

Functoriality of Khovanov homology

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Abstract. In this paper we prove that every Khovanov homology associated to a Frobenius algebra of rank 2 can be modified in such a way as to produce a TQFT on oriented links, that is a monoidal functor from the category of cobordisms of oriented links to the homotopy category of complexes.

Keywords: Frobenius algebra, Khovanov homology, cobordism of oriented links, monoidal functor.

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Introduction.

The Khovanov homology was introduced by Khovanov [Kh1] as a categorification of the Jones polynomial. The main ingredient of this homology is a monoidal functor from the category of cobordisms of oriented 1-manifolds to a category of modules. But such monoidal functors are characterized by commutative Frobenius algebras [Ko]. Therefore the Khovanov homology can be defined for every commutative Frobenius algebra R . The classical Khovanov homology corresponds to the case: $R = \mathbf{Z}[\alpha]/(\alpha^2)$ and the Lee-version of this homology corresponds to the case: $R = \mathbf{Z}[\alpha]/(\alpha^2 - 1)$.

An important problem in this theory is to extend the Khovanov homology to a monoidal functor from the category of cobordisms of oriented links. The first attempt by Khovanov gave a negative answer because of many problems of signs. Functoriality up to sign was conjectured by Khovanov and proved later by Jacobson [Ja], Bar Natan [BN2] and Khovanov [Kh3]. This functoriality up to sign was used by Rasmussen [Ra1] to prove a conjecture of Milnor about the slice genus. Strict functoriality for alternative versions of Khovanov homology were proven by Blanchet [Bl] and Clark, Morrison, Walker [CMW]. In these versions of Khovanov homology, the Kauffman bracket is replaced with the sl_2 -polynomial in [Bl] and the su_2 -polynomial in [CMW].

In this paper we will prove functoriality for the Khovanov homology associated to any Frobenius algebra of rank 2 (and the classical Kauffman bracket).

Suppose K is a commutative ring and R is a Frobenius K -algebra of rank 2. Denote by $u \mapsto \bar{u}$ the involution of the extension $K \subset R$ and by δ the image of 1

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under the composite map:

$$R \xrightarrow{\text{coproduct}} R \otimes R \xrightarrow{\text{product}} R$$

Denote by \mathcal{L} the category of cobordisms of oriented links in \mathbf{R}^3 and by \mathcal{C}_K the homotopy category of K -complexes. These categories are both monoidal. The main result of this paper is the following:

Theorem A: *There exists a monoidal functor Ψ from \mathcal{L} to \mathcal{C}_K satisfying the following property:*

for any diagram D of an oriented link L , $\Psi(L)$ is isomorphic to the classical Khovanov complex of D .

Such functors are not unique. The construction produces a lot of functors (called Khovanov functors) satisfying this property. There is a well defined invertible element in R associated to each Khovanov functor: its weight. Actually if C is an unknotted cobordism of genus 1 from the unknot diagram (without any crossing) to itself, the image of this cobordism under a Khovanov functor of weight π is the multiplication by δ/π (resp. $\bar{\delta}/\bar{\pi}$) from R to R if the unknot is oriented clockwise (resp. counterclockwise).

Theorem B: *For every invertible element $\pi \in R$ there is a Khovanov functor of weight π . Moreover two Khovanov functors with the same weight are isomorphic.*

Consider a closed oriented surface S in \mathbf{R}^4 . This surface may be consider as a cobordism from the empty link to itself. Therefore every Khovanov functor Ψ induces an invariant $\Psi(S) \in K$. In the classical case the functor Ψ was well defined up to sign and $\Psi(S)$ was determined by Tanaka [Ta] and Rasmussen [Ra2], at least for connected surfaces. They prove that $\Psi(S)$ doesn't depend on the embedding $S \subset \mathbf{R}^4$. This result is still true in the general case and $\Psi(S)$ depends only on S and the weight of Ψ and not on the embedding. More precisely we have the following result:

Theorem C: *Let Ψ be a Khovanov functor of weight $\pi \in R^*$ and S be a closed oriented surface in \mathbf{R}^4 . Then we have:*

$$\Psi(S) = \prod_i \varepsilon(\delta^{p_i} \pi^{1-p_i})$$

where the p_i 's are the genus of the components of S .

1. Frobenius algebras of rank 2.

1.1 Definition: *Let K be a commutative ring. A K -algebra of rank 2 is a K -algebra isomorphic to $K[\alpha]/(P)$, where $P \in K[\alpha]$ is a monic polynomial of degree 2.*

The element α is called a K -generator of R and the involution of the extension $K \subset R$ is called the involution of R .

Let R be a K -algebra of rank 2 and α be a K -generator of R . The polynomial P is given by:

$$P(\alpha) = \alpha^2 - s\alpha + p$$

with s and p in K and the involution of R , denoted by $u \mapsto \bar{u}$, is the identity on K and sends α to $\bar{\alpha} = s - \alpha$. So we have:

$$s = \alpha + \bar{\alpha} \quad p = \alpha\bar{\alpha}$$

and, for any $u \in R$, $u + \bar{u}$ and $u\bar{u}$ are in K .

1.2 Proposition: *Let K be a commutative ring and R be a Frobenius K -algebra. Suppose R is a K -algebra of rank 2 and α is a K -generator of R . Then there is a unique invertible element ω in R such that the coproduct and the counit are defined by:*

$$\begin{aligned} \forall u \in R, \quad \Delta(u) &= u \otimes \omega\alpha - u\bar{\alpha} \otimes \omega \\ \varepsilon(\omega) &= 0 \quad \varepsilon(\omega\alpha) = 1 \end{aligned}$$

ω is called the twisting element of R .

Moreover, if ω is any invertible element in R , the coproduct and the counit defined by the formulae above induce a structure of Frobenius K -algebra on R .

Proof: Since Δ is a K -linear map from R to $R \otimes R$, there exist two K -linear map f and g from R to R such that:

$$\forall u \in R, \quad \Delta(u) = f(u) \otimes 1 + g(u) \otimes \alpha$$

Since R is a Frobenius algebra we have the relation: $\Delta(u) = (u \otimes 1)\Delta(1)$ and then:

$$\forall u \in R, \quad f(u) = uf(1) \quad g(u) = ug(1)$$

We have also the relation: $(\alpha \otimes 1 - 1 \otimes \alpha)\Delta(1) = 0$ which implies: $f(1) + \bar{\alpha}g(1) = 0$. By setting: $\omega = g(1)$ we get:

$$\Delta(u) = u\omega \otimes \alpha - u\omega\bar{\alpha} \otimes 1 = u \otimes \omega\alpha - u\bar{\alpha} \otimes \omega$$

On the other hand the counit satisfies the relation: $(1 \otimes \varepsilon)\Delta(u) = u$ which is equivalent to:

$$\varepsilon(\omega) = 0 \quad \varepsilon(\omega\alpha) = 1$$

The last thing to do is to prove that ω is invertible in R . Set: $u = \varepsilon(\alpha) - \bar{\alpha}\varepsilon(1)$. It is easy to see the following:

$$\varepsilon(\omega u) = \varepsilon(1) \quad \varepsilon(\omega u \alpha) = \varepsilon(\alpha)$$

So for any $v \in R$, we have: $\varepsilon((\omega u - 1)v) = 0$. Set: $\omega u - 1 = a + b\alpha$ with a and b in K . Testing this formula with $v = \omega$ and $v = \omega\alpha$ implies: $b = a = 0$. Therefore ω is invertible with inverse u .

The last part of the proposition is easy to check. □

1.3 Remarks: Let R_0 be the ring $\mathbf{Z}[\alpha, \bar{\alpha}, a, b, (a + b\alpha)^{-1}, (a + b\bar{\alpha})^{-1}]$. This ring is equipped with an involution keeping a and b fixed and exchanging α and $\bar{\alpha}$. The ring of invariant elements under this involution is:

$$K_0 = \mathbf{Z}[\alpha + \bar{\alpha}, \alpha\bar{\alpha}, a, b, ((a + b\alpha)(a + b\bar{\alpha}))^{-1}]$$

and R_0 is a K_0 -algebra of rank 2 with K_0 -generator α . By setting: $\omega = a + b\alpha$, we see that R_0 is a Frobenius algebra with twisting element ω . Moreover R_0 is universal in the following sense:

Let R be a Frobenius K -algebra of rank 2 and β be a K -generator of R . Then there exists a unique Frobenius algebra homomorphism from R_0 to R sending α to β .

In particular the endomorphisms of the Frobenius algebra R_0 are ring homomorphisms characterized by:

$$\begin{aligned} \alpha &\mapsto \lambda\alpha + \mu & \bar{\alpha} &\mapsto \lambda\bar{\alpha} + \mu \\ a &\mapsto \lambda^{-1}a - \lambda^{-2}\mu b & b &\mapsto \lambda^{-2}b \end{aligned}$$

where (λ, μ) is any element of $K_0^* \times K_0$.

Another description of the universal algebra R_0 was founded by Khovanov in [Kh4].

1.4 From now on K will be a commutative ring, R a Frobenius K -algebra of rank 2 and α a K -generator of R . The twisting element in R will be denoted by ω . Set:

$$t = \varepsilon(1) \quad \delta = \omega(\alpha - \bar{\alpha}) \quad \theta = \frac{\omega}{\bar{\omega}}$$

It is easy to see that these elements do not depend on the choice of α . Moreover we have the following relations:

$$t\delta = 1 - \theta \quad \bar{t} = t \quad \bar{\delta} = -\theta^{-1}\delta \quad \bar{\theta} = \theta^{-1}$$

Set: $A = \mathbf{Z}[t, \delta, (1 - t\delta)^{-1}]$. This ring is a ring with involution and it is contained in the algebra R_0 . Therefore every Frobenius algebra of rank 2 is an A -algebra.

By setting: $s = \delta\bar{\delta} = -\theta^{-1}\delta^2 = -(1 - t\delta)^{-1}\delta^2$, it is not difficult to see that the ring of elements in A that are fixed by the involution is $\mathbf{Z}[s, t]$ and A is a $\mathbf{Z}[s, t]$ -algebra of rank 2 with $\mathbf{Z}[s, t]$ -generator δ .

1.5 Lemma: *The ring of elements in R_0 which are invariant under every endomorphism of R_0 is the ring A .*

Proof: Suppose that a_1, \dots, a_p are elements in a commutative ring Λ . Then denote by $\Lambda \langle a_1, \dots, a_p \rangle$ the ring obtained by inverting a_1, \dots, a_p in Λ . So we have:

$$R_0 = \mathbf{Z}[\alpha, \bar{\alpha}, a, b] \langle a + b\alpha, a + b\bar{\alpha} \rangle$$

We have the following:

$$\omega = a + b\alpha \quad t = \varepsilon(1) = \frac{-b}{\omega\bar{\omega}} \quad \bar{\omega}(1 - t\delta) = \omega$$

and then

$$\begin{aligned} R_0 &= \mathbf{Z}[\alpha, \alpha - \bar{\alpha}, b, \omega] \langle \omega, \bar{\omega} \rangle = \mathbf{Z}[\alpha, \delta, b, \omega] \langle \omega, \bar{\omega} \rangle \\ &= \mathbf{Z}[\alpha, t, \delta, \omega] \langle \omega, 1 - t\delta \rangle = \left(\mathbf{Z}[\alpha, t, \delta] \langle 1 - t\delta \rangle \right) [\omega^\pm] \end{aligned}$$

Consider the endomorphisms of R_0 sending α to $\alpha + \mu$, for some $\mu \in K_0$. These endomorphisms keep t , δ and ω fixed. Then the ring R_1 of elements in R_0 which are invariant under these endomorphisms is:

$$R_1 = \mathbf{Z}[t, \delta, \omega] \langle \omega, 1 - t\delta \rangle = \left(\mathbf{Z}[t, \delta] \langle 1 - t\delta \rangle \right) [\omega^\pm]$$

But every endomorphism of R_0 keeps t and δ fixed, and multiplies ω by any element $\lambda \in K_0^*$. Since K_0^* contains any power of $\omega\bar{\omega}$, the ring R_2 of elements in R_0 which are invariant under every endomorphism of R_0 is:

$$R_2 = \mathbf{Z}[t, \delta] \langle 1 - t\delta \rangle = A \quad \square$$

Denote by \mathcal{C} the category of cobordisms of oriented curves (i.e. closed oriented 1-dimensional manifolds). The disjoint union induces on \mathcal{C} a monoidal structure. TQFT's for oriented surfaces are in one to one correspondance with commutative Frobenius algebras [Ko]. In particular the Frobenius algebra R induces a monoidal functor Φ from \mathcal{C} to the category of K -modules. This functor has the following properties: it sends \emptyset to K and S^1 to R , unit, counit, product and coproduct of R are the image under Φ of suitable cobordisms and, for every oriented closed surface Σ , $\Phi(\Sigma)$ is an element of K . An easy computation gives the following:

1.6 Lemma: *For every $p \geq 0$ denote by Σ_p an oriented surface of genus p . Then we have the following:*

$$\begin{aligned} \Phi(\Sigma_p) &= \varepsilon(\delta^p) \\ \sum_{p \geq 0} x^p \Phi(\Sigma_p) &= \varepsilon\left(\frac{1}{1 - x\delta}\right) = \frac{t + x(2 - t^2s)}{1 - xts + x^2s} \in K[[x]] \end{aligned}$$

Remark: If R is the ring R_0 and S is a closed oriented surface, $\Phi(S)$ is an element of $\mathbf{Z}[t, s]$ of degree $\chi(S)$ where the degree in $\mathbf{Z}[t, s]$ is defined by: $\partial^\circ t = 2$, $\partial^\circ s = -4$.

Remark: The Khovanov homology is defined by using a Frobenius algebra R . The classical Khovanov homology corresponds to the case: $R = \mathbf{Z}[\alpha]/(\alpha^2)$ and the Lee version of the Khovanov homology corresponds to: $R = \mathbf{Z}[\alpha]/(\alpha^2 - 1)$. In both cases, we have:

$$\omega = 1 \quad t = 0 \quad \theta = 1 \quad \delta = 2\alpha \quad s = -4\alpha^2$$

1.7 The category of mixed cobordisms \mathcal{C}' . The functor Φ is defined on the category \mathcal{C} , but it is possible to extend it to a bigger category \mathcal{C}' .

First of all, if S is a surface, consider the commutative monoid with unit defined by generators and relations. Generators are pairs $(x, a) \in S \times R$ and the relations are the following:

$$(x, a)(x, b) \equiv (x, ab)$$

$$(x, 1) \equiv 1$$

An element of this monoid will be called a R -marking of S .

If C and C' are two closed oriented curves, define a mixed cobordism from C to C' as a triple (S, Γ, u) where:

- S is a compact surface containing a closed curve Γ in its interior
- $S \setminus \Gamma$ is oriented with boundary: $\partial(S \setminus \Gamma) = C' - C$
- when crossing Γ the orientation of $S \setminus \Gamma$ is changed
- u is a R -marking of $S \setminus \Gamma$.

The category \mathcal{C}' is defined as follows: the objects of \mathcal{C}' are the objects of \mathcal{C} , that is the oriented curves. A morphism in \mathcal{C}' from an oriented curve C to an oriented curve C' is the isomorphism class of a mixed cobordism (S, Γ, u) from C to C' . So we get a monoidal category \mathcal{C}' containing \mathcal{C} .

1.8 Proposition: *There is a unique monoidal functor Φ' from \mathcal{C}' to the category of K -modules satisfying the following:*

- 1) Φ' is an extension of the functor Φ
- 2) $\Phi'(S, \Gamma, (x_1, a_1)(x_2, a_2) \dots (x_p, a_p))$ depends only on the isotopy classes of the x_i 's in $S \setminus \Gamma$
- 3) $\Phi'(S, \Gamma, (x, a)u)$ is K -linear with respect to a .
- 4) if $(S, \Gamma, (x, a)u)$ is a mixed cobordism and x' is obtained by making x go through Γ , we have: $\Phi'(S, \Gamma, (x, a)u) = \Phi'(S, \Gamma, (x', \bar{a})u)$
- 5) $\Phi'(S^1 \times [0, 1], \emptyset, (x, a))$ is the multiplication by a , from $R = \Phi'(S^1)$ to R
- 6) $\Phi'(S^1 \times [0, 1], S^1 \times \{1/2\}, 1)$ is the map $a \mapsto \delta \bar{a}$
- 7) $\Phi'(S, \Gamma, u)$ vanishes if S is not orientable.

Proof: Let Φ' and Φ'' be two functors satisfying all these conditions. Because of conditions 1) and 5), Φ' and Φ'' are the same on mixed cobordisms (S, \emptyset, u) . Consider now a mixed cobordism (S, Γ, u) from C to C' . Let N be a small regular neighborhood of Γ and S' be the closure of $S \setminus N$. Then the morphism (S, Γ, u) is the composite of (S', \emptyset, u) from C to $C' \amalg \partial N$ and $1 \times (N, \Gamma, 1)$ from $C' \amalg \partial N$ to C' . Therefore, in order to prove that Φ' and Φ'' are the same on (S, Γ, u) it is enough to prove that Φ' and Φ'' are the same on $(N, \Gamma, 1)$. But that's a consequence of conditions 6) and 7). So, if the functor Φ' exists, it is unique.

For the construction of Φ' , it is enough to consider the universal case: $R = R_0$. Consider a mixed cobordism (S, \emptyset, u) from C_0 to C_1 . This cobordism is a composite of cobordisms on the form $(S', \emptyset, 1)$ or $(C \times [0, 1], \emptyset, u)$. Because of conditions 1) and 5), there is a unique choice for the morphism $\Phi'(S, \emptyset, u)$. Moreover properties of R and Φ imply that this morphism depends only on the cobordism (S, \emptyset, u) .

Consider now any mixed cobordism (S, Γ, u) from C to C' . As before denote by N a small regular neighborhood of Γ and S' be the closure of $S \setminus N$. Then the morphism (S, Γ, u) is the composite of (S', \emptyset, u) from C to $C' \amalg \partial N$ and $1 \times (N, \Gamma, 1)$ from $C' \amalg \partial N$ to C' . So to define $\Phi'(S, \Gamma, u)$, it's enough to define $\Phi'(N, \Gamma, 1) : \Phi'(\partial N) \rightarrow K$. Let Γ_i be the component of Γ and N_i be the component of N containing Γ_i . We must set:

$$\Phi'(N, \Gamma, 1) = \otimes_i \Phi'(N_i, \Gamma_i, 1)$$

If N_i is a Möbius band, we have to set: $\Phi'(N_i, \Gamma_i, 1) = 0$. If not, N_i is a band $\Gamma_i \times [-1, 1]$ and we define the map $\Phi'(N_i, \Gamma_i, 1)$ from $R \otimes R$ to K by: $a \otimes b \mapsto a\bar{b} + b\bar{a}$.

So we get a monoidal functor from \mathcal{C}' to the category of K -modules. It is not difficult to check all the conditions except the last one.

Suppose that (S, Γ, u) is a mixed cobordism from C to C' , where S is nonorientable. Since S isn't orientable, there exists a loop γ starting at some point $x \in S \setminus \Gamma$ and intersecting transversally Γ an odd number of times. If u is a product of (x_i, a_i) , we may suppose that γ doesn't meet any of the x_i 's. Let D be a small disk in $S \setminus (\Gamma \cup \gamma)$ near x and S' be the closure of $S \setminus D$. Then (S, Γ, u) is the composite of (S', Γ, u) from C to $C' \amalg S^1$ and $1 \times (D, \emptyset, 1)$ from $C' \amalg S^1$ to C' and it is enough to prove that $\Phi'(S', \Gamma, u)$ vanishes.

Because of conditions 3) and 4) we have:

$$\Phi'(S', \Gamma, (x, \alpha)u) = \Phi'(S', \Gamma, (x, \bar{\alpha})u) \implies \Phi'(S', \Gamma, (x, \alpha - \bar{\alpha})u) = 0$$

But $\Phi'(S', \Gamma, u)$ is a morphism from $\Phi'(C)$ to $\Phi'(C' \amalg S^1) = \Phi'(C') \otimes R$. So we have:

$$(1 \otimes (\alpha - \bar{\alpha}))\Phi'(S', \Gamma, u)(v) = 0$$

for any $v \in \Phi'(C)$. Since $\alpha - \bar{\alpha}$ is not a zero divisor in $R = R_0$, $\Phi'(S', \Gamma, u)(v)$ vanishes for every $v \in \Phi'(C)$ and $\Phi'(S', \Gamma, u)$ is the zero map. \square

2. Khovanov complexes of diagrams.

In this section we'll construct many Khovanov complexes associated to link diagrams (oriented or not). All these complexes are graded (or bigraded) differential K -modules (or R -modules).

2.1 Notations: Let X be a finite set. Denote by $\Lambda(X)$ the maximal exterior power of the \mathbf{Z} -module freely generated by X . This module is a free \mathbf{Z} -module of rank 1.

Suppose X is a graded set. The grading is defined by a map e from X to \mathbf{Z} and we get a \mathbf{Z} -grading on $\Lambda(X)$ by the rule:

$$\partial^\circ(x_1 \wedge x_2 \wedge \dots \wedge x_p) = \sum_i e(x_i)$$

This graded module will be denoted by $\Lambda^e(X)$. If e is the map $x \mapsto 1$ (resp. $x \mapsto -1$), $\Lambda^e(X)$ will be denoted by $\Lambda^+(X)$ (resp. $\Lambda^-(X)$).

2.2 The Khovanov complex $kh(D)$.

Let D be a link diagram and X be the set of crossings of D . Denote by \widehat{X} the set of maps $s : X \rightarrow \{\pm 1\}$. Such a map is called a state on D .

If $s \in \widehat{X}$ is a state, we can modify D near each crossing x by the rule:

$$\begin{array}{ccc} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & \mapsto & \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} & \text{if } s(x) = 1 \\ \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} & \mapsto & \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} & \text{if } s(x) = -1 \end{array}$$

So we get a new diagram D_s called the s -resolution of D . This diagram is a curve embedded in the plane and there is a unique compact $K \subset \mathbf{R}^2$ such that D_s is the boundary of K . Since K is oriented by the plane, D_s is an oriented curve. Denote by X_s the set $s^{-1}(-1)$. So for every state s we have an oriented curve D_s and a set X_s .

Consider the following graded K -module:

$$E = \bigoplus_s \Lambda^-(X_s) \otimes \Phi(D_s)$$

Let x be a crossing of D . By sending x to -1 and all the other crossings to 1 we get a state e_x . If s is a state denote by s' the state $e_x s$. The manifold $D_{s'}$ is obtained from D_s by a surgery along a curve γ_x near x . So we get an oriented cobordism from D_s to $D_{s'}$ and, via the functor Φ , a map from $\Phi(D_s)$ to $\Phi(D_{s'})$ still denoted by γ_x . Using this map we have a map $x \otimes \gamma_x$ from $\Lambda^-(X_s) \otimes \Phi(D_s)$ to $\Lambda^-(X_{s'}) \otimes \Phi(D_{s'})$ defined by:

$$u \otimes v \mapsto x \wedge u \otimes \gamma_x(v)$$

Notice that this map is trivial on $\Lambda^-(X_s) \otimes \Phi(D_s)$ if x belongs to X_s .

It is easy to see that the map $d = \sum_x x \otimes \gamma_x$ is a differential on E of degree -1 . So we get a complex (E, d) denoted by $kh(D)$ (or $kh(D, R)$).

2.3 Remark: If $R = \mathbf{Z}[\alpha]/(\alpha^2)$ and D is oriented, the classical Khovanov complex of D is essentially isomorphic to some suspension of $kh(D)$.

2.4 The operators T_p .

If D be a link diagram, a point in D which is not a crossing will be called a regular point in D .

Let D be a link diagram and p be a regular point in D . Let a be an element of R . If s is a state on D , there is a unique component C of D_s containing p . Denote by D' the complement of C in D_s . The multiplication by $a \otimes 1$ in $\Phi(D_s) = \Phi(C) \otimes \Phi(D') = R \otimes \Phi(D')$ is an endomorphism f of $\Phi(D_s)$. So the map:

$$1 \otimes f : u \otimes v \mapsto u \otimes f(v)$$

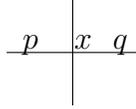
is an endomorphism of $kh(D)$. This endomorphism will be denoted by $T_p(a)$.

2.5 Proposition: *Let D be a link diagram. Then, for every regular point $p \in D$ and every $a \in R$, $T_p(a)$ is a morphism of complexes of degree 0 from $kh(D)$ to itself. Moreover these operators satisfy the following properties:*

- The operators $T_p(a)$ commute.
- The map $a \mapsto T_p(a)$ is a K -algebra homomorphism from R to $\text{End}(kh(D))$.
- Let p and q be two regular points in D . Suppose that these points are the endpoints of a path in D going through exactly one crossing. Then, for every $a \in R$ the two operators $T_p(a)$ and $T_q(\bar{a})$ are homotopic.

Proof: All these properties are easy to check except the last one.

Denote by x the crossing between p and q .



Denote by D^+ (resp. D^-) the diagram obtained from D by a positive (resp. negative) resolution at x . By setting $U = kh(D^+)$ and $V = kh(D^-)$, we have:

$$kh(D) = 1 \otimes U \oplus x \otimes V$$

Denote by γ the surgery homomorphism corresponding to the surgery along the path γ_x . It is a morphism from U to V and from V to U . Denote by k the map from $kh(D)$ to itself defined by:

$$k(1 \otimes u) = 0 \quad k(x \otimes v) = 1 \otimes \gamma(v)$$

We can check that the corresponding homotopy $d(k) = d \circ k + k \circ d$ is the map $1 \otimes \gamma^2$.

Set: $T = T_p \otimes T_q$. This operator is a map from $R \otimes R$ to the algebra of endomorphisms of $kh(D)$. It is easy to check the following:

$$d(k) = T(\Delta(1)) = T_p(\omega)T_q(\alpha) - T_p(\omega\bar{\alpha}) = T_q(\omega)T_p(\alpha) - T_q(\omega\bar{\alpha}) = T_q(\omega)(T_p(\alpha) - T_p(\bar{\alpha}))$$

and $T_p(\alpha)$ is homotopic to $T_p(\bar{\alpha})$ because ω is invertible. Let $a = u + v\alpha$ be any element in R with u and v in K . If \sim is the homotopy relation, we have:

$$T_p(a) = u + vT_p(\alpha) \sim u + vT_p(\bar{\alpha}) = T_p(\bar{a}) \quad \square$$

2.6 The Khovanov complex $kh(D, p)$.

Let D be a link diagram and p be a regular point in D . Such a pair (D, p) will be called a pointed diagram. The operator T_p induces an action of R on the complex $kh(D)$. Using this action $kh(D)$ becomes a graded differential R -module denoted by $kh(D, p)$ (or $kh(D, p, R)$). It is easy to see that $kh(D, p)$ is free over R .

The algebra R may be big: the transcendence degree of R_0 is 4. Nevertheless the complex $kh(D, p)$ can be reduced to a smaller complex.

Consider a graded commutative ring Λ . Denote by $\mathcal{M}_{**}(\Lambda)$ the class of bigraded differential free Λ -modules C satisfying the following:

$$\partial^\circ a = n, \quad \partial^\circ u = (p, q) \implies \partial^\circ(au) = (p, q + n), \quad \partial^\circ(du) = (p - 1, q - 1)$$

for every $a \in \Lambda$ and $u \in C$. The first component of this degree is called the homological degree and the second the q -degree.

Consider the ring $A = \mathbf{Z}[t, \delta, (1 - t\delta)^{-1}]$. This ring is graded by the rule:

$$\partial^\circ t = 2 \quad \partial^\circ \delta = -2$$

Moreover R is an A -algebra.

2.7 Proposition: *There is a correspondance associating to every pointed diagram (D, p) a complex $kh'(D, p) \in \mathcal{M}_{**}(A)$ and an isomorphism:*

$$kh(D, p) \xrightarrow{\sim} R \otimes_A kh'(D, p)$$

compatible with the degree in $kh(D, p)$ and the homological degree in $kh'(D, p)$.

Proof: Let (D, p) be a pointed diagram and X be the set of crossings of D . For any state $s : X \rightarrow \{\pm 1\}$, denote by C_s the set of connected components of D_s . This set is pointed by the component c_0 of D_s containing p . Denote by U_s the set of maps $\lambda : C_s \rightarrow \{\pm 1\}$ sending c_0 to 1. Finally denote by U the set of pairs (s, λ) , where s is a state and λ is an element of U_s .

Because of the universality of R_0 , we may as well suppose that R is the algebra R_0 . In this case, we have a degree in R :

$$\partial^\circ \alpha = \partial^\circ \bar{\alpha} = -2 \quad \partial^\circ \omega = \partial^\circ \bar{\omega} = 0 \implies \partial^\circ t = 2 \quad \partial^\circ \delta = -2$$

For every state s , set:

$$N_s = \bigoplus_{\lambda \in U_s} Re(s, \lambda) \quad M_s = \Lambda^-(X_s) \otimes N_s$$

where the elements $e(s, \lambda)$ are formal vectors in one to one correspondance to the elements of U . We set also:

$$M = \bigoplus_s M_s$$

We put a degree on N_s by:

$$\partial^\circ ae(s, \lambda) = \partial^\circ a + \sum_{c \in C_s} \lambda(c)$$

and a bidegree on M by:

$$\partial^\circ(u \otimes v) = (\partial^\circ u, \partial^\circ v)$$

Let s be a state. Take a numbering of C_s : $C_s = \{c_0, c_1, \dots, c_{q-1}\}$ in such a way that c_0 contains the point p . Every $\lambda \in U_s$ is on the form: $c_i \mapsto \lambda_i$ with $\lambda_0 = 1$.

Consider elements $a_i \in R$ for $0 \leq i < q$. For every $\lambda \in U_s$ we set:

$$a_i(\lambda) = \begin{cases} a_i & \text{if } \lambda_i = 1 \\ \varepsilon(a_i) & \text{if } \lambda_i = -1 \end{cases}$$

Then we get a map φ_s from $\Phi(D_s) = R^{\otimes C_s}$ to N_s defined by:

$$\bigotimes_i a_i \mapsto \sum_{\lambda \in U_s} \left(\prod_i a_i(\lambda) \right) e(s, \lambda)$$

It is easy to check that this map is R -linear and bijective. Its inverse is given by:

$$\varphi_s^{-1}(e(s, \lambda)) = \prod_{0 \leq i < q} b_i$$

with:

$$b_i = \begin{cases} \omega_i \omega_0^{-1} & \text{if } \lambda_i = 1 \\ \omega_i (\alpha_i - \alpha_0) & \text{if } \lambda_i = -1 \end{cases}$$

and: $u_i = 1^{\otimes i} \otimes u \otimes 1^{\otimes (q-i-1)}$ for $u = \omega$ or $u = \alpha$.

The isomorphisms φ_s induce an isomorphism φ from $kh(D, p)$ to M . Via this isomorphism, the differential on $kh(D, p)$ induces a differential d' on M . A straightforward computation shows that the bidegree of d' is $(-1, -1)$ and the entries of the matrix associated to d' are in $\{0, \pm 1, \pm \delta, \pm \theta^{-1}, \pm \theta^{-1}t, \pm \theta^{-1}\delta\}$. That implies the result with:

$$kh'(D, p) = \bigoplus_s \Lambda^-(X_s) \otimes \left(\bigoplus_{\lambda \in U_s} Ae(s, \lambda) \right) \quad \square$$

Using technics in [Kh4] it is possible to get a stronger reduction. Let β be the element $\alpha - \bar{\alpha} = \delta/\omega \in R_0$. Every Frobenius algebra of rank 2 is a $\mathbf{Z}[\beta]$ -algebra. Moreover the map $t \mapsto 0$ and $\delta \mapsto \beta$ induces a ring homomorphism $A \rightarrow \mathbf{Z}[\beta]$ and $\mathbf{Z}[\beta]$ is an A -algebra.

2.8 Proposition: *For each pointed diagram (D, p) , denote by $kh''(D, p)$ the complex $\mathbf{Z}[\beta] \otimes_A kh'(D, p) \in \mathcal{M}_{**}(\mathbf{Z}[\beta])$.*

Then, for each pointed diagram (D, p) , there is an isomorphism of complexes:

$$kh(D, p) \xrightarrow{\sim} R \otimes_{\mathbf{Z}[\beta]} kh''(D, p)$$

compatible with the degree in $kh(D, p)$ and the homological degree in $kh''(D, p)$.

Proof: Let R' be the algebra R equipped with the following coproduct and counit:

$$\Delta'(u) = \Delta(u/\omega) \quad \varepsilon'(u) = \varepsilon(\omega u)$$

It is easy to see that R' is a Frobenius algebra with generator α and twisting element 1. Then R' is an A -algebra and t and δ in A are sent to 0 and β in R . So the A -algebra structure of R' is actually an $\mathbf{Z}[\beta]$ -algebra structure.

Let (D, p) be a pointed diagram. For every state s , D_s is oriented. In particular D_1 (corresponding to the state $x \mapsto 1$) is the oriented boundary of a unique compact K in the plane. Consider a point q near p in the interior of K .

Let s be a state. If c is a component of D_s , c is the oriented boundary of a unique compact $K_s(c)$ in S^2 . So we define an integer $f(s, c)$ by:

$$f(s, c) = \begin{cases} \chi(K_s(c) \cap K) + 1 - \chi(K) & \text{if } q \in K_s(c) \\ \chi(K_s(c) \cap K) & \text{otherwise} \end{cases}$$

where χ is the Euler characteristic.

It is easy to see the following:

Let s be a state, and x be a crossing of D with $s(x) = 1$. Denote by s' the state s modified at x ($s' = se_x$). If the surgery operator γ_x connects two components c and c' of D_s into a component c'' of $D_{s'}$, we have:

$$f(s', c'') = f(s, c) + f(s, c')$$

If the surgery operator disconnects a component c of D_s into two components c' and c'' in $D_{s'}$, we have:

$$f(s, c) + 1 = f(s', c') + f(s', c'')$$

So, following [Kh4], we get an isomorphism between the complexes $kh(D, p, R)$ and $kh(D, p, R')$. This isomorphism is defined as follows:

$$u \otimes \left(\otimes_c v_c \right) \mapsto u \otimes \left(\otimes_c \frac{v_c}{\omega^{f(s,c)}} \right)$$

for $u \in \Lambda^-(X_s)$ and $v_c \in R$. Thus we have:

$$kh(D, p, R) \simeq kh(D, p, R') \simeq R' \otimes_A kh'(D, p)$$

$$\simeq R' \otimes_{\mathbf{Z}[\beta]} \mathbf{Z}[\beta] \otimes_A kh'(D, p) = R' \otimes_{\mathbf{Z}[\beta]} kh''(D, p) \quad \square$$

The only problem with this new reduction is the fact that β is not necessarily stable under the endomorphisms of R .

Another (more serious) problem is the fact that these reductions do not induce any canonical reduction for $kh(D)$.

2.9 The operators \widehat{T}_p .

If a link diagram D is oriented, it is possible to modify the operators T_p in the following way:

Suppose p is a regular point in D . Let q be a point in a neighborhood of p and on the left hand side of D and n be the winding number of D about q . For every $a \in R$, define $\widehat{T}_p(a)$ as the operator $T_p(a)$ if n is even and the operator $T_p(\bar{a})$ if n is odd. So \widehat{T}_p acts on $kh(D)$. It is easy to see that all these operators commute and that the homotopy class of $\widehat{T}_p(a)$ depends only on a and the component of D containing p .

2.10 Khovanov complexes of oriented diagrams.

Let D be an oriented link diagram. Since D is oriented, each crossing of D has a sign. Denote by X_- the set of negative crossings of D . So we define:

$$KH(D) = kh(D) \otimes \Lambda^+(X_-)$$

This is a graded differential K -module. If needed this complex will be denoted by $KH(D, R)$.

The main result of this paper is to prove that the correspondance $D \mapsto KH(D)$ comes from a monoidal functor from the category of cobordisms of oriented links in \mathbf{R}^3 to the homotopy category of K -complexes.

If $R = \mathbf{Z}[\alpha]/\alpha^2$, $KH(D)$ is essentially isomorphic to the classical Khovanov complex. But the isomorphism between these two complexes is not canonical. It is canonical only up to sign.

If p is a regular point in D , the operators $\widehat{T}_p(a)$ induce a structure of R -complex on $KH(D)$. This complex will be denoted by $KH(D, p)$ (or $KH(D, p, R)$).

We have also the complex $KH'(D, p)$: In the case: $\widehat{T}_p(a) = T_p(a)$ the complex $KH'(D, p)$ is the complex $kh'(D, p) \otimes \Lambda^+(X_-)$. In the case: $\widehat{T}_p(a) = T_p(\bar{a})$, $KH'(D, p)$ is the complex $kh'(D', p) \otimes \Lambda^+(X_-)$ where the D' is the diagram D with the opposite orientation. In any case $KH'(D, p)$ is a complex in $\mathcal{M}_{**}(A)$ and $KH(D, p)$ is isomorphic to $R \otimes_A KH'(D, p)$.

As before we have also the complex $KH''(D, p) = \mathbf{Z}[\beta] \otimes_A KH'(D, p)$ in $\mathcal{M}_{**}(\mathbf{Z}[\beta])$.

2.11 Proposition: *Let D be an oriented link diagram. Consider a circle C contained in a half plane disjoint from D and oriented clockwise. Denote by (D°, p) the union $D \cup C$ pointed by a point p in C . Then we have canonical isomorphisms of R -complexes:*

$$R \otimes_K KH(D) \simeq R \otimes_A KH'(D^\circ, p) \simeq R \otimes_{\mathbf{Z}[\beta]} KH''(D^\circ, p)$$

Proof: The R -complex $R \otimes KH(D)$ is obviously isomorphic to $KH(D^\circ, p, R)$. The result follows. \square

2.12 The modules $E(D)$ and $E(D, p)$

Consider an oriented link diagram D . Denote by C the set of components of D , by X the set of crossings of D and by $e : X \rightarrow \{\pm\}$ the map sending each crossing to its sign. For each map $\sigma : C \rightarrow \{\pm\}$ and each sign e denote by D_σ^e the subdiagram of D where σ is equal to e . Denote also by $D(\sigma)$ the diagram D where the orientation on D_σ^- is changed and by $Y(\sigma) \subset X$ the set of crossings between D_σ^+ and D_σ^- .

For any ring B denote by $E(D, B)$ the following graded B -module:

$$E(D, B) = \bigoplus_{\sigma: C \rightarrow \{\pm\}} \Lambda^{-e}(Y(\sigma)) \otimes Bv(\sigma)$$

where the $v(\sigma)$'s are formal vectors in one to one correspondance with the σ 's.

Let p be a regular point in D . Let e_0 be the sign with is equal to $+$ if and only if the operators T_p and \widehat{T}_p are the same on $KH(D)$. The set C is pointed by the component c_0 containing p . Denote by \widehat{C} the set of maps $\sigma : C \rightarrow \{\pm\}$ sending c_0 to e_0 . So, for any ring B denote by $E(D, p, B)$ the following graded B -module:

$$E(D, p, B) = \bigoplus_{\sigma \in \widehat{C}} \Lambda^{-e}(Y(\sigma)) \otimes Bv(\sigma)$$

If B is a graded ring, $E(D, B)$ and $E(D, p, B)$ is bigraded by the rule:

$$\partial^\circ(u \otimes bv(\sigma)) = (\partial^\circ u, \partial^\circ b + b_\sigma)$$

where b_σ the number of components of the oriented resolution of $D(\sigma)$. By equipping $E(D, B)$ and $E(D, p, B)$ with the zero differential, these modules become B -complexes or complexes in $\mathcal{M}_{**}(B)$.

2.13 Lemma: *Let (D, p) be a pointed oriented diagram. Then there are caconical morphisms of complexes:*

$$\varphi(R) : KH(D, p, R) \longrightarrow E(D, p, R)$$

$$\varphi' : KH'(D, p) \longrightarrow E(D, p, A)$$

$$\varphi'' : KH''(D, p) \longrightarrow E(D, p, \mathbf{Z}[\beta])$$

of degree 0 or $(0, 0)$.

Proof: Let σ be an element of \widehat{C} . The oriented resolution of $D(\sigma)$ (i.e. the only resolution compatible with the orientation) is the diagram D_s where s is the state $x \mapsto e(x)\sigma(c)\sigma(c')$ and c and c' the components of D containing x .

Let $Z(\sigma)$ be the complement of $Y(\sigma)$ in X . For each subset H of X we denote by H_+ (resp. H_-) the set of positive crossings (resp. negative crossings) in H . So we have:

$$X_s = Y(\sigma)_+ \cup Z(\sigma)_- \quad X_- = Y(\sigma)_- \cup Z(\sigma)_-$$

For any $u \in R$ and any sign e define the element $u^{(e)}$ by:

$$u^{(+)} = u \quad u^{(-)} = \bar{u}$$

The set C_s of components of D_s has $q = b_\sigma$ elements. Take a numbering of C_s : $C_s = (c_0(s), c_1(s), \dots, c_{q-1}(s))$, such that $c_0(s)$ contains the point p . For each $i < q$, denote by d_i the winding number of D_s about a point to the left of $c_i(s)$ and by a_i the sign $(-1)^{d_i}$.

We have a morphism f_σ from $KH(D)$ to $\Lambda^{-e}(Y(\sigma)) \otimes R$:

This morphism is trivial on $\Lambda^-(X_{s'}) \otimes \Phi(D_{s'}) \otimes \Lambda^+(X_-)$ for $s' \neq s$. For $s' = s$, we have:

$$\Lambda^-(X_s) \otimes \Phi(D_s) \otimes \Lambda^+(X_-) \simeq \Lambda^-(Y(\sigma)_+) \otimes \Lambda^-(Z(\sigma)_-) \otimes R^{\otimes q} \otimes \Lambda^+(Y(\sigma)_-) \otimes \Lambda^+(Z(\sigma)_-)$$

$$\simeq \Lambda^{-e}(Y(\sigma)) \otimes R^{\otimes q}$$

and $\Lambda^-(X_s) \otimes \Phi(D_s) \otimes \Lambda^+(X_-)$ is canonically isomorphic to $\Lambda^{-e}(Y(\sigma)) \otimes R^{\otimes q}$. So the map f_σ is on $\Lambda^-(X_s) \otimes \Phi(D_s) \otimes \Lambda^+(X_-)$ the following:

$$\Lambda^-(X_s) \otimes \Phi(D_s) \otimes \Lambda^+(X_-) \xrightarrow{\sim} \Lambda^{-e}(Y(\sigma)) \otimes R^{\otimes q} \xrightarrow{1 \otimes g} \Lambda^{-e}(Y(\sigma)) \otimes R$$

where g is the map:

$$b_0 \otimes b_1 \otimes \dots \otimes b_{q-1} \mapsto \prod_i b_i^{(a_i)}$$

More precisely, f_σ is the map:

$$u_1 \otimes u_2 \otimes u_3 \otimes u_4 \otimes u_5 \mapsto (-1)^{|u_2||u_4|} \langle u_2, u_5 \rangle u_1 \wedge u_4 \otimes g'(u_3)$$

for every $(u_1, u_2, u_3, u_4, u_5) \in \Lambda^-(Y(\sigma)_+) \times \Lambda^-(Z(\sigma)_-) \times \Phi(D_s) \times \Lambda^+(Y(\sigma)_-) \times \Lambda^+(Z(\sigma)_-)$, where $|u|$ is the degree of u , g' is the map:

$$\Phi(D_s) \xrightarrow{\sim} R^{\otimes q} \xrightarrow{g} R$$

and $\langle ?, ? \rangle$ is the isomorphism $\Lambda^-(Z(\sigma)_-) \otimes \Lambda^+(Z(\sigma)_-) \xrightarrow{\sim} \mathbf{Z}$ defined by:

$$\langle x_1 \wedge x_2 \wedge \dots \wedge x_n, x_1 \wedge x_2 \wedge \dots \wedge x_n \rangle = (-1)^{n(n-1)/2}$$

It is easy to see that this map is a morphism of degree 0 from $KH(D)$ to $\Lambda^{-e}(Y(\sigma)) \otimes R$ inducing a morphism $\varphi_\sigma(R)$ from $KH(D, p, R)$ to $\Lambda^{-e}(Y(\sigma)) \otimes Rv(\sigma)$.

On the level of complexes KH' , we get a morphism φ_σ from $KH'(D, p)$ to $\Lambda^{-e}(Y(\sigma)) \otimes Av(\sigma)$. This morphism sends $e(s, \lambda)$ to 0 if there exists an i with $a_i = e_0$ and $\lambda_i = -$. If it is not the case, the morphism sends $e(s, \lambda)$ to $\theta^{-c}(-\delta)^d v(D')$, where d is the number of i such that: $\lambda_i = -$ and c the number of i such that: $a_i = -e_0$. Because of the degree of $v(\sigma)$, the bidegree of φ_σ is $(0, 0)$.

By taking the sum of all these morphisms we get the desired morphisms $\varphi(R)$, φ' and then φ'' . \square

2.14 Remark: Suppose δ is invertible in R . Then the map $\varphi(R)$ has a section. By using the notations above this section is a morphism of complexes defined by:

$$\Lambda^{-e}(Y(\sigma)) \otimes Rv(\sigma) \xrightarrow{\sim} \Lambda^-(X_s) \otimes Rv(\sigma) \otimes \Lambda^+(X_-) \xrightarrow{1 \otimes g \otimes 1} \Lambda^-(X_s) \otimes \Phi(D_s) \otimes \Lambda^+(X_-)$$

where g is the map:

$$uv(\sigma) \mapsto u_0 \prod_{0 \leq i < j < q} \left(\delta_0^{-1} (\omega_i^{(a_i)} \alpha_j^{(a_j)} - (\omega_i \bar{\alpha}_i)^{(a_i)}) \right)$$

and u_i is, for every $u \in R$, the element $1^{\otimes i} \otimes u \otimes 1^{\otimes (q-i-1)}$.

2.15 Theorem: Let (D, p) be an pointed oriented link diagram. Suppose that the underlying graph of D is connected. Then the morphisms:

$$\varphi(R) : KH(D, p, R) \longrightarrow E(D, p, R)$$

$$\begin{aligned}\varphi' &: KH'(D, p) \longrightarrow E(D, p, A) \\ \varphi'' &: KH''(D, p) \longrightarrow E(D, p, \mathbf{Z}[\beta])\end{aligned}$$

are surjective and the action of β is homotopically nilpotent on their kernels.

Proof: Because the underlying graph of D is connected, the map sending $\sigma \in \widehat{C}$ to the state $s : x \mapsto e(x)\sigma(c)\sigma(c')$ is injective. Therefore all maps φ are surjective, in particular φ'' .

The last thing to do is to prove that the action of β on the kernel U of φ'' is homotopically nilpotent. But that is equivalent to the fact that $\mathbf{Z}[\beta^\pm] \otimes_{\mathbf{Z}[\beta]} U$ is acyclic. Let B be the ring $\mathbf{Z}[\beta^\pm]$ and R_1 be the algebra $B \times B$. Using the diagonal map $B \rightarrow B \times B$, R_1 is a Frobenius B -algebra of rank 2 with generator $(1, 0)$. In this algebra, the involution is: $(u, v) \mapsto (v, u)$, the twisting element is 1 and the counit is: $(u, v) \mapsto u - v$.

It is clear that $B \otimes U$ is acyclic if and only if $R_1 \otimes U$ is acyclic. Therefore it is enough to prove that $\varphi(R_1)$ induces an isomorphism in homology from $KH(D, p, R_1)$ to $E(D, p, R_1)$. But in R_1 we have two orthogonal idempotents:

$$\pi_+ = (1, 0) \quad \pi_- = (0, 1)$$

So we can use the Karoubi completion method of Bar-Natan and Morrison [BM] to prove that $\varphi(R_1)$ is a homology equivalence. \square

2.16 Remark: If the underlying graph of D is not connected the result is still true but the kernels have to be replaced by the homotopy kernels. In particular the morphisms φ are homotopy equivalences if δ is invertible in R .

3. Elementary moves.

Consider an elementary move (i.e. a Reidemeister move or a surgery move) $f : D \rightarrow D'$ transforming a link diagram D into a link diagram D' . If the correspondance $D \mapsto kh(D)$ comes from a functor on the category of cobordisms of links, the move f would have to induce a morphism f_* from $kh(D)$ to $kh(D')$. So the first thing to do is to associate to every elementary move $f : D \rightarrow D'$ a morphism from $kh(D)$ to $kh(D')$.

3.1 Proposition: *There is a correspondance associating to every elementary move $f : D \rightarrow D'$ a morphism $f^0 : kh(D) \rightarrow kh(D')$ such that:*

if f is a Reidemeister move with inverse move g , f^0 is a homotopy equivalence and g^0 is a homotopy inverse of f .

Proof: In order to construct the correspondance $f \mapsto f^0$, we have to consider all types of elementary moves.

Surgery moves: There is three kinds of surgery moves: surgery moves of index 0, 1 or 2. A surgery move $f : D \rightarrow D'$ of index 0 transforms D by adding a circle bounding

a disk in the plane which is disjoint from D and the inverse move $g : D' \rightarrow D$ is a surgery move of index 2.

$$f : \left[\begin{array}{|c|c|} \hline & \bigcirc \\ \hline \end{array} \right]$$

Let D be a link diagram. Consider a path γ embedded in the plane which doesn't meet any crossing point of D and intersects D in its boundary. Such a path is called a surgery path of D .

A surgery move $f : D \rightarrow D'$ of index 1 is a modification $D \mapsto D'$ by surgery along some surgery path γ of D .

$$f : \left[\begin{array}{|c|c|} \hline \text{)} \text{---} \text{(} & \text{---} \text{)} \text{---} \text{(} \\ \hline \end{array} \right]$$

Suppose $f : D \rightarrow D'$ is a surgery move of index 0 and $g : D' \rightarrow D$ its inverse move. Then we have: $kh(D') = kh(D) \otimes R$ and the map f^0 and g^0 are defined by:

$$\forall (u, a) \in kh(D) \times R, \quad f^0(u) = u \otimes 1, \quad g^0(u \otimes a) = u\varepsilon(a)$$

Suppose $f : D \rightarrow D'$ is a surgery move of index 1. This surgery move is defined by a surgery path γ . For every state s the surgery induces, via the functor Φ , a map $\gamma : \Phi(D_s) \rightarrow \Phi(D'_s)$ and these maps induce a map f^0 from $kh(D)$ to $kh(D')$. It is easy to see that f^0 is a morphism of complexes.

Reidemeister moves of type I: There is four kinds of Reidemeister moves of type I depending of two signs. The first sign is $+$ (or 1) if the move creates a new crossing and $-$ (or -1) if it removes one crossing. The second one is the sign of this crossing (for any orientation of the diagram). We say that f is a Reidemeister move of type I_e^e if f is a Reidemeister move where e is the first sign and e' the second one. It is clear that the inverse move of a Reidemeister move of type I_e^e is a Reidemeister move of type $I_{e'}^{-e}$.

Consider a Reidemeister move $f : D \rightarrow D'$ of type I_e^+ and denote by g its inverse move (of type I_e^-). Let x be the created crossing and U be the complex $kh(D)$. Let p be a point in D which is near x . So we have:

$$kh(D') = \begin{cases} U \otimes R \oplus x \otimes U & \text{if } e = + \\ U \oplus x \otimes U \otimes R & \text{if } e = - \end{cases}$$

Suppose $e = +$. Then the move is the following:

$$f : \left[\begin{array}{|c|c|} \hline p \quad | & x \quad \bigcirc \\ \hline \end{array} \right]$$

and we set:

$$\forall (u, a) \in U \times R, \quad f^0(u) = u \otimes \omega \alpha - T_p(\alpha)u \otimes \omega, \quad g^0(x \otimes u) = 0, \quad g^0(u \otimes a) = u\varepsilon(a)$$

Suppose $e = -$. Then the move is the following:

$$f : \left[\begin{array}{|c|c|} \hline p \quad | & x \text{ (loop)} \\ \hline \end{array} \right]$$

and we set:

$$\forall (u, a) \in U \times R, \quad f^0(u) = x \otimes u \otimes 1 \quad g^0(u) = 0, \quad g^0(x \otimes u \otimes a) = T_p(\bar{a})u$$

It is not difficult to see that f^0 and g^0 are morphisms of complexes and that g^0 is a left inverse of f^0 . Moreover the cokernel of f^0 is the mapping cone of an isomorphism. So f^0 is a homotopy equivalence and g^0 is a homotopy inverse of f^0 .

Reidemeister moves of type II: There is two kinds of Reidemeister moves of type II: Reidemeister moves of type II^+ that create two new crossings and their inverses of type II^- .

Consider a Reidemeister move $f : D \rightarrow D'$ of type II^+ and its inverse move $g : D' \rightarrow D$. Let D'' be the diagram obtained from D by a surgery along a path joining the two branches of D modified by f .

$$f : \left[\begin{array}{|c|c|} \hline p \quad | \quad | & x \text{ (crossing)} \\ \hline \end{array} \right] \quad D'' : \begin{array}{c} \cup \\ \cap \end{array}$$

Denote by U and V the complexes $kh(D)$ and $kh(D'')$. Then we have:

$$kh(D') = V \oplus x \otimes U \oplus y \otimes V \otimes R \oplus x \wedge y \otimes V$$

where x and y are the created crossings in D' . Surgery moves from D to D'' and from D'' to D induce morphisms $\gamma : U \rightarrow V$ and $\gamma : V \rightarrow U$. The morphisms f^0 and g^0 are defined by:

$$\forall u \in U, \quad f^0(u) = x \otimes u + y \otimes \gamma(T(\omega^{-1})u) \otimes \omega, \quad g^0(x \otimes u) = u$$

$$\forall (v, a) \in V \times R, \quad g^0(v) = g^0(x \wedge y \otimes v) = 0, \quad g^0(y \otimes v \otimes a) = -\varepsilon(a)\gamma(v)$$

where T is the operator T_p for some p in one of the two branches of D .

We can see that g^0 doesn't depend on p and f^0 and g^0 are morphisms of complexes. Moreover g^0 is a left inverse of f^0 and the cokernel of f^0 is the mapping cone of an isomorphism. So f^0 is a homotopy equivalence and g^0 is a homotopy inverse of f^0 .

Reidemeister moves of type III: There is two kinds of Reidemeister moves of type III depending on a sign. If $f : D \rightarrow D'$ is a Reidemeister move of type III, three crossings of D are modified. These crossings are the vertices of a triangle Θ which is oriented by the orientation of the plane. We number the branches starting with the

top branch and ending with the bottom one. So we get a numbering of the edges of the triangle. We say that the move is of type III_+ if this numbering is compatible with the orientation of Θ and III_- if it is not the case.

Let's denote by x the intersection of the top branch and the bottom branch. The other vertices of Θ are denoted by y and z in such a way that the numbering (x, y, z) is compatible with the orientation of the triangle. We proceed similarly for the diagram D' and the move is the following:

$$f : \begin{array}{|c|c|} \hline \begin{array}{c} \diagup x / \\ \diagdown y / \quad \diagdown z \\ \hline \end{array} & \begin{array}{c} \diagdown z \backslash / y \\ \diagup x / \\ \hline \end{array} \\ \hline \end{array}$$

By resolution of D near the triangle we get five new diagrams:

$$D_0 : \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \quad D_1 : \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \quad D_x : \begin{array}{c} \diagdown \quad \diagup \\ \hline \end{array} \quad D_y : \begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \quad D_z : \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}$$

and, for D' , we get: $D'_0 = D_1$, $D'_1 = D_0$, $D'_x = D_x$, $D'_y = D_y$ and $D'_z = D_z$.

With these new diagrams we get five new complexes: $U = kh(D_0)$, $V = kh(D_1)$, $A = kh(D_x)$, $B = kh(D_y)$, $C = kh(D_z)$.

Consider six points $p_i, i \in \mathbf{Z}/6$, sitted as follows in D :

$$\begin{array}{c} p_1 \quad p_6 \\ \diagdown \quad \diagup \\ p_2 \quad p_5 \\ \diagup \quad \diagdown \\ p_3 \quad p_4 \end{array}$$

All these points are in a circle Γ which bounds a disk Δ . Since the move modifies the diagram D only in Δ , the points p_i are also in diagrams D' , D_0 , D_1 , D_x , D_y and D_z .

The arc of Γ with endpoints p_i and p_{i-1} will be denoted by γ_i . This arc induces a surgery operator still denoted by γ_i . We'll denote also by T_i the operator T_{p_i} . All these operators are well defined on the complexes U, V, A, B and C .

Suppose f is a move of type III_+ . Then the move is the following:

$$f : \begin{array}{|c|c|} \hline \begin{array}{c} \diagup x / \\ \diagdown y / \quad \diagdown z \\ \hline \end{array} & \begin{array}{c} \diagdown z \backslash / y \\ \diagup x / \\ \hline \end{array} \\ \hline \end{array}$$

And we have:

$$kh(D) = V \oplus x \otimes V \otimes R \oplus y \otimes C \oplus z \otimes B \oplus y \wedge z \otimes U \oplus x \wedge z \otimes V \oplus x \wedge y \otimes V \oplus x \wedge y \wedge z \otimes A$$

$$kh(D') = U \oplus x \otimes U \otimes R \oplus y \otimes C \oplus z \otimes B \oplus y \wedge z \otimes V \oplus x \wedge z \otimes U \oplus x \wedge y \otimes U \oplus x \wedge y \wedge z \otimes A$$

In $kh(D)$ and $kh(D')$ the differential d is on the form $d = d_1 + d_2$, where d_1 corresponds to the differentials of the complexes U, V, A, B, C . The map d_2 is defined on $kh(D)$ by the following (for any $r \in R, u \in U, v \in V, a \in A, b \in B, c \in C$):

$$\begin{aligned}
d_2(v) &= x \otimes \gamma_1(v) + y \otimes \gamma_2(v) + z \otimes \gamma_6(v) \\
d_2(x \otimes v \otimes r) &= -x \wedge y \otimes T_3(r)v - x \wedge z \otimes T_5(r)v \\
d_2(y \otimes c) &= x \wedge y \otimes \gamma_1(c) - y \wedge z \otimes \gamma_4(c) \\
d_2(z \otimes b) &= x \wedge z \otimes \gamma_1(b) + y \wedge z \otimes \gamma_4(b) \\
d_2(y \wedge z \otimes u) &= x \wedge y \wedge z \otimes \gamma_1(u) \\
d_2(x \wedge z \otimes v) &= -x \wedge y \wedge z \otimes \gamma_4(v) \\
d_2(x \wedge y \otimes v) &= x \wedge y \wedge z \otimes \gamma_4(v) \\
d_2(x \wedge y \wedge z \otimes a) &= 0
\end{aligned}$$

and, by exchanging U and V and replacing γ_i and T_i by γ_{i+3} and T_{i+3} , we get the differential d_2 on $kh(D')$.

So we define the map f^0 by the following:

$$f^0(v) = 0$$

$$\begin{aligned}
f^0(x \otimes v \otimes r) &= \varepsilon(r\omega) \left(x \otimes T_1(\omega^{-1})\gamma_2\gamma_6(v) \otimes 1 + y \otimes T_1(\omega^{-1})\gamma_2(v) + z \otimes T_1(\omega^{-1})\gamma_6(v) \right) \\
f^0(y \otimes c) &= -x \otimes \gamma_4(c) \otimes 1 - y \otimes c \\
f^0(z \otimes b) &= -z \otimes b \\
f^0(x \wedge y \otimes v) &= y \wedge z \otimes v \\
f^0(x \wedge z \otimes v) &= -y \wedge z \otimes v \\
f^0(y \wedge z \otimes u) &= -x \wedge z \otimes u \\
f^0(x \wedge y \wedge z \otimes a) &= x \wedge y \wedge z \otimes a
\end{aligned}$$

We can check that f^0 is a morphism of complexes of degree 0. Let W be the \mathbf{Z} -module freely generated by $\lambda, d(\lambda), \mu$ and $d(\mu)$. By setting:

$$\partial^\circ \lambda = 0 \quad \partial^\circ \mu = -1$$

W becomes an acyclic graded differential \mathbf{Z} -module. We have maps $\Psi : W \otimes V \rightarrow kh(D)$ and $\Psi' : W \otimes U \rightarrow kh(D')$ defined by:

$$\begin{aligned}
\Psi(\lambda \otimes v) &= v & \Psi(d(\lambda) \otimes v) &= d_2(v) \\
\Psi(\mu \otimes v) &= x \otimes v \otimes 1 & \Psi(d(\mu) \otimes v) &= d_2(x \otimes v \otimes 1) \\
\Psi'(\lambda \otimes u) &= u & \Psi'(d(\lambda) \otimes u) &= d_2(u) \\
\Psi'(\mu \otimes u) &= x \otimes u \otimes 1 & \Psi'(d(\mu) \otimes u) &= d_2(x \otimes u \otimes 1)
\end{aligned}$$

It is easy to see that Ψ and Ψ' are injective morphisms of complexes of degree 0. Since W is acyclic, the images of Ψ and Ψ' are acyclic too. Moreover the image of Ψ is killed by f^0 and the image of Ψ' is therefore killed by g^0 , where g is the inverse move of f . Modulo these images we have:

$$kh(D) \equiv y \otimes C \oplus z \otimes B \oplus y \wedge z \otimes U \oplus x \wedge z \otimes V \oplus x \wedge y \wedge z \otimes A$$

$$kh(D') \equiv y \otimes C \oplus z \otimes B \oplus y \wedge z \otimes V \oplus x \wedge z \otimes U \oplus x \wedge y \wedge z \otimes A$$

and maps f^0 and g^0 are on these quotients:

$$f^0 : \begin{cases} y \otimes c \mapsto -y \otimes c \\ z \otimes b \mapsto -z \otimes b \\ y \wedge z \otimes u \mapsto -x \wedge z \otimes u \\ x \wedge z \otimes v \mapsto -y \wedge z \otimes v \\ x \wedge y \wedge z \otimes a \mapsto x \wedge y \wedge z \otimes a \end{cases} \quad g^0 : \begin{cases} y \otimes c \mapsto -y \otimes c \\ z \otimes b \mapsto -z \otimes b \\ y \wedge z \otimes v \mapsto -x \wedge z \otimes v \\ x \wedge z \otimes u \mapsto -y \wedge z \otimes u \\ x \wedge y \wedge z \otimes a \mapsto x \wedge y \wedge z \otimes a \end{cases}$$

Therefore f^0 and g^0 are homotopy equivalences and g^0 is a homotopy inverse of f^0 . Suppose now the move is of type III₋. The move is the following:

$$f : \begin{array}{|c|c|} \hline \begin{array}{c} x/ \\ \hline y/ \quad z \end{array} & \begin{array}{c} z \quad /y \\ \hline /x \end{array} \\ \hline \end{array}$$

And we have:

$$kh(D) = A \oplus x \otimes U \oplus y \otimes V \oplus z \otimes V \oplus y \wedge z \otimes V \oplus R \oplus x \wedge z \otimes C \oplus x \wedge y \otimes B \oplus x \wedge y \wedge z \otimes V$$

$$kh(D') = A \oplus x \otimes V \oplus y \otimes U \oplus z \otimes U \oplus y \wedge z \otimes U \oplus R \oplus x \wedge z \otimes C \oplus x \wedge y \otimes B \oplus x \wedge y \wedge z \otimes U$$

As before the differentials of $kh(D)$ and $kh(D')$ are on the form $d = d_1 + d_2$, where d_1 corresponds to the differentials of the complexes U, V, A, B, C . The map d_2 is defined on $kh(D)$ by the following:

$$d_2(a) = x \otimes \gamma_2(a) + y \otimes \gamma_5(a) + z \otimes \gamma_5(a)$$

$$d_2(x \otimes u) = -x \wedge y \otimes \gamma_3(u) - x \wedge z \otimes \gamma_5(u)$$

$$d_2(y \otimes v) = x \wedge y \otimes \gamma_6(v) - y \wedge z \otimes \gamma_5(v)$$

$$d_2(z \otimes v) = x \wedge z \otimes \gamma_2(v) + y \wedge z \otimes \gamma_3(v)$$

$$d_2(y \wedge z \otimes v \otimes r) = x \wedge y \wedge z \otimes T_1(r)v$$

$$d_2(x \wedge y \otimes b) = x \wedge y \wedge z \otimes \gamma_1(b)$$

$$d_2(x \wedge z \otimes c) = -x \wedge y \wedge z \otimes \gamma_1(c)$$

$$d_2(x \wedge y \wedge z \otimes v) = 0$$

and, as before, we get the differential d_2 on $kh(D')$ by exchanging U and V and replacing γ_i and T_i by γ_{i+3} and T_{i+3} . So we define the map f^0 by the following:

$$\begin{aligned}
f^0(a) &= a \\
f^0(x \otimes u) &= (y + z) \otimes u \\
f^0(y \otimes v) &= x \otimes v \\
f^0(z \otimes v) &= 0 \\
f^0(y \wedge z \otimes v \otimes r) &= \varepsilon(r)(x \wedge z \otimes \gamma_2(v) + y \wedge z \otimes T_4(\omega^{-1})\gamma_4\gamma_2(v) \otimes \omega) \\
f^0(x \wedge y \otimes b) &= -x \wedge y \otimes b + y \wedge z \otimes T_4(\omega^{-1})\gamma_4(b) \otimes \omega \\
f^0(x \wedge z \otimes c) &= -x \wedge z \otimes c - y \wedge z \otimes T_4(\omega^{-1})\gamma_4(c) \otimes \omega \\
f^0(x \wedge y \wedge z \otimes v) &= 0
\end{aligned}$$

We can check that f^0 is a morphism of complexes of degree 0. We have maps $\Psi : W \otimes V \rightarrow kh(D)$ and $\Psi' : W \otimes U \rightarrow kh(D')$:

$$\begin{aligned}
\Psi(\lambda \otimes v) &= z \otimes v & \Psi(d(\lambda) \otimes v) &= d_2(z \otimes v) \\
\Psi(\mu \otimes v) &= y \wedge z \otimes v \otimes \omega & \Psi(d(\mu) \otimes v) &= d_2(y \wedge z \otimes v \otimes \omega) \\
\Psi'(\lambda \otimes u) &= z \otimes u & \Psi'(d(\lambda) \otimes u) &= d_2(z \otimes u) \\
\Psi'(\mu \otimes u) &= y \wedge z \otimes u \otimes \omega & \Psi'(d(\mu) \otimes u) &= d_2(y \wedge z \otimes u \otimes \omega)
\end{aligned}$$

These two maps are injective morphisms of complexes of degree -1 and their images are acyclic subcomplexes killed by f^0 and g^0 . Modulo these images we have:

$$\begin{aligned}
kh(D) &\equiv A \oplus x \otimes U \oplus y \otimes V \oplus x \wedge y \otimes B \oplus x \wedge z \otimes C \\
kh(D') &\equiv A \oplus x \otimes V \oplus y \otimes U \oplus x \wedge y \otimes B \oplus x \wedge z \otimes C
\end{aligned}$$

and maps f^0 and g^0 are:

$$f^0 : \begin{cases} a \mapsto a \\ x \otimes u \mapsto y \otimes u \\ y \otimes v \mapsto x \otimes v \\ x \wedge y \otimes b \mapsto -x \wedge y \otimes b \\ x \wedge z \otimes c \mapsto -x \wedge z \otimes c \end{cases} \quad g^0 : \begin{cases} a \mapsto a \\ x \otimes v \mapsto y \otimes v \\ y \otimes u \mapsto x \otimes u \\ x \wedge y \otimes b \mapsto -x \wedge y \otimes b \\ x \wedge z \otimes c \mapsto -x \wedge z \otimes c \end{cases}$$

Therefore f^0 and g^0 are homotopy equivalences and g^0 is a homotopy inverse of f^0 . \square

3.2 Remark: For every elementary move f , the map f^0 is a morphism of degree 0 except if f is a Reidemeister move of type I_e^- or II^e (with $e = \pm$). In these cases f^0 is a morphism of degree $-e$.

3.3 Elementary moves for oriented link diagrams. Consider an oriented link diagram D . Denote by w the function sending each point x outside of D to the winding number of D about x . The sign $(-1)^{w(x)}$ will be called the D -sign of x . If x

is a crossing of D , the function w takes three values $n - 1, n, n + 1$ near x . The sign $(-1)^n$ will also be called the D -sign of x .

In order to describe the orientation of a curve we'll use the following convention: suppose a is a sign. Then the figure



means that the curve is oriented by the arrow if $a = +$ and by the opposite orientation if $a = -$.

Let $f : D \rightarrow D'$ be a Reidemeister move of type I_e^+ between oriented diagrams and g be its inverse move. Let a be the winding number of the created loop and h be the D' -sign of the created crossing. We say that f is a Reidemeister move of type $I^+(e, a, h)$ and that g is a Reidemeister move of type $I^-(e, a, h)$. In this case the diagram D' is oriented as follows:



Let $f : D \rightarrow D'$ be a Reidemeister move of type II^+ between oriented diagrams and g be its inverse move. Let Γ be the created loop. This loop is oriented by the orientation of the plane. Let a (resp. b) be the sign which is equal to $+$ if and only if the orientation of the top branch (resp. the bottom branch) agrees with the orientation of Γ . Let h be the common D' -sign of the created crossings. We say that f is a Reidemeister move of type $II^+(a, b, h)$ and that g is a Reidemeister move of type $II^-(a, b, h)$. In this case the diagram D' is oriented as follows:



For technical reasons we'll say also that f is a Reidemeister move of type $II^e(e', a, b, h)$ if f is a Reidemeister move of type $II^+(a, b, h)$ (resp. $II^-(b, a, h)$) if $e' = +$ (resp. $e' = -$).

Let $f : D \rightarrow D'$ be a Reidemeister move of type III_e between oriented diagrams. Let Θ be the triangle in D modified by f . This triangle is oriented by the orientation of the plane. Let a (resp. b, c) be the sign which is equal to $+$ if and only if the orientation of the top branch (resp. the middle branch, the bottom branch) agrees with the orientation of the triangle. Let h be the D -sign of the center of Θ . We say that f is a Reidemeister move of type $III(e, a, b, c, h)$. The diagram D is oriented as follows:



depending if $e = +$ or $e = -$.

It is easy to see that the inverse move of f is a Reidemeister move of type $III(e, -a, -b, -c, -h)$.

Let $f : D \rightarrow D'$ be a surgery move of index 0 between oriented diagrams and g be its inverse move. Let a be the winding number of the created circle and h be the D -sign of a point in this circle. We say that f is a surgery move of type $(0, a, h)$ and that g is a surgery move of type $(2, a, h)$.

Let $f : D \rightarrow D'$ be a surgery move of index 1 between oriented diagrams. Let γ be a path inducing this surgery. Let a be the sign which is equal to $+$ if and only if γ is on the left of D and h be the D -sign of the middle point in γ . We say that f is a surgery move of type $(1, a, h)$. It is easy to see that the inverse move of f is a surgery move of type $(1, -a, -h)$.

3.4 The correspondance $f \mapsto f^1$.

Consider an elementary move $f : D \rightarrow D'$ where D and D' are oriented link diagrams. This move is called an elementary move of oriented diagrams if D and D' have the same orientation outside the modification area.

Here we'll associate to each elementary move $f : D \rightarrow D'$ of oriented diagrams a morphism f^1 from $KH(D)$ to $KH(D')$.

Denote by X_- and X'_- the set of negative crossings of D and D' .

Suppose the move is a Reidemeister move of type I_+^+ or II^+ . In this case X'_- is the union of X_- and one crossing x and the map f^1 is defined by:

$$\forall (u, v) \in kh(D) \times \Lambda^+(X_-), \quad f^1(u \otimes v) = (-1)^{|u|} f^0(u) \otimes x \wedge v$$

where $|u|$ is the degree of u .

Suppose the move is a Reidemeister move of type I_-^- or II^- . In this case X'_- is the union of X'_- and one crossing x and the map f^1 is defined by:

$$\forall (u, v) \in kh(D) \times \Lambda^+(X'_-), \quad f^1(u \otimes x \wedge v) = -(-1)^{|u|} f^0(u) \otimes v$$

In all other cases, X_- and X'_- are the same and the map f^1 is defined by:

$$\forall (u, v) \in kh(D) \times \Lambda^+(X_-), \quad f^1(u \otimes v) = f^0(u) \otimes v$$

It is not difficult to see that each map f^1 is a morphism of complexes of degree 0. Moreover if f is a Reidemeister move with inverse move g , g^1 is a homotopy inverse of f^1 .

Notice that each map f^0 and each map f^1 are invariant under any endomorphism of R .

3.5 The correspondance $f \mapsto f^{\mathcal{K}}$.

Consider elements $A(e, a, h), X(a, h), Y(a, h), Z(a, h)$ in R^* , elements $B(a, b, h)$ in $(R \otimes R)^*$ and elements $C(e, a, b, c)$ in $(R \otimes R \otimes R)^*$, depending on signs e, a, b, c, h . Such a data $\mathcal{K} = (A, B, C, X, Y, Z)$ will be called a Khovanov data. Using this data we'll construct a correspondance associating to each elementary move $f : D \rightarrow D'$ a morphism $f^{\mathcal{K}} : KH(D) \rightarrow KH(D')$.

Let $f : D \rightarrow D'$ be a Reidemeister move of type $I^+(e, a, h)$ and g be its inverse move. Let $p \in D$ be a point in the modified branch of D . The morphisms $f^{\mathcal{K}}$ and $g^{\mathcal{K}}$ are defined by:

$$f^{\mathcal{K}} = f^1 \circ \widehat{T}_p(A(e, a, h)) \quad g^{\mathcal{K}} = \widehat{T}_p(A(e, a, h)^{-1}) \circ g^1$$

Let $f : D \rightarrow D'$ be a Reidemeister move of type $II^+(a, b, h)$ and g be its inverse move. Let p be a point in the top branch and q be a point in the bottom branch. The morphisms $f^{\mathcal{K}}$ and $g^{\mathcal{K}}$ are defined by:

$$f^{\mathcal{K}} = f^1 \circ \widehat{T}(B(a, b, h)) \quad g^{\mathcal{K}} = \widehat{T}(B(a, b, h)^{-1}) \circ g^1$$

where \widehat{T} is the map: $u \otimes v \mapsto \widehat{T}_p(u)\widehat{T}_q(v)$.

Let $f : D \rightarrow D'$ be a Reidemeister move of type $III(e, a, b, c, h)$ and g be its inverse move. The move g is a Reidemeister move of type $III(e, -a, -b, -c, -h)$. Let B_1 (resp. B_2, B_3) be the top branch (resp. the middle branch, the bottom branch) near the modified triangle in D . For $i = 1, 2, 3$, denote by p_i a point in B_i and denote by \widehat{T} the map: $u \otimes v \otimes w \mapsto \widehat{T}_{p_1}(u)\widehat{T}_{p_2}(v)\widehat{T}_{p_3}(w)$. Suppose that $D(e, a, b, c, h)$ are elements in $(R \otimes R \otimes R)^*$ then we can set: $\widehat{f} = f^1 \circ \widehat{T}(D(e, a, b, c, h))$. But the condition: $\widehat{f} \circ \widehat{g} \sim \text{Id}$ is equivalent to the condition:

$$D(e, a, b, c, h)D(e, -a, -b, -c, -h) = 1$$

and that's equivalent to the fact that $D(e, a, b, c, h)$ is on the form $D'(e, ah, bh, ch)^h$. So we define $f^{\mathcal{K}}$ by:

$$f^{\mathcal{K}} = f^1 \circ \widehat{T}(C(e, ah, bh, ch)^h)$$

Let $f : D \rightarrow D'$ be a surgery move of type $(0, a, h)$ and g be its inverse move. Let p be a point in the created circle. Then we define $f^{\mathcal{K}}$ and $g^{\mathcal{K}}$ by:

$$f^{\mathcal{K}} = \widehat{T}_p(X(a, h)) \circ f^1 \quad g^{\mathcal{K}} = g^1 \circ \widehat{T}_p(Z(a, h))$$

Let $f : D \rightarrow D'$ be a surgery move of type $(1, a, h)$. Let p be a point in the boundary of the surgery path. Then $f^{\mathcal{K}}$ is defined by:

$$f^{\mathcal{K}} = f^1 \circ \widehat{T}_p(Y(a, h))$$

It is easy to see that, for every elementary move f , $f^{\mathcal{K}}$ is a morphism of degree 0 well defined up to homotopy. Moreover, if g is the inverse move of a Reidemeister move f , $g^{\mathcal{K}}$ is a homotopy inverse of $f^{\mathcal{K}}$.

4. Movie moves.

In all this section $\mathcal{K} = (A, B, C, X, Y, Z)$ is a given Khovanov data. For technical reasons we define the elements $B^e(a, b, h)$ by:

$$B(a, b, h) = u \otimes v \implies B^0(a, b, h) = uv \quad B^+(a, b, h) = u \otimes v \quad B^-(b, a, h) = v \otimes u$$

Let \mathcal{D} be the set of link diagrams or the set of oriented link diagrams and A be a correspondance associating to each $D \in \mathcal{D}$ a K -complex $A(D)$.

In \mathcal{D} we have elementary moves and every elementary move $f : D \rightarrow D'$ has an inverse move: $\bar{f} : D' \rightarrow D$.

Consider diagrams D_0, D_1, \dots, D_n in \mathcal{D} and elementary moves $f_i : D_{i-1} \rightarrow D_i$. Such a sequence $\varphi = (D_0, D_1, \dots, D_n)$ (or $\varphi = (f_1, f_2, \dots, f_n)$) will be called a movie sequence from D_0 to D_n . If $D = D_0$ and $D' = D_n$ are the diagrams of two links L and L' , the movie sequence φ induces a cobordism from L to L' and this cobordism is oriented in the oriented case. By replacing every elementary moves by its inverse, we get a new movie sequence: $\bar{\varphi} = (\bar{f}_n, \dots, \bar{f}_1)$ from D' to D and the cobordism associated to $\bar{\varphi}$ is isotopic to the opposite of the cobordism associated to φ .

Suppose all the elementary moves of a movie sequence φ are Reidemeister moves. Then the movie sequence induces an isotopy from L to L' . Suppose also D is equal to D' and the isotopy from L to $L' = L$ is isotopic to the identity. In this case the movie sequence φ will be called a movie move of type I.

Consider $\varphi = (D_0, D_1, \dots, D_p)$ and $\psi = (D'_0, D'_1, \dots, D'_q)$ two movie sequences from $D_0 = D'_0$ to $D_p = D'_q$. The diagrams D_0 and D_p are the diagrams of two links L and L' and the two movie sequences induce two cobordisms from L to L' . Suppose these two cobordisms are isotopic. Then the pair (φ, ψ) will be called a movie move of type II.

Consider a correspondance $f \mapsto \hat{f}$ associating to every elementary move $f : D \rightarrow D'$ in \mathcal{D} a morphism \hat{f} from $A(D)$ to $A(D')$ and suppose that this correspondance has the following property: for every Reidemeister move f with inverse move g , \hat{g} is a homotopy inverse of \hat{f} .

Let $\varphi = (f_1, f_2, \dots, f_n)$ be a movie sequence from D to D' . Then we have a map $\hat{\varphi} = \hat{f}_n \circ \dots \circ \hat{f}_1$ from $A(D)$ to $A(D')$.

Suppose this correspondance is coming from a functor from the category of cobordisms of links to the homotopy category of K -complexes. In this case, for each movie move φ of type I, the morphism $\hat{\varphi}$ is homotopic to the identity and for each movie move (φ, ψ) of type II, the two morphisms $\hat{\varphi}$ and $\hat{\psi}$ are homotopic.

Let φ be a movie move of type I. Then the morphism $\hat{\varphi}$ is a homotopy inverse of $\hat{\varphi}$ and the condition " $\hat{\varphi}$ is homotopic to the identity " implies the same condition for the morphism $\hat{\varphi}$.

But if (φ, ψ) is a movie move of type II, we get two conditions: $\hat{\varphi}$ and $\hat{\psi}$ are homotopic and $\hat{\varphi}$ and $\hat{\psi}$ are also homotopic.

So a movie move of type I induces one condition on the correspondance and a movie move of type II induces two conditions.

We'll apply this construction in three cases:

— \mathcal{D} is the set of link diagrams, $A(D)$ is the Khovanov complex $kh(D)$ and the correspondance is: $f \mapsto f^0$.

— \mathcal{D} is the set of oriented link diagrams, $A(D)$ is the Khovanov complex $KH(D)$ and the correspondance is: $f \mapsto f^1$.

— \mathcal{D} is the set of oriented link diagrams, $A(D)$ is the Khovanov complex $KH(D)$ and the correspondance is: $f \mapsto f^{\mathcal{K}}$.

4.0 Notation: From now on we'll use the following notation: for every element $u \in R$ and every sign e we set:

$$u^{(e)} = \begin{cases} u & \text{if } e = + \\ \bar{u} & \text{if } e = - \end{cases}$$

4.1 Movie moves of type MVM_0 . Consider a link diagram D and two elementary moves f and g modifying D on disjoint areas. Let D_1 be the diagram D modified by f and D_2 be the diagram D modified by g . The move f (resp. g) induces a well defined move f' (resp. g') on D_2 (resp. D_1) and we have a commutative diagram:

$$\begin{array}{ccc} D & \xrightarrow{f} & D_1 \\ g \downarrow & & g' \downarrow \\ D_2 & \xrightarrow{f'} & D' \end{array}$$

So we get two movie sequences: $\varphi = (f, g')$ and $\psi = (g, f')$ and a movie move (φ, ψ) of type II. This movie move will be called a movie move of type MVM_0 .

4.1.a Lemma: Let (φ, ψ) be a movie move of type MVM_0 associated to elementary moves f and g . Then we have:

$$\varphi^0 = (-1)^{pq} \psi^0$$

where p and q are the degrees of the map f^0 and g^0 .

In the oriented case we have:

$$\varphi^1 = \psi^1 \quad \varphi^{\mathcal{K}} = \psi^{\mathcal{K}} \quad \bar{\varphi}^{\mathcal{K}} = \bar{\psi}^{\mathcal{K}}$$

Proof: It is easy to see that we have some commutativity in the graded sense and that implies the first formula and therefore the other ones.

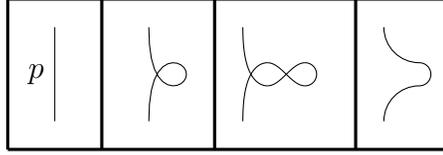
For example, consider the case where f and g are Reidemeister moves of type $I^+(-, a, h)$ and $I^+(-, b, k)$. Let x (resp. y) be the crossing created by f (resp. g) and p (resp. q) be a regular point in D near x (resp. near y). Let X_- be the set of negative crossings of the diagram D . So, for every $u \in kh(D)$ and every $v \in \Lambda^+(X_-)$, we have:

$$\begin{aligned} \varphi^0(u) &= f^0(y \otimes u \otimes 1) = x \wedge y \otimes u \otimes 1 \otimes 1 \\ \psi^0(u) &= g^0(x \otimes u \otimes 1) = y \wedge x \otimes u \otimes 1 \otimes 1 \implies \psi^0 = -\varphi^0 \\ \varphi^1(u \otimes v) &= f^1(g^0(u) \otimes y \wedge v) = \varphi^0(u) \otimes x \wedge y \wedge v \\ \psi^1(u \otimes v) &= g^1(f^0(u) \otimes x \wedge v) = \psi^0(u) \otimes y \wedge x \wedge v \implies \psi^1 = \varphi^1 \end{aligned}$$

$$\varphi^{\mathcal{K}} = \varphi^1 \circ \widehat{T}_p(A(-, a, h))\widehat{T}_q(A(-, b, k)) = \psi^1 \circ \widehat{T}_q(A(-, b, k))\widehat{T}_p(A(-, a, h)) = \psi^{\mathcal{K}}$$

The last formula $\overline{\varphi}^{\mathcal{K}} = \overline{\psi}^{\mathcal{K}}$ is just the formula $\varphi^{\mathcal{K}} = \psi^{\mathcal{K}}$ applied for Reidemeister moves \overline{g} and \overline{f} . \square

4.2 Movie moves of type MVM_1 . Consider a link diagram D and a regular point p in D . Let e be a sign. We may apply a Reidemeister move of type I_e^+ on D near p , then a new Reidemeister move of type I_{-e}^- in the created loop. By applying a Reidemeister move of type II^- we may remove the two created crossings and recover the diagram D . So we have a movie move of type I. This move will be called a movie move of type $MVM_1(e)$ near p .



4.2.a Lemma: Let φ be a movie move of type $MVM_1(e)$ near a point $p \in D$. Then we have:

$$\varphi^0 = -eT_p\left(\frac{\omega}{\omega(e)}\right)$$

In the oriented case, suppose the first move is a Reidemeister move of type $I^+(e, a, h)$. Then we have:

$$\begin{aligned} \varphi^1 &= -eT_p\left(\frac{\omega}{\omega(e)}\right) = -e\widehat{T}_p\left(\frac{\omega^{(ah)}}{\omega^{(eah)}}\right) \\ \varphi^{\mathcal{K}} &= -e\widehat{T}_p\left(\frac{\omega^{(ah)}}{\omega^{(eah)}} \frac{A(e, a, h)A(-e, -a, h)}{B^0(a, a, h)}\right) \end{aligned}$$

Proof: A straightforward computation gives the following:

$$\begin{aligned} e = + &\implies \varphi^0 = -\text{Id} \\ e = - &\implies \varphi^0 = T_p\left(\frac{\omega}{\overline{\omega}}\right) \end{aligned}$$

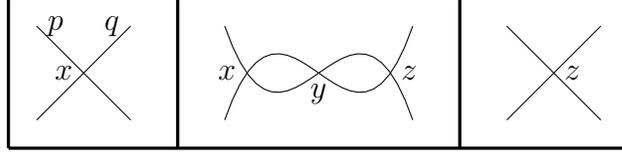
So we get the first formula. The second formula: $\varphi^1 = -eT_p(\omega/\omega^{(e)})$ is easy to deduce.

Since the first move is a Reidemeister move of type $I^+(e, a, h)$, the orientation of D is given by the following:

$$\begin{array}{c} p \\ \downarrow \\ a \end{array}$$

So the second move is a Reidemeister move of type $I^+(-e, -a, h)$ and the last one is a Reidemeister move of type $II^-(a, a, h)$. The result follows. \square

4.3 Movie moves of type MVM_2 . Let D be a link diagram and x be a crossing of D . We may modify D by a Reidemeister move of type II^+ which creates two new crossings y and z and then remove x and y by a Reidemeister move of type II^- .



So we get a movie move. We say that this move is a movie move of type $MVM_2(e)$ near (p, q) where $e = +$ (resp. $e = -$) if the branch containing p is over (resp. under) the branch containing q .

4.3.a Lemma: Let e be a sign and φ be a movie move of type $MVM_2(e)$ near (p, q) . Then we have:

$$\varphi^0 = eT_p(\omega)T_q(\omega^{-1})$$

In the oriented case, suppose the first move is a Reidemeister move of type $II^+(e, a, b, h)$. Then we have:

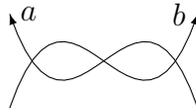
$$\varphi^1 = -abT_p(\omega)T_q(\omega^{-1}) = -ab\widehat{T}(\omega^{(-bh)} \otimes \frac{1}{\omega^{(ah)}})$$

$$\varphi^{\mathcal{K}} = -ab\widehat{T}\left(\left(\omega^{(-bh)} \otimes \frac{1}{\omega^{(ah)}}\right) \frac{B^e(a, b, h)}{B^e(-a, -b, h)}\right)$$

where \widehat{T} is the map $u \otimes v \mapsto \widehat{T}_p(u)\widehat{T}_q(v)$.

Proof: A straightforward computation gives the first formula. Let f and g be the Reidemeister moves in φ and X_- be the set of negative crossings of D .

So it is easy to see that the orientation of the second diagram is given by:



and the two moves are Reidemeister moves of type $II^+(e, a, b, h)$ and $II^-(e, -a, -b, h)$. Moreover the signs of the crossings x, y, z are: $-eab, eab, -eab$.

If $eab = +$, x is in X_- and we have, for every $u \in kh(D)$ and every $x \wedge v \in \Lambda^+(X_-)$:

$$\begin{aligned} \varphi^1(u \otimes x \wedge v) &= (-1)^{|u|}g^1(f^0(u) \otimes z \wedge x \wedge v) = -(-1)^{|u|}g^1(f^0(u) \otimes x \wedge z \wedge v) \\ &= (-1)^{|f^0(u)|}(-1)^{|u|}g^0(u) \otimes z \wedge v = -\varphi^0(u) \otimes z \wedge v \end{aligned}$$

and that implies:

$$\varphi^1 = -abT_p(\omega)T_q(\omega^{-1})$$

If $eab = -$, x is not in X_- and we have, for every $u \in kh(D)$ and every $v \in \Lambda^+(X_-)$:

$$\varphi^1(u \otimes v) = (-1)^{|u|}g^1(f^0(u) \otimes y \wedge v) = -(-1)^{|f^0(u)|}(-1)^{|u|}g^0(u) \otimes v = \varphi^0(u) \otimes v$$

and that implies again:

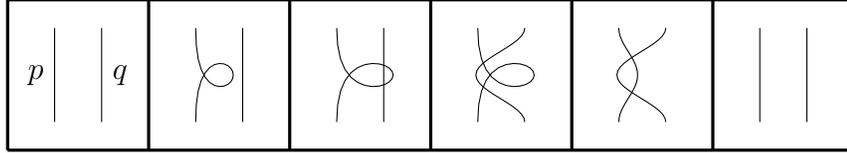
$$\varphi^1 = -abT_p(\omega)T_q(\omega^{-1})$$

On the other hand, the D -sign of a point in the plane between p and q is abh . So we get:

$$\varphi^1 = -ab\widehat{T}_p(\omega^{(-bh)})\widehat{T}_q\left(\frac{1}{\omega^{(ah)}}\right) = -ab\widehat{T}(\omega^{(-bh)}) \otimes \frac{1}{\omega^{(ah)}}$$

and the last formula follows easily. \square

4.4 Movie moves of type MVM_3 . Let D be a link diagram and p and q be two regular points in D . Suppose that these two points may be joined by a path outside of D . Then we have a movie move described by the following figure:



This movie move depends on two signs e and e' . The sign e is the sign of the first Reidemeister move of type I and e' is $+$ if and only if the loop goes over the branch containing q . We say that this move is a movie move of type $MVM_3(e, e')$ near (p, q) .

4.4.a Lemma: *Let e and e' be two signs. Let p and q be two points in a link diagram D and φ be a movie move of type $MVM_3(e, e')$ near (p, q) . Then we have:*

$$\varphi^0 = T_p\left(\frac{\omega}{\omega^{(e)}}\right)$$

In the oriented case, let a, b, h be the signs such that the first move of φ is a Reidemeister move of type $I^+(e, a, h)$ and a is equal to b if and only if the two branches are oriented in the same way. Then we have:

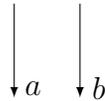
$$\begin{aligned} \varphi^1 &= eT_p\left(\frac{\omega}{\omega^{(e)}}\right) = e\widehat{T}_p\left(\frac{\omega^{(ah)}}{\omega^{(eah)}}\right) \\ \varphi^{\mathcal{K}} &= e\widehat{T}\left(\left(\frac{\omega^{(ah)}}{\omega^{(eah)}} \otimes 1\right) \left(\frac{A(e, a, h)}{A(e, a, -h)} \otimes 1\right) \frac{B^{e'}(a, b, -abh)}{B^{e'}(-a, b, abh)} C\right) \end{aligned}$$

where \widehat{T} is the map $u \otimes v \mapsto \widehat{T}_p(u)\widehat{T}_q(v)$ and C is defined by:

$$\begin{aligned} e' = +, \quad C(e, -ah, -ah, bh) &= u \otimes v \otimes w \implies C = (uv)^{-h} \otimes w^{-h} \\ e' = -, \quad C(e, bh, -ah, -ah) &= u \otimes v \otimes w, \implies C = (uv)^{-h} \otimes u^{-h} \end{aligned}$$

Proof: A straightforward computation gives the first formula.

The diagram D is oriented as follows:



Denote by $(f_1, f_2, f_3, f_4, f_5)$ the sequence φ . Then the types of f_1, f_2, f_3, f_4, f_5 are:

$$I^+(e, a, h), II^+(a, b, -abh), III(e, a, a, -b, -h), I^-(e, a, -h), II^-(-a, b, abh)$$

if $e' = +$ and:

$$I^+(e, a, h), II^+(b, a, -abh), III(e, -b, a, a, -h), I^-(e, a, -h), II^-(b, -a, abh)$$

if $e' = -$.

In the case: $e = +$, only f_2 and f_5 modify the set of negative crossings. Then, during the move φ , only one negative crossing appears and then disappears. So we get:

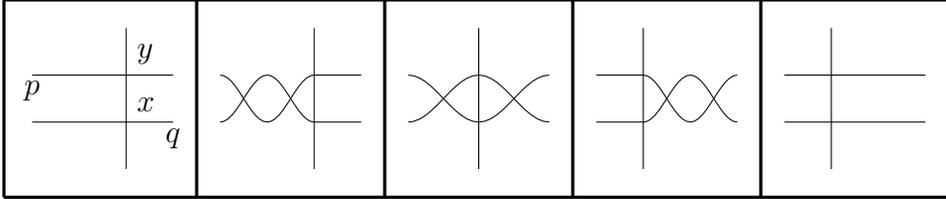
$$\varphi^1 = T_p\left(\frac{\omega}{\omega^{(e)}}\right)$$

In the case: $e = -$, f_1 creates a negative crossing x , f_2 creates a negative crossing y , f_4 removes x and f_5 removes y . This operation produces a sign and we have:

$$\varphi^1 = -T_p\left(\frac{\omega}{\omega^{(e)}}\right)$$

So we get the second formula. The other formulae are easy to check. \square

4.5 Movie moves of type MVM_4 . Consider a link diagrams and two consecutive crossings x and y . Then we have a movie move described by the following figure:



Up to replacing this movie move by its inverse we may as well suppose that the horizontal branch containing y goes over the other horizontal branch after the first Reidemeister move. So this move depends only on an element $e \in \{0, +, -\}$ where $e = +$ (resp. $e = 0, e = -$) if the vertical branch is over (resp. between, under) the two other branches. We say that this move is a movie move of type $MVM_4(e)$ near (p, q) .

4.5.a Lemma: *Let p and q be two points in a link diagram D . Let φ be a movie move of type $MVM_4(e)$ near (p, q) . Then the morphism φ^0 is homotopic to the operator:*

$$(1 - 2e^2)T_p(\omega^{-1})T_q(\omega)$$

In the oriented case, suppose that the first move in φ is a Reidemeister move of type $II^+(a, b, h)$ and that $c = +$ (resp. $c = -$) if the vertical branch is oriented downward (resp. upward). Then we have:

$$\varphi^1 \sim (1 - 2e^2)T_p(\omega^{-1})T_q(\omega) = (1 - 2e^2)\widehat{T}_p\left(\frac{1}{\omega^{(bh)}}\right)\widehat{T}_q(\omega^{(-ah)})$$

$$\varphi^{\mathcal{X}} \sim (1 - 2e^2) \widehat{T} \left(\left(\frac{1}{\omega(bh)} \otimes \omega^{(-ah)} \otimes 1 \right) \left(\frac{B(a, b, h)}{B(a, b, -h)} \otimes 1 \right) C^{-abh} \right)$$

where \widehat{T} is the map $u \otimes v \otimes w \mapsto \widehat{T}_p(u) \widehat{T}_q(v) \widehat{T}_r(w)$ (for any r in the vertical branch) and C is defined by:

$$\begin{aligned} e = -, \quad C(+, bh, ah, abch) C(-, -bh, -ah, abch) &= u \otimes v \otimes w \implies C = u \otimes v \otimes w \\ e = 0, \quad C(-, bh, abch, ah) C(+, -bh, abch, -ah) &= u \otimes v \otimes w \implies C = u \otimes w \otimes v \\ e = +, \quad C(+, abch, bh, ah) C(-, abch, -bh, -ah) &= u \otimes v \otimes w \implies C = w \otimes u \otimes v \end{aligned}$$

Proof: The union of the branches is a graph with six points in its boundary. Take a counterclockwise numbering of these points beginning with the point p . So we have: $p_1 = p$, $p_4 = q$. The operator T_{p_i} will be denoted by T_i .

There are four complexes U , V , W and H such that:

$$kh(D) = 1 \otimes U \oplus x \otimes V \oplus y \otimes W \oplus x \wedge y \otimes H$$

Using small paths near x or y , we get surgery maps γ_x and γ_y . These maps are morphisms of complexes. The morphism γ_x is a map from U to V , from V to U , from W to H and from H to W . The morphism γ_y is a map from U to W , from W to U , from V to H and from H to V . A straightforward computation shows the following:

If φ is a movie move of type $\text{MVM}_4(-)$, we have (for $u \in U$, $v \in V$, $w \in W$, $h \in H$):

$$\begin{aligned} \varphi^0(1 \otimes u) &= -1 \otimes u \\ \varphi^0(x \otimes v) &= -x \otimes T_1(\bar{\omega}) T_3(\omega^{-1})(v) + \varepsilon(\omega^2) y \otimes T_1(\omega^{-2}) \gamma_x \gamma_y(v) \\ \varphi^0(y \otimes w) &= y \otimes \left(-1 + T_1(\omega^{-1}) T_3(\bar{\omega}) - T_1(\omega^{-1}) T_3(\omega) \right)(w) - \varepsilon(\omega^2) x \otimes T_1(\omega^{-2}) \gamma_x \gamma_y(w) \\ \varphi^0(x \wedge y \otimes h) &= -x \wedge y \otimes h \end{aligned}$$

If φ is a movie move of type $\text{MVM}_4(0)$, we have:

$$\begin{aligned} \varphi^0(1 \otimes u) &= 1 \otimes T_3(\omega) T_5(\bar{\omega}^{-1})(u) \\ \varphi^0(x \otimes v) &= x \otimes T_1(\omega) T_5(\bar{\omega}^{-1})(v) - \varepsilon(1) y \otimes \gamma_x \gamma_y(v) \\ \varphi^0(y \otimes w) &= y \otimes w \\ \varphi^0(x \wedge y \otimes h) &= x \wedge y \otimes T_3(\omega^{-1}) T_5(\omega)(h) \end{aligned}$$

If φ is a movie move of type $\text{MVM}_4(+)$, we have:

$$\begin{aligned} \varphi^0(1 \otimes u) &= -1 \otimes T_1(\bar{\omega}) T_3(\omega^{-1})(u) \\ \varphi^0(x \otimes v) &= -x \otimes T_1(\omega^{-1}) T_5(\omega)(v) \\ \varphi^0(y \otimes w) &= -y \otimes T_1(\bar{\omega}) T_3(\omega^{-1})(w) + \varepsilon(\omega^2) x \otimes T_1(\omega^{-2}) \gamma_x \gamma_y(w) \\ \varphi^0(x \wedge y \otimes h) &= -x \wedge y \otimes h \end{aligned}$$

Consider the maps k_x and k_y from $kh(D)$ to $kh(D)$ that vanish on $1 \otimes U \oplus x \otimes V \oplus y \oplus W$ and satisfy the following:

$$k_x(x \wedge y \otimes h) = x \otimes \gamma_y(h) \quad k_y(x \wedge y \otimes h) = y \otimes \gamma_x(h)$$

for every $h \in H$. These maps induce the following homotopies:

$$h_x = d(k_x) = d \circ k_x + k_x \circ d \quad h_y = d(k_y) = d \circ k_y + k_y \circ d$$

and we have:

$$\begin{aligned} h_x(1 \otimes u) &= 0 & h_y(1 \otimes u) &= 0 \\ h_x(x \otimes v) &= -x \otimes \gamma_y^2(v) & h_y(x \otimes v) &= -y \otimes \gamma_x \gamma_y(v) \\ h_x(y \otimes w) &= x \otimes \gamma_x \gamma_y(w) & h_y(y \otimes w) &= y \otimes \gamma_x^2(w) \\ h_x(x \wedge y \otimes h) &= -x \wedge y \otimes \gamma_y^2(h) & h_y(x \wedge y \otimes h) &= x \wedge y \otimes \gamma_x^2(h) \end{aligned}$$

Moreover one checks that γ_x^2 (resp. γ_y^2) is on W and H (resp. on V and H) the map $T_2(\omega)T_4(\alpha) - T_2(\omega\bar{\alpha})$ (resp. the map $T_1(\omega)T_5(\alpha) - T_1(\omega\bar{\alpha})$). So we get:

$$\varphi^0 = \varepsilon(1)T_1(\bar{\omega})T_3(\omega^{-1})h_x - \varepsilon(\omega^2)T_1(\omega^{-2})h_y - T_2(\omega^{-1})T_5(\omega)$$

in the first case,

$$\varphi^0 = \varepsilon(1)h_y + T_4(\omega)T_5(\bar{\omega}^{-1})$$

in the second case and:

$$\varphi^0 = \varepsilon(\omega^2)T_1(\omega^{-2})h_x - T_1(\bar{\omega})T_2(\omega^{-1})$$

in the last case. If we denote by \sim the homotopy relation, we get:

$$\varphi^0 \sim -T_2(\omega^{-1})T_5(\omega) \sim -T_4(\bar{\omega}^{-1})T_1(\bar{\omega}) = -T_1(\omega^{-1})T_4(\omega)$$

in the first case,

$$\varphi^0 \sim T_4(\omega)T_5(\bar{\omega}^{-1}) \sim T_1(\omega^{-1})T_4(\omega)$$

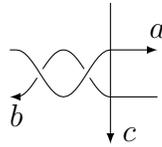
in the second case and:

$$\varphi^0 \sim -T_1(\bar{\omega})T_2(\omega^{-1}) \sim -T_1(\bar{\omega})T_4(\bar{\omega}^{-1}) = -T_1(\omega^{-1})T_4(\omega)$$

in the last case. So we get the first formula.

The movie move creates one negative crossing and then remove it. Therefore we get the same formula for φ^1 .

The second diagram D' is oriented as follows:



Denote by (f_1, f_2, f_3, f_4) the sequence φ . Then the types of f_1, f_2, f_3, f_4 are:

$$\text{II}^+(a, b, h), \text{III}(+, -a, -b, -c, -abh), \text{III}(-, a, b, -c, -abh), \text{II}^-(a, b, h)$$

if $e = -$,

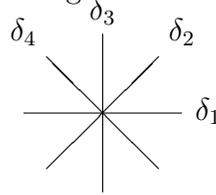
$$\text{II}^+(a, b, h), \text{III}(-, -a, -c, -b, -abh), \text{III}(+, a, -c, b, -abh), \text{II}^-(a, b, h)$$

if $e = 0$ and

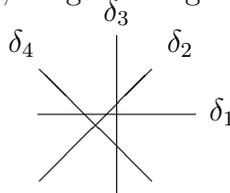
$$\text{II}^+(a, b, h), \text{III}(+, -c, -a, -b, -abh), \text{III}(-, -c, a, b, -abh), \text{II}^-(a, b, h)$$

if $e = +$. The result follows. □

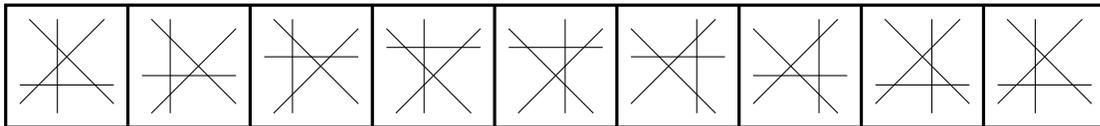
4.6 Movie moves of type MVM_5 . Consider a link in \mathbf{R}^3 that projects on the plane by an immersion and suppose that every crossing is simple except one and the image of the link looks like the following near the singular crossing:



If we move the link a little bit, we get a diagram like:



For $\{i, j\} \subset \{1, 2, 3, 4\}$, denote by x_{ij} the intersection of δ_i and δ_j . One checks the following fact: if, during a small isotopy, $x_{24} - x_{13}$ makes one counterclockwise turn around 0, $x_{14} - x_{23}$ and $x_{34} - x_{12}$ make one clockwise turn around 0. Therefore any small isotopy such that $x_{ij} - x_{kl}$ makes one turn around 0 is equivalent to an isotopy where δ_3 and δ_4 are fixed and δ_1 and δ_2 are moving by translations. During this isotopy, we get diagrams D_z where $z = x_{12}$ and z makes one counterclockwise turn around $x = x_{34}$. If z is not in one of the initial δ_i 's, D_z is a link diagram. So we get a movie move which is the composite of eight Reidemeister moves of type III.



This move depends essentially on the heights c_1, c_2, c_3, c_4 of the lines $\delta_1, \delta_2, \delta_3, \delta_4$. We say that this move is a movie move of type $\text{MVM}_5(c_1, c_2, c_3, c_4)$.

4.6.a Lemma: *Let φ be a movie move of type MVM_5 . Then φ^0 is homotopic to the identity.*

In the oriented case, φ^1 is also homotopic to the identity.

Proof: Consider a movie move φ of type $\text{MVM}_5(c_1, c_2, c_3, c_4)$. The morphism φ^0 acts on $kh(D)$ for some link diagram D . The modification area of φ involves six crossings and, to describe φ^0 , we have to decompose $kh(D)$ into a direct sum of 64 complexes. So an explicit description of φ^0 is almost impossible, at least by hand. But it is possible to compute φ^0 using a program on a computer and we check that φ^0 is an idempotent (i.e. $\varphi^0 \circ \varphi^0 = \varphi^0$) when the sequence (c_i) is increasing or decreasing. This property is may be related to the following fact which is easy to check: all the Reidemeister moves of φ are of type III_+ if (c_i) is increasing and of type III_- if (c_i) is decreasing.

Since φ^0 is a homotopy equivalence it has a homotopy inverse ψ . So when (c_1, c_2, c_3, c_4) is increasing or decreasing, we have:

$$\varphi^0 \sim \psi \circ \varphi^0 \circ \varphi^0 = \psi \circ \varphi^0 \sim \text{Id}$$

and φ^0 is homotopic to the identity.

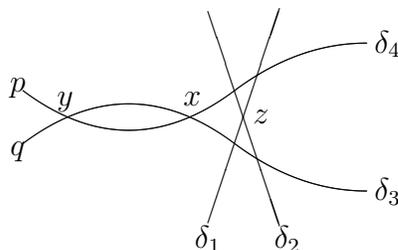
In the other cases, we have to proceed differently. Let S be the set of sequences (c_1, c_2, c_3, c_4) of distinct reals such that the lemma is true for every movie move of type $\text{MVM}_5(c_1, c_2, c_3, c_4)$. So we have:

$$c_1 > c_2 > c_3 > c_4 \implies (c_1, c_2, c_3, c_4) \in S$$

On the other hand, it is easy to see that a movie move of type $\text{MVM}_5(c_1, c_2, c_3, c_4)$ is conjugate to a movie move of type $\text{MVM}_5(c_2, c_3, c_4, c_1)$. So we have:

$$(c_1, c_2, c_3, c_4) \in S \iff (c_2, c_3, c_4, c_1) \in S$$

Let c_1, c_2, c_3, c_4 be distinct reals and φ be a movie move of type $\text{MVM}_5(c_1, c_2, c_3, c_4)$ acting on a diagram D . It is possible to modify D by a Reidemeister move of type II in such a way to get the following configuration in D :



In this figure δ_i have constant height c_i and, during the isotopy, δ_3 and δ_4 are fixed and δ_1 and δ_2 are moving by translation. So the isotopy of the figure is determined by the isotopy of z .

If z makes one counterclockwise turn around x , we get a movie move of type $\text{MVM}_5(c_1, c_2, c_3, c_4)$. If z makes one counterclockwise turn around y , we get a movie move of type $\text{MVM}_5(c_1, c_2, c_4, c_3)$.

Let Δ be the loop containing x and y . Suppose that (c_1, c_2, c_4, c_3) is in S . Then the move φ of type $\text{MVM}_5(c_1, c_2, c_3, c_4)$ is equivalent to the move φ' obtained by moving z around Δ . So we have:

$$\varphi^0 = D \circ C \circ B \circ A$$

where A is obtained by moving δ_1 through Δ , B by moving δ_2 through Δ , C by moving back δ_1 through Δ and D by moving back δ_2 through Δ . Denote by f the Reidemeister move of type II which creates crossings x and y and by g its inverse move. So the morphism φ^0 is homotopic to:

$$f^0 \circ g^0 \circ D \circ f^0 \circ g^0 \circ C \circ f^0 \circ g^0 \circ B \circ f^0 \circ g^0 \circ A \circ f^0 \circ g^0$$

But each morphism $g^0 \circ X \circ f^0$ (with $X = A, B, C, D$) is obtained by a type III Reidemeister move and a movie move of type MVM_4 : Let ψ (resp. ψ') be the Reidemeister move obtained by pushing up z through δ_4 (resp. pushing down z through δ_3). If a, b, c are distinct reals, we set $\mu(a, b, c) = 1$ if a is between b and c and $\mu(a, b, c) = -1$ otherwise. So we have:

$$\begin{aligned} g^0 \circ A \circ f^0 &\sim \mu(c_1, c_3, c_4)T(\omega^{-1} \otimes \bar{\omega})\psi \\ g^0 \circ B \circ f^0 &\sim \mu(c_2, c_3, c_4)\psi^{-1}T(\bar{\omega}^{-1} \otimes \omega) \\ g^0 \circ C \circ f^0 &\sim \mu(c_1, c_3, c_4)T(\bar{\omega} \otimes \omega^{-1})\psi' \\ g^0 \circ D \circ f^0 &\sim \mu(c_2, c_3, c_4)\psi'^{-1}T(\omega \otimes \bar{\omega}^{-1}) \end{aligned}$$

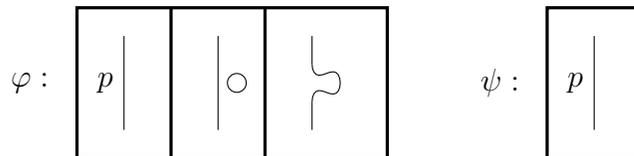
where T is the map $u \otimes v \mapsto T_p(u)T_q(v)$. Therefore φ^0 is homotopic to the identity and (c_1, c_2, c_3, c_4) belongs to S . So we have:

$$(c_1, c_2, c_3, c_4) \in S \iff (c_1, c_2, c_4, c_3) \in S$$

and S contains all sequences (c_i) . So the first part of the lemma is proven and the second one follows easily. \square

Remark: The general formula for $\varphi^{\mathcal{K}}$ is too complicated to be written down if φ is a movie move of type MVM_5 .

4.8 Movie moves of type MVM_6 . Let D be a link diagram. We can modify D by adding a small circle near a branch of D and then connect the circle to D near a point p in the branch. So we get a movie sequence $\varphi = (f, g)$ where f is a surgery move of index 0 and g a surgery move of index 1. The second sequence is $\psi = \text{Id}$. The move (φ, ψ) is called a movie move of type MVM_6 near p .



An easy computation gives the following:

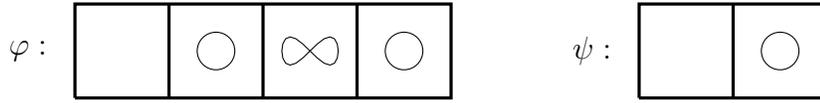
4.8.a Lemma: *Let (φ, ψ) be a movie move of type MVM_6 . Then we have:*

$$\varphi^0 = \psi^0 \quad (\bar{\varphi})^0 = (\bar{\psi})^0$$

In the oriented case, suppose that the first move in φ is a surgery move of type $(0, a, h)$. then we have:

$$\begin{aligned} \varphi^1 &= \psi^1 & (\bar{\varphi})^1 &= (\bar{\psi})^1 \\ \varphi^{\mathcal{K}} &= \widehat{T}_p(X(a, h)Y(-a, h))\psi^{\mathcal{K}} & (\bar{\varphi})^{\mathcal{K}} &= (\bar{\psi})^{\mathcal{K}}\widehat{T}_p(Z(a, h)Y(a, -h)) \end{aligned}$$

4.9 Movie moves of type MVM_7 . Let D be a link diagram and e be a sign. We have a movie move (φ, ψ) described as follows:



We add a small circle to D by a surgery of index 0. Then we apply a Reidemeister move of type I_e^+ which creates a loop in the right part of the circle and we remove the other loop in the circle by a Reidemeister move of type I_e^- . So we get the movie sequence φ . The movie sequence ψ is a surgery move of index 0. The move (φ, ψ) is called a movie move of type $MVM_7(e)$.

An easy computation gives the following:

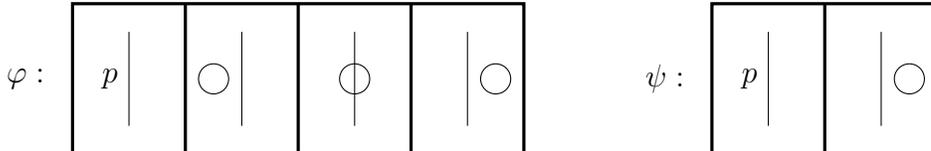
4.9.a Lemma: *Let e be a sign and (φ, ψ) be a movie move of type $MVM_7(e)$. Let p be a point in the created circle. Then we have:*

$$\varphi^0 = -eT_p\left(\frac{\omega^{(e)}}{\bar{\omega}}\right)\psi^0 \quad (\bar{\varphi})^0 = e(\bar{\psi})^0T_p\left(\frac{\omega}{\omega^{(e)}}\right)$$

In the oriented case, suppose that the first move in φ is a surgery move of type $(0, a, h)$. Then we have:

$$\begin{aligned} \varphi^1 &= -e\widehat{T}_p\left(\frac{\omega^{(-eah)}}{\omega^{(ah)}}\right)\psi^1 & (\bar{\varphi})^1 &= e(\bar{\psi})^1\widehat{T}_p\left(\frac{\omega^{(eah)}}{\omega^{(ah)}}\right) \\ \varphi^{\mathcal{K}} &= \widehat{T}_p\left(X(a, h)\frac{A(e, -a, h)}{A(e, a, h)}\right)\varphi^1 & \psi^{\mathcal{K}} &= \widehat{T}_p\left(X(-a, h)\right)\psi^1 \\ (\bar{\varphi})^{\mathcal{K}} &= (\bar{\varphi})^1\widehat{T}_p\left(\frac{A(e, a, h)}{A(e, -a, h)}Z(a, h)\right) & (\bar{\psi})^{\mathcal{K}} &= (\bar{\psi})^1\widehat{T}_p\left(Z(-a, h)\right) \end{aligned}$$

4.10 Movie moves of type MVM_8 . Let D be a link diagram and e be a sign. We have a movie move (φ, ψ) described as follows:



The move φ is obtained by creating a small circle of the left hand side of a branch of D with a surgery move of index 0 and then moving this circle through the branch with two Reidemeister moves of type II. The move ψ is a surgery move of index 0 creating the circle on the right hand side of the branch. This move depends on a sign e where $e = +$ if and only if the circle goes over the branch. Let p be a point in the branch. We say that (φ, ψ) is a movie move of type $MVM_8(e)$ near p .

4.10.a Lemma: *Let D be a link diagram and e be a sign. Let (φ, ψ) be a movie move of type $MVM_8(e)$ near a point $q \in D$. Let p be a point in the created circle. Then we have:*

$$\varphi^0 = T_p\left(\frac{\omega}{\bar{\omega}}\right)\psi^0 \quad (\bar{\varphi})^0 = -(\bar{\psi})^0$$

In the oriented case, suppose that the first move of φ is a surgery move of type $(0, a, h)$. Let b be the sign such that $b = +$ if and only if the vertical branch is oriented downward. Then we have:

$$\varphi^1 = \widehat{T}\left(\frac{\omega^{(ah)}}{\omega^{(-ah)}} \otimes 1\right)\psi^1 \quad (\bar{\varphi})^1 = -(\bar{\psi})^1$$

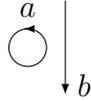
$$\varphi^{\mathcal{K}} = \widehat{T}\left((X(a, h) \otimes 1) \frac{B^e(a, b, -abh)}{B^e(a, -b, -abh)}\right)\varphi^1 \quad \psi^{\mathcal{K}} = \widehat{T}\left(X(a, -h) \otimes 1\right)\psi^1$$

$$(\bar{\varphi})^{\mathcal{K}} = (\bar{\varphi})^1 \widehat{T}\left(\frac{B^e(a, -b, -abh)}{B^e(a, b, -abh)}(Z(a, h) \otimes 1)\right) \quad (\bar{\psi})^{\mathcal{K}} = \widehat{T}\left(Z(a, -h) \otimes 1\right)(\bar{\psi})^1$$

where \widehat{T} is the map $u \otimes v \mapsto \widehat{T}_p(u)\widehat{T}_q(v)$.

Proof: A straightforward computation gives the desired expressions of φ^0 and $(\bar{\varphi})^0$ and therefore the expressions of φ^1 and $(\bar{\varphi})^1$.

The orientation of the diagram modified by the first move is described as follows:



The types of the moves in φ are:

$$(0, a, h), \quad \text{II}^+(e, a, b, -abh), \quad \text{II}^-(e, a, -b, -abh)$$

and the types of the moves in $\bar{\varphi}$ are:

$$\text{II}^+(e, a, -b, -abh), \quad \text{II}^-(e, a, b, -abh), \quad (2, a, h)$$

The result follows. □

4.11 Movie moves of type MVM_9 . Let D be a link diagram and e be a sign. We have a movie move (φ, ψ) described as follows:



Both moves φ and ψ are obtained by a Reidemeister move of type I_e^+ followed by a surgery move of index 1. We say that (φ, ψ) is a movie move of type $MVM_9(e)$ near (p, q) .

4.11.a Lemma: *Let D be a link diagram and e be a sign. Let (φ, ψ) be a movie move of type $MVM_9(e)$ near (p, q) . Then we have:*

$$\varphi^0 T_p \left(\frac{\omega^{(e)}}{\bar{\omega}} \right) \sim -e\psi^0 \quad T_p \left(\frac{\omega}{\omega^{(e)}} \right) (\bar{\varphi})^0 \sim e(\bar{\psi})^0$$

In the oriented case, suppose the first move of φ is a Reidemeister move of type $I^+(e, a, h)$. Then we have:

$$\begin{aligned} \varphi^1 \widehat{T}_p \left(\frac{\omega^{(eah)}}{\omega^{(-ah)}} \right) &\sim -e\psi^1 & \widehat{T}_p \left(\frac{\omega^{(ah)}}{\omega^{(eah)}} \right) (\bar{\varphi})^1 &\sim e(\bar{\psi})^1 \\ \varphi^{\cancel{\mathcal{K}}} &\sim \widehat{T}_p \left(A(e, a, h) Y(-a, h) \right) \varphi^1 & \psi^{\cancel{\mathcal{K}}} &\sim \widehat{T}_p \left(A(e, -a, h) Y(a, h) \right) \psi^1 \\ (\bar{\varphi})^{\cancel{\mathcal{K}}} &\sim (\bar{\varphi})^1 \widehat{T}_p \left(\frac{Y(a, -h)}{A(e, a, h)} \right) & (\bar{\psi})^{\cancel{\mathcal{K}}} &\sim (\bar{\psi})^1 \widehat{T}_p \left(\frac{Y(-a, -h)}{A(e, -a, h)} \right) \end{aligned}$$

Proof: Set: $\varepsilon = (1 + e)/2$. It is easy to check the following:

$$\begin{aligned} \varphi^0 T_p(\omega^\varepsilon) &= -e\psi^0 T_q(\omega^\varepsilon) \\ T_p(\omega^{1-\varepsilon}(\bar{\varphi})^0) &= eT_q(\omega^{1-\varepsilon}(\bar{\psi})^0) \end{aligned}$$

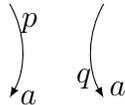
So we get:

$$\begin{aligned} -e\psi^0 &= \varphi^0 T_p(\omega^\varepsilon) T_q(\omega^{-\varepsilon}) = T_q(\omega^{-\varepsilon}) \varphi^0 T_p(\omega^\varepsilon) \sim T_p(\bar{\omega}^{-\varepsilon}) \varphi^0 T_p(\omega^\varepsilon) \\ &= \varphi^0 T_p(\bar{\omega}^{-\varepsilon} \omega^\varepsilon) = \varphi^0 T_p \left(\frac{\omega^{(e)}}{\bar{\omega}} \right) \\ e(\bar{\psi})^0 &= T_q(\omega^{\varepsilon-1}) T_p(\omega^{1-\varepsilon})(\bar{\varphi})^0 = T_p(\omega^{1-\varepsilon})(\bar{\varphi})^0 T_q(\omega^{\varepsilon-1}) \sim T_p(\omega^{1-\varepsilon})(\bar{\varphi})^0 T_p(\bar{\omega}^{\varepsilon-1}) \\ &= T_p(\bar{\omega}^{\varepsilon-1} \omega^{1-\varepsilon})(\bar{\varphi})^0 = T_p \left(\frac{\omega}{\bar{\omega}^{(e)}} \right) (\bar{\varphi})^0 \end{aligned}$$

and we get the first formulae and therefore the following:

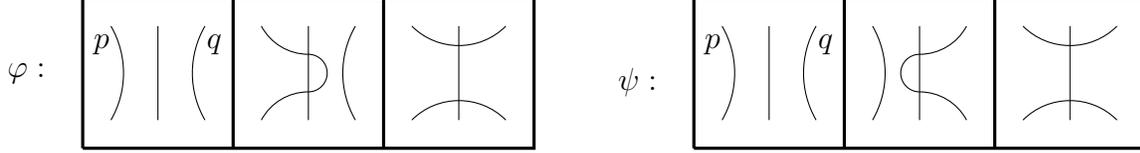
$$\varphi^1 T_p \left(\frac{\omega^{(e)}}{\bar{\omega}} \right) \sim -e\psi^1 \quad T_p \left(\frac{\omega}{\omega^{(e)}} \right) (\bar{\varphi})^1 \sim e(\bar{\psi})^1$$

Since the first move of φ is a Reidemeister move of type $I^+(e, a, h)$, the orientation of D is as follows:



and the types of the moves in φ (resp. ψ) are $I^+(e, a, h)$ and $(1, -a, h)$ (resp. $I^+(e, -a, h)$ and $(1, a, h)$). For $\bar{\varphi}$ (resp. $\bar{\psi}$), the types are $(1, a, -h)$ and $I^-(e, a, h)$ (resp. $(1, -a, -h)$ and $I^-(e, -a, h)$). The result follows. \square

4.12 Movie moves of type MVM_{10} . Consider a movie move (φ, ψ) acting on a link diagram D as follows:



Both moves φ and ψ are obtained by a Reidemeister move of type II^+ followed by a surgery move of index 1. Let e be the sign such that $e = +$ (resp. $e = -$) if the middle branch is under (resp. over) the other ones in the modified diagram. We say that (φ, ψ) is a movie move of type $MVM_{10}(e)$ near (p, q) .

4.12.a Lemma: Let (φ, ψ) be a movie move of type $MVM_{10}(e)$ near (p, q) . Then we have:

$$\varphi^0 T_p\left(\frac{\omega}{\bar{\omega}}\right) \sim \psi^0 \quad (\bar{\varphi})^0 = -(\bar{\psi})^0$$

In the oriented case, suppose that the first move of φ is a Reidemeister move of type $II^+(e, a, b, h)$. Then we have:

$$\varphi^1 \widehat{T}\left(\frac{\omega^{(bh)}}{\omega^{(-bh)}} \otimes 1\right) \sim \psi^1 \quad (\bar{\varphi})^1 = -(\bar{\psi})^1$$

$$\varphi^{\mathcal{K}} \sim \varphi^1 \widehat{T}(B^e(a, b, h)(Y(-a, abh) \otimes 1)) \quad \psi^{\mathcal{K}} \sim \psi^1 \widehat{T}(B^e(a, -b, h)(Y(-a, -abh) \otimes 1))$$

$$(\bar{\varphi})^{\mathcal{K}} \sim \widehat{T}\left(\frac{Y(a, -abh) \otimes 1}{B^e(a, b, h)}\right) (\bar{\varphi})^1 \quad (\bar{\psi})^{\mathcal{K}} \sim \widehat{T}\left(\frac{Y(a, abh) \otimes 1}{B^e(a, -b, h)}\right) (\bar{\psi})^1$$

where r is a point in the middle branch and \widehat{T} the map $u \otimes v \mapsto \widehat{T}_p(u) \widehat{T}_r(v)$.

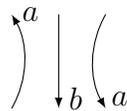
Proof: It is easy to check the following:

$$\psi^0 = \varphi^0 T_p(\omega) T_q(\omega^{-1}) \quad (\bar{\varphi})^0 = -(\bar{\psi})^0$$

and that implies:

$$\psi^0 = T_p(\omega) T_q(\omega^{-1}) \varphi^0 \sim T_p(\omega) T_p(\bar{\omega}^{-1}) \varphi^0 = \varphi^0 T_p(\omega \bar{\omega}^{-1})$$

By assumption the diagram D is oriented as follows:



Therefore the types of the moves in φ (resp. ψ) are:

$\text{II}^+(e, a, b, h)$ and $(1, b, abh)$ (resp. $\text{II}^+(e, a, -b, abh)$ and $(1, -a, -abh)$).
 For the move $\overline{\varphi}$ (resp. $\overline{\psi}$), the types are:
 $(1, -b, -abh)$ and $\text{II}^-(e, a, b, h)$ (resp. $(1, a, abh)$ and $\text{II}^-(e, a, -b, abh)$).
 The results follows. □

4.13 Remark: The conditions induced by all these movie moves are essentially the same as the conditions induced by all the movie moves of Carter and Saito [CS].

Denote by $C(i)$ the condition induced on the correspondance $f \mapsto f^{\mathcal{K}}$ by all the movie moves of type MVM_i and by $C'(j)$ the condition induced on it by all the movie moves of type MM_j (by using the Bar Natan classification of these moves [BN2]).

The conditions C'_j for $j = 1, 2, 3, 4, 5$ are automatically satisfied here because, for every Reidemeister move f with inverse move g , $g^{\mathcal{K}}$ is a homotopy inverse of $f^{\mathcal{K}}$.

The movie moves of type $\text{MVM}_1, \text{MVM}_2, \text{MVM}_4, \text{MVM}_5, \text{MVM}_6, \text{MVM}_9, \text{MVM}_{10}$ are essentially the movie moves of type $\text{MM}_7, \text{MM}_9, \text{MM}_6, \text{MM}_{10}, \text{MM}_{11}, \text{MM}_{13}$ and MM_{15} . The movie moves of type MVM_7 and MVM_8 are more or less equivalent to the movie moves of type MM_{12} and MM_{14} , that is the conditions $C(7)$ and $C(8)$ are exactly the conditions $C'(12)$ and $C'(14)$. Finally the condition $C(3)$ is equivalent, modulo the conditions $C(0)$ and $C(4)$, to the condition $C'(8)$.

5. Functoriality.

5.1 The category of cobordisms of oriented links.

Denote by π the projection map $(x, y, z) \mapsto (x, y)$ from \mathbf{R}^3 onto \mathbf{R}^2 and by \mathcal{E} the space of oriented links in \mathbf{R}^3 . We say that a link $L \in \mathcal{E}$ is generic if π sends L by an immersion onto a link diagram.

The space \mathcal{E} is a Fréchet manifold and the space $Z \subset \mathcal{E}$ of non generic links is a closed stratified subspace of \mathcal{E} .

Let L_0 and L_1 be two oriented links. A cobordism C from L_0 to L_1 is an oriented compact surface C contained in $\mathbf{R}^3 \times [0, 1]$ and meeting transversally $\mathbf{R}^3 \times \{0, 1\}$ in $\partial C = L_1 \times \{1\} \cup (-L_0) \times \{0\}$. Two cobordisms C and C' from L_0 to L_1 are called isotopic if there exist an isotopy $f_t : \mathbf{R}^3 \times [0, 1] \rightarrow \mathbf{R}^3 \times [0, 1]$, with $0 \leq t \leq 1$, such that:

$$\begin{aligned} f_0 &= \text{Id} & f_1(C) &= C' \\ \forall t \in [0, 1], & f_t \text{ is the identity on } \mathbf{R}^3 \times \{0, 1\} \end{aligned}$$

Let L_0 and L_1 be two generic oriented links and D_0 and D_1 be the corresponding link diagrams. Let $f : D_0 \rightarrow D_1$ be an elementary move. This move induces a cobordism C from L_0 to L_1 which is well defined up to isotopy. Such cobordisms are called elementary cobordisms.

Let L_0 and L_1 be two generic oriented links and C be a cobordism from L_0 to L_1 . We say that C is generic if there is finitely many elements $t_i \in (0, 1)$ such that: $C \cap \mathbf{R}^3 \times \{t\}$ is generic for all $t \in [0, 1]$ except the t_i 's and the cobordism $C \cap \mathbf{R}^3 \times [t_i - \varepsilon, t_i + \varepsilon]$ is elementary for each i and ε small enough.

By transversality we see that every link $L \in \mathcal{E}$ is isotopic to a generic link by a small isotopy. Moreover every cobordism between two generic links in \mathcal{E} is isotopic to a generic cobordism by a small isotopy.

Denote by \mathcal{L} the category of cobordisms of oriented links in \mathbf{R}^3 . The objects of \mathcal{L} is the links in \mathcal{E} and the morphisms are the isotopy classes of cobordisms. Every morphism in \mathcal{L} from a generic link L_0 to a generic link L_1 is a composite of elementary morphisms and can be described by a movie sequence.

5.2 Definition: Let \mathcal{K} be a Khovanov data. This data is said to be functorializable if there exists a monoidal functor Ψ from the category of cobordisms of oriented links \mathcal{L} to the homotopy category of K -complexes satisfying the following:

- if D is the diagram of a generic oriented link L , we have: $\Psi(L) = KH(D)$
- if C is an elementary cobordism from L_0 to L_1 associated to an elementary move $f : D_0 \rightarrow D_1$, $\Psi(C)$ is homotopic to the morphism $f^{\mathcal{K}}$.

If these conditions are satisfied the functor Ψ will be called a Khovanov functor associated to \mathcal{K} .

5.3 Lemma: Let \mathcal{K} be a Khovanov data. Suppose \mathcal{K} is functorializable. Then all the Khovanov functors associated to \mathcal{K} are isomorphic.

Proof: Let \mathcal{L}_0 be the full subcategory of \mathcal{L} generated by generic links. It is clear that the inclusion $\mathcal{L}_0 \subset \mathcal{L}$ is a equivalence of categories. The result follows. \square

5.4 Lemma: Let \mathcal{K} be a Khovanov data. Then \mathcal{K} is functorializable if and only if all conditions $C(i)$, for $i = 0, \dots, 10$ are satisfied for the correspondance $f \mapsto f^{\mathcal{K}}$.

Proof: For every movie move (φ, ψ) , the condition $\varphi^{\mathcal{K}} \sim \psi^{\mathcal{K}}$ will be denoted by $\mathcal{K}(\varphi, \psi)$. It is clear that, if the correspondance $f \mapsto f^{\mathcal{K}}$ comes from a functor, every movie move induces a trivial condition. So all conditions $C(i)$ are satisfied.

Conversely, suppose all conditions $C(i)$, for $i = 0, \dots, 10$ are satisfied. The only thing to do is to define the functor Ψ on the category of cobordisms of generic oriented links \mathcal{L}_0 . This functor is well defined on the objects.

Let L_0 and L_1 be two generic oriented links and C_0 be a cobordism from L_0 to L_1 . This cobordism represents a morphism f from L_0 to L_1 . We have to define $\Psi(f)$. Since C_0 is isotopic to a generic cobordism, $\Psi(f)$ is homotopic to a morphism $\varphi^{\mathcal{K}}$, where φ is some movie sequence. So we don't have any choice for $\Psi(f)$. The only thing to do is to prove that $\varphi^{\mathcal{K}}$ depends only on the isotopy class of C_0 .

Let $M = \mathcal{E}(C_0)$ be the space of cobordisms isotopic to C_0 and $Z_0 = Z(C_0)$ be the space of cobordisms in M which are not generic. The space M is a connected Fréchet manifold and Z_0 is a closed stratified subspace of M of codimension 1. Every $C \in M$ which is not in Z_0 is determined by a movie sequence $\varphi = \widehat{C}$, and $\Psi(C) = \varphi^{\mathcal{K}}$ is well defined. The last thing to do is to prove the following: if C and C' are in $M \setminus Z_0$, $\Psi(C)$ and $\Psi(C')$ are homotopic.

Let C be a cobordism in M . For every $t \in [0, 1]$ denote by C_t the intersection $C \cap \mathbf{R}^3 \times \{t\}$. We say that t is C -regular if C is transverse to $\mathbf{R}^3 \times \{t\}$ and C_t is a generic link in $\mathbf{R}^3 \times \{t\} \simeq \mathbf{R}^3$. We say that t is C -singular if t is not C -regular.

If $C \in M$ is generic, there is only finitely many C -singular elements in $[0, 1]$, and every C -singular parameter t correspond to a Reidemeister move if C is transverse to $\mathbf{R}^3 \times \{t\}$ and a surgery move if it is not the case.

Since M is connected, there is a path γ in M joining C and C' . Up to modify γ we may as well suppose that γ is transverse to Z_0 and we have to prove that $\Psi(\gamma(s))$ doesn't change when $\gamma(s)$ goes through Z_0 . Let s_0 be a parameter such that $\gamma(s_0)$ is in Z_0 . Since γ is transverse to Z_0 , $\gamma(s_0)$ is in a stratum $S \subset Z_0$ of codimension 1. For ε small enough, $\gamma(s_0 - \varepsilon)$ and $\gamma(s_0 + \varepsilon)$ correspond to movie sequences φ and ψ and we get a movie move (φ, ψ) . So every codimension 1 stratum of Z_0 induces a movie move (φ, ψ) . The only condition to check is the conditions $\mathcal{K}(\varphi, \psi)$ for all these movie moves.

Let S be a codimension 1 stratum in Z_0 . An element in S is a cobordism C which is not generic but for only one reason. So there is a unique $t_0 \in (0, 1)$ such that $C \cap (\mathbf{R}^3 \times [t_0 - \varepsilon, t_0 + \varepsilon])$ is not generic. Let (φ, ψ) be the movie move associated to C . The cobordism $C \cap (\mathbf{R}^3 \times [t_0 - \varepsilon, t_0 + \varepsilon])$ induces a movie move (φ_0, ψ_0) and there are two movies sequences α and β such that: $\varphi = \alpha \circ \varphi_0 \circ \beta$ and $\psi = \alpha \circ \psi_0 \circ \beta$. So the condition $\mathcal{K}(\varphi_0, \psi_0)$ implies the condition $\mathcal{K}(\varphi, \psi)$ and it is enough to consider the case where t_0 is the only C -singular element in $[0, 1]$ and C_t is generic for every $t \neq t_0$.

Suppose C is transverse to $\mathbf{R}^3 \times \{t_0\}$. Then the problem reduces to an isotopy problem and then to an isotopy corresponding to a loop around a codimension 2 stratum of $Z \subset \mathcal{E}$. These loops correspond to all movie moves of type MVM_i for $0 \leq i \leq 5$ involving only Reidemeister moves.

Suppose C is not transverse to $\mathbf{R}^3 \times \{t_0\}$. Then the function $C \subset \mathbf{R}^3 \times [0, 1] \rightarrow [0, 1]$ has critical points u_i in $\mathbf{R}^3 \times \{t_0\}$. If one of these points is degenerated, the problem reduces to a movie move of type MVM_6 . If there is at least 2 critical points, the problem reduces to a movie move of type MVM_0 involving two surgery moves. In the other cases, we have only one critical point u and this point is non degenerated with index d . But we have an extra condition because C belongs to Z_0 . There is two possibilities for this condition: u is a critical point for $\pi : C \rightarrow \mathbf{R}^2$ or $\pi(u)$ is a multiple point in $\pi(C_{t_0})$. In the first case, we get a movie move of type MVM_7 if $d = 0$ or $d = 2$ and a movie move of type MVM_9 if $d = 1$. In the second case, we get a movie move of type MVM_0 involving one Reidemeister move and one surgery move or a movie move of type MVM_8 if $d = 0$ or $d = 2$ and a movie move of type MVM_{10} if $d = 1$.

Thus, if all conditions $C(i)$ are satisfied, the functor Ψ is well defined. \square

For simplicity we'll identify the sign $+$ with 1 and the sign $-$ with -1 and we define a map $\langle ?|? \rangle$ from $\{\pm\}^2$ to $\{\pm\}$, a map μ from $\{\pm\}$ to R^* and a map δ from $\{\pm\}^2$ to R^* by:

$$\begin{aligned} \langle (-1)^p | (-1)^q \rangle &= (-1)^{pq} \\ \mu(e) &= \omega^{(1+e)/2} = \begin{cases} \omega & \text{if } e = + \\ 1 & \text{if } e = - \end{cases} \\ \delta(a, b) &= \begin{cases} \omega\bar{\omega} & \text{if } a = b = + \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

The map $\langle ? | ? \rangle$ is symmetric and satisfy the following property:

$$\forall a, b, c \in \{\pm\}, \quad \langle a | bc \rangle = \langle a | b \rangle \langle a | c \rangle$$

If $H(h)$ is an element of R^* depending on a sign h , we set:

$$\tilde{H} = \frac{H(+)}{H(-)}$$

5.5 Theorem: Let $\mathcal{K} = (A, B, C, X, Y, Z)$ be a Khovanov data. Then \mathcal{K} is functorializable if and only if there exist elements $\sigma(h)$ in K^* , $E_e(h)$, $F(e)$ in R^* , $W(h)$ in $(R \otimes R)^*$ depending on signs e and h such that the following holds, for every signs e, h, a, b, c :

$$\begin{aligned} \sigma(+)\sigma(-) &= \omega\bar{\omega} \\ E_+(h)E_-(h) &= \omega F(+)\omega F(-)U(h)V(h) \\ A(e, a, h) &= \langle e | a \rangle \langle e | h \rangle \langle a | h \rangle \left(\frac{\sigma(h)}{\omega(a)} \right)^{(1-eh)/2} \frac{E_e(h)}{F(a)} \\ B(a, b, h) &= \frac{\langle a | b \rangle}{\delta(ah, bh)} \sigma(h)^{(1+ab)/2} U(h)\mu(bh) \otimes V(h)\mu(ah) \\ X(a, h) &= \langle a | -h \rangle \left(\frac{\sigma(h)}{\omega(a)} \right)^{(1+h)/2} F(-a) \\ Y(a, h) &= X(-a, h)^{-1} \quad Z(a, h) = X(-a, -h) \\ C(e, a, b, c) &= -ac(1 \otimes (\tilde{E}_{-e})^{-1} H^{(a+b)(b+c)/4} \otimes 1) \hat{U}^{a(b+c)/2} \hat{V}^{(a+b)c/2} \end{aligned}$$

with: $W(h) = U(h) \otimes V(h)$, $H = \tilde{\sigma}(\tilde{U}\tilde{V})^2$, $\hat{U} = \tilde{U} \otimes \tilde{U}^{-1} \otimes 1$, $\hat{V} = 1 \otimes \tilde{V}^{-1} \otimes \tilde{V}$.

Remark: Suppose \mathcal{K} is functorializable. Denote by Ψ the associated functor. The system (σ, E, F, W) will be called a parametrization of the Khovanov functor Ψ . It is easy to see that such a parametrization is unique.

5.6 Remark: In the classical case, ω is equal to 1 and the Khovanov data given by:

$$\begin{aligned} A(e, a, h) &= \langle e | a \rangle \langle e | h \rangle \langle a | h \rangle & B(a, b, h) &= \langle a | b \rangle & C(e, a, b, c) &= -ac \\ X(a, h) &= \langle a | -h \rangle & Y(a, h) &= \langle -a | -h \rangle & Z(a, h) &= \langle -a | h \rangle \end{aligned}$$

is functorializable.

Proof: Because of lemma 4.3.a, the condition $C(2)$ is equivalent to:

$$\frac{B^e(-a, -b, h)}{B^e(a, b, h)} = -ab\omega^{(-bh)} \otimes \frac{1}{\omega^{(ah)}}$$

for every signs e, a, b, h . It is easy to see that these conditions are equivalent to:

$$\frac{B(-a, -b, h)}{B(a, b, h)} = -ab\omega^{(-bh)} \otimes \frac{1}{\omega^{(ah)}}$$

for every a, b, h . Define the elements $B'(a, b, h)$ in $(R \otimes R)^*$ by:

$$B(a, b, h) = \frac{\langle a|b \rangle}{\delta(ah, bh)} (\mu(bh) \otimes \mu(ah)) B'(a, b, h)$$

With these new elements, the condition $C(2)$ is equivalent to:

$$B'(-a, -b, h) = B'(a, b, h)$$

and $C(2)$ is equivalent to the fact that $B'(a, b, h)$ depends only on ab and h :

$$B'(a, b, h) = B''(ab, h)$$

Because of lemma 4.10.a, the condition $C(8)$ is equivalent to:

$$\frac{B^e(a, b, -abh)}{B^e(a, -b, -abh)} = \left(\frac{X(a, -h)}{X(a, h)} \frac{\omega^{(-ah)}}{\omega^{(ah)}} \right) \otimes 1$$

$$\frac{B^e(a, b, -abh)}{B^e(a, -b, -abh)} = - \left(\frac{Z(a, h)}{Z(a, -h)} \right) \otimes 1$$

These conditions imply the following:

$$\frac{B(a, b, h)}{B(a, -b, h)} \in R^* \otimes 1 \quad \frac{B(a, b, h)}{B(-a, b, h)} \in 1 \otimes R^*$$

and that's equivalent to the fact that

$$\frac{B''(e, h)}{B''(-e, h)}$$

lies in $R^* \otimes 1$ and in $1 \otimes R^*$ and therefore in $K^*(1 \otimes 1)$.

Set:

$$W(h) = B''(-, h) \quad \sigma(h) = B''(+, h)W(h)^{-1}$$

The elements $\sigma(h)$ are in K^* and we have:

$$B(a, b, h) = \frac{\langle a|b \rangle}{\delta(ah, bh)} \sigma(h)^{(1+ab)/2} (\mu(bh) \otimes \mu(ah)) W(h)$$

With this expression, we have:

$$\frac{B^e(a, b, -abh)}{B^e(a, -b, -abh)} = a \sigma(-abh)^{ab} (\omega^{(bh)})^{-ab} \otimes 1$$

and the condition $C(8)$ is equivalent to:

$$a \sigma(-abh)^{ab} (\omega^{(bh)})^{-ab} = \frac{X(a, -h)}{X(a, h)} \frac{\omega^{(-ah)}}{\omega^{(ah)}} = - \frac{Z(a, h)}{Z(a, -h)}$$

So the left hand side term is independant of b and we get: $\sigma(+)\sigma(-) = \omega\bar{\omega}$. Using that, the condition $C(8)$ is equivalent to:

$$\frac{X(a, h)}{X(a, -h)} = a \frac{\sigma(h)}{\omega^{(ah)}} \quad \frac{Z(a, h)}{Z(a, -h)} = -a \frac{\sigma(-h)}{\omega^{(ah)}}$$

Set: $F(a) = X(-a, -)$. Then we have:

$$X(a, h) = \langle a | -h \rangle \left(\frac{\sigma(h)}{\omega^{(a)}} \right)^{(1+h)/2} F(-a)$$

The condition $C(6)$ is equivalent to:

$$X(a, h)Y(-a, h) = 1 \quad Z(a, h)Y(a, -h) = 1$$

and B, X, Y, Z can be describe in term of σ, W, F . Using these descriptions one can check that all conditions $C(i)$, for $i = 0, 2, 6, 8, 10$, are satisfied.

Define the elements $A'(e, a, h)$ by:

$$A(e, a, h) = \langle e | a \rangle \langle ea | h \rangle \left(\frac{\sigma(h)}{\omega^{(a)}} \right)^{(1-eh)/2} \frac{A'(e, a, h)}{F(a)}$$

Using these new elements, the condition $C(7)$ becomes: $A'(e, a, h) = A'(e, -a, h)$ and $A'(e, a, h)$ depends only on (e, h) . So by setting: $A'(e, a, h) = E_e(h)$, we have:

$$A(e, a, h) = \langle e | a \rangle \langle ea | h \rangle \left(\frac{\sigma(h)}{\omega^{(a)}} \right)^{(1-eh)/2} \frac{E_e(h)}{F(a)}$$

Using that, all conditions $C(i)$, for $i = 0, 2, 6, 7, 8, 9, 10$, are now satisfied. The condition $C(1)$ becomes the relation:

$$E_+(h)E_-(h) = \omega F(+)F(-)U(h)V(h)$$

with: $U(h) \otimes V(h) = W(h)$ and the last thing to do is to compute the elements $C(e, a, b, c)$ and verify the conditions $C(i)$, for $i = 3, 4, 5$.

Because of lemma 4.4.a, the condition $C(3)$ is equivalent to:

$$C(e, -ah, -ah, bh) = u \otimes v \otimes w \implies uv \otimes w = ab\tilde{\sigma}^{(1-ab)/2} \tilde{E}_e \otimes 1(\tilde{U} \otimes \tilde{V})^{-ab}$$

$$C(e, bh, -ah, -ah) = u \otimes v \otimes w \implies wv \otimes u = ab\tilde{\sigma}^{(1-ab)/2} \tilde{E}_e \otimes 1(\tilde{V} \otimes \tilde{U})^{-ab}$$

In the case: $b = -a$, we get (with: $C(e, -ah, -ah, -ah) = u \otimes v \otimes w$):

$$uv \otimes w = -\tilde{\sigma}(\tilde{E}_e \otimes 1)(\tilde{U} \otimes \tilde{V})$$

$$wv \otimes u = -\tilde{\sigma}(\tilde{E}_e \otimes 1)(\tilde{V} \otimes \tilde{U})$$

and that implies:

$$C(e, -ah, -ah, -ah) = -\tilde{U} \otimes \tilde{\sigma} \tilde{E}_e \otimes \tilde{V}$$

and then:

$$C(e, a, a, a) = -\tilde{U} \otimes \tilde{\sigma} \tilde{E}_e \otimes \tilde{V}$$

Consider the condition $C(4)$ given by lemma 4.5.a. This condition depends on signs a, b, c, h and on an element e in $\{+, -, 0\}$. We consider this condition in the case: $a = b = 1$.

If $e = -$, this condition is the following:

$$C(+, h, h, ch)C(-, -h, -h, ch) = -\tilde{\sigma}(\tilde{U} \otimes \tilde{V} \otimes 1)$$

or:

$$\begin{aligned} C(c, ch, ch, ch)C(-c, -ch, -ch, ch) &= -\tilde{\sigma}(\tilde{U} \otimes \tilde{V} \otimes 1) \\ \implies C(-c, -ch, -ch, ch) &= 1 \otimes \frac{\tilde{V}}{\tilde{E}_c} \otimes \frac{1}{\tilde{V}} \end{aligned}$$

So we get:

$$C(e, a, a, -a) = 1 \otimes \frac{\tilde{V}}{\tilde{E}_{-e}} \otimes \frac{1}{\tilde{V}}$$

If $e = 0$ the condition is:

$$C(+, h, ch, h)C(-, -h, ch, -h) = \tilde{\sigma}(\tilde{U} \otimes 1 \otimes \tilde{V})$$

or:

$$\begin{aligned} C(c, ch, ch, ch)C(-c, -ch, ch, -ch) &= \tilde{\sigma}(\tilde{U} \otimes 1 \otimes \tilde{V}) \\ \implies C(-c, -ch, ch, -ch) &= -1 \otimes \frac{1}{\tilde{E}_c} \otimes 1 \end{aligned}$$

and we have:

$$C(e, a, -a, a) = -1 \otimes \frac{1}{\tilde{E}_{-e}} \otimes 1$$

If $e = +$ the condition is:

$$C(+, ch, h, h)C(-, ch, -h, -h) = -\tilde{\sigma}(1 \otimes \tilde{U} \otimes \tilde{V})$$

or:

$$\begin{aligned} C(c, ch, ch, ch)C(-c, ch, -ch, -ch) &= -\tilde{\sigma}(1 \otimes \tilde{U} \otimes \tilde{V}) \\ \implies C(-c, ch, -ch, -ch) &= \frac{1}{\tilde{U}} \otimes \frac{\tilde{U}}{\tilde{E}_c} \otimes 1 \end{aligned}$$

and we have:

$$C(e, -a, a, a) = \frac{1}{\tilde{U}} \otimes \frac{\tilde{U}}{\tilde{E}_e} \otimes 1$$

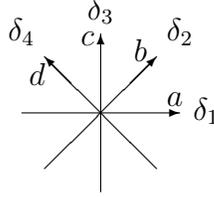
So we get the general formula:

$$C(e, a, b, c) = -ac(1 \otimes (\tilde{E}_{-e})^{-1} H^{(a+b)(b+c)/4} \otimes 1) \hat{U}^{a(b+c)/2} \hat{V}^{(a+b)c/2}$$

with: $H = \tilde{\sigma}(\tilde{U}\tilde{V})^2$, $\hat{U} = \tilde{U} \otimes \tilde{U}^{-1} \otimes 1$, $\hat{V} = 1 \otimes \tilde{V}^{-1} \otimes \tilde{V}$.

Using this expression, it is easy to check the conditions $C(3)$ and $C(4)$ and the last condition to check is $C(5)$.

Consider a movie move of type $\text{MVM}_5(c_1, c_2, c_3, c_4)$, where the c_i 's are distinct reals. This move involves four lines $\delta_1, \delta_2, \delta_3, \delta_4$ and each c_i is the height of δ_i . Since we are consider oriented links, each line has to be oriented. Let a, b, c, d be four signs and suppose the singular link diagram is oriented as follows:



This figure induces a movie move of type $\text{MVM}_5(c_1, c_2, c_3, c_4)$ and then an element $M(a, b, c, d, h) \in (R^4)^*$ depending on the signs a, b, c, d and the D -sign of the center of the triangle corresponding to the first Reidemeister move. So we have to prove that $M(a, b, c, d, h)$ is always equal to 1.

First of all, consider the case where the sequence (c_1, c_2, c_3, c_4) is decreasing.

For each k in $\{1, 2, 3, 4\}$ denote by $F_k : R^{\otimes 3} \rightarrow R^{\otimes 4}$ the tensorization by 1 at the k -th position. For example, F_2 is the map:

$$u \otimes v \otimes w \mapsto u \otimes 1 \otimes v \otimes w$$

We have the following:

$$\begin{aligned} M(a, b, c, d, h) &= F_4(C(-, ah, -bh, ch)^h) F_3(C(-, -ah, bh, -dh)^{-h}) \times \\ &F_2(C(-, ah, -ch, dh)^h) F_1(C(-, -bh, ch, -dh)^{-h}) \times \\ &F_4(C(-, -ah, bh, -ch)^h) F_3(C(-, ah, -bh, dh)^{-h}) \times \\ &F_2(C(-, -ah, ch, -dh)^h) F_1(C(-, bh, -ch, dh)^{-h}) \end{aligned}$$

But $C(e, ah, bh, ch)$ doesn't depend on h . So we have:

$$M(a, b, c, d, h)^h = M^2$$

with:

$$M = \frac{F_4(C(-, a, -b, c)) F_2(C(-, a, -c, d))}{F_3(C(-, a, -b, d)) F_1(C(-, b, -c, d))}$$

A straightforward computation gives the following:

$$\begin{aligned} M &= \left(1 \otimes H^{(a-b)(c-d)/4} \otimes H^{-(a-b)(c-d)/4} \otimes 1 \right) \left(1 \otimes \tilde{U}^{-(a-b)(c-d)/2} \otimes \tilde{U}^{(a-b)(c-d)/2} \otimes 1 \right) \times \\ &\left(1 \otimes \tilde{V}^{-(a-b)(c-d)/2} \otimes \tilde{V}^{(a-b)(c-d)/2} \otimes 1 \right) = 1 \otimes X \otimes X^{-1} \otimes 1 \end{aligned}$$

with:

$$X = H^{(a-b)(c-d)/4} \tilde{U}^{-(a-b)(c-d)/2} \tilde{V}^{-(a-b)(c-d)/2} = \left(H \tilde{U}^{-2} \tilde{V}^{-2} \right)^{(a-b)(c-d)/4} = \tilde{\sigma}^{(a-b)(c-d)/4}$$

Therefore M is equal to 1 and the condition $C(5)$ is always satisfied when the sequence (c_i) is decreasing. For a general sequence, we use the fact that the condition $C(4)$ is always satisfied and we check the condition with exactly the same proof as the proof of lemma 4.6.a. So all the conditions $C(i)$ are satisfied and the theorem is proven. Therefore theorem A is also proven. \square

Let (σ, E, F, W) be a parametrization of a Khovanov functor Ψ . Consider the following element in R^* :

$$\pi = -\frac{\omega F(+)F(-)}{\sigma(-)}$$

This element is called the weight of Ψ . By construction, every invertible element in R is the weight of a Khovanov functor. For example the weight of the functor described in remark 5.6 is -1 .

5.7 Proposition: *Let Ψ be a Khovanov functor and π be its weight. For every integer $p \geq 0$, denote by Σ_p an unknotted oriented surface of genus p in \mathbf{R}^4 . Then we have:*

$$\Psi(\Sigma_p) = \varepsilon(\delta^p \pi^{1-p})$$

$$\sum_{p \geq 0} x^p \Psi(\Sigma_p) = \varepsilon \left(\frac{\pi}{1 - x\delta/\pi} \right) = \frac{\varepsilon(\pi) + x(2 - \varepsilon(\pi)^2 u)}{1 - x\varepsilon(\pi)u + x^2 u} \in K[[x]]$$

with: $u = -\omega\bar{\omega}(\alpha - \bar{\alpha})^2 / (\pi\bar{\pi}) = \delta\bar{\delta} / (\pi\bar{\pi})$.

5.8 Remark: For $\pi = 1$, this formula is exactly the same as the formula in the lemma 1.6. Actually, the right hand side part of this formula is the image of the corresponding part in lemma 1.6 by the morphism sending $\varepsilon(1)$ to $\varepsilon(\pi)$ and δ to δ/π .

Proof: Let (A, B, C, X, Y, Z) be a Khovanov data such that Ψ is the corresponding functor. Let $p \geq 0$ be an integer. An unknotted surface of genus p in $\mathbf{R}^3 \times [0, 1]$ can be describe by the following movie sequence:

$$\varphi = (f, g, g', \dots, f')$$

where (g, g') is repeated p times. In this sequence, f, g, g' and f' are surgery moves of type $(0, -, +)$, $(1, -, -)$, $(1, +, +)$ and $(2, -, +)$. So we get:

$$\Psi(\Sigma_p) = \varepsilon \left(X(-, +)Z(-, +) \left(Y(-, -)Y(+, +)\omega(\alpha - \bar{\alpha}) \right)^p \right)$$

But it is easy to check the following:

$$X(-, +)Z(-, +) = X(-, +)X(+, -) = \pi$$

$$Y(-, -)Y(+, +) = \left(X(+, -)X(-, +) \right)^{-1} = \frac{1}{\pi}$$

So we have:

$$\Psi(\Sigma_p) = \varepsilon(\delta^p \pi^{1-p})$$

$$\sum_{p \geq 0} x^p \Psi(\Sigma_p) = \varepsilon \left(\frac{\pi}{1 - x\delta/\pi} \right) = \frac{\varepsilon(\pi - x\bar{\delta}\pi/\bar{\pi})}{(1 - x\delta/\pi)(1 - x\bar{\delta}/\bar{\pi})} = \frac{\varepsilon(\pi - x\bar{\delta}\pi/\bar{\pi})}{1 - x\varepsilon(\pi)\sigma\bar{\sigma} + x^2\sigma\bar{\sigma}}$$

with: $\sigma = \delta/\pi$. So we get the desired formula. \square

5.9 Theorem: *Two Khovanov functors with the same weight are isomorphic.*

Proof: Consider two Khovanov functors Ψ and Ψ' with the same weight π . They are described by two Khovanov data (A, B, C, X, Y, Z) and (A', B', C', X', Y', Z') . Since these two Khovanov data are functorializable there exist elements $\sigma(h)$ in K^* , $E_e(h), F(e)$ in R^* and $W(h)$ in $(R \otimes R)^*$ such that:

$$\sigma(+)\sigma(-) = 1$$

$$E_+(h)E_-(h) = F(+)\sigma(h)U(h)V(h)$$

$$A'(e, a, h) = A(e, a, h)\sigma(h)^{(1-eh)/2} \frac{E_e(h)}{F(a)}$$

$$B'(a, b, h) = B(a, b, h)\sigma(h)^{(1+ab)/2} W(h)$$

$$X'(a, h) = X(a, h)\sigma(h)^{(1+h)/2} F(-a)$$

$$C'(e, a, b, c) = C(e, a, b, c)(1 \otimes \tilde{E}_e^{-1} H^{(a+b)(b+c)/4} \otimes 1) \hat{U}^{a(b+c)/2} \hat{V}^{(a+b)c/2}$$

with: $W(h) = U(h) \otimes V(h)$, $H = \tilde{\sigma}(\tilde{U}\tilde{V})^2$, $\hat{U} = \tilde{U} \otimes \tilde{U}^{-1} \otimes 1$, $\hat{V} = 1 \otimes \tilde{V}^{-1} \otimes \tilde{V}$.

Because of the first relation, there exists an element $\sigma \in K^*$ with: $\sigma(h) = \sigma^h$. So we have:

$$A'(e, a, h) = A(e, a, h)\sigma^{(h-e)/2} \frac{E_e(h)}{F(a)}$$

$$B'(a, b, h) = B(a, b, h)\sigma^{h(1+ab)/2} W(h)$$

$$X'(a, h) = X(a, h)\sigma^{(1+h)/2} F(-a)$$

$$H = (\sigma\tilde{U}\tilde{V})^2$$

Moreover, since Ψ and Ψ' have the same weight, we have also:

$$F(+)\sigma F(-) = 1$$

So, by setting: $F = F(+)$, we have:

$$F(e) = F^e \sigma^{(e-1)/2}$$

Consider elements $x_\varepsilon(e, h)$ in R^* depending on signs ε , e and h .

Let D be an oriented link diagram and \hat{D} be the oriented resolution of D . Denote by $d(D)$ the winding number of D . For each component \hat{C} of \hat{D} denote by $g'(\hat{C})$ the $(\hat{D} \setminus \hat{C})$ -sign of a point in \hat{C} . Denote also by $g(D)$ the sum of all $g'(\hat{C})$. It is clear that $d(D)$ and $g(D)$ are both congruent to the number of components of \hat{D} mod 2 and $d(D) - g(D)$ is even.

For each signs e and h , denote by $X(e, h)$ the set of crossings of D with sign e and D -sign h . If C is a component of D , denote by $N_+(e, h, C)$ (resp. $N_-(e, h, C)$) the number of crossing x in $X(e, h)$ such that the over branch (resp. the under branch) containing x is in C .

Set:

$$G(D) = \sigma^{(d(D)-g(D))/2} \otimes_C \left(F^{d(C)} \prod_{\varepsilon, e, h} x_\varepsilon(e, h)^{N_\varepsilon(e, h, C)} \right)$$

Let D_0 be the set of components of D . The element $G(D)$ belongs to $R^{\otimes D_0}$ and induces, via the maps \widehat{T} , an automorphism $A(D) : KH(D) \rightarrow KH(D)$ well defined up to homotopy.

By conjugation with these automorphisms, the functor Ψ' is transformed into a new functor Ψ'' . So, for each morphism $f : D \rightarrow D'$, we have a diagram which is commutative up to homotopy:

$$\begin{array}{ccc} KH(D) & \xrightarrow{\Psi'(f)} & KH(D') \\ A(D) \downarrow & & \downarrow A(D') \\ KH(D) & \xrightarrow{\Psi''(f)} & KH(D') \end{array}$$

This new functor is clearly isomorphic to Ψ' . Let's choose the elements $x_*(*, *)$ such that:

$$\begin{aligned} x_+(+, h)x_+(-, h) &= \frac{1}{U(h)} & x_- (+, h)x_- (-, h) &= \frac{1}{V(h)} \\ x_+(e, h)x_-(e, h) &= \frac{\sigma^{(e-1)/2}}{E_e(h)} \end{aligned}$$

It is easy to see that this choice is possible. In this case, a straightforward computation shows that Ψ and Ψ'' agree on every elementary move and therefore on the category \mathcal{L}_0 . Thus Ψ' is isomorphic to Ψ and theorem 5.9 (and theorem B) is proven. \square

5.10 Remark: Consider two isomorphic Khovanov functors of weight π and π' . Because of Proposition 5.7, we have: $\varepsilon(\pi') = \varepsilon(\pi)$. In the case $R = R_0$, that implies: $\pi' = \pi$ or $\pi' = -\theta\pi$. So we may ask the question:

Two Khovanov functors of weight π and $-\theta\pi$ are they isomorphic?

5.11 Khovanov functors and Frobenius endomorphisms. If f is an endomorphism of the Frobenius algebra R , it acts on the complexes $KH(D)$ and transforms a Khovanov functor Ψ to a Khovanov functor Ψ' . Actually, if (σ, E, F, W) and (σ', E', F', W') are parametrizations of Ψ and Ψ' , we have:

$$\begin{aligned} f(\sigma(h)) &= \lambda\sigma'(h) & f(E_e(h)) &= E'_e(h) \\ f(F(h)) &= F'(h) & f(W(h)) &= \frac{1}{\lambda}W'(h) \end{aligned}$$

where λ is the element in K^* such that: $f(\omega) = \lambda\omega$.

Therefore, if π is the weight of Ψ , the weight of Ψ' is $f(\pi)$. Another consequence is the fact that there is no Khovanov functor invariant under the endomorphisms of R (at least if R is the universal Frobenius algebra R_0).

5.12 Extensions of Khovanov functors. In section 1.7, a category of mixed cobordisms \mathcal{C}' was introduced. This category is a monoidal category containing the category of cobordisms of closed oriented curves \mathcal{C} . Moreover the functor associated to R extends to this category. Actually it is possible to define in the same way a category of mixed cobordisms of oriented links \mathcal{L}' containing the category \mathcal{L} . So we may ask the following:

5.13 Question: Is it possible to extend a Khovanov functor to a monoidal functor from the category \mathcal{L}' of mixed cobordisms of oriented links to the homotopy category of K -complexes?

If such an extension Ψ exists, the morphism $\Psi(f)$ associated to a mixed cobordism f would be a chain map with a non necessarily zero degree. Notice that, if L is an oriented link and L' is the same link but where the orientation of a sublink L_1 of L is changed, there is a homotopy equivalence from $KH(L)$ to $KH(L')$ of degree 2λ , where λ is the linking number between L_1 and $L \setminus L_1$.

Actually there is another category \mathcal{C}'' between \mathcal{C} and \mathcal{C}' : the category of decorated cobordisms, where a decorated cobordism is a mixed cobordism on the form (C, \emptyset, u) or equivalently a pair (C, u) where C is a cobordism decorated by a map $u : \pi_0(C) \rightarrow R$. Similarly there is a category \mathcal{L}'' with: $\mathcal{L} \subset \mathcal{L}'' \subset \mathcal{L}'$. Using the operators \widehat{T} it is easy to extend every Khovanov functor to the category \mathcal{L}'' , but the extension to \mathcal{L}' is much more problematic.

6. Invariant of knotted surfaces.

This section is devoted to the proof of theorem C.

Consider a closed oriented surface S contained in \mathbf{R}^4 . Up to isotopy, we may as well suppose that S is contained in $\mathbf{R}^3 \times (0, 1)$. So this surface is a cobordism in \mathcal{L} from the empty link to itself. Therefore, any Khovanov functor Ψ sends this surface to a morphism from K to K which is the multiplication by an element $\Psi(S) \in K$. Because of proposition 5.7, we have: $\Psi(S) = \varepsilon(\delta^p \pi^{1-p})$ if S is an unknotted connected surface of genus p and π is the weight of Ψ . That proves theorem C if S is unknotted.

In the classical case, the functor Ψ was well defined up to sign and Tanaka [Ta] and Rasmussen [Ra2] proved that $\Psi(S)$ is ± 2 if S is the torus and 0 if S is any other connected surface. Notice that $\varepsilon(\delta^p)$ in the classical case is equal to 2 if $p = 1$ and to 0 otherwise.

Let's say that (R, π) is special if the following conditions hold:

- δ is invertible in R
- π is a square in R
- R has a twisting element which is the square of a element $x \in R$ with: $\bar{x} = x^{-1}$.

6.1 Lemma: *Suppose theorem C is true if (R, π) is special. Then the theorem is true in any case.*

Proof: Consider a Frobenius algebra R and a Khovanov functor Ψ with weight $\pi \in R^*$. Let R_1 be the following ring:

$$R_1 = \mathbf{Z}[\alpha, \bar{\alpha}, a, b, c, d, (a + b\alpha)^{-1}, (a + b\bar{\alpha})^{-1}, (c + d\alpha)^{-1}, (c + d\bar{\alpha})^{-1}]$$

This ring is a localization of a polynomial ring with 6 variables. It is equipped with an involution keeping a, b, c, d fixed and exchanging α and $\bar{\alpha}$. Set:

$$\omega_1 = a + b\alpha \quad \pi_1 = c + d\alpha$$

Then R_1 is a Frobenius algebra with generator α and twisting element ω_1 . It is easy to see that there is a unique morphism of Frobenius algebras f from R_1 to R sending α, ω_1, π_1 to α, ω, π respectively.

Consider the following ring:

$$R_2 = \mathbf{Z}[i][p, q, p_1, q_1, p_2, q_2, p^{-1}, (1 + q^2)^{-1}, 1/2, q_1^{-1}, (p_2^2 + q_2^2)^{-1}] \subset \mathbf{C}(p, q, p_1, q_1, p_2, q_2)$$

The complex conjugation induces an involution on R_2 and R_2 is a Frobenius algebra with generator $(p_1 + iq_1)/p$ and twisting element $p(1 + iq)^2(1 - iq)^{-2}$. It is easy to see that there is a unique morphism g of Frobenius algebras from R_1 to R_2 such that:

$$g(\alpha) = \frac{1}{p}(p_1 + iq_1) \quad g(\omega_1) = p \left(\frac{1 + iq}{1 - iq} \right)^2$$

$$g(\pi_1) = \pi_2 = (p_2 + iq_2)^2$$

It is clear that the fraction field of R_2 is, via g , an algebraic extension of the fraction field of R_1 . Then g is injective.

On the other hand, R_2 has iq_1 as generator and the corresponding twisting element is: $\omega_2 = x^2$ with:

$$x = \frac{1 + iq}{1 - iq}$$

and (R_2, π_2) is special.

Consider a Frobenius functor Ψ_1 associated with the Frobenius algebra R_1 with weight π_1 . The morphisms f and g send Ψ_1 to functors Ψ' and Ψ_2 with weight π and π_2 .

If the theorem is true for special pairs, it is true for Ψ_2 . Since $g : R_1 \rightarrow R_2$ is injective, the theorem is also true for Ψ_1 and then for Ψ' . But Ψ' is isomorphic to any Khovanov functor with weight π . Therefore the theorem is true for Ψ . \square

Using this lemma, we may as well suppose that Ψ is a Khovanov functor of weight π and that (R, π) is special. So $\beta = \alpha - \bar{\alpha}$ is invertible and there exist x and y in R^* such that:

$$\omega = x^2 \quad x\bar{x} = 1 \quad \pi = y^2$$

and we may also suppose that Ψ is parametrized by (σ, E, F, W) with:

$$\sigma(h) = 1 \quad E_e(h) = eh y x^{-1} \quad F(a) = -a y x^{-1} \quad W(h) = x^{-1} \otimes x^{-1}$$

From now on, we will suppose that all these properties are satisfied. The Khovanov data (A, B, C, X, Y, Z) associated to Ψ will be denoted by \mathcal{K} .

Consider an oriented link diagram D . Let D° be the union of D and a trivial circle oriented clockwise and contained in half a plan disjoint from D . Let p be a point in this circle. The complex $KH(D^\circ, p, R)$ is naturally isomorphic to $R \otimes_K KH(D, R)$ and the graded module $E(D^\circ, p, R)$ is naturally isomorphic to $E(D, R)$. Denote by φ_D the composite map:

$$R \otimes_K KH(D, R) \xrightarrow{\sim} KH(D^\circ, p, R) \xrightarrow{\varphi_D^{(R)}} E(D^\circ, p, R) \xrightarrow{\sim} E(D, R)$$

Because of theorem 2.15, this map is a homotopy equivalence and we have an explicit homotopy inverse of it (see remarks 2.14 and 2.16).

Then for every elementary move $f : D \rightarrow D'$ (compatible with the orientations) the morphism $f^{\mathcal{K}} : KH(D) \rightarrow KH(D')$ induces a well defined R -linear map $\hat{f} : E(D, R) \rightarrow E(D', R)$. Since \mathcal{K} is functorializable the correspondance $f \mapsto \hat{f}$ extends to composite of elementary moves (i.e. to movie sequences).

Consider an oriented curve Γ in the plane and a component C of Γ . We set:

$$\lambda(\Gamma, C) = \omega^{a(1+h)/2} \langle a|h \rangle$$

where a is the winding number of C and h is the $(\Gamma \setminus C)$ -sign of any point in C .

If D is an oriented link diagram, we set:

$$g(D) = \beta^{(m+q-n)/2} \prod_C \lambda(\tilde{D}, C)$$

where the product holds for every component C of the oriented resolution \tilde{D} of D , q being the algebraic number of crossings of D and m (resp. n) the number of components of \tilde{D} (resp. D).

Consider an elementary move $f : D \rightarrow D'$ between oriented diagrams. Denote by $\pi_0(D)$ the set of components of D and by \hat{D} the set of maps from $\pi_0(D)$ to $\{\pm\}$. Such a map is called a D -state. For each state $\sigma \in \hat{D}$, denote by $Y(\sigma)$ the set of crossings of D between two components c_1 and c_2 with $\sigma(c_1) \neq \sigma(c_2)$, by $\tilde{D}(\sigma)$ the oriented resolution of $D(\sigma)$ and by $d(\sigma)$ the number of components of $\tilde{D}(\sigma)$. By using the same notations with the diagram D' , we have sets $\pi_0(D')$ and \hat{D}' and, for each $\sigma \in \hat{D}'$, a set $Y'(\sigma)$, a diagram $\tilde{D}'(\sigma)$ and an integer $d'(\sigma)$.

This elementary move corresponds to a cobordism C between links associated to D and D' . So we have two maps from \hat{C} to \hat{D} and \hat{D}' , where \hat{C} is the set of C -states that is the set of maps from $\pi_0(C)$ to $\{\pm\}$. Let σ (resp. σ') be a D -state (resp. a D' -state). We say that σ and σ' are compatible (or: $\sigma \sim \sigma'$) if they are coming from a state of C . A straightforward computation shows the following:

6.2 Lemma: Let $f : D \rightarrow D'$ and σ (resp. σ') be a D -state (resp. a D' -state). Suppose that σ and σ' aren't compatible. Then the morphism:

$$\Lambda^{-e}(Y(\sigma)) \otimes Rv(\sigma) \rightarrow E(D, R) \xrightarrow{\hat{f}} E(D', R) \rightarrow \Lambda^{-e}(Y'(\sigma')) \otimes Rv(\sigma')$$

is trivial.

An immediate consequence of this lemma is the following:

6.3 Lemma: Let $f : D \rightarrow D'$ be a movie sequence represented by a cobordism C . Let σ and σ' be a D -state and a D' -state. Suppose the composite map:

$$\Lambda^{-e}(Y(\sigma)) \otimes Rv(\sigma) \rightarrow E(D, R) \xrightarrow{\hat{f}} E(D', R) \rightarrow \Lambda^{-e}(Y'(\sigma')) \otimes Rv(\sigma')$$

is not trivial. Then the two states σ and σ' are coming from a C -state.

Let $f : D \rightarrow D'$ be a movie sequence represented by a cobordism C and τ be a C -state. This state restricts to a D -state σ and a D' -state σ' . Define the map $\hat{f}(\tau)$ by:

$$\hat{f}(\tau) = pr' \circ \hat{f} \circ pr$$

where pr (resp. pr') is the projection $E(D, R) \rightarrow \Lambda^{-e}(Y(\sigma)) \otimes Rv(\sigma) \subset E(D, R)$ (resp. $E(D', R) \rightarrow \Lambda^{-e}(Y'(\sigma')) \otimes Rv(\sigma') \subset E(D', R)$). The map $\hat{f}(\tau)$ vanishes on $\Lambda^{-e}(Y(\sigma_1)) \otimes Rv(\sigma_1)$ for $\sigma_1 \neq \sigma$ and is the composite:

$$\Lambda^{-e}(Y(\sigma)) \otimes Rv(\sigma) \rightarrow E(D, R) \xrightarrow{\hat{f}} E(D', R) \rightarrow \Lambda^{-e}(Y'(\sigma')) \otimes Rv(\sigma') \subset E(D', R)$$

on $\Lambda^{-e}(Y(\sigma)) \otimes Rv(\sigma)$.

We have clearly the following:

$$\hat{f} = \sum_{\tau} \hat{f}(\tau)$$

If f corresponds to a closed surface S , $\hat{f}(\tau)$ is the multiplication by an element $\hat{S}(\tau) \in R$. Clearly $\Psi(S)$ is the sum of all $\hat{S}(\tau)$.

Consider now an elementary move $f : D \rightarrow D'$ corresponding to a cobordism C . Consider a C -state τ . This state restricts to a D -state σ and a D' -state σ' . A straightforward computation, case by case, gives the following:

6.4 Lemma: Suppose f is a Reidemeister move of type $I^+(e, a, h)$. Denote by c the component of D which is modified by f . Then the morphism $\hat{f}(\tau)$ is the map:

$$u \otimes v(\sigma) \mapsto u \otimes wv(\sigma')$$

with:

$$w = \left(\frac{g(D')}{g(D)} \right)^{(\sigma(c))} = \frac{g(D'(\sigma'))}{g(D(\sigma))} \langle \sigma(c) | -eh \rangle$$

6.5 Lemma: Suppose f is a Reidemeister move of type $II^+(a, b, h)$. Let x_+ and x_- be the created crossings with sign $+$ and $-$. Denote by c_+ (resp. c_-) the component of D containing the over (resp. under) branch of the move. Then the morphism $\widehat{f}(\tau)$ is the map:

$$u \otimes v(\sigma) \mapsto u \otimes wv(\sigma')$$

with:

$$w = \left(\frac{g(D')}{g(D)} \right)^{(\sigma(c_+))} = \frac{g(D'(\sigma'))}{g(D(\sigma))} \langle \sigma(c_+) | -ab \rangle$$

if $\sigma(c_+) = \sigma(c_-)$ and the map:

$$u \otimes v(\sigma) \mapsto x_+ \wedge x_- \wedge u \otimes wv(\sigma')$$

with:

$$w = \frac{g(D'(\sigma'))}{g(D(\sigma))} \langle \sigma(c_+) | b \rangle \langle \sigma(c_-) | a \rangle \langle \sigma(c_+) | \sigma(c_-) \rangle x^{-h(a\sigma(c_+) + b\sigma(c_-))(1 - \langle \sigma(c_+) | \sigma(c_-) | ab \rangle)}$$

otherwise.

6.6 Lemma: Suppose f is a Reidemeister move of type $III(e, a, b, c, h)$. Let c_1 (resp. c_2, c_3) be the over branch (resp. the middle branch, the under branch) of the move. Then the morphism $\widehat{f}(\tau)$ is the map:

$$u \otimes v(\sigma) \mapsto u \otimes wv(\sigma')$$

with:

$$w = \left(\frac{g(D')}{g(D)} \right)^{(\sigma(c_1))} = \frac{g(D'(\sigma'))}{g(D(\sigma))}$$

if $\sigma(c_1) = \sigma(c_2) = \sigma(c_3)$, and:

$$w = \frac{g(D'(\sigma'))}{g(D(\sigma))} \sigma(c_1) \sigma(c_3)$$

in the general case.

6.7 Lemma: Suppose f is a surgery move of type $(0, a, h)$. Let c be the created circle. Then the morphism $\widehat{f}(\tau)$ is the map:

$$u \otimes v(\sigma) \mapsto u \otimes wv(\sigma')$$

with:

$$w = \left(\frac{g(D')}{g(D)} yx^{-1} \right)^{(\sigma'(c))} = \frac{g(D'(\sigma'))}{g(D(\sigma))} \left(yx^{-1} \right)^{(\sigma'(c))}$$

6.8 Lemma: Suppose f is a surgery move of type $(1, a, h)$. Let c be the component of the cobordism containing the modified branches. Let: $e = 1$ (resp. $e = 0$) if the

surgery increases (resp. decreases) the number of components of the link. Then the morphism $\widehat{f}(\tau)$ is the map:

$$u \otimes v(\sigma) \mapsto u \otimes wv(\sigma')$$

with:

$$w = \left(\frac{g(D')}{g(D)} \beta^e xy^{-1} \right)^{(\tau(c))} = \frac{g(D'(\sigma'))}{g(D(\sigma))} \beta^e (xy^{-1})^{(\tau(c))} < \tau(c) | h >$$

6.9 Lemma: Suppose f is a surgery move of type $(2, a, h)$. Let c be the component removed by the surgery. Then the morphism $\widehat{f}(\tau)$ is the map:

$$u \otimes v(\sigma) \mapsto u \otimes wv(\sigma')$$

with:

$$w = \left(\frac{g(D')}{g(D)} \beta^{-1} yx^{-1} \right)^{(\sigma(c))} = \frac{g(D'(\sigma'))}{g(D(\sigma))} \beta^{-1} (yx^{-1})^{(\sigma(c))} < \sigma(c) | -h >$$

An easy consequence of these lemmas is the following:

6.10 Lemma: Let $f : D \rightarrow D'$ be a movie sequence corresponding to a cobordism C . Let τ be a constant C -state sending each component of C to a sign e . Let n_0 (resp. n_2) be the number of surgery moves in f of index 0 (resp. 2). Let n_1^+ (resp. n_1^-) be the number of surgery moves in f of index 1 which increases (resp. decreases) the number of components of the link. Then the morphism $\widehat{f}(\tau)$ is the map:

$$u \otimes v(\sigma) \mapsto u \otimes w^{(e)}v(\sigma')$$

with:

$$w = \frac{g(D')}{g(D)} \beta^{n_1^+ - n_2} (yx^{-1})^{n_0 + n_2 - n_1^+ - n_1^-}$$

6.11 Corollary: Let S be a surface in \mathbf{R}^4 . Let p_i be the genus of the i -th component of S . Let τ be a constant S -state sending each component of S to a sign e . Then we have:

$$\widehat{S}(\tau) = \left(\prod_i \left(\frac{\delta}{\pi} \right)^{p_i - 1} \right)^{(e)}$$

Proof: Let k be the number of components of S and p be the sum of the p_i 's. Numbers n_0, n_1^+, n_1^-, n_2 are related with a handle decomposition of S and there exist two integers $a, b \geq 0$ such that:

$$n_0 = k + a \quad n_1^+ = p + b \quad n_1^- = p + a \quad n_2 = k + b$$

So we have:

$$\beta^{n_1^+ - n_2} (yx^{-1})^{n_0 + n_2 - n_1^+ - n_1^-} = \beta^q (xy^{-1})^{2q} = (\beta x^2 y^{-2})^q = (\delta/\pi)^q$$

with $q = p - k = \sum_i (p_i - 1)$. The result follows. \square

Let $f : D \rightarrow D'$ be a movie sequence corresponding to a cobordism C and τ be a C -state. Define the sign $s(f, \tau)$ by the following:

Suppose f is a Reidemeister move of type III. Denote by c_1 (resp. c_2, c_3) the component of C containing the top branch (resp. the middle branch, the bottom branch) of the move. In this case we set: $s(f, \tau) = -1$ if $\tau(c_1) = \tau(c_3) = -\tau(c_2)$ and $s(f, \tau) = 1$ otherwise.

If f is another elementary move we set: $s(f, \tau) = 1$.

If f is a movie sequence: $f = (f_1, f_2, \dots, f_p)$, we set: $s(f, \tau) = \prod_i s(f_i, \tau)$.

If S is a closed oriented surface in \mathbf{R}^4 , S corresponds to a movie sequence f and we set: $s(S, \tau) = s(f, \tau)$.

6.12 Lemma: *Let S be a surface in \mathbf{R}^4 and τ be a S -state. Let S_i be the i -th component of S , p_i be the genus of S_i and e_i be the sign $\tau(S_i)$. Then we have:*

$$\widehat{S}(\tau) = s(S, \tau) \prod_i \left(\left(\frac{\delta}{\pi} \right)^{p_i - 1} \right)^{e_i}$$

Proof: Denote by S_+ (resp. S_-) the submanifold of S where τ is equal to $+$ (resp. $-$). By moving down S_- along the vertical axis in \mathbf{R}^3 , we get a new surface S' which is isotopic to $S_+ \amalg S_-$. The S -state τ induces a S' -state τ' . Because of the corollary we have:

$$\widehat{S}'(\tau') = \widehat{S}_+(+) \widehat{S}_-(-) = \prod_i \left(\left(\frac{\delta}{\pi} \right)^{p_i - 1} \right)^{e_i}$$

Suppose the morphism f corresponding to S is a movie sequence: $f = (f_1, \dots, f_k)$ where f_i is an elementary move from a diagram D_{i-1} to a diagram D_i . Then the morphism corresponding to S' is a movie sequence: $f' = (f'_1, f'_2, \dots, f'_k)$ where f'_i is an elementary move from a diagram D'_{i-1} to a diagram D'_i . For every $i = 1, 2, \dots, k$ we have:

$$\begin{aligned} \widehat{f}_i(\tau) &= \frac{g(D_i(\sigma_i))}{g(D_{i-1}(\sigma_{i-1}))} \varphi_i(\tau) \\ \widehat{f}'_i(\tau') &= \frac{g(D'_i(\sigma'_i))}{g(D'_{i-1}(\sigma'_{i-1}))} \varphi'_i(\tau') \end{aligned}$$

Using lemmas 6.4 to 6.9, we check that $\varphi_i(\tau)$ and $\varphi'_i(\tau')$ are always the same except for type III Reidemeister moves. In these cases, we have:

$$\varphi'_i(\tau') = s(f_i, \tau) \varphi_i(\tau)$$

Thus we have:

$$\widehat{S}(\tau) = \prod_i \varphi_i(\tau) = \prod_i s(f_i, \tau) \varphi'_i(\tau') = s(f, \tau) \prod_i \varphi'_i(\tau') = s(S, \tau) \widehat{S}'(\tau')$$

and the result follows. \square

6.13 Lemma: *Let S be a closed oriented surface in \mathbf{R}^4 and τ be a S -state. Then we have:*

$$s(S, \tau) = 1$$

Proof: Because of lemma 6.12, $s(S, \tau)$ is invariant under isotopy. But $s(S, \tau)$ is also invariant under surgery moves. Therefore $s(S, \tau)$ depends only on the cobordism class of (S, τ) in the group Ω of cobordisms of bicolored surfaces in \mathbf{R}^4 . The Pontryagin-Thom construction implies:

$$\Omega = \pi_4(MSO_2 \vee MSO_2) = \pi_4(MU_1 \vee MU_1) = \pi_4(BU_1 \vee BU_1)$$

Denote by E the space of paths in BU_1 ending at the base point. The homotopy fiber F of the inclusion $BU_1 \vee BU_1 \subset BU_1 \times BU_1$ is the following:

$$F = E \times \Omega BU_1 \cup_{\Omega BU_1 \times \Omega BU_1} \Omega BU_1 \times E$$

But we have a homotopy equivalence: $(E, \Omega BU_1) \sim (B^2, S^1)$. So we get a homotopy equivalence:

$$F \sim B^2 \times S^1 \cup_{S^1 \times S^1} S^1 \times B^2 = S^3$$

and we have:

$$\Omega \simeq \pi_4(F) \simeq \pi_4(S^3) \simeq \mathbf{Z}/2$$

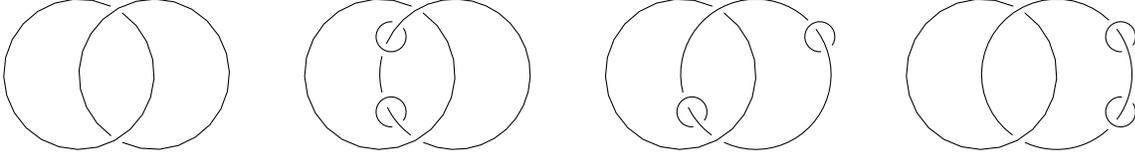
Since $s(S, \tau)$ depends only on the class of (S, τ) in Ω , we have a map $\varphi : \Omega \rightarrow \{\pm\}$ such that:

$$s(S, \tau) = \varphi([S, \tau])$$

where $[S, \tau]$ is the cobordism class of (S, τ) . By testing this formula for the empty surface we get: $\varphi(0) = 1$. Then the last thing to do is to determine $s(S, \tau)$ for some bicolored surface which is not trivial in Ω . Following [CKSS], the non zero element in Ω is represented by two tori T_+ and T_- where T_+ intersects $\mathbf{R}^3 \times \{0\}$ in a Hopf link H and T_- is the boundary of a regular neighborhood of one component of H in $\mathbf{R}^3 \times \{0\}$.

Consider a movie sequence f from the empty diagram to the diagram D of a Hopf link, given by a 0-surgery, two Reidemeister moves of type I_+^+ and a 1-surgery. Denote by \bar{f} the inverse move. So T_+ is represented by the movie move (f, \bar{f}) . Consider a movie sequence f_1 from D to a diagram D_1 given by a 0 surgery, two Reidemeister moves of type II^+ and a 1-surgery. This movie creates two circles C_1 and C_2 in a neighborhood of a component C of the diagram D . By moving C_1 around C , this circle goes through the other component of D and we have a movie sequence f_2 from D_1 to a diagram D_2 given by a Reidemeister move of type II^+ , two Reidemeister moves of type III and a Reidemeister move of type II^- . By moving C_2 around the other part of C , we get a movie sequence f_3 from D_2 to D_3 given also by a Reidemeister move of type II^+ , two Reidemeister moves of type III and a Reidemeister move of

type II^- . Finally we have a movie sequence f_4 from D_3 to D given by a 1-surgery, two Reidemeister moves of type II^- and a 2-surgery. The diagrams D , D_1 , D_2 and D_3 are the following:



The movie sequence $(f, f_1, f_2, f_3, f_4, \bar{f})$ represents a bicolored surface S which is not zero in Ω . On the other hand, among the movies f , \bar{f} and the f_i , only f_2 and f_3 contains some type III Reidemeister moves. So we have:

$$s(S, \tau) = s(f_2, \tau)s(f_3, \tau) = (-1)(-1) = 1$$

Then the map φ is trivial on Ω and the result follows. \square

Now we are able to prove theorem C. Let S be a closed oriented surface in \mathbf{R}^4 . Denote by S_i the i -th component of S and by p_i the genus of S_i . Denote also by u_i the element $(\delta/\pi_i)^{p_i-1}$. A S -state is characterized by the signs $e_i = \tau(S_i)$. So we have:

$$\Psi(S) = \sum_{\tau} \widehat{S}(\tau) = \sum_{e_*} \prod_i \left(\left(\frac{\delta}{\pi} \right)^{p_i-1} \right)^{(e_i)} = \sum_{e_*} \prod_i u_i^{(e_i)} = \prod_i (u_i + \bar{u}_i)$$

and the desired result follows from the obvious relation:

$$\forall u \in R, \quad u + \bar{u} = \varepsilon(\delta u) \quad \square$$

6.14 Remark: We can extend the functor Ψ to the category \mathcal{L}'' of decorated cobordisms of links. Consider a closed oriented surface S in \mathbf{R}^4 decorated by u . This decoration sends each component S_i of S to an element $u_i \in R$. In this case we have:

$$\Psi(S, u) = \prod_i \varepsilon(u_i \delta^{p_i} \pi^{1-p_i})$$

where p_i is the genus of S_i . In any case $\Psi(S, u)$ doesn't depend on the embedding $S \subset \mathbf{R}^4$.

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