ON REPEATED GAMES WITH COMPLETE INFORMATION*

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We consider $N$ person repeated games with complete information and standard signalling. We first prove several properties of the sets of feasible payoffs and Nash equilibrium payoffs for the $n$-stage game and for the $\lambda$-discounted game. In the second part we determine the set of equilibrium payoffs for the Prisoner’s Dilemma corresponding to the critical value of the discount factor.

0. Introduction. We consider $N$-person repeated games with complete information and standard signalling. We introduce the $n$-stage game, the $\lambda$-discounted game and the infinitely repeated game; then we prove several properties concerning the sets of feasible payoffs and of Nash equilibrium payoffs.

The properties studied are mainly the relation between convexity and stationarity and the simply-connectedness of the set of feasible payoffs.

The second part of the paper is devoted to the study of the $\lambda$-discounted Prisoner’s Dilemma. If $\lambda$ is greater than a critical value $\lambda_0$ the only Nash equilibrium payoff is the usual one (like in any finite repetition). Then we determine exactly the set of Nash equilibrium in the game with this discount factor $\lambda_0$, and this is a connected set of dimension 2 which differs from the set of individually rational feasible payoffs.

1. Notations and preliminaries. Let $G_i$ be an $N$-person game in normal form with finite pure strategy sets $T_i$, $i \in N$ and payoff function $X$ from $T = \prod_{i=1}^N T_i$ into $R^N$. We denote by $\mathcal{E}_i$ the set of mixed strategies of player $i$. We associate to $G_i$ a repeated game with perfect recall played as follows: at each stage $m$, knowing the previous history $h_m$ (i.e. the sequence of moves of all players up to stage $m - 1$), each player $i$ chooses a move $t_i$ in $T_i$ and this choice is told to all players.

We denote by $S_i$ (resp. $\Sigma_i$) the set of pure (resp. mixed) strategies of player $i$ in this repeated game and $S = \prod_{i=1}^N S_i$, $\Sigma = \prod_{i=1}^N \Sigma_i$. We now define 3 games according to the following payoffs:

\begin{align*}
(1/n) \cdot \sum_{m=1}^n x_m, & \quad n \in N \text{ for } G_n \text{ (n-stage repeated game)}, \\
\lambda \cdot \sum_{m=1}^\infty (1 - \lambda)^{m-1} x_m, & \quad \lambda \in (0, 1] \text{ for } G_\lambda \text{ (\lambda-discounted game)}, \\
L((1/n) \cdot \sum_{m=1}^n x_m), & \quad \text{for } G_\infty \text{ (L-infinitely repeated game)},
\end{align*}

where $x_m$ is the payoff at stage $m$ and $L$ a Banach limit.\(^1\)

Let us now define $D_n$ (resp. $D_\lambda, D_\infty$) to be the set of feasible payoffs using mixed strategies and $E_n$ (resp. $E_\lambda, E_\infty$) to be the set of Nash equilibrium payoffs in $G_n$ (resp. $G_\lambda, G_\infty$).

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Remark that the payoff in the $L$-infinitely repeated game is defined as the $L$-limit of the expectation. Nevertheless in our set-up the results would be the same by taking the expectation of the $L$-limit (this is no longer true for games with incomplete information).
Note that $G_n$ and $G_\lambda$ are special cases of games $\tilde{G}:(\tilde{S}_i, \tilde{\Sigma}_i, f_i, i \in N)$ where $\tilde{S}_i$ are compact strategy spaces, $\tilde{\Sigma}_i$ regular probabilities on $S_i$ and $f_i$ continuous (real) functions on $\tilde{S} = \prod_{i=1}^N \tilde{S}_i$. The (vector) payoff function is defined on $\tilde{\Sigma} = \prod_{i=1}^N \tilde{\Sigma}_i$ by

$$F(\sigma) = \int_{\tilde{S}} f(s) \prod_{i=1}^N \sigma_i (ds_i).$$

It follows that $D_n$ and $D_\lambda$ will share all the properties of $\tilde{D}$ (set of feasible payoffs in $\tilde{G}$) and similarly for $E_n$ and $E_\lambda$ with respect to $\tilde{E}$ (set of equilibrium payoffs in $\tilde{G}$).

In particular we have:

1. $\tilde{D}$ is a nonempty, path-connected, compact set,
2. $\tilde{E}$ is a nonempty compact set (Nash theorem).

Recall that $\tilde{D}$ is usually not convex and $\tilde{E}$ not connected.

Let $F$ be the finite set of feasible payoffs in pure strategies in $G_1$ and let $C = \text{co} F$ denote the convex hull of $F$. Hence $C$ is the set of payoffs achievable by using correlated strategies in $G_1$.

Finally define $a_i$ to be the individually rational level of player $i$ and $\Delta$ to be the set of individually rational payoffs in $C$, namely:

$$\Delta = \left\{ y \mid y \in C, y_i \geq a_i = \min_{\sigma'} \max_{\tau_i} X_i(\tau_i, \tau_i) \forall i, \text{ where } \sigma' = \prod_{j \neq i} \sigma_j \right\}.$$

Then the following asymptotic properties hold:

1. $D_n$ (resp. $D_\lambda$) converges in the Hausdorff topology as $n$ goes to $\infty$ (resp. as $\lambda$ goes to 0) to $C$ and $D_\infty$ equals $C$ (see [2], [6] and Proposition 4 below).
2. $E_\lambda$ converges in the Hausdorff topology, as $\lambda$ goes to 0, to $\Delta$ (see [2] or Lemma 2 below)\(^2\) and $E_\infty$ equals $\Delta$ (Folk theorem see [1] or [6]). It is well known that $E_n$ does not necessarily converge to $\Delta$, see e.g. example in §3.

Thus Property (4) shows an important difference with zero-sum two-person repeated games; in this framework the asymptotic behaviour of $v_n$ (value of $G_n$) and $v_\lambda$ (value of $G_\lambda$) is the same, even for stochastic games (where it converges to $v_\infty$ (value of $G_\infty$), see [4] and [8]) or for a large class of games with incomplete information (where $v_\infty$ may not exist, see [9]).

2. Study of $G_n$ and $G_\lambda$. We first recall and prove briefly easy results.

**Lemma 1.** (5) $F \subset D_1 \subset D$,
(6) $D \subset C$,
(7) $D$ convex $\iff D = C$,
where $D$ stands for $D_n$ or $D_\lambda$.

(8) $E_1 \subset E \subset \Delta$ where $E$ stands for $E_n$ or $E_\lambda$.

**Proof.** If an $N$-tuple $\tau$ of strategies in $\prod_{i=1}^N \tilde{\sigma}_i$ generates the payoff $x$ in $D_1$, then $\sigma(\tau)$ defined in $\Sigma$ by playing $\tau$ i.i.d. at each stage gives the same payoff in $D$, hence (5).

Now each payoff in $D$ is the expectation of barycenters of (random) points in $F$, hence lies in $C$ (6).

Finally since the extreme points of $C$ lie in $F$, (5) and (6) imply (7). The first inclusion in (8) is proved like in (5). The second follows from the fact that at each stage $m$, conditionally to the history $h_m$, each player can obtain an individually rational payoff.  \(\blacksquare\)

**Lemma 2.** $E_\lambda$ converges in the Hausdorff topology to $\Delta$, as $\lambda$ goes to 0.  \(^3\)

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\(^2\) A condition is needed, see added in proof.

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PROOF. By (8) it is enough to prove that in any neighbourhood of a point from $\Delta$ lies a point from $E_\lambda$, for $\lambda$ small enough.

Let $x$ in $\Delta$ and assume first $x_i - a_i > \epsilon > 0, \forall i = 1, \ldots, N$. Then we can write $x = \sum_{k=1}^{\infty} \alpha_k x^k$ with $x^k$ in $F$, $\alpha_k$ in $[0,1]$ and $\sum_k \alpha_k = 1$. Hence there exist $n_k$ in $\mathbb{N}$ such that, if $\sum_k n_k = R$ and $\sum_k (n_k/R)x^k = y$, we have: $y_i > a_i + \epsilon/2$ and $|y_i - x_i| < \epsilon/2, \forall i$.

Choose now $\lambda$ such that $(1 - \lambda)^{R-1} > 1 - \epsilon/4$. It follows then that by playing $n_i$ times a move inducing $x_i$, . . ., $n_k$ times a move inducing $x^k$ and so on and starting again at stage $R + 1$, the payoff in $G_\lambda$ will be some $z$ with: $|x_i - z_i| < \epsilon$ and $z_i > a_i + \epsilon/4$, for $\lambda < \lambda$.

We now claim that this payoff can be obtained by equilibrium strategies for $\lambda$ small enough. In fact since the strategies described above are pure any deviation can be observed and the deviator's payoff reduced to $a_i$.

Defining by $L$ the greatest absolute value of the payoffs it follows that the gain by deviating is at most: $2L(1 - (1 - \lambda)^{R+1}) - (\epsilon/4)(1 - \lambda)^{R+1}$ which is negative for $\lambda$ small enough. This ends the proof if $\Delta$ is full dimensional.

If now, for some $i$, $x_i = a_i$, for all $x$ in $\Delta$, player $i$ will always play a best reply and no profitable deviation for him is profitable. It is then enough to specify the strategies of the other players and the proof goes by induction. •

Note that contrary to the "Perfect Folk Theorem" (see [2]) the previous result does not extend to perfect equilibria, for a counterexample see [5].

For any set $X$ and any $t$ in $\mathbb{N}$ we define:

$tX = \{tx; x \in X\}$,
$t* X = \{y; y = \sum_{m = 1}^{t} x_m, x_m \in X\}$.

**Lemma 3.** Let $n = mp + r$ in $\mathbb{N}$, then

(9) $nD_n \supset m * (pD_p) + rD_r$,

(10) $nE_n \supset m * (pE_p) + rE_r$.

**Proof.** Let $a_0$ in $D_p$ and $a_j$ in $D_p, j = 1, \ldots, m$, be obtained by the $N$-tuple of strategies $a(j), j = 0, \ldots, m$. Then the strategy $a$ in $\Sigma$, defined by: play $a(0)$ up to stage $r$, $a(j)$ from stage $r + (j - 1)p$ up to stage $r + jp - 1$ (independently from the history at stage $r + (j - 1)p$), induces a payoff in $G_n$ equal to $n^{-1}(ra_0 + \sum_{j=1}^{m} pa_j)$ hence (9).

Now if $a(0)$ is an equilibrium strategy in $G$, and similarly for $a(j)$ in $G_p, j = 1, \ldots, m$, then the strategy $a$ defined above is still an equilibrium in $G_n$ hence (10).

In particular this gives $D_n \subset D_\infty \forall k > 1, k \in \mathbb{N}$ hence $D_\infty \subset D_n$ for some $k > 1$ implies $D_n$ convex and similarly for $E_n$.

Nevertheless there are games for which:

(11) the sequences $D_n$ and $E_n$ are not monotonic.

**Example 1.** $G_1$ is a 2-person game defined by the following payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>(1,0)</th>
<th>(0,0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0,1)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that $(1/2,1/2) = 1/2(1,0) + 1/2(0,1)$ belongs to $E_2$ hence to $D_2$. Obviously $(1/2,1/2)$ is not in $D_1$.

Now since this payoff is Pareto Optimal, the only way to achieve it in $G_3$ is to play a pure strategy at each stage. This gives the payoffs $(n/3,1 - n/3), n = 0,1,2,3$ and $(1/2,1/2) \notin D_3$. Since $E_n \subset D_n$ (11) follows.

Note in this example that $D_n \neq C$ for all $n$. Remark also that by duplicating one
strategy of one of the players, $D_1$ and $E_1$ will not change, but $D_2$ will increase and $(\frac{1}{2}, \frac{1}{2})$ will belong to $D_3$.

Moreover the variations of $D_n$ and $E_n$ are not related:

(12) $D_n = D_{n+1}$ does not imply $E_n = E_{n+1}$.

Example 2.

<table>
<thead>
<tr>
<th></th>
<th>(1,0)</th>
<th>(2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0,1)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In this game $D_1 = C$ hence $D_1 = D_n$ for all $n$. $E_1$ is reduced to $(2,2)$ since each player has a strictly dominating strategy. Now we claim that $(1,1)$ belongs to $E_2$.

In fact this payoff is achievable through the following equilibrium strategies: (Bottom, Left) at the first stage, and at the second stage:

— for player I: Bottom if player II played Right at the first stage.

  Top otherwise.

— for player II: Left if player I played Top at the first stage.

  Right otherwise.

Similarly we have:

(13) $E_n = E_{n+1}$ does not imply $D_n = D_{n+1}$.

Example 3.

<table>
<thead>
<tr>
<th></th>
<th>(1,0)</th>
<th>(1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1,0)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$E_1 = \{(1,x); x \in [0,1]\} = E_n$ for all $n$ and $(\frac{1}{2}, \frac{1}{2}) \in D_2 \setminus D_1$. Note that Example 2 shows also:

(14) $E_n$ is not contained in the convex hull of $E_1$.

Moreover:

(15) $E_{n+1} \subseteq E_n$ does not imply $E_{n+2} \subseteq E_n$.

Example 4.

<table>
<thead>
<tr>
<th></th>
<th>(m,0)</th>
<th>(m+1,m+1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td></td>
<td>(0,m)</td>
</tr>
</tbody>
</table>

Since by playing first Bottom player I can achieve at most $(n - 1)(m + 1)/n$ in $G_n$, the fact that he can guarantee $m$ by playing always top implies by induction that $E_n$ is reduced to $(m + 1, m + 1)$ for all $n < m$.

Now it is easy to see that $(m,m)$ belongs to $E_{m+1}$ (play $(0,0)$ once then $(m + 1, m + 1)$, see Example 2). As for the game $G_\lambda$ we have, as in (11):

(16) the nets $D_\lambda$ and $E_\lambda$ are not monotonic.

Example 1 (revisited). By playing once $(1,0)$ and then always $(0,1)$, the players achieve $(7/8, 1/8)$ in $E_7/8$.

It is clear that this payoff is not in $D_1$. To prove that it does not belong to $D_{3/4}$ note that since it is Pareto optimal it can only be achieved by using pure strategies. The payoff for player I in $G_{3/4}$ is at most $\frac{1}{4}$ if $X_1 = (0,1)$ hence $X_1$ has to be $(1,0)$. Now if $X_2 = (1,0)$ player I get at least $\frac{13}{16}$ and at most $\frac{1}{2}$ if $X_2 = (0,1)$.

We shall now focus on the sets of feasible payoffs and study properties of convexity and stationarity.

For small values of $\lambda$ the description of $D_\lambda$ is easy since we have the following (compare with (3) and example 1 where $D_n \neq C \forall n$):
PROPOSITION 4.

(17) \( D_\lambda = C \) for all \( \lambda < 1/N \).

PROOF. By (5) and (6) \( C \) is the convex hull of \( D_1 \) and \( D_1 \) is connected (1). A theorem of Fenchel (see e.g. [10, p. 169, Proposition 3.3]) now implies that each point of \( C \) is a convex combination of at most \( N \) points of \( D_1 \). Thus given \( x \) in \( C \), there exist \( x_i \) in \( D_1 \) and \( \lambda_i \) in \( [0,1] \), \( i = 1, \ldots, N \), with \( x = \sum_{i=1}^N \lambda_i x_i \).

Now we can assume \( \lambda_i > 1/N \) and we can introduce \( x' \) in \( C \) defined by:

\[
x' = \frac{1}{1-\lambda} \left( (\lambda_1 - \lambda)x_1 + \sum_{i>1} \lambda_i x_i \right),
\]

such that \( x = \lambda x_1 + (1-\lambda)x' \).

Doing the same decomposition for \( x' \) we obtain inductively:

\[
x = \lambda \sum_{m=0}^{\infty} (1-\lambda)^{m} x_{1}^{(m)} \quad \text{with} \quad x_{1}^{(m)} \text{in} \ D_1 \quad \text{for all} \quad m = 0, 1, \ldots
\]

This implies that \( x \) is in \( D_\lambda \), by playing at stage \( m + 1 \) a strategy in \( \prod_{i=1}^{N} \varepsilon_i \), achieving \( x_{1}^{(m)} \).

Note that this bound is the best one:

EXAMPLE 5. \( T_i = \{1, \ldots, N\} \) for all \( i = 1, \ldots, N \). The payoff function \( X \) from \( T \) to \( R^N \) is defined by:

\[
X(t_1, \ldots, t_N) = e_j \quad \text{\( (j\)-unit vector in \( R^N \)) \quad \text{if} \quad t_i = j \quad \text{for all} \quad i,
\]

\[
\geq 0 \quad \text{otherwise}.
\]

Then \( (1/N, \ldots, 1/N) \) does not belong to \( D_\lambda \) for \( \lambda > 1/N \).

PROPOSITION 5.

(18) If \( D_\lambda \) is convex then \( D_{n+1} = D_n \), hence \( D_m = C \) for all \( m > n \).

PROOF. Let \( x \) in \( D_n \) be induced by an \( N \)-tuple of strategies \( \sigma \) and let \( x_m, m = 1, \ldots, n \) be the corresponding expected payoff at stage \( m \). It follows that \( nx = \sum_{m=1}^{n} x_m \) with \( x_1 \) in \( D_1 \) and \( x_m \) in \( C \) for all \( m \).

Now \( y = (\sum_{m>1} x_m)/(n-1) \) still belongs to the convex set \( C \) which equals \( D_n \) by (7).

By (5) this implies that the line segment \([x_1, y]\) lies in \( D_n \) hence: \( z = x_1/n^2 + (1-1/n^2)y \) belongs to \( D_1 \) and is induced by some \( \tau \).

Since we have \( x = (x_1 + nz)/(n+1) \) it follows that \( x \) is achievable in \( G_{n+1} \) by playing \( \sigma \) at the first stage and then \( \tau \).

Reciprocally the following obviously holds:

(19) \( D_m = D_n \) for all \( m > n \) implies \( D_n = C \) (by (3) or (9)).

Nevertheless we have:

(20) \( D_n \) convex does not imply \( D_{n-1} \) convex.

EXAMPLE 6. Let \( G_1 \) be the following two-person game:

\[
\begin{array}{cccc}
(0,1) & (1,1) & (2,0) & (3,0) \\
(0,0) & (1,0) & (2,1) & (3,1) \\
\end{array}
\]

\((3/2,1)\) does not belong to \( D_1 \) (a payoff 1 to player II implies that player I is using a pure strategy) but \( D_1 \) contains the two squares \( C' = \text{co}((0,0),(0,1),(1,1),(1,0)) \) and \( C'' = \text{co}((2,0),(3,0),(3,1),(2,1)) \). Thus we have

\[
C = \frac{1}{2}(C' + C'') \subset \frac{1}{2}(D_1 + D_1) \subset D_2.
\]
Remark that for a two-person game where each player has only two pure strategies, either \( D_1 = C \) (see Example 2) or \( D_n \neq C \) for all \( n \) (see Example 1).

In a similar way one can prove:

**Proposition 6.**

(21) If \( D_{\lambda} \) is convex then \( D_\delta = C \) for all \( 0 < \delta < \lambda \).

**Proof.** Let \( x \) in \( D_\lambda \) be induced by some \( \sigma \) and denote by \( x_m \) the expected payoff at stage \( m \). Here also \( x_1 \) is in \( D_1 \) and \( x_m \) is in \( C \) with \( x = \lambda \sum_{m=1}^{\infty} (1 - \lambda)^{m-1} x_m \). Define \( y \) to be \( \lambda \sum_{m=2}^{\infty} (1 - \lambda)^{m-2} x_m \), then \( y \) belongs to \( C = D_\lambda \) and \( x = \lambda x_1 + (1 - \lambda)y \).

\( D_1 \) being included in the convex set \( D_\lambda \) it follows that \( x' \) defined to be \( \left( (\lambda - \delta)/(1 - \delta) \right) x_1 + \left( (1 - \lambda)/(1 - \delta) \right) y \) belongs to \( D_\lambda \) and \( x = \delta x_1 + (1 - \delta)x' \). Doing the same decomposition for \( x' \) we obtain by induction \( x = \delta \sum_{m=0}^{\infty} (1 - \delta)^{m} x_1(m) \) with \( x_1(m) \) in \( D_1 \) for all \( m = 0, 1, \ldots \). By playing \( \sigma_m \) at stage \( m + 1 \), where \( \sigma_m \) achieves \( x_1(m) \) in \( G_1 \), the players can obtain \( x \) in \( G_\delta \) hence \( x \) belongs to \( D_\delta \).

Reciprocally we have:

(22) \( D_\lambda = D_\delta \) for all \( 0 < \delta < \lambda \) implies \( D_\delta = C \) (by (17)).

Recall that \( C = \text{co} F \) is a convex polyhedron. Denote by \( L \) a one-dimensional face of \( C \). Then by (5), \( L \cap D_\delta \) and \( L \cap D_n \) are nonempty for all \( \lambda \) in \( (0, 1] \) and all \( n > 1 \).

We now consider the feasible payoffs lying on \( L \) and prove that if this set is decreasing then it contains all \( L \). For \( N = 2 \), this property has interesting consequences (see Corollary 12).

**Proposition 7.**

(23) If for some \( \delta, 0 < \delta < \lambda \), \( D_\delta \cap L \) is included in \( D_\lambda \cap L \) then \( L \) is included in \( D_\delta \).

**Proof.** Let us suppose that there exists a point in \( L \) which is not in \( D_\delta \). Without loss of generality we can assume that \( L = \{z, 0, \ldots, 0\} \) with \( z = (z_0, z_1, \ldots, 0) \) in \( R^N \), and \( z_0, z_1 \) belonging to \( F \subset D_\delta \).

For each point \( Z \) in \( L \), let \( d(Z) \) denotes its distance to the compact set \( D_\delta \cap L \). The maximum of \( d(Z) \) on \( L \), denoted by \( \overline{d} \), is taken at some point \( Z = (z, 0, \ldots, 0) \) and is strictly positive by hypothesis. Let us introduce: \( X = (x, 0, \ldots, 0) \) and \( Y = (y, 0, \ldots, 0) \) with \( x = z - \overline{d} \) and \( y = z + \overline{d} \). Then we have:

(\( \ast \)) \( X \) and \( Y \) belong to \( D_\delta \cap L \) and \( (X, Y) \cap D_\delta \) is empty.

(\( \ast \ast \)) No other couple of points \( X', Y' \) with \( ||X' - Y'|| > 2\overline{d} \) satisfy (\( \ast \)).

Let \( X \) be induced by \( \sigma \). Since \( X \) lies on a face of \( C \), at each stage the random payoff induced by \( \sigma \) will belong to this face. Hence it is enough to consider the first component of the payoff.

Let \( H \) be the set of histories at stage 2, having positive probability \( p(h) \), under \( \sigma \). For each \( h \) in \( H \), let \( \sigma(h) \) be the strategy from stage 2 on defined by \( \sigma \) conditionally on \( h \).

Denote by \( x_1 \) the expected payoff at stage 1 and by \( x_2(h) \) the payoff induced in \( G_\delta \) by \( \sigma(h) \), for each \( h \) in \( H \). Thus:

\[ x = \delta x_1 + (1 - \delta) \sum_{h \in H} p(h) x_2(h). \]

(a) If for some \( h_0 \) in \( H \), \( x_2(h_0) \) is strictly less than 1, then by (\( \ast \ast \)) there exists \( Z = (z, 0, \ldots, 0) \) in \( D_\delta \) with: \( x_2(h_0) < z < x_2(h_0) + 2\overline{d} \).

If \( Z \) is achievable by \( \tau \) in \( G_\delta \), then the following strategy: play \( \sigma \), unless the history at stage 2 is \( h_0 \) and from this stage on use \( \tau \), gives a payoff \( w \) with:

\[ w = \delta x_1 + (1 - \delta) \left( p(h_0)z + \sum_{h = h_0}^{h \neq h_0} p(h) x_2(h) \right). \]
Note that $0 < w - x < (1 - \delta)2\delta; \text{ thus } W = (w, 0, \ldots, 0) \text{ belongs to } D_\delta \cap (X, Y) \text{ contradicting } (*)$.

(b) Since we can do the same construction starting from $Y$ it remains to consider the case where:

$$x = \delta x_1 + (1 - \delta), \quad y = \delta y_1.$$ 

We now use the fact that $D_\delta \cap L$ is included in $D_\lambda \cap L$, hence $x$ can be written as $\lambda u_1 + (1 - \lambda)u_2$ with $U_1 = (u_1, 0, \ldots, 0)$ in $D_1$. Since $u_2$ is less than one and $\delta < \lambda$ we have $u_1 > x_1$. Hence:

$$\delta u_1 < \lambda u_1 < x,$$

$$\delta u_1 + (1 - \delta) > x.$$ 

Let us consider the following set: $A = (\delta u_1 + (1 - \delta)t; T = (t, 0, \ldots, 0) \text{ is in } D_\delta \cap L)$. By (**) it follows that there exists $z \in A$ satisfying: $0 < z - x < (1 - \delta)2\delta$. Now if $z$ is $\delta u_1 + (1 - \delta)t$, let $U_1$ be induced by $\sigma$ (in $G_1$) and $T$ be induced by $\tau$ (in $G_\delta$).

The strategy defined by playing $\sigma$ at stage 1 and $\tau$ from stage 2 on gives as a payoff in $G_\delta$, $Z = (z, 0, \ldots, 0)$ contradicting (•).

As for the feasible payoffs in the finitely repeated game $G_n$ we have:

**Proposition 8.**

Let $n > Nm$, then $D_{n+m} \subset D_n \implies D_{n+m} = C.$

**Proof.** The proof goes by induction on the dimension of the faces of $C$ and follows obviously from the following:

**Proposition 9.**

Let $P$ be a face of $C$ of dimension $p$ ($p < N$). If $n > pm$ and $D_{n+m} \cap P \subset D_n \cap P$ then $P \subset D_{n+m}.$

**Proof.** By induction (the proof follows from (5) if $p = 0$) we assume that each face of $P$ of dimension at most $p - 1$ is in $D_{n+m}$ and we write $D'_{m}$ for $D_{m} \cap P$, for all $m$.

Note that by (9) we can and shall assume $m < n.$ Suppose that $P$ is not included in $D_{n+m}.$ For each point $Z$ in $P$, $d(Z)$ denotes its distance to the compact $D'_{m}$ and the maximum, $\delta > 0$, is taken at some $Z.$

Let $B = B(Z, d) \cap P$ where $B(Z, d)$ is the closed ball in $R^N$ with center $Z$ and radius $\delta$.

We first need the following:

**Lemma 10.** $Z$ belongs to the convex hull of $B \cap D'_{n+m}.$

**Proof.** By definition of $\delta$, $B \cap D'_{n+m}$ is not empty. Define $H$ to be the convex hull of $B \cap D'_{n+m}$, $H$ is a compact convex set. If $Z$ is not in $H$, let $Y$ be a closest point to $Z$ in $H.$ Thus:

$$\langle Z - Y, Z \rangle > \langle Z - Y, T \rangle \quad \text{for all } T \text{ in } H. \quad (\ast)$$

For every $\epsilon > 0$ let $B_\epsilon = B(Z, \epsilon) \cap P$ where $Z_\epsilon = Z + \epsilon(Z - Y).$ Since by induction the frontier of $P$ is in $D'_{n+m}$, $Z_\epsilon$ is in $P$ for $\epsilon$ small enough, hence $B_\epsilon \cap D'_{n+m}$ is not empty. Note now that if $T$ belongs to $B_\epsilon$ and

$$\langle Z - Y, T \rangle < \langle Z - Y, Z + \frac{\epsilon}{2}(Z - Y) \rangle$$

then $T$ belongs to the interior $B$ of $B$. By the choice of $Z$, $B \cap D'_{n+m}$ is empty hence there exists $T$ in $B \cap D'_{n+m}$ with $\langle Z - Y, T \rangle > \langle Z - Y, Z \rangle.$ By compacity we thus obtain a point $T$ in $B \cap D'_{n+m}$ satisfying $\langle Z - Y, T \rangle > \langle Z - Y, Z \rangle$ contradicting (•).
Using Caratheodory’s theorem we can now introduce $X^k$ in $B \cap D_{n+m}$, $k = 1, \ldots, q_0, q_0 < p + 1$, such that $\overline{Z}$ lies in the convex hull of the $X^k$, and this family is minimal with respect to this property. If $X^k$ is generated by $\sigma^k$ in $G_{n+m}$, let us denote by $S^k$ the average expected payoff up to stage $m$ and for each history in $H_k$: set of histories at stage $m + 1$ having positive probability $p(h)$ under $\sigma^k$, let $U^k(h)$ be the average expected payoff for the remaining $n$ stages in $G_{n+m}$, conditionally on $h$. Thus:

\[(n + m)X^k = mS^k + n \sum_{h \neq h_0} p(h)U^k(h).\]  

Since $X^k$ belongs to the face $P$, $S^k$ and $U^k(h)$ have the same property.

(a) Assume that there exists $h_0$ in $H_k$ such that:

\[\langle U^k(h_0), X^k - \overline{Z} \rangle > \min_{T \in P} \langle T, X^k - \overline{Z} \rangle = a^k.\]  

Since the frontier of $P$ is in $D_{n+m}$, the intersection of $D_{n+m}$ with the closed ball $B^k(h_0)$ centered at $U^k(h_0) - X^k + \overline{Z}$ and of radius $\bar{d}$ is not empty.

Using (***) there exists a point $Z^k(h_0)$ in this intersection and different from $U^k(h_0)$.

Since $D_{n+m}$ is included in $D_n$, $Z^k(h_0)$ is in $D_n$ hence $\overline{X}$ defined by

\[\overline{X} = \frac{1}{n + m} \left( mS^k + n \sum_{h \neq h_0} p(h)Z^k(h_0) + \sum_{h \neq h_0} p(h)U^k(h) \right)\]

belongs to $D_{n+m}$ (see the proof of Proposition 7).

It remains to compute the distance from this new point to $\overline{Z}$. But

\[\|X^k - \overline{Z}\|^2 = \|\overline{X} - X^k\|^2 + \|X^k - Z\|^2 + 2\langle \overline{X} - X^k, X^k - \overline{Z} \rangle.\]

Note that

\[\langle \overline{X} - X^k, X^k - \overline{Z} \rangle = p(h_0) \frac{n}{n + m} \langle Z^k(h_0) - U^k(h_0), X^k - \overline{Z} \rangle\]

hence this quantity is negative.

Moreover, since $Z^k(h_0)$ is in $B^k(h_0)$:

\[|\langle Z^k(h_0) - U^k(h_0), X^k - \overline{Z} \rangle| > \frac{1}{2} \|Z^k(h_0) - U^k(h_0)\|^2.\]

Thus:

\[\|X^k - \overline{Z}\|^2 < \bar{d}^2 + \left( p^2(h_0) \frac{n^2}{(n + m)^2} - p(h_0) \frac{n}{n + m} \right) \|Z^k(h_0) - U^k(h_0)\|^2 < \bar{d}^2\]

which contradicts the definition of $\overline{Z}$ and $\bar{d}$.

(b) We are now left the case where for each $k$ and each $h$ in $H_k$

\[\langle U^k(h), X^k - \overline{Z} \rangle = a^k.\]  

Let $L$ be the linear space generated by the $X^k$ and denote by $Q$ the projection on $L$ of the points $T$ in $\mathbb{R}^n$ satisfying: $\langle T, X^k - \overline{Z} \rangle > a^k$ for all $k$. Note that $Q$ contains the projection of $P$ on $L$ and that $Q$ is homeomorphic to a simplex of dimension $q - 1 < p$.

We shall write $\overline{T}$ for the projection of $T$ on $L$ and introduce barycentric coordinates $(\alpha^1, \ldots, \alpha^q)$ for the points in $Q$ such that the set of $\alpha^k$ with $\alpha^k = 0$ corresponds to the set of $\overline{T}$ in $Q$ with $\langle \overline{T}, X^k - \overline{Z} \rangle = a^k$. Let $(\bar{\alpha}^1, \ldots, \bar{\alpha}^q)$ corresponding to $\overline{Z}$. It follows
from (**) and (•••) that \( \bar{a}^k < m/(m + n) \) for all \( k = 1, \ldots, q \). Since \( \sum q a^k = 1 \), this inequality implies \( pm > n \) contradicting the assumption. •

In order to obtain more precise results for \( N = 2 \) we shall prove and use the following property (recall that \( \tilde{D} \) is the set of feasible payoffs in a game \( \tilde{G} \)):

**Proposition 11.**

(27) If \( N = 2 \) then \( \tilde{D} \) is simply connected.

**Proof.** Let \( \gamma \) be a closed continuous path in \( \tilde{D} \) (i.e. \( \gamma \) is a continuous map from \([0,1]\) to \( \tilde{D} \) with \( \gamma(0) = \gamma(1) \)) and assume that there exists \( y \) in \( \mathbb{R}^2 \setminus \tilde{D} \) such that:

\[
\text{Ind}(y, \gamma) \neq 0. \quad (*)
\]

For each \( t \) in \([0,1]\) and each \( \sigma \) (resp. \( \tau \)) strategy of player I (resp. player II) in \( \tilde{G} \) such that \( X(\sigma, \tau) = \gamma(t) \) we define a closed continuous path \( \Gamma[t; \sigma, \tau] \) as follows:

Fix \( \sigma_0, \tau_0 \), such that \( X(\sigma_0, \tau_0) = \gamma(0) \). Now \( \Gamma[t; \sigma, \tau] \) coincides with \( \gamma \) on \( \{ \gamma(0), \gamma(t) \} \).

Starting from \( \gamma(t) \) it follows the two line segments:

- first \( X(\sigma, \sigma_0 + (1 - s) \tau) \) where \( s \) goes from 0 to 1,
- then \( X(u \sigma_0 + (1 - u) \tau, \tau_0) \) where \( u \) goes from 0 to 1.

By construction we have \( \text{Ind}(y, \Gamma[0, \sigma_0, \tau_0]) = \text{Ind}(y, \gamma(0)) = 0 \) and, since \( \gamma(0) = \gamma(1) \),

\[
\text{Ind}(y, \Gamma[1, \sigma_0, \tau_0]) = \text{Ind}(y, \gamma) \neq 0.
\]

Using the continuity of \( \Gamma[t; \sigma, \tau] \) and the compactness of the strategies' sets we obtain the existence of two couples of strategies \( (\sigma, \tau) \) and \( (\sigma', \tau') \) and of a point \( t \) in \([0,1]\) such that:

\[
\gamma(t) = X(\sigma, \tau) = X(\sigma', \tau') \text{ and } \text{Ind}(y, \Gamma[t; \sigma, \tau]) \neq \text{Ind}(y, \Gamma[t; \sigma', \tau']).
\]

Defining \( \tilde{\gamma} \) by \( \Gamma[t; \sigma, \tau] - \Gamma[t; \sigma', \tau'] \) we obviously have: \( \text{Ind}(y, \tilde{\gamma}) \neq 0 \). The idea of the proof now is to introduce a new path \( \gamma^* \), such that \( \text{Ind}(y, \gamma^*) = \text{Ind}(y, \gamma) \), with the additional property that \( \gamma^* \) will be the image under \( X \) of a path in the strategy's space. The latter being simply connected (in fact contractible) this will imply \( \text{Ind}(y, \gamma^*) = 0 \), hence the contradiction.

Recall that \( \tilde{\gamma} \) is defined by:

\[
\gamma(0) = X(\sigma_0, \tau_0) \rightarrow X(\sigma', \tau_0) \rightarrow X(\sigma', \tau')
\]

\[
\gamma(t) = X(\sigma, \tau) \rightarrow X(\sigma, \tau_0) \rightarrow X(\sigma_0, \tau_0).
\]

We define \( \gamma^* \) by adding to \( \tilde{\gamma} \) from the point \( \gamma(t) \) the closed path \( \rho \)

\[
\gamma(t) = X(\sigma', \tau') \rightarrow X(\sigma, \tau') \rightarrow X(\sigma, \tau) = \gamma(t).
\]

Note that, since \( X \) is linear in each variable, \( \rho \) consists of a line segment in \( D \) in both directions hence \( \text{Ind}(y, \gamma^*) = \text{Ind}(y, \tilde{\gamma}) \). Obviously \( \gamma^* \) is now the image under \( X \) of the following closed continuous path in the strategy's space:

\[
(\sigma_0, \tau_0) \rightarrow (\sigma', \tau_0) \rightarrow (\sigma', \tau') \rightarrow (\sigma, \tau') \rightarrow (\sigma, \tau) \rightarrow (\sigma_0, \tau_0),
\]

hence the result. •

**Corollary 12.**

If \( N = 2, \quad 0 < \delta < \lambda, \quad D_\delta \subseteq D_\lambda \) implies \( D_\delta = C \),

\[
\text{m > 0,} \quad D_{n+m} \subseteq D_n \text{ implies } D_{n+m} = C. \tag{28}
\]

**Proof.** Using (23) \( D_\delta \) contains the frontier of \( C \) hence is equal to \( C \) by (27). The proof is similar for \( D_{n+m} \) by using (26) with \( p = 1 \), then (9) to reduce to the case \( m < n \), and finally (27). •

Open problem: is \( D \) simply connected or even contractible for \( N > 2 ? \)
3. Study of the prisoner's dilemma. In this part we shall study the following two-person game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>(4,4)</td>
<td>(0,5)</td>
</tr>
<tr>
<td>B</td>
<td>(5,0)</td>
<td>(1,1)</td>
</tr>
</tbody>
</table>

We first remark that $D_1 = C$ hence $D_n = C$ for all $n$ and that $\Delta = \{ x = (x_1, x_2) \mid x \in C, x_i > 1, i = 1, 2 \}$. Moreover $E_1 = \{(1,1)\}$ since $B$ and $R$ are strictly dominating strategies in $G_1$.

This game has been widely analyzed and it is well known that $E_n = \{(1,1)\}$, see e.g. [7, pp. 95–102]. Nevertheless this property is not a consequence of the existence of strictly dominating strategies (see Example 4) and backwards induction arguments lead only to perfect Nash equilibrium payoffs.

A more general class of games for which an analog property holds is described by the following result: (recall that $a_i = \min_{\sigma'} \max_{T'} X_i(\sigma', t_i)$.)

**Proposition 13.** Let $G_1$ be an $N$-person game such that $E_1 = \{a\}$ then $E_n = \{a\}$ for all $n$.

**Proof.** Let $\sigma$ be a Nash equilibrium $N$-tuple of strategies in $G_n$ corresponding to a payoff different from $a$. Denote by $H_m(\sigma)$ the set of histories up to stage $m$ having a positive probability under $\sigma$.

Obviously, since $a$ is the only one-stage Nash equilibrium payoff, the payoff induced by $\sigma$ at stage $n$ conditionally to any history in $H_n(\sigma)$ is $a$. Hence there exists a stage $m$ and an history $h_m$ in $H_m(\sigma)$ such that:

- the payoff induced by $\sigma$ at stage $m$ conditionally to $h_m$ is different from $a$,
- the payoff at any further stage $k > m + 1$ conditionally to any $h_k$ that follows $h_m$ and belongs to $H_k(\sigma)$ is $a$.

In particular this implies that $\sigma$ is not in equilibria at stage $m$, conditionally to $h_m$; hence we can assume that player 1 can strictly increase his payoff at that stage by using some $T_i$.

Now, by definition of $a_1$, whatever being $\sigma'$, player 1 can obtain at least $a_1$ for the remaining stages, which was his payoff under $\sigma$.

It follows that by deviating at stage $m$ if $h_m$, player can strictly increase his average payoff; since $h_m$ belongs to $H_m(\sigma)$ we obtain a contradiction. ■

Note that this condition is also necessary since a recent result states that for $N = 2$, $E_1 \neq \{a\}$ implies that $E_n$ converges to $\Delta$ (see [3]).

We now turn to the study of the discounted game.

The following result was already announced in [2].

**Proposition 14.** $E_\lambda$ is reduced to $\{(1,1)\}$ for all $\lambda$ in $(0, 1]$.

**Proof.** Let $(\sigma, \tau)$ be an equilibrium pair in $G_\lambda$. $H_n$ will denote the set of histories up to stage $n$ and $H^*_n$ those histories in $H_n$ having positive probability under $(\sigma, \tau)$. We write $a_n$ for the random payoff of player 1 at stage $n$, and $s_n(h_n)$ (resp. $t_n(h_n)$) for the probability of playing $T$ (resp. $L$) at stage $n$, conditionally on $h_n$, $\sigma$ and $\tau$.

The equilibrium condition can be written as follows:

For each $h_n$ in $H^*_n$ and each $\sigma'$ which coincides with $\sigma$ up to stage $n - 1$:

$$E_{\sigma', \tau} \left( \lambda \sum_{m=0}^{\infty} (1 - \lambda)^m a_{n+m} | h_n \right) > E_{\sigma', \tau} \left( \lambda \sum_{m=0}^{\infty} (1 - \lambda)^m a_{n+m} | h_n \right)$$

(and similarly for player II).
In particular if $\sigma'$ is after $h_n$, play always bottom:

$$E_{\sigma',r}(a_n \mid h_n) - E_{\sigma',r}(a_n \mid h_n) = -s_n(h_n) \quad \text{and} \quad E_{\sigma',r}(a_{n+m} \mid h_n) > 1 \quad \forall m.$$ 

(*) now implies

$$E_{\sigma',r} \left( \sum_{1}^{\infty} (1 - \lambda)^m (a_{n+m} - 1) \mid h_n \right) \geq s_n(h_n) \quad \text{for all } h_n \text{ in } H_n^*.$$ 

Define: $\gamma_{n+m}(h_n) = E_{\sigma',r}(a_{n+m} + b_{n+m} \mid h_n) - 2$ and note that

$$\gamma_n(h_n) = (4 + 4)s_n(h_n) + (5 + 0)(1 - s_n(h_n))t_n(h_n) + (0 + 5)s_n(h_n)(1 - t_n(h_n)) + (1 + 1)(1 - s_n(h_n))(1 - t_n(h_n)) - 2$$

$$= 3(s_n(h_n) + t_n(h_n)).$$

From (*) and the similar inequality for player II we obtain:

$$\sum_{1}^{\infty} (1 - \lambda)^m \gamma_{n+m}(h_n) \geq \frac{1}{3} \gamma_n(h_n) \quad \text{for all } h_n \text{ in } H_n^*.$$ 

(\text{**})

Let us introduce, for all $m$, $\overline{\gamma}_m = E_{\sigma',r}(a_m + b_m - 2)$. Since $P_{\sigma',r}(H_n^*) = 1$, integrating (\text{**}) gives

$$\sum_{1}^{\infty} (1 - \lambda)^m \overline{\gamma}_{n+m} \geq \frac{1}{3} \overline{\gamma}_n \quad \text{for all } n > 1.$$ 

Letting $\gamma$ be the supremum of the $\overline{\gamma}_n$ it follows that $\overline{\gamma} \sum_{1}^{\infty} (1 - \lambda)^m > \frac{1}{3} \overline{\gamma}$. Thus

$$\overline{\gamma} \left( \frac{1 - \lambda}{\lambda} \right) > \frac{1}{3} \overline{\gamma}.$$ 

Hence either $\gamma = 0$, i.e. $\overline{\gamma}_m = 0$ for all $m$, and the payoff is always $(1,1)$ or $\lambda < \frac{3}{4}$. \end{proof}

In the last proposition we shall describe explicitly the set of equilibrium payoffs in $G_{3/4}$.

We first define $S$ to be the square with extreme points $(1,1), (1,4), (4,4), (4,1)$ and $A$ to be the union of $S$ with the two line segments: $[(4,1),(19/4,1)], [(1,4), (1,19/4)]$:

![Figure 1. Equilibrium payoffs in the prisoner's dilemma with discount factor $3/4$.](image-url)
We can now state the result:

**Proposition 15.** $E_{3/4} = A$.

**Proof.**

*First part: $A \subseteq E_{3/4}$.** Straightforward computation shows that the extreme points of $S$ are equilibrium payoffs (we describe the pure equilibrium strategies by their sequence of payoffs on $H^*$, both players using their dominating strategies outside $H^*$):

- for $(4,4)$ always $(4,4)$,
- for $(1,1)$ always $(1,1)$,
- for $(4,1)$ alternating sequence of $(5,0), (0,5) \ldots$ starting from $(5,0)$ and symmetrically for $(1,4)$.

Now given $(a, b)$ in $S$, write $a$ as $4t + 1 - t$ and $b$ as $4s + 1 - s$, $t$ and $s$ being in $[0,1]$. The strategies are now:

- At the first stage, for player I plays $T$ with probability $s$, for player II plays $L$ with probability $t$.
- From stage 2 on:

<table>
<thead>
<tr>
<th>Payoff</th>
<th>Strategy 1</th>
<th>Strategy 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(4,4)$</td>
<td>$(4,4)$</td>
<td>$(5,0)$</td>
</tr>
<tr>
<td>$(5,0)$</td>
<td>$(0,5)$</td>
<td>$(0,5)$</td>
</tr>
<tr>
<td>$(0,5)$</td>
<td>$(5,0)$</td>
<td>$(0,5)$</td>
</tr>
<tr>
<td>$(1,1)$</td>
<td>$(1,1)$</td>
<td>$(1,1)$</td>
</tr>
</tbody>
</table>

It is easy to see that the payoffs of both players are independent of their first moves. Since from stage 2 on no deviation is profitable the above description gives an equilibrium and it is easy to see that the corresponding payoff is $(a, b)$.

Finally, in order to obtain a payoff equal to $(4 + \frac{1}{2}a, 1)$, $a \in [0,1]$ in $E_{3/4}$ the strategies are:

- play $(5,0)$ at the first stage, then achieve the equilibrium payoff $(1 + 3a, 4)$, which belongs to $S$, in $G_{3/4}$ starting from stage 2. Here also none of the players has incentive to deviate at stage 1, hence the equilibrium with the right payoff: $\frac{1}{2}(5,0) + \frac{1}{2}(1 + 3a, 4)$.

*Second part: $E_{3/4} \subseteq A$.** Obviously $E_{3/4}$ is included in $\Delta$, hence it remains to prove, by symmetry that there exists no $(a, b)$ in $E_{3/4}$ with $a > 4$, $b > 1$ and $(a - 4)(b - 1) = \lambda(a,b) > 0$.

We shall prove that if there is such a payoff this implies the existence of another payoff $(a', b')$ in $E_{3/4}$ with $a' > 4$, $b' > 1$ and $\lambda(a', b') > 4\lambda(a,b)$, hence the contradiction.

(a) Let $\sigma, \tau$ be the equilibrium strategies corresponding to $(a,b)$. We define $a_1$ to be the maximal payoff that player I can achieve in $G_{3/4}$ if player II is using $\tau(h_2)$ with $h_2 = (T,L)$.

In words, $a_1$ corresponds to the normalized payoff from stage 2 on if player I uses a best response to $\tau$, conditionally on $(T,L)$ at stage 1.

$b_1$ is defined in the same way for player II and similarly $(a_2, b_2)$ correspond to $(T,R)$, $(a_3, b_3)$ to $(B,L)$ and $(a_4, b_4)$ to $(B,R)$. (Note that $a_j > 1, b_i > 1$.) Hence the players face a matrix with current and future payoffs
Let \( s \) and \( t \) be the strategies induced by \( \sigma \) and \( \tau \) at the first stage. If we define \( f(\tilde{s}, \tilde{t}) \) for every \( \tilde{s}, \tilde{t} \) by:

\[
f(\tilde{s}, \tilde{t}) = \frac{1}{4} \left( 4\tilde{s}\tilde{t} + 5(1 - \tilde{s})\tilde{t} + (1 - \tilde{s})(1 - \tilde{t}) \right)
+ \frac{1}{4} \left( \tilde{s}(\tilde{a}_1 + \tilde{s}(1 - \tilde{t})a_2 + (1 - \tilde{s})\tilde{a}_3 + (1 - \tilde{s})(1 - \tilde{t})a_4) \right),
\]

then the equilibrium condition implies:

\[
f(s, t) > f(\tilde{s}, \tilde{t}) \quad \text{for all } \tilde{s}
\]

(and similarly for a function \( g \) corresponding to player II's payoff) and

\[
s, t > 0 \Rightarrow (a_1, b_1) \in E_{3/4}, \quad s(1 - t) > 0 \Rightarrow (a_2, b_2) \in E_{3/4},
\]

\[
(1 - s)t > 0 \Rightarrow (a_3, b_3) \in E_{3/4}, \quad (1 - s)(1 - t) > 0 \Rightarrow (a_4, b_4) \in E_{3/4}.
\]

(b) We can assume \( t > 0 \) otherwise \( a \) is less than \( 4 \); and \( s < 1 \) otherwise by playing \( \tilde{t} = 0 \) player II obtains \( \frac{1}{2}5 + \frac{1}{4}b_2 > \frac{1}{2}5 + \frac{1}{4} = 4 \) hence \( a \) is again less than \( 4 \).

Using (\*) we now obtain, with \( a = 4 + x, \ b = 1 + y, \)

(1) \( 4 + x = \frac{1}{2}[5t + (1 - t)] + \frac{1}{4}(ta_3 + (1 - t)a_4), \)

(2) \( 1 + y = \frac{3}{4}(4s) + \frac{1}{4}[sb_1 + (1 - s)b_3] \)

As \( a_4 < 5 \) we obtain from (1) that:

(3) \( a_2 > 1 + 4x. \)

(c) If \( s = 0 \). By (2) we have \( b_3 = 4 + 4y \). By (**), this implies that \( (a_3, b_3) \) is in \( E_{3/4} \), hence by symmetry \( (b_3, a_3) \) also. Now \( b_3 > 4, \ a_2 > 1 \) and \( \lambda(b_3, a_3) > 16xy = 16\lambda(a, b) \).

(d) Assume now \( s > 0 \). (\*) implies:

(4) \( 4 + x = \frac{1}{2}(4t) + \frac{1}{4}(ta_1 + (1 - t)a_2), \)

(5) \( 1 + y = \frac{3}{4}(5s + (1 - s)) + \frac{1}{4}(sb_2 + (1 - s)b_4). \)

From (2) and (5) it follows that

(6) \( sb_1 + (1 - s)b_3 \geq 3 + sb_2 + (1 - s)b_4 > 4; \)

hence using again (2)

(7) \( s < y/3. \)

Finally from (4) we get:

(8) \( a_1 > 4 + 4x. \)

We now consider two cases:

—either \( b_1 > 1 + y \)

Then by (8) we obtain \( \lambda(a_1, b_1) > 4xy = 4\lambda(a, b) \) hence the result.

—or \( b_1 < 1 + y. \)

Using again (2) we have:

\[
4 + 4y < s(12 + 1 + y - b_3) + b_3
\]

\[
< \frac{y}{3} (12 + 1 + y - b_3) + b_3 \quad \text{by (7).}
\]

This inequality gives:

(9) \( (4 + y)(3 - y) < b_2(3 - y). \)

Since \( x > 0 \) implies \( y < 3 \) it follows that

(10) \( b_2 > 4 + y. \)

By (3) and (10) we obtain \( \lambda(b_3, a_3) > 4\lambda(a, b) \) and this achieves the proof. ■
CONCLUDING REMARKS. (1) The computations made in the proof of Proposition 14 show also that for the following values of the parameters

\[
\begin{array}{|c|c|}
\hline
\beta - x, \beta - x & \alpha - x, \beta \\
\hline
\beta, \alpha - x & \alpha, \alpha \\
\hline
\end{array}
\] with $\beta - x > \alpha$

$x > 0$

similar results hold with a critical value $\lambda = (\beta - \alpha - x)/(\beta - \alpha)$.

(2) One can also prove that the analog of Proposition 15 holds, at least for $\beta > \max\{1 + \alpha, 1 + 2x\}$.

(3) To compute $E_{\lambda}$ for other values of the discount factor seems quite difficult. It is nevertheless easy to see that $E_{\lambda}$ is not monotonic: there are denumbrably many points on the Pareto boundary for $1/2 < \lambda < 3/4$.

(4) We use deeply the fact that the "gain of deviating" was uniform, namely $x$. In the more general case:

\[
\begin{array}{|c|c|}
\hline
\beta - y, \beta - y & \alpha - x, \beta \\
\hline
\beta, \alpha - x & \alpha, \alpha \\
\hline
\end{array}
\] with $\beta - y > \alpha$

$x > 0, \ y > 0$

the critical value is $\lambda = \max\{\beta - \alpha - x, \beta - \alpha - y\}/(\beta - \alpha)$.

In fact if $y > x$ an alternate sequence $(\beta, \alpha - x), (\alpha - x, \beta), \ldots$ gives an equilibrium at $\lambda$; and similarly if $x > y$ a stationary sequence of $(\beta - y, \beta - y)$ is an equilibrium. Now if for some $\lambda > \lambda_0$ $(\sigma, \tau)$ is an equilibrium, it keeps this property as $x$ or $y$ decrease, in particular for $x' = y' = \min(x, y)$ contradicting Remark 1. The explicit computation of $E_{\lambda}$ seems more delicate.

Added in proof. Lemma 2 holds under the following additional assumption: $\Delta$ is full dimensional or $N = 2$, as used in the proof. Forges, Mertens and Neyman have a counterexample where $N = 3$ and $\Delta$ is 2-dimensional.

References


