F)$  is calculated as follows: write  $F(z) = (F_1(z_1, z_2, \dots, z_r), \dots, F_r(z_1, z_2, \dots, z_r))$ . Then  $Jac(F) = \det((\frac{\partial}{\partial z_i} F_j)_{i,j})$ . We assume  $Jac(F)(0)$  does not vanish. For any  $f$  in  $\hat{R}_\Delta$ , the following change of variable formula, which will be useful in our calculations later on, holds in  $S_\Delta$ :

$$Tres(f) = Tres(Jac(F)(F^* f)).$$

Note that the total residue of a rational function is again a rational function. By definition, this function can be expressed as a linear combination

of the simple fractions  $f_\sigma(z)$ . If  $f \in S_\Delta$ , then  $Tres(f)$  is just equal to  $f$ . We also know that  $Tres$  vanishes on homogeneous rational functions of degree  $m$ , whenever  $m \neq -r$  and that  $Tres$  vanishes on derivatives. If  $f = \frac{P}{\prod_k \langle \phi_k, z \rangle}$  (with  $P$  a polynomial in  $r$  variables) has a denominator product of linear forms  $\langle \phi_k, z \rangle$  which do not generate, then it is easy to see that  $f$  is a derivative and the total residue of  $f$  is equal to 0. We are now ready to fix our notation and recall the key formulas.

**Definition 9** For  $a \in \mathbb{R}^r$ , define

$$J_\Phi(a)(z) = Tres\left(\frac{e^{\langle a, z \rangle}}{\prod_{k=1}^N \langle \phi_k, z \rangle}\right) = \frac{1}{(N-r)!} Tres\left(\frac{\langle a, z \rangle^{N-r}}{\prod_{k=1}^N \langle \phi_k, z \rangle}\right)$$

and its “periodic” version

$$K_\Phi(a)(z) = Tres\left(\frac{e^{\langle a, z \rangle}}{\prod_{k=1}^N 1 - e^{-\langle \phi_k, z \rangle}}\right).$$

The equality:

$$Tres\left(\frac{e^{\langle a, z \rangle}}{\prod_{k=1}^N \langle \phi_k, z \rangle}\right) = \frac{1}{(N-r)!} Tres\left(\frac{\langle a, z \rangle^{N-r}}{\prod_{k=1}^N \langle \phi_k, z \rangle}\right)$$

follows right away from the fact that the total residue vanishes on homogeneous rational functions of degree  $m$ , whenever  $m \neq -r$ .

By definition,  $J_\Phi(a)(z)$  and  $K_\Phi(a)(z)$  are rational functions of  $z$  homogeneous in  $z$  of degree  $-r$ . They are polynomial functions of  $a$  of degree  $N-r$  and the homogeneous part in  $a$  of degree  $(N-r)$  in  $K_\Phi(a)(z)$  is  $J_\Phi(a)(z)$ .

**Example 10** Let us compute  $J_\Phi(a)(z)$  and  $K_\Phi(a)(z)$  in the case of the Pitman-Stanley polytope associated to  $\Phi_G$  of Example 4. The matrix  $\Phi_G$  is a 3 by 4 matrix of rank 2. Deleting the last row leads to

$$\Phi = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}.$$

$$\begin{aligned} \text{Then } J_\Phi(a_1, a_2)(z_1, z_2) &= Tres\left(\frac{e^{(a_1 z_1 + a_2 z_2)}}{(z_1 - z_2) z_1 z_2^2}\right) \text{ is} \\ &= \frac{1}{2!} Tres\left(\frac{(a_1 z_1 + a_2 z_2)^2}{(z_1 - z_2) z_1 z_2^2}\right) \\ &= \frac{a_1^2}{2} Tres\left(\frac{z_1^2}{(z_1 - z_2) z_1 z_2^2}\right) + a_1 a_2 Tres\left(\frac{z_1 z_2}{(z_1 - z_2) z_1 z_2^2}\right) + \frac{a_2^2}{2} Tres\left(\frac{z_2^2}{(z_1 - z_2) z_1 z_2^2}\right) \\ &= \frac{a_1^2}{2} Tres\left(\frac{z_1}{(z_1 - z_2) z_2^2}\right) + a_1 a_2 Tres\left(\frac{1}{(z_1 - z_2) z_2}\right) + \frac{a_2^2}{2} Tres\left(\frac{1}{(z_1 - z_2) z_1}\right). \end{aligned}$$

Now  $\frac{1}{(z_1-z_2)z_2}$  and  $\frac{1}{(z_1-z_2)z_1}$  are simple elements so that they are equal to their respective total residue. To compute the total residue of  $\frac{z_1}{(z_1-z_2)z_2^2}$ , we write  $z_1$  as a linear combination of linear forms in the denominator, in order to reduce the degree of denominator:

$$\frac{z_1}{(z_1-z_2)z_2^2} = \frac{(z_1-z_2) + z_2}{(z_1-z_2)z_2^2} = \frac{1}{z_2^2} + \frac{1}{(z_1-z_2)z_2}.$$

The total residue of  $\frac{1}{z_2^2}$  is 0, as  $\frac{1}{z_2^2} = -\frac{\partial}{\partial z_2} \frac{1}{z_2}$  is a derivative, thus  $Tres\left(\frac{z_1}{(z_1-z_2)z_2^2}\right) = \frac{1}{(z_1-z_2)z_2}$ . We finally obtain:

$$J_{\Phi}(a_1, a_2)(z_1, z_2) = \frac{1}{2} \frac{a_1^2 + 2a_1a_2}{(z_1-z_2)z_2} + \frac{1}{2} \frac{a_2^2}{(z_1-z_2)z_1}.$$

We now compute:

$$K_{\Phi}(a_1, a_2)(z_1, z_2) = Tres \left( \frac{e^{(a_1z_1+a_2z_2)}}{(1-e^{-(z_1-z_2)})(1-e^{-z_1})(1-e^{-z_2})^2} \right).$$

This is

$$Tres \left( \frac{1}{(z_1-z_2)z_1z_2^2} e^{(a_1z_1+a_2z_2)} \frac{(z_1-z_2)}{(1-e^{-(z_1-z_2)})} \frac{z_1}{(1-e^{-z_1})} \frac{z_2^2}{(1-e^{-z_2})^2} \right).$$

We replace the analytic function

$$e^{(a_1z_1+a_2z_2)} \frac{(z_1-z_2)}{(1-e^{-(z_1-z_2)})} \frac{z_1}{(1-e^{-z_1})} \frac{z_2^2}{(1-e^{-z_2})^2}$$

by its Taylor series at  $z_1 = 0, z_2 = 0$ , and keep only its term  $N(a_1, a_2)(z_1, z_2)$  of homogeneous degree 2 in  $z_1, z_2$  which is

$$\left(\frac{5}{12} + a_1 + \frac{1}{2}a_1^2\right)z_1^2 + \left(\frac{7}{12} + a_2 + \frac{1}{2}a_1 + a_1a_2\right)z_1z_2 + \left(\frac{1}{2}a_2 + \frac{1}{2}a_2^2\right)z_2^2.$$

Thus  $K_{\Phi}(a_1, a_2)(z_1, z_2)$  is equal to

$$Tres \left( \frac{N(a_1, a_2)(z_1, z_2)}{(z_1-z_2)z_1z_2^2} \right).$$

Arguing as for  $J_{\Phi}$ , we finally obtain that  $K_{\Phi}(a_1, a_2)(z_1, z_2)$  is equal to

$$\frac{1}{2} \frac{a_1^2 + 2a_1a_2 + 3a_1 + 2a_2 + 2}{(z_1-z_2)z_2} + \frac{1}{2} \frac{a_2^2 + a_2}{(z_1-z_2)z_1}.$$

We are now ready to write the formulas to compute the volume and number of integral points. See [2, Section 2] for details. To each chamber  $\mathfrak{c}$  of the subdivision of  $C(\Delta^+)$  is associated a linear form  $f \rightarrow \langle\langle \mathfrak{c}, f \rangle\rangle$  on  $S_\Delta$ . If the system  $\Phi$  is unimodular, as is the case for networks, it takes value 1 or 0 on  $f_\sigma$  whether or not  $\mathfrak{c}$  is contained in  $C(\sigma)$ .

**Theorem 11 (Baldoni-Vergne [2])** *Let  $\mathfrak{c}$  be a chamber of the subdivision of  $C(\Delta^+)$*

1. *For  $a \in \bar{\mathfrak{c}}$ , the volume of  $P(\Phi, a)$  is given by*

$$v(\Phi, a) = \langle\langle \mathfrak{c}, J_\Phi(a) \rangle\rangle.$$

2. *If the system  $\Phi$  is unimodular, then for  $a \in \bar{\mathfrak{c}} \cap \mathbb{Z}\Phi$ , the number of integral points in  $P(\Phi, a)$  is given by*

$$k(\Phi, a) = \langle\langle \mathfrak{c}, K_\Phi(a) \rangle\rangle.$$

3. *The function  $a \mapsto v(\Phi, a)$  is polynomial on a chamber  $\mathfrak{c}$ .*
4. *If the system  $\Phi$  is unimodular, as is the case for networks, the Ehrhart function  $a \mapsto k(\Phi, a)$  is polynomial on a specified neighborhood of a chamber  $\mathfrak{c}$ .*

A more general formula for arbitrary  $\Phi$  spanning a lattice  $\mathbb{Z}\Phi$  in  $\mathbb{R}^r$  is given in [23]. Now, the question is how to apply these two formulas for the computations with flow polytopes. The calculation of total residues will simplify considerably.

## 2 Counting Integer Flows in Networks

In this section we will focus on flow polytopes for acyclically directed graphs. We already justified in the introduction this makes sense, as other networks can be reduced to acyclic uncapacitated networks. Consider a  $r + 1$  real dimensional vector space. Let  $A_r^+$  (the positive root system of  $A_r$ ) be defined by

$$A_r^+ = \{(e_i - e_j) | 1 \leq i < j \leq (r + 1)\}.$$

Consider  $E_r$  the vector space spanned by the elements  $(e_i - e_j)$ , then

$$E_r = \{a \in \mathbb{R}^{r+1} | a = a_1 e_1 + \dots + a_r e_r + a_{r+1} e_{r+1} \text{ with } a_1 + a_2 + \dots + a_r + a_{r+1} = 0\}.$$

The vector space  $E_r$  is of dimension  $r$  and the map

$$(1) \quad f : \mathbb{R}^r \longrightarrow E_r$$

defined by

$$a = (a_1, a_2, \dots, a_r) \longmapsto \mathbf{a} = a_1 e_1 + \dots + a_r e_r - (a_1 + \dots + a_r) e_{r+1}$$

explicitly provides an isomorphism of  $E_r$  with the Euclidean space  $\mathbb{R}^r$ . Let, as before,  $\Phi = \{\phi_1, \phi_2, \dots, \phi_N\}$  denote a sequence of non-zero linear forms belonging to  $A_r^+$ . We assume that the vector space spanned by  $\Phi$  is  $E_r$ . This sequence is completely specified by the multiplicity  $m_{i,j}$  of the vector  $e_i - e_j$  in  $\Phi$ . Explicitly for the transportation polytope  $T_{m,n}(d, c)$ , if we denote by  $\Phi_{m,n} \subset A_{m+n-1}^+$  the roots associated to it, then we have  $\Phi_{m,n} = \{(e_i - e_j) | 1 \leq i < m, m+1 \leq j \leq m+n\}$  and thus  $m_{i,j} = 1$  if  $1 \leq i \leq m, m+1 \leq j \leq m+n, m_{i,j} = 0$  otherwise.

It is clear that the polytope  $P(\Phi, a)$  is the polytope associated to the uncapacitated network with  $(r+1)$  nodes, where the arc  $i \mapsto j$  ( $i < j$ ) appears  $m_{i,j}$  times ( $m_{i,j}$  can be 0 for some arcs), and with excess function  $a_i$  at each node  $1, 2, \dots, r$  and  $-(a_1 + a_2 + \dots + a_r)$  at the last node  $r+1$ . Indeed we have seen in Remark 3 that the columns of the matrix corresponding to  $P(\Phi, a)$  are vectors of the form  $e_i - e_j$  for some  $i$  and  $j$ .

The hyperplane arrangement (setting  $z_{r+1} = 0$ ) generated by  $A_r^+$  is given by the following set of hyperplanes:

$$\{z_i | 1 \leq i \leq r\} \cup \{(z_i - z_j) | 1 \leq i < j \leq r\}.$$

A function in  $R_{A_r}$  is thus a rational function  $f(z_1, z_2, \dots, z_r)$  on  $\mathbb{C}^r$ , with poles on the hyperplanes  $z_i = z_j$  or  $z_i = 0$ . The following result is proved by induction in [2], Proposition 14.

**Lemma 12** *Let  $\Sigma_r$  be the set of permutations on  $\{1, 2, \dots, r\}$  and  $f_\pi, f_w, w \in \Sigma_r$  be defined by*

$$f_\pi(z_1, z_2, \dots, z_r) = \frac{1}{(z_1 - z_2)(z_2 - z_3) \cdots (z_{r-1} - z_r) z_r}$$

and

$$f_w(z_1, \dots, z_r) = w \cdot f_\pi(z_1, \dots, z_r) = \frac{1}{(z_{w(1)} - z_{w(2)})(z_{w(2)} - z_{w(3)}) \cdots (z_{w(r-1)} - z_{w(r)}) z_{w(r)}}$$

then

$$(2) \dim S_{A_r} = r!$$

and

$$(3) \{f_w(z_1, \dots, z_r) = w \cdot f_\pi(z_1, \dots, z_r), \quad w \in \Sigma_r\}$$

is a basis for  $S_{A_r}$

The cone  $C(A_r^+)$  generated by positive roots is the cone  $a_1 \geq 0, a_1 + a_2 \geq 0, \dots, a_1 + a_2 + \dots + a_r \geq 0$ . We denote by  $\mathfrak{c}^+$  the open set of  $C(A_r^+)$  defined by

$$\mathfrak{c}^+ = \{a \in C(A_r^+) \mid a_i > 0, i = 1, \dots, r\}.$$

It is a chamber of our subdivision, and will be called the *nice chamber*. The importance of this chamber is that its “permutations” form a “basis” for the formulas that express volume and number of integral points. If  $\mathfrak{c}$  is a chamber for  $C(\Phi)$  then there exists a unique chamber of  $C(A_r^+)$  that contains  $\mathfrak{c}$ .

**Definition 13 ([2])** Let  $m_{i,j}$  ( $i < j$ ) be the multiplicity of the vector  $e_i - e_j$  in  $\Phi$  (i.e. this is the number of times the arc  $i, j$  is present in the network). Let  $N = \sum_{i,j} m_{i,j}$  the total number of arcs. We explicitly write down the functions  $J_\Phi(a)$  and  $K_\Phi(a)$  for our choice of  $\Phi, a$ . Recalling that  $z_{r+1} = 0$ , we have that

$$\begin{aligned} \bullet J_\Phi(a)(z_1, \dots, z_r) &= \frac{1}{(N-r)!} \text{Tres} \left( \frac{(a_1 z_1 + \dots + a_r z_r)^{N-r}}{z_1^{m_{1,r+1}} z_2^{m_{2,r+1}} \dots z_r^{m_{r,r+1}} \prod_{1 \leq i < j \leq r} (z_i - z_j)^{m_{ij}}} \right), \\ \bullet K_\Phi(a)(z_1, \dots, z_r) &= \text{Tres} \left( \frac{e^{a_1 z_1} e^{a_2 z_2} \dots e^{a_r z_r}}{\prod_{i=1}^r (1 - e^{-z_i})^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (1 - e^{-(z_i - z_j)})^{m_{ij}}} \right). \end{aligned}$$

We now write these functions in two specific examples.

**Example 14** We consider the polytope associated to a complete bipartite graph with 3 nodes on each side. Recall that in this case the matrix that determines the polytope is given by the vectors  $\Phi = \{e_1 - e_4, e_1 - e_5, e_1 - e_6, e_2 - e_4, e_2 - e_5, e_2 - e_6, e_3 - e_4, e_3 - e_5, e_3 - e_6\}$ . So

$$\begin{cases} m_{i,j} = 1 & \text{if } 1 \leq i \leq 3 \text{ and } 4 \leq j \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

and

- $J_{\Phi}(a)(z_1, \dots, z_5) = \frac{1}{4!} \text{Tr}es \left( \frac{(a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4 + a_5 z_5)^4}{z_1 z_2 z_3 \prod_{\substack{1 \leq i < j \leq 3 \\ 4 \leq j \leq 5}} (z_i - z_j)} \right),$
- $K_{\Phi}(a)(z_1, \dots, z_5) = \text{Tr}es \left( \frac{e^{a_1 z_1} e^{a_2 z_2} e^{a_3 z_3} e^{a_4 z_4} e^{a_5 z_5}}{\prod_{i=1}^3 (1 - e^{-z_i}) \prod_{\substack{1 \leq i < j \leq 3 \\ 4 \leq j \leq 5}} (1 - e^{-(z_i - z_j)})} \right).$

**Example 15** We consider the polytope determined by the complete graph  $K_5$ , in other words  $\Phi = A_4^+$ . We obtain

- $J_{\Phi}(a)(z_1, \dots, z_4) = \frac{1}{6!} \text{Tr}es \left( \frac{(a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4)^6}{z_1 z_2 z_3 z_4 \prod_{1 \leq i < j \leq 4} (z_i - z_j)} \right),$
- $K_{\Phi}(a)(z_1, \dots, z_4) = \text{Tr}es \left( \frac{e^{a_1 z_1} e^{a_2 z_2} e^{a_3 z_3} e^{a_4 z_4}}{\prod_{i=1}^4 (1 - e^{-z_i}) \prod_{1 \leq i < j \leq 4} (1 - e^{-(z_i - z_j)})} \right).$

In handling the formulas that we have for computing the volume and the number of integral points, the first problem is that of computing the total residue. This is in general a very difficult task. On the other hand, as we have seen, there is a very nice basis in  $S_{A_r}$  and this will allow us to rewrite the formulas in terms of iterated residue, which are certainly more tractable. The point is that one needs to find some, but not all, simplicial cones that contain the chamber determined by  $a$ . This is a step that allows the complexity of the algorithm to be reduce. We are now going to introduce the iterated residue for  $A_r$ .

Recall that, via the identification (1) of  $E_r$  with  $\mathbb{R}^r$ , a function in  $R_{A_r}$  is a rational function  $f(z_1, z_2, \dots, z_r)$  on  $\mathbb{C}^r$ , with poles on the hyperplanes  $z_i = z_j$  or  $z_i = 0$ . For a permutation  $\sigma \in \Sigma_r$  define the linear form on  $R_{A_r}$

$$\begin{aligned} \text{Ires}_{z=0}^{\sigma} f &= \text{Res}_{z_{\sigma(1)}=0} \text{Res}_{z_{\sigma(2)}=0} \cdots \text{Res}_{z_{\sigma(r)}=0} f(z_1, z_2, \dots, z_r) = \\ &\text{Res}_{z_1=0} \text{Res}_{z_2=0} \cdots \text{Res}_{z_r=0} f(z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, \dots, z_{\sigma^{-1}(r)}). \end{aligned}$$

In particular for  $\sigma = id$  the linear form  $f \mapsto \text{Ires}_{z=0} f$  defined by

$$\begin{aligned} &\text{Ires}_{z=0} f \\ &= \text{Res}_{z_1=0} \text{Res}_{z_2=0} \cdots \text{Res}_{z_r=0} f(z_1, z_2, \dots, z_r) \end{aligned}$$

is called the *iterated residue*.

**Remark**

- the linear form  $f \mapsto \text{Ires}_{z=0}^{\sigma} f$  on  $R_{A_r}$  induces a linear form on  $S_{A_r}$ , since it vanishes on the vector space of derivatives  $\sum_{i=1}^r \partial_i R_{A_r}$ .



- $Ires_{z=0}^\sigma f_w = \delta_w^\sigma$ .
- the  $r!$  linear forms  $Ires_{z=0}^\sigma f, \sigma \in \Sigma_r$ , on  $S_{A_r}$  are dual to the basis  $f_w$ .

Iterated residues are easier to understand, and we will see shortly how to use them in connection to our formulas. Let  $w \in \Sigma_r$  and  $n(w)$  be the number of elements  $i$  such that  $w(i) > w(i+1)$  (this is called the number of *descents* of the permutation  $w$  in [20]). We denote by  $C_w^+ \subset C(A_r^+)$  the simplicial cone generated by the vectors

$$\epsilon(1)(e_{w(1)} - e_{w(2)}), \epsilon(2)(e_{w(2)} - e_{w(3)}), \dots, \epsilon(r-1)(e_{w(r-1)} - e_{w(r)}), (e_{w(r)} - e_{r+1}),$$

where  $\epsilon(i)$  is 1 or -1 depending whether  $w(i) < w(i+1)$  or not. When  $w = 1$ , then  $C_1 = C(A_r^+)$ . The following lemma is easy to see.

**Lemma 16** *Let  $a = \sum_{j=1}^{r+1} a_j e_j$  in  $E_r$ . The cone  $C_w^+ \subset E_r$  is given by the following system of inequalities  $\sum_{j=1}^i a_{w(j)} \geq 0$ , for all  $i$  such that  $w(i) < w(i+1)$ , but  $\sum_{j=1}^i a_{w(j)} \leq 0$  if  $w(i) > w(i+1)$ .*

From Theorem 11 we obtain:

**Theorem 17 ([2])** *Let  $\mathfrak{c}$  be a chamber of  $C(\Phi)$ . Consider the set of elements  $w \in \Sigma_r$  such that  $\mathfrak{c} \subset C_w^+$ . Then, for  $f \in S_{A_r}$ ,*

$$\langle \langle \mathfrak{c}, f \rangle \rangle = \sum_{w \in \Sigma_r, \mathfrak{c} \subset C_w^+} (-1)^{n(w)} Ires_{z=0} w^{-1} f.$$

*In particular for  $f = J_\Phi(a)$  we obtain*

**Formula 1:** *for  $a \in \bar{\mathfrak{c}}$ , we have*

$$v(\Phi, a) = \langle \langle \mathfrak{c}, J_\Phi(a) \rangle \rangle = \sum_{w \in \Sigma_r, \mathfrak{c} \subset C_w^+} (-1)^{n(w)} Ires_{z=0}^w J_\Phi(a).$$

The formula is a direct consequence of the fact that  $Ires_{z=0}^w$  is the dual basis of  $f_w$ . We have seen that to compute the number of integral points of our polytope we need to compute  $K_\Phi(a)$ . Let  $t_j = m_{j,j+1} + \dots + m_{j,r+1} - 1$ , where we recall that  $m_{i,j}$  is the multiplicity of the root  $e_i - e_j$  in  $\Phi$ . After a change of variable for the total residue, we obtain:

**Theorem 18** Let  $a = \sum_{i=1}^{r+1} a_i e_i$  in  $E_r \cap \mathbb{Z}^{r+1}$ . Let

$$f_\Phi(a)(z) = \frac{(1+z_1)^{a_1+t_1} (1+z_2)^{a_2+t_2} \dots (1+z_r)^{a_r+t_r}}{z_1^{m_{1,r+1}} z_2^{m_{2,r+2}} \dots z_r^{m_{r,r+1}} \prod_{1 \leq i < j \leq r} (z_i - z_j)^{m_{ij}}}.$$

Then **Formula 2:** for  $a \in \bar{\mathfrak{c}}$ ,

$$k(\Phi, a) = \sum_{w \in \Sigma_r, \mathfrak{c} \subset C_w^+} (-1)^{n(w)} \text{Res}_{z=0}^w f_\Phi(a).$$

We now want to give an even more explicit formulation of the above result suited to be directly implemented. For this purpose we need to introduce some more notations. For  $a \in E_r$ , let  $\text{def}(a)$  be defined by  $\text{def}(a) = a + \epsilon \sum_{\alpha \in \Phi} \alpha + \epsilon^2 (\sum_{i=1}^r e_i - r e_{r+1})$  with  $\epsilon = \frac{1}{2mr^2}$  and  $m$  the maximum of the multiplicities  $m_{ij}$ .

A wall of  $A_r^+$  is a hyperplane generated by  $r-1$  linearly independent elements of  $A_r^+$ . The cells in  $C(A_r^+) \setminus \mathcal{H}$  ( $\mathcal{H}$  being the set of hyperplanes for  $A_r^+$ ) are open cells, interior of polyhedral cones. We will call these open cells *topes*. We will say that  $a \in C(A_r^+)$  is *regular* if  $a$  is not on any wall for  $A_r^+$ . The walls of  $A_r^+$  are easily characterized since they are the kernel of a linear form as  $\sum_{i \in J} a_i$  where  $J$  is a subset of  $\{1, 2, \dots, r\}$ . It is then easy to decide whether a vector  $a$  is regular or not.

If  $a$  is a regular element we let  $\mathfrak{c}$  denote the unique chamber of  $C(A_r^+)$  containing it. Then the set  $Sp(a) = \{w \in \Sigma_r \mid \mathfrak{c} \subset C_w^+\}$  can be computed without explicit knowledge of the chamber. In fact one can easily see that the set  $Sp(a)$  consists of those  $w \in \Sigma_r$  that satisfy the following conditions:

$$\left. \begin{array}{l} \text{if } a_{w(1)} \geq 0 \text{ then} \\ \text{if } a_{w(1)} + a_{w(2)} \geq 0 \text{ then} \\ \dots \\ \text{if } a_{w(1)} + \dots + a_{w(i)} \geq 0 \text{ then} \\ \dots \\ \text{if } a_{w(1)} + \dots + a_{w(r-1)} \geq 0 \text{ then} \end{array} \right\} \begin{array}{l} w(1) < w(2) \text{ else } w(1) > w(2) \\ w(2) < w(3) \text{ else } w(2) > w(3) \\ \\ w(i) < w(i+1) \text{ else } w(i) > w(i+1) \\ \\ w(r-1) < w(r) \text{ else } w(r-1) > w(r) \end{array}$$

An element of  $Sp(a)$  will be called a *special permutation*.

Remark that if  $a_i \geq 0$  for all  $i \leq r$ , then  $a = \sum_{i=1}^r a_i e_i - (\sum_{i=1}^r a_i) e_{r+1}$  belongs to the closure of the *nice chamber*  $\mathfrak{c}^+$  and  $Sp(a) = \{id\}$ .

Now we can state Theorem 18 as follows:

**Theorem 19** Let  $\Phi \subset A_r^+$  be a system generating  $E_r$ . Let  $a = \sum_{i=1}^{r+1} a_i e_i \in E_r$ ,  $a_{r+1} = -(a_1 + \dots + a_r)$ ,  $a_i \in \mathbb{Z}$  and assume that  $a \in C(A_r^+)$ .

Write

$$f_{\Phi}(a_1, a_2, \dots, a_r)(z) = \frac{(1+z_1)^{a_1+t_1}(1+z_2)^{a_2+t_2} \dots (1+z_r)^{a_r+t_r}}{z_1^{m_{1,r+1}} z_2^{m_{2,r+2}} \dots z_r^{m_{r,r+1}} \prod_{1 \leq i < j \leq r} (z_i - z_j)^{m_{ij}}}.$$

Then

- **Formula 2A:** if  $a$  is regular then

$$k(\Phi, a) = \sum_{w \in Sp(a)} (-1)^{n(w)} \text{Ires}_{z=0}^w f_{\Phi}(a).$$

- **Formula 2B:** if  $a$  is not regular then

$$k(\Phi, a) = \sum_{w \in Sp(def(a))} (-1)^{n(w)} \text{Ires}_{z=0}^w f_{\Phi}(a).$$

**Remark** Formula 2B in the theorem follows by observing that the chamber containing the regular element  $def(a)$  contains  $a$  in its closure. The deformation has to be done with care to deal with some *border* cases. The following lemma, that we state for completeness, shows that the deformation with  $a_i$  integers is small enough to take care of such cases.

**Lemma 20** Given  $a \in C(A_r^+) \cap \mathbb{Z}^{r+1}$ , define  $def(a) := a + \epsilon \sum_{\alpha \in \Phi} \alpha + \epsilon^2 (\sum_{i=1}^r e_i - r e_{r+1})$ ,  $\epsilon = \frac{1}{2mr^2}$  where  $m$  is the maximum of the multiplicities  $m_{ij}$ . Then the following holds:

- $def(a)$  is regular, i.e. it belongs to a chamber.
- if  $\tau$  is a tope and  $a \in \tau$  then  $def(a) \in \tau$
- $a \in C(A_r^+)$  if and only if  $def(a) \in C(A_r^+)$
- In general if  $\Phi$  is a subset of  $A_r^+$ ,  $a \notin C(\Phi)$  if and only if  $def(a) \notin C(\Phi)$ .

For example, we obtain the following formula for the complete network  $K_{r+1}$  on  $r+1$  nodes, with excess vector  $a_1, a_2, \dots, a_r, a_{r+1} = -\sum_{i=1}^r a_i$ . In this case, the function  $k(A_r^+, a)$  is the so-called Kostant partition function and has special importance for the representation theory of the group  $GL(r+1, \mathbb{C})$ .

**Corollary 21** For  $a \in C(A_r^+) \cap \mathbb{Z}^{r+1}$ , the Kostant partition function is given by:

$$k(A_r^+, a) = \sum_{w \in Sp(a')} (-1)^{n(w)} \text{Ires}_{z=0}^w \frac{(1+z_1)^{a_1+r-1} (1+z_2)^{a_2+r-2} \dots (1+z_r)^{a_r}}{z_1 \cdots z_r \prod_{1 \leq i < j \leq r} (z_i - z_j)}$$

where

$$a' = \begin{cases} a & \text{if } a \text{ is regular} \\ \text{def}(a) & \text{otherwise} \end{cases}$$

In particular, if  $a_i \geq 0$  for  $1 \leq i \leq r$ , we have

$$k(A_r^+, a) = \text{Res}_{z_1=0} \text{Res}_{z_2=0} \cdots \text{Res}_{z_r=0} \left( \frac{(1+z_1)^{a_1+r-1} (1+z_2)^{a_2+r-2} \dots (1+z_r)^{a_r}}{z_1 \cdots z_r \prod_{1 \leq i < j \leq r} (z_i - z_j)} \right).$$

Similarly we may write a formula for the transportation polytope  $T_{m,n}(d, c)$ .

**Corollary 22** Let  $a = \sum_{i=1}^m d_i e_i - \sum_{j=1}^n c_j e_{m+j}$ , with  $d_i$  and  $c_j$  non negative integers. Then the number of integral points in  $T_{m,n}(d, c)$  is equal to

$$\sum_{w \in Sp(a')} (-1)^{n(w)} \text{Ires}_{z=0}^w \times \frac{(1+z_1)^{d_1+n-1} (1+z_2)^{d_2+n-1} \dots (1+z_m)^{d_m+n-1} (1+z_{m+1})^{-c_1-1} \dots (1+z_{m+n-1})^{-c_{n-1}-1}}{z_1 \cdots z_m \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n-1}} (z_i - z_{m+j})}$$

where

$$a' = \begin{cases} a & \text{if } a \text{ is regular} \\ \text{def}(a) & \text{otherwise} \end{cases}$$

## 2.1 The Algorithm for Counting Integral Flows.

Scope of this section is a brief description of the various algorithmic procedures that were implemented with the symbolic language `Maple` and that achieve the formula for the number of integral points described in Theorem 18. This software is available at [www.math.ucdavis.edu/~totalresidue](http://www.math.ucdavis.edu/~totalresidue). The initial data are an  $r$  by  $N$  matrix  $A$  whose columns are the elements of  $\Phi$  and an element  $a = \{a_1, \dots, a_r\} \in \mathbb{Z}^r$  that determines the polytope. The ingredients that we need to compute are:

1. The element  $a' = def(a)$  obtained by deforming the initial parameter  $a$ .
2. The set of permutations that appear in the formula, that is the set of special permutations  $Sp(a')$ .
3. The residues that appear in Formula 2.

We will discuss the ingredients for each one of these steps listing the various algorithms that are related to the part we are describing.

First of all we want to check if our vector is in  $C(A_r^+)$ , that is in the cone generated by  $\{(e_1 - e_2), (e_2 - e_3), \dots, (e_{(r-1)} - e_r), e_r\}$  because otherwise the polytope is empty and there is nothing to do. To be in the cone,  $a$  must satisfy  $a_1 \geq 0$ ,  $a_1 + a_2 \geq 0$ ,  $\dots$ ,  $a_1 + a_2 + \dots + a_r \geq 0$ . The procedure **check-vector** verifies whether this is true or not. In fact because of Lemma 20 we may use  $def(a)$  instead of  $a$  and we do this to simplify the procedures. We compute the element  $def(a)$  via the **Maple** procedure **def-vector**. The vector  $def(a)$  is used in all the formulas defining  $Sp(a)$  instead of  $a$ , whether or not  $a$  is regular. This takes care of the first part.

For finding the subset  $Sp(a)$  of  $\Sigma_r$ , we use the procedure **special-permutations**. We stress that using the **Maple** function `combinat[permute]` is impractical and does not go very far because of memory limitations. Our approach constructs recursively the permutations subject to our conditions, thus we save much memory in listing only those permutations. The set  $Sp(a)$  depends strongly on the element  $a$ . We do not have upper bound estimates on the subset  $Sp(a) \subset \Sigma_r$ , but it seems that this set is small compared to  $\Sigma_r$ . One of the worst experimental cases for the complete graph  $K_{10}$  on 10 nodes (the case of  $A_9^+$ ) is the case of the vector  $a = [30201, 59791, 70017, 41731, 58270, -81016, -68993, -47000, -43001, -20000]$  where the number  $Sp(a) \subset \Sigma_9$  is 9572, certainly much smaller than  $9!$ . Experiments show that the time spent to compute this set is rather small.

Each permutation  $w \in Sp(a)$  gives rise to the simplicial cone  $C_w^+$  containing  $a$ , this corresponds to a vertex of the polytope  $P(A_r^+, a)$ . However, clearly the cardinality of  $Sp(a)$  is much smaller than the number of vertices of the partition polytope  $P(A_r^+, a)$ . For example, for  $a = [a_1, a_2, \dots, a_r, -(\sum_{i=1}^r a_i)]$  with  $a_i > 0$ , we have already remarked that the cardinality of  $Sp(a)$  is 1, as  $Sp(a)$  is reduced to the identity permutation.

Finally, for the last step we need to compute the residue. Recall that we need to compute

$$Ires_{z=0}^w \frac{(1+z_1)^{a_1+t_1}(1+z_2)^{a_2+t_2} \dots (1+z_r)^{a_r+t_r}}{z_1^{m_{1,r+1}} z_2^{m_{2,r+2}} \dots z_r^{m_{r,r+1}} \prod_{1 \leq i < j \leq r} (z_i - z_j)^{m_{ij}}}$$

with  $w$  one of the special permutations. Let us denote by  $F$  the function appearing in the formula above. The function  $F$  is a product of a certain number of functions. This allows us to take the residues by introducing little by little the part of the function  $F$  containing the needed variable. To make things simpler we assume that  $w$  is the identity permutation. We start by taking the residue at  $z_r = 0$  of the function  $g := \frac{(1+z_r)^{(a_r+t_r)}}{z_r^{m_{r,r+1}} \prod_{j=1}^{r-1} (z_j - z_r)^{m_{jr}}}$ . Suppose  $g_r(z_1, z_2, \dots, z_{r-1})$  is the result. We continue by taking the residue in  $z_{r-1}$  of the function  $g_r$  multiplied by all the factors of the original function  $F$  that involve the variables  $z_{r-1}$  and so on. The way we compute the residue in one variable  $z$  of a function  $g(z) = F(z)/z^u$ , where  $F$  is analytic, is by computing the Taylor expansion of  $F$  up to the estimate we have for the order  $u$  of the pole of the function  $g$  and then taking the coefficient of  $1/z$ . The argument just described is implemented via different procedures: **coex, invi, trunc-next-function** and **RRK**. Finally, the procedure **number-kostant** adds up, with a sign (the appropriate sign is computed using **segnop**), all residues coming from the different special permutations, thus getting Formula 2. The procedure **polynomial-kostant** computes the polynomial  $a \mapsto k(\Phi, a)$  on the chamber determined by  $a$ .

As we pointed out we need an uniform estimate for the order of poles appearing. The result for the order of pole is the content of the subsection that follows and it is implemented in procedure **E**.

## 2.2 Estimates for the order of poles

Let  $G_r$  be a Laurent polynomial in the  $r$  variables  $z = (z_1, z_2, \dots, z_r)$  and let  $D_r = \prod_{1 \leq i < j \leq r} (z_i - z_j)$ . We have seen that we need to compute iterated residues of the form :

$$Res_{z_1=0} Res_{z_2=0} \dots Res_{z_r=0} G_r / D_r^m.$$

The following key lemma will handle the situations that will appear in computing the estimate we are looking for.

**Lemma 23** *Assume that  $G_r = \frac{F(z_1, \dots, z_r)}{(z_1 z_2 \dots z_r)^g} H_r(\frac{1}{z_1}, \dots, \frac{1}{z_r})$  where  $F$  is analytic and  $H_r$  is a homogeneous polynomial of degree  $h$ , then*

$$Res_{z_r=0} G_r / D_r^m$$

is a linear combination of functions of the form  $G_{r-1}/D_{r-1}^m$  with

$$G_{r-1} = \frac{F(z_1, \dots, z_{r-1})}{(z_1 z_2 \dots z_{r-1})^{(g+m)}} H_{r-1}\left(\frac{1}{z_1}, \dots, \frac{1}{z_{r-1}}\right)$$

where  $H_{r-1}$  is a homogeneous polynomial of degree at most  $g + h - 1$  and  $F(z_1, \dots, z_{r-1})$  is analytic.

**Proof:** Let us prove the lemma for a monomial  $H_r = z_1^{i_1} \dots z_{r-1}^{i_{r-1}} z_r^{i_r}$  where  $i_1, i_2, \dots, i_r$  are non-negative integers such that  $i_1 + i_2 + \dots + i_r = h$ . We write  $\prod_{1 \leq i \leq r-1} (z_i - z_r)^m = (z_1 z_2 \dots z_{r-1})^m \prod_{1 \leq i \leq r-1} (1 - \frac{z_r}{z_i})^m$ .

The Taylor expansion of  $\frac{1}{\prod_{1 \leq i \leq r-1} (1 - \frac{z_r}{z_i})^m}$  at  $z_r = 0$  is

$$\sum_{U_1, \dots, U_{r-1}} z_1^{-|U_1|} z_2^{-|U_2|} \dots z_{r-1}^{-|U_{r-1}|} z_r^{|U_1| + |U_2| + \dots + |U_{r-1}|}$$

where  $U_s = \{j_1^s, j_2^s, \dots, j_m^s\}$  varies over the  $m$  tuples of non negative integers. Write also  $F(z_1, \dots, z_r) = \sum_k F_k(z_1, \dots, z_{r-1}) z_r^k$ . Thus we obtain

$$\begin{aligned} \text{Res}_{z_r=0} \frac{G_r}{D_r^m} &= \\ \frac{z_1^{-i_1} \dots z_{r-1}^{-i_{r-1}}}{(z_1 z_2 \dots z_{r-1})^{g+m}} \frac{1}{D_{r-1}^m} \text{Res}_{z_r=0} \frac{F(z_1, \dots, z_r)}{z_r^{g+i_r} \prod_{i=1}^{r-1} (1 - \frac{z_r}{z_i})^m} &= \\ \left( \frac{z_1^{-i_1} \dots z_{r-1}^{-i_{r-1}}}{(z_1 z_2 \dots z_{r-1})^{g+m}} \frac{1}{D_{r-1}^m} \right) \times \\ \sum_{k=0}^{g-1+i_r} (F_k(z_1, \dots, z_{r-1}) \sum_{U_1, \dots, U_{r-1}: |U_1| + \dots + |U_{r-1}| = g-1+i_r-k} z_1^{-|U_1|} \dots z_{r-1}^{-|U_{r-1}|}) \end{aligned}$$

For  $0 \leq k \leq i_r + g - 1$ , the monomial

$$z_1^{-i_1} \dots z_{r-1}^{-i_{r-1}} z_1^{-|U_1|} z_2^{-|U_2|} \dots z_{r-1}^{-|U_{r-1}|}$$

is such that

$$i_1 + \dots + i_{r-1} + |U_1| + \dots + |U_{r-1}| = i_1 + \dots + i_{r-1} + i_r + g - 1 - k \leq h + g - 1$$

and we obtain the lemma.

Observe that if  $F = 1$  then the same proof shows that  $H_r$  is homogeneous of degree precisely  $h + g - 1$ . Now starting from  $G_r = \frac{F(z_1, \dots, z_r)}{(z_1 \dots z_r)^m}$  we want to compute

$$Res_{z_{k+1}=0} Res_{z_{k+2}=0} \cdots Res_{z_{r-1}=0} Res_{z_r=0} G_r / D_r^m.$$

Applying the lemma with  $h = 0$ , we obtain that

$$Res_{z_r=0} G_r / D_r^m$$

is a linear combination of functions of the form  $\frac{G_{r-1}}{D_{r-1}^m}$  where

$$G_{r-1} = \frac{F(z_1, \dots, z_{r-1})}{(z_1 \cdots z_{r-1})^{2m}} H\left(\frac{1}{z_1}, \dots, \frac{1}{z_{r-1}}\right)$$

and  $H$  is homogeneous of degree at most  $m - 1$ , thus at the next residue we get again a linear combination of functions of the form  $\frac{G_{r-2}}{D_{r-2}^m}$  where

$$G_{r-2} = \frac{F(z_1, \dots, z_{r-2})}{(z_1 \cdots z_{r-2})^{3m}} H\left(\frac{1}{z_1}, \dots, \frac{1}{z_{r-2}}\right)$$

with  $H$  homogeneous of degree at most  $2m + m - 1 - 1 = 3m - 2$ , so finally the last residue in  $z_{k+1} = 0$  leaves a linear combination of functions of the form

$$\frac{G_k}{D_k^m}$$

with

$$G_k = \frac{F(z_1, \dots, z_k)}{(z_1 \cdots z_k)^{(r-k+1)m}} H\left(\frac{1}{z_1}, \dots, \frac{1}{z_k}\right).$$

Here  $H$  is homogeneous of degree at most  $\frac{(r-k)(r-k+1)m}{2} - (r-k)$ . In particular, considering  $H(\frac{1}{z_1}, \dots, \frac{1}{z_k})$  we have the estimate on poles we were looking for.

**Corollary 24** 1. Let  $G_r = \frac{F(z_1, \dots, z_r)}{(z_1 \cdots z_r)^m}$ , with  $F$  analytic. Then the function

$$Res_{z_{k+1}=0} Res_{z_{k+2}=0} \cdots Res_{z_{r-1}=0} Res_{z_r=0} G_r / D_r^m$$

has a pole in  $z_k$  of order at most  $\frac{m(r-k)(r-k+1)}{2} - (r-k)$ .

2. In particular with the notation as in Theorem 18, if  $m = \max_{i,j} m_{ij}$  then the pole in  $\sigma(z_k)$  of the function

$$Res_{z_{\sigma(k+1)}=0} \cdots Res_{z_{\sigma(r)}=0} f_{\Phi}(a_1, a_2, \dots, a_r)(z) = \frac{(1+z_1)^{a_1+t_1} (1+z_2)^{a_2+t_2} \cdots (1+z_r)^{a_r+t_r}}{z_1^{m_{1,r+1}} z_2^{m_{2,r+2}} \cdots z_r^{m_{r,r+1}} \prod_{1 \leq i < j \leq r} (z_i - z_j)^{m_{ij}}}$$

has at most order  $\frac{m(r-k)(r-k+1)}{2} - (r-k)$  independently from  $\sigma \in \Sigma_r$ .



### 3 The Chamber Complex

In this section we discuss the chambers and how to compute them. It is important to emphasize that everything that we present in this section is valid for general matrices, not necessarily unimodular. There is an implementation of these ideas in the Maple program **chambers** available at [www.math.ucdavis.edu/~totalresidue](http://www.math.ucdavis.edu/~totalresidue). Let  $\Delta^+$  the set of distinct vectors  $\{\Phi\}$ . Recall the *chamber complex* is the polyhedral subdivision of the cone  $C(\Delta^+)$  of nonnegative linear combinations of  $\Delta^+$ . It is defined as the common refinement of the simplicial cones  $C(\sigma)$  running over all possible basic subsets  $\sigma$  of  $\Delta^+$ . To be more precise we introduce now notation and the key definitions. In what follows, when we consider a subset  $I = \{s_1, s_2, \dots, s_k\}$ , where the elements  $s_i$  of  $I$  are subsets of a set  $X$ , we assume there is a partial order on  $I$  by containment. Thus the set of minimal elements of  $I$  is denoted by  $\text{minimalize}(I)$ . We adopt the convention that the intersection of an empty family of subsets of  $X$  is  $X$  itself.

Let  $\Delta^+$  be the set  $\{\phi_1, \phi_2, \dots, \phi_N\}$  of vectors in  $\mathbb{R}^r$ . Recall that a *wall* is a hyperplane in  $\mathbb{R}^r$  spanned by  $(r-1)$  vectors of  $\Delta^+$ . Each wall  $W$  partitions the set of indices  $\{1, 2, \dots, N\}$  into three sets:  $\text{zeros}(W) = \{i | \phi_i \in W\}$ , and two disjoint subsets  $\text{pos}(W)$ ,  $\text{neg}(W)$  whose union  $\text{pos}(W) \cup \text{neg}(W)$  is precisely the subset of  $\{1, 2, \dots, N\} \setminus \text{zeros}(W)$ . We consider the set  $\{\text{pos}(W), \text{neg}(W)\} = \{\text{neg}(W), \text{pos}(W)\}$ . We denote by  $\mathcal{B}$  the set of subsets  $\sigma$  of  $\{1, 2, \dots, N\}$  such that  $\sigma$  is of cardinality  $r$  and the set of vectors  $\{\phi_i | i \in \sigma\}$  are linearly independent. For convenience, we continue to call such  $\sigma$  a *basic subset* of  $\Delta^+$ , thinking of  $\sigma$  as a subset of integers or as a subset of elements of  $\Phi$  labeled by indices.

For  $\sigma \in \mathcal{B}$ , we consider the closed cone  $C(\sigma)$  generated by  $\sigma$ . If  $I$  is a subset of  $\mathcal{B}$ , let  $F(I) = \bigcap_{\sigma \in I} C(\sigma)$  be the intersection of the cones  $C(\sigma)$ , when  $\sigma$  runs in  $I$ . We will say that  $I$  is a *feasible subset* of  $\mathcal{B}$  if the interior of  $F(I)$  is non empty. A *combinatorial chamber*  $I$  is a maximal feasible subset of  $\mathcal{B}$ . The polyhedral cone  $F(I)$  will be called a *geometric chamber*. The actual chamber  $\text{Chamber}(I)$  is the interior of  $F(I)$ . Reciprocally, the collection  $I$  is entirely determined by  $F(I)$ . We have  $I = \{\sigma \in \mathcal{B} | F(I) \subset C(\sigma)\}$ . The collection of all geometric chambers and their faces forms a polyhedral complex that partitions the cone  $C(\Delta^+)$ , the so called *chamber complex* [1, 5, 10].

Figure 5 shows an example, the chamber complex for the cone associated to the acyclic complete graph  $K_4$  we discussed in the previous section. The picture represents a 2-dimensional slice of the cone decomposition (the cone is 3-dimensional and pointed at the origin). The 6 dots labeled  $(e_i - e_j)$  on

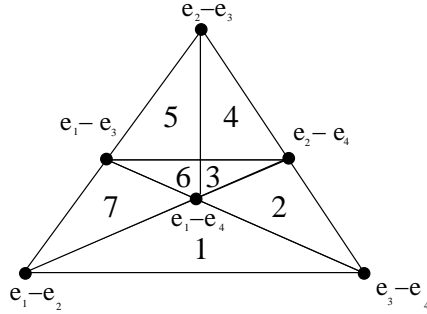


Figure 5: A slice of the chamber complex for  $K_4$

the drawing are the intersections of the rays  $\mathbb{R}^+(e_i - e_j)$  with the hyperplane  $(3x_1 + x_2 - x_3 - 3x_4) = 2$ . Seven chambers, numbered from 1 to 7, are present. In the configuration of vectors of Figure 5 there are seven walls, one for each of the distinct lines obtained from the vectors in the configuration.

Let  $\mathcal{H}$  denote the hyperplane arrangement consisting of all walls.  $\mathcal{H}$  contains as a subset the walls of the chambers. The cells in  $C(\Delta^+) \setminus \mathcal{H}$  are open cells, interior of polyhedral cones. We will call these open cells *topes* (following the oriented matroid terminology [7]). Note that the set of topes is (typically) a much finer subdivision of  $C(\Delta^+)$  than its chambers. See Figure 6 for a comparison between the chamber complex and the tope complex of the hyperplane arrangement  $\mathcal{H}$  associated with the example in Figure 5.

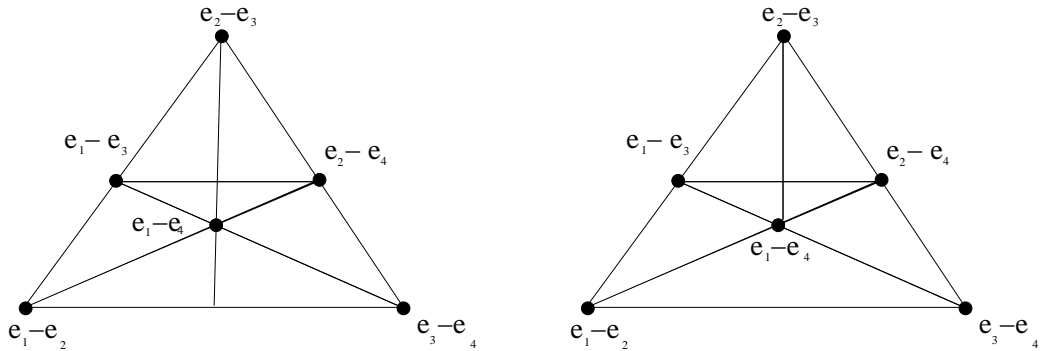


Figure 6: 8 topes (left) versus 7 chambers (right)

A tope  $\tau$  of  $C(\Delta^+)$  does not touch any wall of  $\Delta^+$ . Then, for each

wall  $W$ , we denote by  $pos(W, \tau)$  the set of elements  $i \in \{1, 2, \dots, N\}$  such that  $\phi_i \in \Delta^+$  lies on the same open half-space determined by  $W$  than the tope  $\tau$ . We say that  $pos(W, \tau)$  is a *non-face* (this terminology is justified because these are the non-faces of a certain simplicial complex in the sense of Chapter two of [21]). We denote by  $Chamber(\tau)$  the chamber containing the tope  $\tau$ .

To each tope  $\tau$ , we associate the family of positive non-faces determined by the tope  $\tau$  (we have a non-face for each wall). Let us call this full family  $Polarized(\tau)$ . Consider the family  $MNF(\tau)$  of minimal elements of  $Polarized(\tau)$ . This is the family  $MNF(\tau) = minimize(Polarized(\tau))$ . The first main observation is that we can reconstruct the chamber  $Chamber(\tau)$  containing the tope  $\tau$  from the set  $MNF(\tau)$ . This is very useful to construct one initial chamber. Later all others will be found from it.

The set  $MNF(\tau)$  is a set of non-faces. Let  $f$  be the cardinality of the set  $MNF(\tau)$ . Let us list  $MNF(\tau) := \{p_1, p_2, \dots, p_f\}$ . Each  $p_i$  is a non-face. We construct the family  $\mathcal{P}(\tau)$  of sets  $\nu$  of the form  $\nu := \{i_1, i_2, \dots, i_f\}$  with  $i_1 \in p_1, i_2 \in p_2, \dots, i_f \in p_f$ . These we call *transversals* of a family of sets. This family is denoted by  $transversal(MNF(\tau))$  in the computer program we present. Again  $\mathcal{P}(\tau)$  is a set whose elements are sets of indexes, its elements being subsets of  $\{1, 2, \dots, N\}$ . The cardinality of a set  $\nu \in \mathcal{P}(\tau)$  may be smaller than  $f$ , as the family  $MNF(\tau)$  does not consists of disjoint sets. It is important to observe that if  $\nu$  is in  $\mathcal{P}(\tau)$ , then for any wall  $W$ , the intersection  $\nu \cap pos(W, \tau)$  is not empty. We have the theorem.

**Theorem 25** *The minimal elements of the family  $\mathcal{P}(\tau) := transversal(MNF(\tau))$  are exactly the basic subsets  $\sigma$  of  $\Delta^+$  such that  $\tau \subset C(\sigma)$ .*

In other words, given the set  $MNF(\tau)$  associated to a tope  $\tau$ , the family of basic subsets  $\sigma$  of  $\Delta^+$  such that  $\tau$  is contained in  $C(\sigma)$  is precisely the set  $minimize(transversal(MNF(\tau)))$ . We are going to prove this theorem. We start by a lemma.

**Lemma 26** *Every  $\nu \in \mathcal{P}(\tau)$  is such that the set of vectors  $\{\phi_i | i \in \nu\}$  generates  $\mathbb{R}^r$ .*

**Proof:** Let us see that a set  $\nu \in \mathcal{P}(\tau)$  generates  $\mathbb{R}^r$ . Indeed, if not, the set of vectors  $\{\phi_i | i \in \nu\}$  would be contained in a wall  $W$ . Consider the set  $pos(W, \tau)$  and a minimal element  $p$  of the family  $MNF(\tau) := minimize(Polarized(\tau))$  contained in  $pos(W, \tau)$ . Then  $p$  (meaning the set of elements  $\phi_i$  indexed by  $p$ ) is contained in one of the open half-space

determined by  $W$ . Thus, contrary to our hypothesis, we would have  $\nu \cap p = \emptyset$ .

**QED**

We go on proving Theorem 25.

**Proof:** Let  $\sigma$  be a basic subset of  $\Delta^+$  ( $\sigma$  (elements indexed by  $\sigma$ ) generates a simplicial cone). We now prove that if  $\tau \subset C(\sigma)$ , then  $\sigma \in \mathcal{P}(\tau)$  and is a minimal element in the family of transversal sets  $\mathcal{P}(\tau)$ .

For each wall  $W$ , the set  $\sigma \cap \text{pos}(W, \tau)$  is non empty. Otherwise  $\sigma$  would be contained in the closed half space determined by  $W$ , but would be on the opposite to  $\tau$  with respect to  $W$ , and the cone  $C(\sigma)$  will not contain  $\tau$ . Let us pick for each  $p \in \text{MNF}(\tau)$  an element  $\phi_p \in \sigma \cap p$ . It follows that  $\sigma$  contains necessarily the set  $\nu := \{\phi_p | \phi_p \in \sigma \cap p; p \in \text{MNF}(\tau)\}$ , belonging to the family  $\mathcal{P}(\tau)$ . But then  $\sigma = \nu$ , as  $\sigma$  is a basic subset of  $\Delta^+$  and  $\nu$  indexes a set of generators of  $\mathbb{R}^r$  by Lemma 26. Furthermore  $\sigma$  is minimal, as all sets belonging to the family  $\mathcal{P}(\tau)$  have cardinality at least equal to  $r$ .

We now prove the converse. Let  $\nu$  be a minimal set of  $\mathcal{P}(\tau)$ . We claim that  $\tau$  is contained in the cone  $C(\nu)$ . Otherwise, there would be a wall  $W$  separating  $\tau$  and  $C(\nu)$ . But by construction of  $\nu$  there is an element  $p \in \nu$  contained in  $\text{pos}(W, \tau)$ ; a contradiction with  $W$  separating  $C(\nu)$  and  $\tau$ . Now all we have to prove is that  $\nu$  has cardinality  $r$ .

Let  $x$  be a point in  $\tau$ . By Caratheodory theorem, there is a basic subset  $\sigma$  contained in  $\nu$  such that  $x \in C(\sigma)$ . Then the tope  $\tau$  is entirely contained in  $C(\sigma)$  because a tope is, by definition, not separated in two by any hyperplane. The set  $\sigma$  belongs to  $\mathcal{P}(\tau)$  by the preceding discussion. But  $\sigma \subset \nu$  and  $\nu$  is minimal, thus  $\nu = \sigma$ .

So we conclude that the set  $\text{Chamber}(\tau)$  of basic subsets  $\sigma$  of  $\Delta^+$  such that  $\tau \subset C(\sigma)$  is the set  $\text{minimalize}(\mathcal{P}(\tau))$  of minimal elements of  $\mathcal{P}(\tau) = \text{transversal}(\text{MNF}(\tau))$ . **QED**

The *lexicographic tope* is the tope containing the vector  $\xi = \phi_1 + \epsilon\phi_2 + \epsilon^2\phi_3 + \dots$  where  $\epsilon$  is a small number. The lexicographic chamber is the chamber that contains the lexicographic tope.

**Corollary 27** *The following algorithm determines the  $r$ -simplicial cones  $C(\sigma)$  that contain the lexicographic chamber associated with a particular labeling of the elements of  $\Delta^+$ , by finding the basic sets  $\sigma$  that define them.*

1. *Create the list  $L$  of lexicographic nonfaces  $\text{pos}(W, \tau)$  where  $\tau$  is the lexicographic tope, and  $W$  runs over all possible walls of  $\Delta^+$ .*
2. *Let  $F = \{A_1, A_2, \dots, A_m\}$  be the minimal non-faces from  $L$ .*
3. *Find the transversal sets to the family  $F$  then minimize the set of transversals. The result is  $\sigma_1, \dots, \sigma_k$  the desired basic sets.*

Now we are concerned with producing all other chambers from one initial chamber, such as the lexicographic chamber. For this we need to understand the polyhedron  $F(I)$ . This is a pointed polyhedral cone. We recall, say from Chapter 8 in the book [19], that for a polyhedron  $P$  (e.g.  $F(I)$ ) given by a finite set of inequalities  $Ax \leq b$ , a *supporting hyperplane* is an affine hyperplane  $\{x|cx = d\}$  such that  $d = \max\{cx|Ax \leq b\}$ . A subset of  $P$  is a *face* if  $F = P$  or  $F$  is the intersection of  $P$  with a supporting hyperplane of  $P$ . A *facet* of  $P$  is a maximal face distinct from  $P$ . We say a wall  $W$  is an *essential wall* of the geometric chamber  $F(I)$ , if  $F(I) \cap W$  is a facet of the pointed polyhedral cone  $F(I)$ . This is equivalent to  $W$  being a supporting hyperplane of  $F(I)$  and  $\dim(F(I) \cap W) = r - 1$ . We say that two geometric chambers  $F(I)$  and  $F(I')$  are *W-adjacent* if they share a common essential wall  $W$  and  $\dim(F(I) \cap F(I') \cap W) = r - 1$ . In particular, the wall  $W$  is an interior wall. In what follows, unless is necessary to avoid ambiguity, we will simply refer to “adjacent chambers” without specifying the wall they share. We present now an operation that allows us to move, under certain conditions, from a geometric chamber to another adjacent geometric chamber. Since the geometric chambers form a connected polyhedral complex, we can then apply some standard search procedure, such as depth-first search, to enumerate and list all chambers.

We denote by  $\mathcal{W}$  the set of subsets  $\nu$  of  $\{1, 2, \dots, N\}$  such that  $\nu$  is of cardinality  $r - 1$  and the set of vectors  $\{\phi_i|i \in \nu\}$  are linearly independent. In other words, if  $\nu$  is in  $\mathcal{W}$ , the vector space  $\mathcal{L}(\nu)$  spanned by the vectors  $\{\phi_i|i \in \nu\}$  is a wall  $W$ . If  $W$  is a wall we denote by  $\mathcal{W}(W)$  the subset of  $\mathcal{W}$  with elements those  $\nu$  such that  $\mathcal{L}(\nu) = W$ .

If  $\nu$  is in  $\mathcal{W}$ , we consider the subsets  $zeros(\nu)$ ,  $pos(\nu)$  and  $neg(\nu)$ . If  $i$  is not in  $zeros(\nu)$ , then  $\nu \cup \{i\}$  is an element of  $\mathcal{B}$ . We denote by  $\delta^+(\nu)$  the subset of  $\mathcal{B}$  consisting of elements  $\sigma = \nu \cup \{i\}$  where  $i$  runs in  $pos(\nu)$ ; denote  $\delta^-(\nu)$  the subset of  $\mathcal{B}$  consisting of elements  $\sigma = \nu \cup \{i\}$  where  $i$  runs in  $neg(\nu)$ ;

If  $W$  is a wall, and  $\sigma$  a subset of  $\{1, 2, \dots, N\}$  we denote by  $\sigma \cap W = \sigma \cap zeros(W)$ . We denote by  $\mathcal{B}(W|facet)$  the subset of  $\mathcal{B}$  consisting of those elements  $\sigma$  such that  $\sigma \cap W$  is of cardinality  $(r - 1)$ . In other words,  $W$  is spanned by a facet of the cone  $C(\sigma)$ . We denote by  $\mathcal{B}(W|cut)$  the subset of  $\mathcal{B}$  consisting of elements  $\sigma$  such that both sets  $\sigma \cap pos(W)$  and  $\sigma \cap neg(W)$  are non empty. For any subset  $I$  of  $\mathcal{B}$ , we denote by  $I(W|facet) = I \cap \mathcal{B}(W|facet)$  and by  $I(W|cut) = I \cap \mathcal{B}(W|cut)$ .

Let  $I$  be a combinatorial chamber which is a maximal feasible subset of  $\mathcal{B}$ . Let  $W$  be a wall, we define  $B(W, I) = \{\sigma \cap W|\sigma \in I(W|facet)\}$ . This is a subset of  $\mathcal{W}(W) = \{\nu \in \mathcal{W}|\mathcal{L}(\nu) = W\}$ . If  $W$  is an essential wall of

$F(I)$ , then (as we will see later) for each subset  $\nu \in B(W, I)$  either  $\delta^+(\nu)$  is contained in  $I$  or  $\delta^-(\nu)$  is contained in  $I$ , but not both.

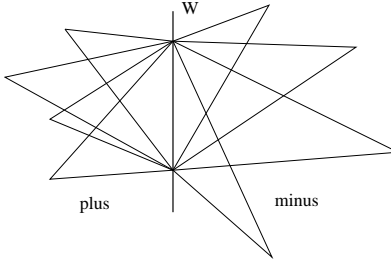


Figure 7: A reflexion exchanges the simplicial cones supported on opposite sides of a wall.

If  $W$  is an interior wall then define the *reflexion operation*, this is a new combinatorial chamber denoted by  $reflexion(I, W)$ . We keep in  $reflexion(I, W)$  all elements  $\sigma \in I(W|cut)$ , while we replace each subset  $\delta^+(\nu) \subset I(W|facet)$  by its opposite  $\delta^-(\nu)$ . The operation of reflexion has also received the name of *flip* by several authors. Applying a reflexion over any wall may not yield an adjacent chamber, as we see in the example of Figure 8

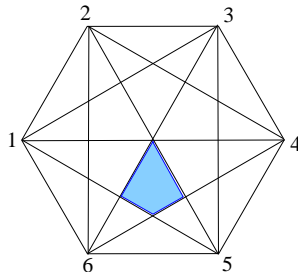


Figure 8: A reflexion using the wall 1,4 does not give a chamber

The important fact is that if one performs the reflexions over essential walls the result is the desired one:

**Lemma 28** *If  $W$  is an essential interior wall of  $F(I)$ , and let  $reflexion(I, W)$  the geometric chamber obtained by reflexion of  $I$  along the essential wall  $W$ . Then the set  $reflexion(I, W)$  is the combinatorial chamber associated to the  $W$ -adjacent chamber sharing  $W$  with  $F(I)$ .*

Clearly all elements  $\sigma \in I(W|cut)$  and elements in  $\delta^-(\nu)$ , when  $\nu$  runs over  $B(W, I)$ , give rise to simplicial cones containing the  $W$ -adjacent chamber. Conversely, any  $\sigma$  in  $\mathcal{B}$  such that the cone  $C(\sigma)$  contains the  $W$ -adjacent chamber is either in  $I(W|cut)$  or in a set of the form  $\delta^-(\nu)$ , with  $\nu \in B(W, I)$ .

The above lemma stresses the importance of determining the essential walls and that is what we describe next. Each essential wall  $W$  is described by a linear inequality, that reaches equality at  $F(I) \cap W$ . The chamber is contained in the corresponding half-space. The presentation we have of the chamber is as the intersection of simplicial cones, their facets provide us with a system of inequalities whose solution is precisely the chamber. The trouble is that this system contains *redundant* inequalities. An inequality is redundant if it is implied by the other constraints in the system, so redundant inequalities can be removed.

Our algorithm for finding the essential walls is based in the following statement, which is essentially Theorem 8.1 in page 101 of [19]. Here we state it for full-dimensional polyhedra (thus no equality constraints are present):

**Theorem 29** *If no inequality in the system  $Ax \leq b$  defining the full-dimensional polyhedron  $P$  is redundant, then there exists a one-to-one correspondence between the facets of a polyhedron and the inequalities in  $Ax \leq b$  given by  $F = \{x \in P | a_i x = \beta_i\}$ , for any facet  $F$  of  $P$  and any inequality  $a_i x \leq \beta_i$  from the system  $Ax \leq b$ .*

So if we manage to remove redundant inequalities from the original system of inequalities associated to  $F(I)$  we would have found the essential facets of the pointed polyhedral cone  $F(I)$ . To do this let us describe a direct method. Let  $Ax \leq b, s^T x \leq t$  be a given system of  $m + 1$ -inequalities in  $d$ -variables  $x = (x_1, x_2, \dots, x_d)^T$ . We want to test whether the subsystem of first  $m$  inequalities  $Ax \leq b$  implies the last inequality  $s^T x \leq t$ . If so, the inequality  $s^T x \leq t$  is redundant and can be removed from the system. A linear programming formulation of this is rather simple:

$$\begin{aligned} f^* = \quad & \text{maximize} && s^T x \\ & \text{subject to} && Ax \leq b \\ & && s^T x \leq t + 1. \end{aligned}$$

Then the inequality  $s^T x \leq t$  is redundant if and only if the optimal value  $f^*$  is less than or equal to  $t$ . By successively solving this LP for each untested inequality against the remaining inequalities, one would finally obtain an equivalent non-redundant system. Thus the algorithm to recover all the essential walls as follows.

1. Find the inequalities of each of the simplicial cones in  $F(I)$ .
2. Remove redundant inequalities using linear programming until there is no redundant inequality left. By the previous theorem the wall is uniquely determined by setting to equality the inequalities.

Thus to find all the chambers, we have

**Corollary 30** *The following algorithm finds all the chambers of the vector set  $\Delta_+$ :*

1. Find the lexicographic chamber  $I_{initial}$ . Put that as the first element of a list of chambers  $L$ .
2. Pick an element  $I$  of  $L$  for which we have not yet found its adjacent chambers. Determine its essential walls  $W$  using the method above.
3. Perform the reflexions  $reflexion(I, W) = I(W)$  for each essential interior wall  $W$ .
4. Add the  $I(W)$  to the list  $L$  of existing chambers if not already there, and continue until we have found adjacent chambers for all elements in  $L$ .

Although we have a concrete algorithm now to generate all chambers for practical reasons it is highly desirable to improve the speed on recognizing the essential walls. For this we prove some necessary conditions of the essential walls of a chamber:

**Proposition 31** *Let  $I$  be a combinatorial chamber (a maximal feasible subset of  $\mathcal{B}$ ). Let  $W$  be a wall of  $\Delta^+$ . If  $W$  is an essential wall of  $F(I)$ , then the following conditions hold true:*

1.  $I = I(W|facet) \cup I(W|cut)$ .
2.  $I(W|facet) \neq \emptyset$ .
3. For each  $\nu \in \mathcal{W}$ , either  $\delta^+(\nu) \cap I \neq \emptyset$ . Then  $\delta^+(\nu) \subset I$  and  $\delta^-(\nu) \cap I = \emptyset$ ; or  $\delta^-(\nu) \cap I \neq \emptyset$ . Then  $\delta^-(\nu) \subset I$  and  $\delta^+(\nu) \cap I = \emptyset$ ;
4. Assume  $I(W|cut)$  is not empty. Then  $\bigcap_{\sigma \in I(W|cut)} C(\sigma)$  intersects  $W$  in an  $(r - 1)$  dimensional set.



We start the proof. Let  $\{v_1, \dots, v_{r-1}\}$  be independent vectors in  $\mathbb{R}^r$ , generating a cone contained in  $F(I) \cap W$ . If  $\sigma = \{\phi_1, \dots, \phi_r\} \in I$  we denote by  $A_\sigma$  the matrix expressing  $\{v_1, \dots, v_{r-1}\}$  in terms of  $\sigma$ , that is  $v_i = \sum_{j=1}^r a_{ji} \phi_j$ . The matrix  $a_{ji}$  has non negative entries for any  $\sigma \in I$ .

Denote by

$$A_{i,\sigma} = [a_{i,1}, \dots, a_{i,r-1}]$$

the components of  $v_1, v_2, \dots, v_{r-1}$  on  $\phi_i$ . These are the columns vectors of  $A_\sigma$ .

**Lemma 32** *Assume  $W$  is an essential wall of  $F(I)$ . Suppose  $W$  is spanned by the vector set  $\{v_1, \dots, v_{r-1}\}$ . Then for each  $\sigma \in I$ , either*

a)  $A_{i,\sigma} \neq 0$  for all  $i$ ,

or

b) there exists an index  $k$  such that  $A_{k,\sigma} = 0$  while  $A_{s,\sigma} \neq 0, s \neq k$ .

*If  $\sigma$  verifies the condition a), then  $\sigma \in I(W|cut)$ . If  $\sigma$  verifies the condition b), then  $\sigma \in I(W|facet)$ .*

**Proof:**

Indeed, suppose that by rearranging the indices  $A_{1\sigma} = 0, \dots, A_{q\sigma} = 0$ , then the vectors  $\{v_1, \dots, v_{r-1}\}$  belong to the linear space spanned by  $\{\phi_{q+1}, \dots, \phi_r\}$  forcing  $r - q \geq r - 1$  that is  $q = 1$ . Thus if  $\sigma \in I$ ,  $\sigma$  verifies either a) or b).

Suppose we are in the first case. We now prove that  $\sigma \in I(W|cut)$ . Let us see that if all the  $A_{i,\sigma}$  are non zero vectors, then there *exists* an element  $X \in W \cap \overset{\circ}{C}(\sigma)$  which would force that  $\sigma \in I(W|cut)$ . Let  $X = \sum_{i=1}^{r-1} f_i v_i$  with  $f_i > 0$ , then  $X \in W$ . On the other hand  $X = \sum_{i=1}^{r-1} f_i \sum_{j=1}^r a_{ji} \phi_j = \sum_{j=1}^r (\sum_{i=1}^{r-1} f_i a_{ji}) \phi_j = \sum_{j=1}^r b_j \phi_j$ . Because all the vectors  $A_{i,\sigma}$  are nonzero, then  $b_j > 0$  for all  $j$ , thus  $X$  belongs to the open simplicial cone spanned by  $\sigma$ .

Consider the second case. Suppose for simplicity that  $i_0 = 1$ . Then  $\{v_1, \dots, v_{r-1}\}$  is a subset of the linear span  $\mathcal{L}\{\phi_2, \dots, \phi_r\}$ , therefore  $W = \mathcal{L}\{\phi_2, \dots, \phi_r\}$  and hence  $\sigma \in I(W|facet)$ .

**Lemma 33** *Let  $W$  be an essential wall of  $F(I)$ . Then the relative interior of the cone generated by  $F(I) \cap W$  (in  $W$ ) is contained in  $\cap_{\sigma \in I(W|cut)} \overset{\circ}{C}(\sigma) \cap W$ .*

**Proof:** For  $X$  in the relative interior of the cone  $F(I) \cap W$ , we choose, as in the preceding proof,  $\{v_1, \dots, v_{r-1}\}$  independent vectors in  $\mathbb{R}^r$ , generating a cone contained in  $F(I) \cap W$  and such that  $X = \sum_{i=1}^{r-1} f_i v_i$ , with  $f_i > 0$ . Arguing as before, we see that  $X$  is in the interior of  $C(\sigma)$  for any  $\sigma \in I(W|cut)$ .

**Lemma 34** *Let  $W$  be an essential wall of  $F(I)$ . Then  $I(W|facet)$  is not empty.*

**Proof:** If  $I(W|facet)$  was empty, then the open chamber  $Chamber(I)$  would be  $Chamber(I) = \bigcap_{\sigma \in I(W|cut)} C(\sigma)$  and would intersect  $W$  due to the preceding lemma, and would not be contained on a half-space of  $W$ .

If  $W$  is an essential wall of  $F(I)$ , then  $F(I)$  is on one side of  $W$ , thus we have a distinguished non face  $pos(W, I)$ . For each  $\nu \in \mathcal{W}$  spanning  $W$ , we have a distinguished set  $\delta^+(\nu) = \{\nu \cup \{i\} | i \in pos(W, I)\}$  of elements of  $\mathcal{B}$ , while  $\delta^-(\nu) = \{\nu \cup \{i\} | i \in neg(W, I)\}$  is disjoint from  $I$ :

**Lemma 35** *If  $W$  is an essential wall of  $F(I)$ , then for every  $\nu \in \mathcal{W}$  such that  $\mathcal{L}(\nu) = W$ , then*

- 1) *We have  $\delta^-(\nu) \cap I = \emptyset$*
- 2) *If  $\delta^+(\nu) \cap I \neq \emptyset$ , then  $\delta^+(\nu) \subset I$ .*

**Proof:** Condition 1 is clear, otherwise  $F(I)$  would be on the wrong side of  $W$ . Now let  $x \in Chamber(I)$  very closed to  $W \cap F(I)$ , and  $X$  in the interior of  $W \cap F(I)$ . Assume that  $\sigma = \nu \cup \{i\}$  belongs to  $I$ . Then the point  $X$  is in the interior of  $C(\nu)$ . The line  $[x, X]$  is in the chamber  $Chamber(I)$  except at the last point  $X$ . It cannot cross any boundary of any simplicial cone. Thus we see that it stays entirely in the interior of any simplicial cone spanned by  $\nu$  and a vector  $\phi_k$  with  $\phi_k$  on the same side than  $x$ , as clearly its beginning  $(1 - \epsilon)X + \epsilon x$  is inside this simplicial cone.

Thus we have proven that if  $W$  is an essential wall of  $F(I)$ , the wall  $W$  satisfies 1, 2, 3, 4 in the statement of Proposition 31.

**Corollary 36** *If  $W$  is a wall of  $F(I)$  satisfying 1), 2), 3) and not 4); then  $reflexion(I, W)$  is not a feasible subset of  $\mathcal{B}$ .*

**Proof:** Assume  $W$  verifies 1) 2) 3). Let  $I' = reflexion(I, W)$ . If  $W$  does not satisfy 4), the set  $F(cut) = \bigcap_{\sigma \in I(W|cut)} C(\sigma)$  does not cut  $W$  in an open set. Thus is contained in one side of the hyperplane  $W$ . The set

$I(W|cut)$  is left stable under the procedure reflexion. Clearly, the other cone  $F'(facet) = \cap_{\sigma \in I'(W|facet)} C(\sigma)$  is on the other side of the hyperplane  $W$ . Thus the set  $I'$  is not feasible.

The following result justifies the difficulty of finding the combinatorial chamber that contains an input vector:

**Proposition 37** *Let  $A$  be an integral matrix. Let a vector  $b$  in the cone  $C(A)$  generated by the columns of  $A$  and a list  $F$  of simplicial cones with rays in the columns of  $A$  such that all elements of  $F$  contain  $b$ . Deciding whether  $F$  includes all simplices that contain  $b$ , i.e. whether  $F$  determines the combinatorial chamber that contains  $b$ , is NP-hard.*

**Proof:** One well-known NP-complete problem is that of given a complete graph with positive integral weights on the edges to decide whether there is a hamiltonian tour of cost less than  $\beta$ . We will explain now why this can be transformed of the problem of deciding whether a list simplicial cones is already enough to determine a chamber.

We will use a theorem by K. Murty (see Theorem 2.1 in [16]): Consider a complete bipartite graph  $K_{n,n}$ . Orient the edges all in the same direction and assign excess 1 to the tail nodes and -1 to the head nodes of each arc. It is well known that the associated Network polytope is the famous Birkhoff-Von Neumann polytope of doubly stochastic matrices we saw in the introduction. This polytope is embedded in  $R^{n^2}$  and the coordinates are in correspondence with the arcs of the bipartite network. The associated network matrix has rank  $2n - 1$ ,  $2n$  rows and  $n^2$  columns one per arc in the network and we label them  $(1, 1), (1, 2), \dots, (n - 1, n), (n, n)$ .

Extend the above network matrix by adding a row of costs, where  $c_{i,j}$ ,  $i \neq j$ , is the cost to go from  $i$  to  $j$ , except for the entry associated to the arc  $i, i$  where one can put a huge integer value  $M$ , much larger than the sum of the  $n$  largest  $c_{i,j}$ 's. On the righthandside of the matrix equation we add an entry of value  $\beta$ . Written in terms of equations we have

$$\begin{aligned} \sum_{i=1}^n x_{i,j} &= 1, j = 1..n \\ \sum_{j=1}^n -x_{i,j} &= -1, i = 1..n \\ \sum_{i=1}^n \sum_{j=1}^n c_{i,j} x_{i,j} &= \beta \end{aligned}$$

$$x_{ij} \geq 0, \text{ for all } i, j.$$

This system has now rank  $2n$ . The important point is: If the set of columns  $\{(1, 1), (2, 2), \dots, (n, n), (i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$  defines a simplicial cone containing the vector  $b = (1, 1, 1, \dots, -1, -1, -1, \beta)$ , then  $(i_1, j_1), \dots, (i_n, j_n)$  must be a traveling salesman tour with cost less or equal to  $\beta$ . Thus if we take as  $F$  the set of all simplicial cones of bases that do not use all columns  $\{(1, 1), (2, 2), \dots, (n, n)\}$  and contain  $b = (1, 1, \dots, -1, -1, \beta)$ , the remaining job of deciding whether any other cone contains the vector  $b$  is then at least as hard as the solution of the traveling salesman problem.

To conclude this section it is worth mentioning that one can abstractly apply reflexions to the non-essential walls satisfying 1) 2) and 3). The interior of the resulting “chamber” may actually have empty interior in that case and thus is not useful for us here. Nevertheless this phenomenon plays an important role in the theory under the name of *virtual chambers*. In fact, there is another characterization of the chambers using the triangulations of the Gale diagram of the original vectors (see [25] for an introduction to Gale diagrams and triangulations).

**Lemma 38 (See [5, 10])** *The face lattice of the chamber complex of a vector configuration  $A$  is anti-isomorphic to the face lattice of the secondary polyhedron of the Gale transform of  $A$ ,  $\hat{A}$ . The vertices of the polyhedron are the regular triangulations of  $\hat{A}$ .*

Thus generating the chambers of a network cone is the same as generating the distinct regular triangulations of the Gale diagram of an extended network matrix. Such calculations can be also be done using the software *topcom*.

## 4 Computational Experiments

Now we present some computational experiments. All experiments were done in a 1 GHZ pentium computer running Linux using Maple 7. All our software is available at [www.math.ucdavis.edu/~totalresidue](http://www.math.ucdavis.edu/~totalresidue). We present our experiments in three tables. We begin with Table 1 and Table 2 that deal with Kostant’s partition function, this is the case of acyclic complete graphs. As we saw in Lemma 1, all other networks can be embedded into this case. We did examples in the cases of  $K_4 (A_3^+)$ ,  $K_5 (A_4^+)$  in the first table and in the second table we have bigger examples for the

cases  $A_6^+$ ,  $A_7^+$ ,  $A_8^+$ ,  $A_9^+$  and  $A_{10}^+$ . We show computation times in both tables and Table 2 also shows the cardinality of the special permutation sets. The computations show that the total residue method is faster than brute force enumeration and the current implementation of software `LattE` [11] by one or two orders of magnitude. `LattE`, on the other hand, is the only software that deals with arbitrary rational convex polyhedra.

As it is clear on the two first tables, the computation time does not increase significantly when the weights on nodes are very large. In contrast, computation time becomes quickly very large, when the number of nodes on the graph is growing. In the second table it is evident that for a fixed number of nodes, time of computation depends strongly of the cardinality of the set  $Sp(a)$ , i.e. the signs of weights on the nodes (when all weights are positive, except the last, the cardinality of  $Sp(a)$  is 1).

Let us stress that one of the features of our method is that it can directly compute the polynomial  $k_{\Phi}(a)$  giving the number of lattice points in the polytope  $P(\Phi, a)$  in the chamber determined by  $a$ . In particular, the Ehrhart polynomial of the polytope  $P(\phi, a)$ , i.e. the function  $t \mapsto k_{\Phi}(ta)$  is also computed easily from our algorithm. For example, corresponding to the first line of Table 2:

$$k_{A_r^+}(t, 2t, 3t, 4t, 5t, -15t) = \frac{1}{120960}(6t+1)(t+4)(t+3)(t+2)(t+1) \times \\ (64921t^5 + 233897t^4 + 307649t^3 + 184639t^2 + 50574t + 5040)$$

which was computed in 0.55 seconds. In contrast, the polynomial function  $k_{\Phi}(a_1, a_2, a_3, a_4, a_5)$  (with  $a_5 = -(a_1 + a_2 + a_3 + a_4)$ ) in the chamber  $\{a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0\}$  is computed in 0.48 seconds.

The Ehrhart polynomials for the second, third and fourth examples in Table 2, i.e.  $k_{A_r^+}(21128 * t, 45716 * t, 79394 * t, -76028 * t, -31176 * t, 66462 * t)$ ,  $k_{A_r^+}(82275t, 33212t, 91868t, -57457t, 47254t, -64616t, 94854t)$ , and  $k_{A_r^+}(31994t, -12275t, 55541t, 72295t, 26697t, -3212t, -38225t, 6916t)$ , were computed in 1.36 seconds, 18.54 seconds, and 93.36 seconds respectively. It is also amusing to check the program on the value of the Kostant partition for  $A_r^+$  on the vector  $a = [1, 2, 3, 4, \dots, r, -r(r+1)/2]$ . As proven by Zeilberger [24], this value is  $\prod_{i=1}^r \frac{(2i)!}{i!(i+1)!}$ .

The last table is dedicated to  $4 \times 4$  transportation matrices. In the case of transportation polytopes, i.e. complete bipartite graphs. Here we also able to compare our speed to the special purpose  $C^{++}$  program written by Beck and Pixton [4]. Both `LattE` and Beck-Pixton's software are faster than our `Maple` implementation, with Beck-Pixton's significantly so, but it

Weights on nodes	# of flows	secs
[6, 8, -5, -9]	223	0.1
[9, 11, -12, -8]	330	0.1
[1000, 1, -1000, -1]	3002	0.009
[4383, -886, -2777, -720]	785528058	0.1
[4907, 2218, -3812, -3313]	20673947895	0.1
[47896, 30744, -46242, -32398]	19470466783680	0.01
[69295, 62008, -28678, -102625]	179777378508547	0.1
[3125352, 6257694, -926385, -8456661]	34441480172695101274	0.01
[6860556, 1727289, -934435, -7653410]	91608082255943644656	0.1
[12, 8, -9, -7, -4]	14805	0.081
[125, 50, -75, -33, -67]	6950747024	0.020
[763, 41, -227, -89, -488]	222850218035543	0.019
[11675, 88765, -25610, -64072, -10758]	563408416219655157542748	0.011
[78301, 24083, -22274, -19326, -60784]	1108629405144880240444547243	0.029
[52541, 88985, -1112, -55665, -84749]	3997121684242603301444265332	0.010
[71799, 80011, -86060, -39543, -26207]	160949617742851302259767600	0.010
[45617, 46855, -24133, -54922, -13417]	15711217216898158096466094	0.21
[54915, 97874, -64165, -86807, -1817]	102815492358112722152328	0.060
[69295, 62008, -28678, -88725, -13900]	65348330279808617817420057	0.010
[8959393, 2901013, -85873, -533630, -11240903]	6817997013081449330251623043931489475270	0.010
[2738090, 6701290, -190120, -347397, -8901863]	277145720781272784955528774814729345461	0.010
[6860556, 1727289, -934435, -818368, -6835042]	710305971948234346520365668331191134724	0.060

Table 1: Testing for the complete graphs  $K_4$  and  $K_5$ . Time is given in seconds. Excess vectors are in the first column.

Weights on nodes	# of flows	secs	$ Sp(a) $
[1,2,3,4,5, -15 ]	5880	0.02	1
[21128,45716,79394,-76028,-31176,66462,-105496]	58733548560911702671 16780821466940568432 553474831987566395925	0.22	8
[82275,33212, 91868, -57457,47254,-64616,94854,-227390]	22604049468113537772 228176193404009135 6424181	2.14	26
[31994,-12275, 55541, 72295,26697,-3212,-38225,6916,-139731]	11446847479255704222 87042245223206779226 01568734727431018393 069006356672309031382 51984519069399479632 6644137066000	7.94	24
[12275,55541, 72295, 26697,-3212,-38225,6916,92409,9528, -234224]	12970047729476531166 58326881685949118367 16319862924094634125 27856414458487356258 66474206451882923253 41990044115208492747 58896993761880000897 382293730	21.31	16
[1,2, 3,4,5,6,7,8,9,10, -55]	38883505145515430400	5	1
[46398,36794, 92409,-16156,29524,-68385,93335,50738,75167, -54015, -285809 ]	20889867895116832060 28578373441423712122 50684806890637191792 33590765780756053509 92237184823590262176 29560725791309259479 21077842421668832691 54404688022155977982 34585056426719876125 028873152	2193.23	322

Table 2: Testing for complete graphs  $K_n$  with  $n = 6, 7, 8, 9, 10, 11$ . Time is given in seconds.

must still be emphasized that our calculations for transportation polytopes makes use of the fact that they are embedded inside the complete graph for large enough number of nodes. For example the case of  $4 \times 4$  transportation polytopes is treated via the complete graph  $K_8$ . The same kind of embedding can be done for other networks.

If we consider the case of 4 times 5 matrices with weights on nodes [3046, 5173, 6116, 10928], [182, 778, 3635, 9558, 11110], the number of lattice points is 23196436596128897574829611531938753 calculated in 11.15 seconds. The number of special permutations for this vector is 540 while the number of vertices of the corresponding polytope is 912. These same example takes 7.8 seconds in `LattE` and 0.1 seconds in Beck-Pixton program.

Ehrhart polynomial  $k_{\Phi_{4,5}}((3046 * t, 5173 * t, 6116 * t, 10928 * t, -182 * t, -778 * t, -3635 * t, -9558 * t, -11110 * t))$  is computed in 30.72 seconds.

If we consider the case of 5 times 5 matrices with weights on nodes [30201, 59791, 70017, 41731, 58270], [81016, 68993, 47000, 43001, 20000], the number of lattice points is

24640538268151981086397018033422264050757251133401758112509495633028,

which we computed in 23 minutes. The number of special permutations needed is 9572 while the number of vertices of the corresponding polytope is 13150. This example took 20 minutes with `LattE` and just 4 seconds with Beck-Pixton program.

Transportation polytopes were treated by Beck and Pixton [4] in a special purpose `C++` program dedicated for this particular family of flow polytopes. Their computation is also via residues and is the fastest at the moment. It is important to remark that their use of residues is quite different from ours; our main theorem can be thought of as a multidimensional analogue of the fact that sums of the residues of a rational function on  $P_1(\mathbb{C})$  is zero. It is to be expected that in a forthcoming `C++` implementation the timings discussed here will be considerable faster than those from this preliminary `Maple` implementation. Besides obvious implementation speed ups, the ideas presented in this paper could still be improved when the total residue method is applied directly to the bipartite graph, not as a subnetwork of  $K_n$ .

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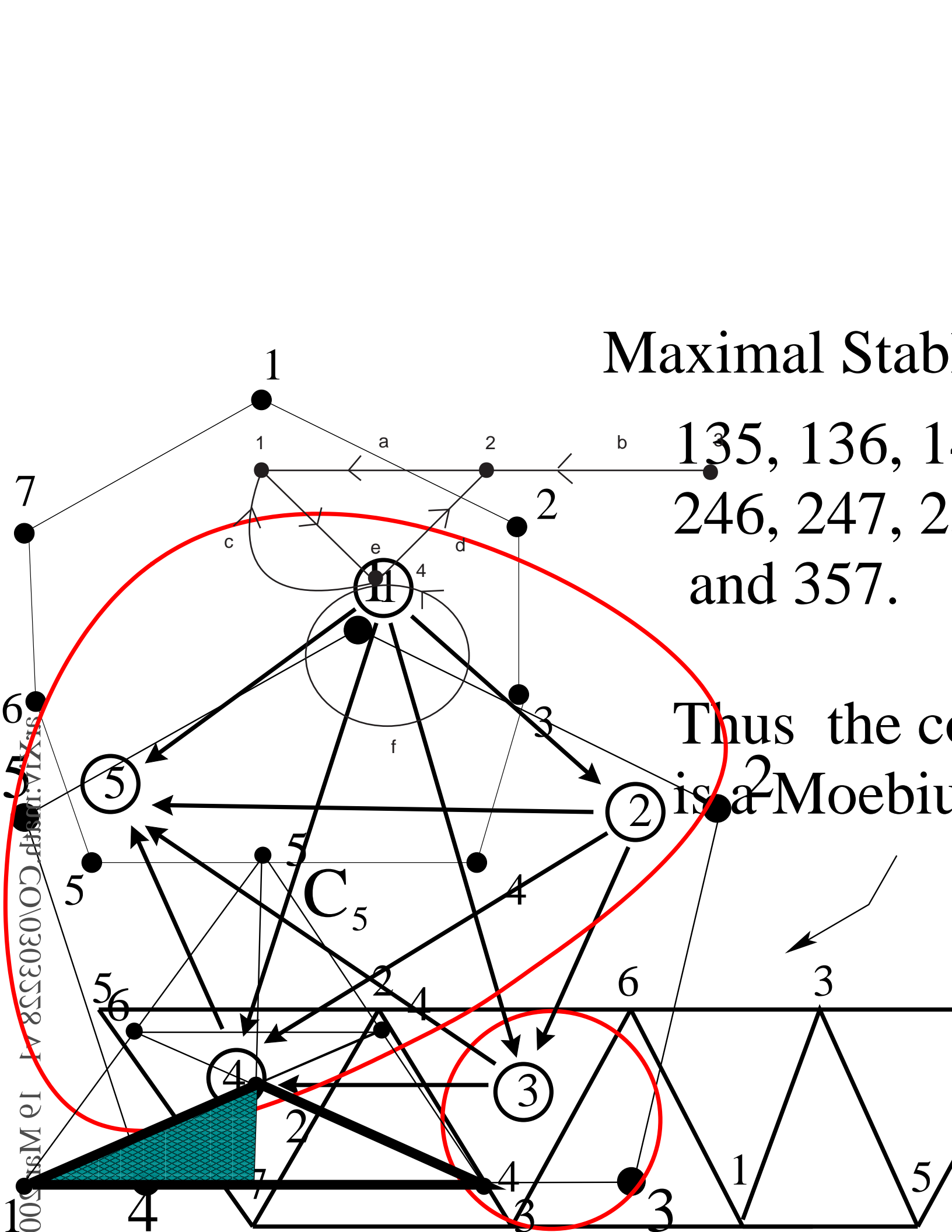


Margins	# of lattice points	secs
[220, 215, 93, 64], [108, 286, 71, 127]	1225914276768514	5.04
[109, 127, 69, 109], [119, 86, 108, 101]	993810896945891	10.43
[72, 67, 47, 96], [70, 70, 51, 91]	25387360604030	6.5
[179909, 258827, 224919, 61909], [190019, 90636, 276208, 168701]	13571026063401838164668296635065899923152079	5.87
[229623, 259723, 132135, 310952], [279858, 170568, 297181, 184826]	646911395459296645200004000804003243371154862	16.1
[249961, 232006, 150459, 200438], [222515, 130701, 278288, 201360]	319720249690111437887229255487847845310463475	16.1
[140648, 296472, 130724, 309173], [240223, 223149, 218763, 194882]	322773560821008856417270275950599107061263625	11.7
[65205, 189726, 233525, 170004], [137007, 87762, 274082, 159609]	6977523720740024241056075121611021139576919	9.0
[251746, 282451, 184389, 194442], [146933, 239421, 267665, 259009]	861316343280649049593236132155039190682027614	15
[138498, 166344, 187928, 186942], [228834, 138788, 189477, 122613]	63313191414342827754566531364533378588986467	19.4
[20812723, 17301709, 21133745, 27679151], [28343568, 18410455, 19751834, 20421471]	665711555567792389878908993624629379187969880179721169068827951	15.6
[15663004, 19519372, 14722354, 22325971], [17617837, 25267522, 20146447, 9198895]	63292704423941655080293971395348848807454253204720526472462015	27.4
[13070380, 18156451, 13365203, 20567424], [12268303, 20733257, 17743591, 14414307]	43075357146173570492117291685601604830544643769252831337342557	14.8

Table 3: Testing for  $4 \times 4$  transportation polytopes.

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