LATTICES IN CRYSTALLINE REPRESENTATIONS

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0. Introduction

Let K be a p-adic field, which means it is a field of characteristic 0 and complete with respect to a discrete valuation with a perfect residue field k of characteristic $p > 0$. Fixing an algebraic closure $\overline{K}$ of K, we denote the Galois group by $G_K = Gal(\overline{K}/K)$. p-adic Hodge theory deals with $Rep_{\mathbb{Q}_p}(G_K)$, the category of the p-adic Galois representations (i.e. finite dimensional $\mathbb{Q}_p$-vector spaces equipped with a linear and continuous G_K-action). Among them, what we call crystalline representations are very important. The aim of this mémoire is mainly to introduce the work of Kisin in his paper [Kis] which gives a classification of crystalline representations and gets some important results concerning the G_K-stable lattices of crystalline representations.

The category $Rep_{\mathbb{Q}_P}(G_K)$ is very large. p-adic representations arising from geometry catch most interest. The étale cohomology $H^i_{ét}(X_{\overline{K}}, \mathbb{Q}_p)$ for a proper and smooth variety X over K, is really the most interesting example. Faltings proved an important result conjectured by Tate.

Theorem 0.1 ([Fal88]). *Let K be a p-adic field. For smooth proper K-schemes X, there is a canonical isomorphism*

$$C \otimes_{\mathbb{Q}_p} H^n_{ét}(X_{\overline{K}}, \mathbb{Q}_p) \simeq \bigoplus_q (C(-q) \otimes_K H^{n-q}(X, \Omega^q_{X/K}))$$

in $Rep_C(G_K)$ ($C = \widehat{\overline{K}}$), where the G_K-action on the right side is defined through the action on each $C(-q) = C \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-q)$. In particular, non-canonically

$$C \otimes_{\mathbb{Q}_p} H^n_{ét}(X_{\overline{K}}, \mathbb{Q}_p) \simeq \bigoplus_q C(-q)^{h^{n-q,q}}$$

with $h^{p,q} = \dim_K H^p(X, \Omega^q_{X/K})$.

This is the p-adic analogue of Hodge decomposition in the classic sense. It is one of the important comparison theorems in p-adic Hodge theory, which links p-adic étale cohomology to other cohomology theories. The remarkable fact is that in general, the Galois action on $H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_p)$ is very difficult to know. But after tensoring by C, the Galois action is very easy. This idea is a vague description of Fontaine's theory of period rings.

In the first part of this article, we will introduce the general formalism of admissible representations using period rings, focusing on B_{HT}, B_{dR}, B_{cris} and the categories of their associated admissible representations $Rep^{HT}_{G_K}, Rep^{dR}_{G_K}, Rep^{cris}_{G_K}$. As we mentioned, the category $Rep_{\mathbb{Q}_p}(G_K)$ is too large. The aim of constructing period rings is to select good classes of representations, especially containing those from geometry. The three categories are full subcategories of $Rep_{\mathbb{Q}_p}(G_K)$ and they have the inclusion relation: $Rep^{cris}_{G_K} \subset Rep^{dR}_{G_K} \subset Rep^{HT}_{G_K}$. In addition, it is very useful to relate these good categories to various target categories of semilinear algebraic objects with different structures, like filtration, Frobenius operator and Galois action. It is often easier to operate on these semilinear algebras. One remarkable result is that when V is a crystalline representation, then we can recover V with the three structures.

Theorem 0.2. *The space $(Fil^0 B_{cris})^{\varphi=1} = \{b \in Fil^0(B_{cris}) | \varphi(b) = b\}$ of φ-invariant elements in the 0-th filtered piece of B_{cris} is equal to $\mathbb{Q}_p$. In particular,*

$$Fil^0(B_{cris} \otimes_{K_0} D_{cris}(V))^{\varphi=1} \simeq V.$$

In the general formalism of admissible representaions, the p-adic étale cohomology $H^i_{ét}(X_{\overline{K}}, \mathbb{Q}_p)$ has been proved to be a de Rham representation. When X has a good reduction, it is also a crystalline representation. We then get another two comparision theorems proved by Fontaine-Messing [FM87], T.Tsuji [Tsu99] under various assumptions.

Theorem 0.3 (The de Rham conjecture (C_{dR})). *Let X be a proper smooth variety over K. Then there exists an isomorphism compatible with Galois action and filtrations.*

$$B_{dR} \otimes_{\mathbb{Q}_p} H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_p) \simeq B_{dR} \otimes_K H^i_{dR}(X/K).$$

In particular, $D_{dR}(H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p)) \simeq H^i_{dR}(X/K)$.

Theorem 0.4 (The crystalline conjecture (C_{cris}).). *Let* X_K *be a proper smooth variety over* K*,* X *be a proper smooth model of* X_K *over* $\mathcal{O}_K$*.* X_k *is the special fiber of* X*. Then there exists a canonical isomorphism, which is compatible with the Galois action and Frobenius endomorphism.*

$$B_{cris} \otimes_{\mathbb{Q}_p} H^m_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{cris} \otimes_{W(k)} H^m_{cris}(X_k/W(k)).$$

One important result is the equivalence between the category of crystalline representations and the category of weakly admissible filtered φ-modules.

Theorem 0.5 (Colmez-Fontaine [CF00a]). *The functor* D_{cris} *defined by* $D_{cris}(V) = (B_{cris} \otimes V)^{G_K}$ *induces an equivalence of categories*

$$Rep^{cris}_{G_K} \to MF^{\varphi, w.a}_K$$

where $MF^{\varphi,w.a}_K$ *is the category of weakly admissible filtered* φ*-modules. The inverse is given by* $V_{cris}(D) = Fil^0(B_{cris} \otimes D_K)^{\varphi=1}$.

With this theorem, studying crystalline representations is equivalent to studying their associated weakly admissible representations, just as Kisin did in his paper. In the second part of the article, we will introduce Kisin's work.

Let $W = W(k)$ be the ring of Witt vectors and $K_0 = W(k)[1/p]$. Then K is a finite totally ramified extension of K_0. Fix a uniformizer $\pi \in K$, and let $E(u) \in W(k)[u]$ be the Eisenstein polynomial for π. Set $\mathfrak{S} = W[[u]]$. The whole story lies in the following diagram.

$$\begin{array}{ccc} Mod^{\varphi,N,Fil\geq 0,w.a}_K & \dashrightarrow & Mod^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p \\ {\scriptstyle M}\downarrow\simeq & & \simeq\uparrow \\ Mod^{\varphi,N_\nabla,0}_{/\mathcal{O}} & \longrightarrow & Mod^{\varphi,N,0}_{/\mathcal{O}} \end{array}$$

Kisin constructs the functor M from the category of effective filtered (φ, N)-modules to $Mod^{\varphi,N_\nabla}_{/\mathcal{O}}$,the category of (φ, N_∇)-modules $\mathfrak{M}$ over $\mathcal{O}$ of finite E-height (i.e. the cokernel of the map $\varphi^*\mathcal{M} \to \mathcal{M}$ is killed by some power of $E(u)$) and proves the two categories are in fact equivalent. By using Kedlaya's theory of slopes, Kisin proves the weak admissibility of an effective filtered (φ, N)-module D is equivalent to that $\mathcal{M}(D)$ is pure of slope 0. $Mod^{\varphi}_{/\mathfrak{S}}$ is the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ of finite $E(u)$-height.

By letting $N = 0$, we then have the following theorem.

Theorem 0.6. *The category of crystalline representations with all Hodge-Tate weights* ≤ 0*, admits a fully faithful embedding into the isogeny category* $Mod^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$ *of* $Mod^{\varphi}_{/\mathfrak{S}}$.

Let $K_n = K(\zeta_n)$, in which ζ_n is a primitive p^n-th root of unity. Write K_∞ to be the union of K_n for all $n \in \mathbb{N}$. Denote by Rep_{G_∞} the category of continuous p-adic representations of G_{K_∞}. Then one result conjectured by Breuil can be proved using Kisin's theory.

Corollary 0.7. *The functor* $Rep^{cris}_{G_K} \to Rep_{G_\infty}$ *obtained by restriction is fully faithful.*

As we have seen, the comparison theorems above are all about the cohomology with $\mathbb{Q}_p$ coefficient. In fact, the integral case is much more difficult. Integral p-adic Hodge theory gives back classical p-adic Hodge theory by inverting p, but it also gives rise to completely new characteristic p phenomena by reducing modulo p. Thus it is richer than p-adic Hodge theory. After giving a classification of crystalline representations, Kisin's theory can also be used to describe the $\mathbb{Z}_p$-lattices in crystalline representations.

Denote by $Rep_{G_K}^{cris\circ}$ the category of G_K-stable $\mathbb{Z}_p$-lattices spanning crystalline representations. Then we have the following theorem given by Kisin.

Theorem 0.8 ([Kis10])**.** *There exists a fully faithful functor*

$$\mathfrak{M} : Rep_{G_K}^{cris\circ} \to Mod_{/\mathfrak{S}}^{\varphi}.$$

1. Formalism of Admissible Representations

In p-adic Hodge theory, there are some functors from certain categories of p-adic representations of the Galois group $G_K := Gal(\bar{K}/K)$ of a p-adic field K to semilinear algebra categories. By using period rings, we can construct a general formalisim of such functors.

In this section, we will give the general formalism of admissible representations and talk about some main results. The reference is [BC09, Section 5]. In the next several sections, we will construct concrete period rings like B_{HT}, B_{dR} and B_{cris}. Since these rings have different structures, something interesting will happen in addition to the general results.

Let F be a field and G be a group. Let B be an F-algebra domain equipped with a F-linear G-action, and assume that the F-subalgebra $E = B^G$ is a field. Let C be the fraction field of B, then we have a natural G-action on C.

Definition 1.1. *we say B is (F, G) regular if*

(i) $C^G = B^G$.
(ii) *every nonzero $b \in B$ whose F-linear span $F \cdot b$ is G-stable is a unit in B.*

Note that if B is a field then it obviously satisfies above conditions. But the cases we are interested in are far from fields.

Under the assumption that B is a (F, G)-regular domain, $E = B^G$, for any V in the category of $Rep_F(G)$ of finite dimensional F-linear representations of G, we define the functor D_B

$$D_B(V) = (B \otimes_F V)^G.$$

G acts on $B \otimes_F V$ by $g(\lambda \otimes v) = g(\lambda) \otimes g(v)$, so $D_B(V)$ is a E-vector space, and the inclusion $D_B(V) \subset B \otimes_F V$ induces a canonical map

$$\alpha_V : B \otimes_E D_B(V) \to B \otimes_E (B \otimes_F V) = (B \otimes_E B) \otimes_F V \to B \otimes_F V$$

This is a B-linear G-equivariant map.

Definition 1.2. *We say $V \in Rep_F(G)$ is a B-admissible representation if* $\dim_F(V) = \dim_E(D_B(V))$.

It is not obvious that $D_B(V)$ is a finite dimensional E-vector space. But the following theorem tells us this is true.

Theorem 1.3. *Fix $V \in Rep_F(G)$.*

(i) *The map α_V is always injective and $\dim_E D_B(V) \leq \dim_F(V)$, with equality if and only if α_V is an isomorphism.*

(ii) *Let $Rep_F^B(G) \subset Rep_F(G)$ be the full subcategory of B-admissible representations. The covariant functor $D_B : Rep_F^B(G) \to Vec_E$ to the category of finite-dimensional E-vector spaces is exact and faithful, and any subrepresentaion or quotient of a B-admissible representation is still B-admissible.*

Proof. (1)We first assume α_V is injective. Since $B \to C = Frac(B)$ is flat (indeed we can regard C as a localization of B), by extending the scalars we have $C \otimes D_B(V)$ as a C-subspace of $C \otimes_F V$. $C \otimes_F V$ is of finite dimension, then $\dim_E D_B(V) \leq \dim_F V$.

Suppose their dimensions are both d. $D_B(V)$ has a E-basis $(e_i)_{1\leq i\leq d}$ and V has a F-basis $(f_j)_{1\leq j\leq d}$. Then α_V can be described by a matrix $(b_{ij})_{1\leq i,j\leq d}$ with $b_{ij} \in B$. After tensoring by C, we get an injection between two C-vector spaces of the same dimension. Then it is must be an isomorphism. This means $\det(b_{ij})$ is in $B - \{0\}$. To prove α_V is an isomorphism, we need to prove $\det(b_{ij}) \in B^\times$. As B is (F, G)-regular, it suffices to prove $F \cdot \det(b_{ij})$ is G-stable.

Since $D_B(V)$ is G-invariant of $B \otimes_F V$, we consider the d-th exterior power

$$\wedge^d(\alpha_V)(e_1 \wedge \cdots \wedge e_d) = \det(b_{ij}) f_1 \wedge \cdots \wedge f_d.$$

It is a G-invariant vector because e_i is G-invariant. But G acts on $f_1 \wedge \cdots \wedge f_d$ by some character

$$\eta : G \to F^\times.$$

This means G acts on $\det(b_{ij})$ by η^{-1}. So $F \cdot \det(b_{ij})$ must be G-stable and $\det(b_{ij})$ is a unit. So what left to prove is that α_V is injective.

To prove the injectivity of α_V, we consider the following commutative diagram

$$\begin{array}{ccc} B \otimes_E D_B(V) & \xrightarrow{\alpha_V} & B \otimes_F V \\ \downarrow & & \downarrow \\ C \otimes_E D_C(V) & \longrightarrow & C \otimes_F V \end{array}$$

in which the vertical arrows are injective. So it suffices to prove the bottom is injective. The proof reduces to the case when B is a field. We now need to prove if $x_1, \cdots, x_n \in B \otimes_F V$ are E-linearly independent and G-invariant then they are B-linearly independent. If they are not, we assume for some r

$$x_r = \sum_{i<r} b_i \cdot x_i$$

which is a relation of minimum length. For $g \in G$, we also have

$$x_r = g(x_r) = \sum g(b_i) \cdot x_i$$

The minimum length forces that $b_i = g(b_i)$ for all $i < r$. Then $b_i \in E$. This means $x_1, \cdots, x_n$ are not E-linearly independent. By contradiction, α_V is injective.

(2) Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence in $Rep_F(G)$. Then we have a left exact sequence $0 \to D_B(V_1) \to D_B(V_2) \to D_B(V_3)$ in Vec_E. If V_2 is B-admissible, then $\dim_E D_B(V_2) \leq \dim_E D_B(V_1) + \dim_E D_B(V_3) = \dim_F(V_2)$, so $\dim_E D_B(V_2) \leq \dim_E D_B(V_1) + \dim_E D_B(V_3)$. The sequence should be exact

$$0 \to D_B(V_1) \to D_B(V_2) \to D_B(V_3) \to 0$$

This implies D_B is exact and the subrepresentation or quotient of a B-admissible representation is still B-admissible. Moreover if $f \in Hom_{Rep_F^B(G)}(V, V_1)$, then $D_B(\mathrm{Ker}(f)) = \mathrm{Ker}(D_B(f))$. So $D_B(f) = 0$ implies $f = 0$. The faithfulness has been proved.

□

Remark 1.4. We do not impose any topological structure on F and G in the general definition, but in most cases we are interested in they need additional topological structures. For example, when $F = \mathbb{Q}_p$ and $G = Gal(\bar{K}/K)$, then F is a topological field and G is a profinite group, in this case, $Rep_F(G)$ consists of all continuous F-representations of G.

2. Hodge-Tate repersentations

In classical Hodge theory, there is a natural Hodge decomposition for any compact Kahler manifold X, i.e.,

$$H^i_{dR}(X) \cong H^i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{j=0}^{i} H^{i-j}(X, \Omega^j_X).$$

Let K be a p-adic field and fix a choice of algebraic closure $\overline{K}$ of K. We denote by C the completion of the field $\overline{K}$. Tate discovered for an abelian variety $A/\mathcal{O}_K$ there is a decomposition of étale cohomology with coefficients in C. He conjectured that there existed similar Hodge-like decomposition for the étale cohomology with coefficient in C for a scheme X_C coming from a scheme X projective and smooth over $\mathcal{O}_K$, or even over K. The case that X is a projective smooth scheme over $\mathcal{O}_K$ has been proved by Fontaine and Messing in [FM87] with the hypothesis $\dim X < p$. The case that X is a proper smooth scheme over K has been proved by Faltings [Fal88]. In fact, the étale cohomology for a proper smooth K-scheme is a Hodge-Tate representation which we will introduce later.

In this section, we will construct the period ring B_{HT} which has a decomposition and put it into the general formalisim we introduce in the Section 1 and see it is actually $(\mathbb{Q}_p, G_K)$-regular. We first give some basic notions and properties of C.

2.1. Properties of C.

The Hodge-Tate representation will involve the Galois action on $C \otimes V$ for $V \in Rep(G_K)$. In this subsection, we will give some fundamental properties of the field C.

Definition 2.1 (p-adic cyclotomic character)**.** *Fix $\bar{K}$ as a algebraic closure of K, and let p be a prime. $\mu_{p^n} = \mu_{p^n}(\bar{K})$, which is the group of p^nth roots of unity in $\bar{K}^\times$, and let μ_{p^∞} be the union of μ_{p^n}. We also denote the inverse limit of μ_{p^n} by $\mathbb{Z}_p(1)$. The action G_K on $\mathbb{Z}_p(1)$ is given by $g(\zeta) = \zeta^{\chi(g)}$ for a unique $\chi(g) \in \mathbb{Z}_p^\times$. We call χ the p-adic cyclotomic character of K.*

$\mathbb{Z}_p(1)$ is a free $\mathbb{Z}_p$-module of rank 1. A choice of its basis is equivalent to choose a compatible system $(\zeta_{p^n})_{n\geqslant 1}$ of primitive p-power roots of unity, i.e. $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \geqslant 1$. Fixing a choice of basis of $\mathbb{Z}_p(1)$, we can view it as $\mathbb{Z}_p$ endowed with a G_K-action by χ.

Definition 2.2 (Tate twist)**.** *Define $\mathbb{Z}_p(r) = \mathbb{Z}_p(1)^{\otimes r}$ and $\mathbb{Z}_p(-r) = \mathbb{Z}_p(r)^\vee$. Fix a choice of basis of $\mathbb{Z}_p(1)$, we can view $\mathbb{Z}_p(r)$ as $\mathbb{Z}_p$-module $\mathbb{Z}_p$ with G_K-action χ^r. If M is a $\mathbb{Z}_p[G_K]$-module, write $M(r) = \mathbb{Z}_p(r) \otimes_{\mathbb{Z}_p} M$ with its natural G_K-action. If we* *choose a basis of $\mathbb{Z}_p(1)$, then we can view $M(r)$ as M endowed with a G_K-action given by $g \cdot m = \chi(g)^r g(m)$ for $g \in G_K$ and $m \in M$.*

Lemma 2.3 (Krasner's Lemma)**.** *Let K be the fraction field of a complete DVR and let $\alpha, \beta \in \overline{K}$ an algebraic closure of K with α separable. If $|\beta - \alpha| < |\alpha' - \alpha|$ for all conjugates α' of α, $\alpha' \neq \alpha$, then $\alpha \in K(\beta)$.*

Proof. Suppose not, then $\alpha \notin K(\beta)$, so there is an automorphism $\sigma \in Aut_{K(\beta)}(\bar{K}/K(\beta))$ for which $\sigma(\alpha) \neq \alpha$. Here we are using the separability of α, the extension $K(\alpha, \beta)/K(\beta)$ is separable and nontrivial, so there must be an element in $Hom_{K(\beta)}(K(\alpha, \beta), \bar{K})$ that changes α. For any $\sigma \in Aut_{K(\beta)}(\bar{K}/K(\beta))$ we have

$$|\beta - \alpha| = |\sigma(\beta - \alpha)| = |\sigma(\beta) - \sigma(\alpha)| = |\beta - \sigma(\alpha)|,$$

sicne σ fixes β. But this contradicts the hypothesis.

□

Corollary 2.4. *Let K be a complete nonarchimedean field , K^s be a separable closure of K, $\overline{K}$ be an algebraic closure of K containing K^s. Then $\widehat{K^s} = \widehat{\overline{K}}$ and it is an algebraically closed field.*

Proof. Let $C = \widehat{K^s}$, we shall prove

(i) If char$K$=p, then for any $a \in C$, there exists $\alpha \in C$, such that $\alpha^p = a$.
(ii) C is separably closed.

For the proof of (i), we choose $\pi \in m_K$, choose a valuation v such that $v(\pi) = 1$. Then

$$\mathcal{O}_{K^s} = \{a \in K^s | v(a) \geq 0\}, \mathcal{O}_C = \varprojlim \mathcal{O}_{K^s}/\pi^n \mathcal{O}_{K^s}$$

and $C = \mathcal{O}_C[1/\pi]$. Thus $\pi^{mp} a \in \mathcal{O}_C$ for $m \gg 0$, we may assume $a \in \mathcal{O}_C$. Choose a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{O}_{K^s}$, such that $a \equiv a_n \mod \pi^n$.Let

$$P_n(X) = X^p - \pi^n X - a_n \in K^s[X],$$

then P_n is separable. Let b_n be a root of P_n in K^s, $b_n \in \mathcal{O}_{K^s}$. Then

$$b_{n+1}^p - b_n^p = \pi^{n+1} b_{n+1} - \pi^n b_n + a_{n+1} - a_n,$$

one has $v(b_{n+1}^p - b_n^p) \geq n$. Since $b_{n+1}^p - b_n^p = (b_{n+1} - b_n)^p$, $v(b_{n+1} - b_n) \geq n/p$, which means (b_n) converges in $\mathcal{O}_C$. Denote b the limit of (b_n), then $b^p = \lim_{n\to+\infty} b_n^p = a$ since $v(b_n^p - a) \neq n$.

For the second part of proof, let

$$P(X) = a_0 + a_1 X + a_2 X^2 + \cdots + a_{d-1} X^{d-1} + X^d$$

be an arbitrary separable polynomial in $C[X]$. We need to prove $P(X)$ has a root in C. We may assume $a_i \in \mathcal{O}_C$. Let D be the splitting field of P over C, let $r = \max v(b_i - b_j)$, where b_i and b_j are disinct roots of P in D. Let

$$P_1 = c_0 + c_1 X + c_2 X w^2 + \cdots + c_{d-1} X^{d-1} + X^d \in K^s[X]$$

with $c_i \in K^s$, and $v(c_i - a_i) > rd$. Hence there exists $\beta \in C$, such that $P_1(\beta) = 0$. Choose $\alpha \in D$, a root of P, such that $|\beta - \alpha'| \geq |\beta - \alpha|$ for any $\alpha' \in D$ and $P(\alpha') = 0$. Since $P(\beta) = P(\beta) - P_1(\beta)$, and $v(\beta) > 0$, we have $v(P(\beta)) > rd$. On the other hand,

$$P(\beta) = \prod_{i=1}^{d} (\beta - b_i),$$

thus, $v(P(\beta)) = \sum_{i=1}^{d} v(\beta - b_i) > rd$. It follows that $v(\beta - \alpha) > r$. Applying Krasner's Lemma, we get $\alpha \in C$. □

Another property is related to the G_K-action on C.

Proposition 2.5. *Let H be a closed subgroup of G_K. Then C^H is the completion $\widehat{L}$ of $L = \bar{K}^H$ for the valuation v. In particular, if H is an open subgroup of G_K,then C^H is the finite extension $\bar{K}^H$ of K, and $\widehat{L} \cap \bar{K} = L$.*

Proof. See [BC09, Prop2.1.2]. □

Definition 2.6. *A C-representation of G_K is a finite dimensional C-vector space W equipped with a continuous G_K-action map which is semilinear, i.e., $g(cw) = g(c)g(w)$ for any $c \in C$ and $w \in W$. We write $Rep_C(G_K)$ to denote the category of such representations. The morphisms are C-linear G_K-equivariant morphisms.*

In concrete terms, if W is a C-representation, choose a C-basis $(e_1, \cdots, e_n)$. For any $g \in G_K$, we write $g(e_j) = \sum_i a_{ij}(g)e_i$. Then the map $\mu : G_K \to Mat(C)$ defined by $g \to (a_{ij}(g))$ is a continuous map and it satisfies $\mu(1) = id$ and $\mu(gh) = \mu(g) \cdot g(\mu(h))$ for all $g, h \in G_K$. The most common examples come from the category $Rep_{\mathbb{Q}_p}(G_K)$. If $V \in Rep_{\mathbb{Q}_p}(G_K)$, then $C \otimes V$ is an object in $Rep_C(G_K)$.

At the end of this subsection, we state an important theorem which will be useful in Hodge-Tate decomposition. General statement and proof can be seen in [FO08, Chap3].

Theorem 2.7 (Tate-Sen)**.** *For any p-adic field K, we have $K = C^{G_K}$ and $C(r)^{G_K} = 0$ for $r \neq 0$. Also, $H^1_{cont}(G_K, C(r)) = 0$ if $r \neq 0$ and $H^1_{cont}(G_K, C)$ is 1-dimensional over K. More generally, if $\eta : G_K \to \mathcal{O}_K^\times$ is a continuous character such that $\eta(G_K)$ is finite or contains $\mathbb{Z}_p$ as an open subgroup and if $C_K(\eta)$ denotes C_K with the twisted G_K-action $g \cdot c = \eta(g)g(c)$ then $H^i_{cont}(G_K, C_K(\eta)) = 0$ for $i = 0, 1$ when $\eta(I_K)$ is infinite and these cohomologies are 1-dimensional over K when $\eta(I_K)$ is finite.*

This theorem will play an important role in the proof of the regularity of periods rings.

2.2. **Hodge-Tate decomposition.** For $W \in Rep_C(G_K)$ and $q \in \mathbb{Z}$, define a K-vector space

$$W\{q\} := W(q)^{G_K} \simeq \{w \in W | g(w) = \chi(g)^{-q} w \ for \ all \ g \in G_K\}$$

the isomorphism depends on a choice of basis of $\mathbb{Z}_p(1)$ as we stated before. Note that $W\{q\}$ is not a C-subspace of $W(q)$ when it is nonzero. We have a natural G_K-equivariant K-linear multipication map

$$K(-q) \otimes_K W\{q\} \hookrightarrow K(-q) \otimes_K W(q) \simeq W$$

extending scalars we get a map

$$C(-q) \otimes_K W(-q) \to W$$

in $Rep_C(G_K)$ for all $q \in \mathbb{Z}$.

Lemma 2.8 (Serre-Tate)**.** *For $W \in Rep_C(G_K)$, the natural C-linear G_K-equivariant map*

$$\alpha_W : \bigoplus_q (C(-q) \otimes_K W\{q\}) \to W$$

is injective. In particular, $W\{q\} = 0$ for all but finitely many q and $\dim_K W\{q\} < \infty$ for all q, wih $\sum \dim_K W\{q\} \leq \dim_C W$,equality holds if and only if α_W is an isomorphism.

Definition 2.9. *Before we use period ring, a representation $W \in Rep_C(G_K)$ is called Hodge-Tate if α_W is an isomorphism. We say $V \in Rep_{\mathbb{Q}_p}(G_K)$ is Hodge-Tate if $C \otimes V$ is Hodge-Tate.*

Definition 2.10. *For those $W \in Rep_C(G_K)$ which are Hodge-Tate, we can define the Hodge-Tate weights to be those q such that $W\{q\}$ is nonzero. Then we call $h_q := \dim_K W\{q\}$ the multiplicity of q as a Hodge-Tate weight.*

An example is that $C(q)$ has a unique Hodge-Tate weight $-q$ by Tate-Sen theorem. We now define the period ring B_{HT}.

Definition 2.11. *The Hodge-Tate ring of K is the C-algebra $B_{HT} = \bigoplus_{q\in\mathbb{Z}} C(q)$.*

We have a natural G_K-action on B_{HT}. If we fix a basis of $\mathbb{Z}_p(1)$, then $B_{HT} \cong C[t, t^{-1}]$ which is the ring of Laurent polynomials in t with coefficients in C.

Proposition 2.12. $B_{HT}^{G_K} = K$

Proof. $B_{HT}^{G_K} = (\bigoplus_{q\in\mathbb{Z}} C(q))^{G_K} = \bigoplus C(q)^{G_K}$. By Tate-Sen theorem, we have $B_{HT}^{G_K} = K$. □

Proposition 2.13. *B_{HT} is actually a $(\mathbb{Q}_p, G_K)$-regular ring.*

Proof. As we mentioned, after fixing a basis of $\mathbb{Z}_p(1)$, then $B_{HT} = C[t, t^{-1}]$ with G-acting through the p-adic cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$ via $g(\sum a_n t^n) = \sum g(a_n)\chi(g)^n t^n$. Then in this case $\mathrm{Frac}(B_{HT}) = C(t)$.

(1)$B_{HT}^{G_K} = K$, we need to prove $\mathrm{Frac}(B_{HT})^{G_K} = K$. In fact, there is a G_K-equivariant injection from $C(t)$ into the formal Laurent series field $C((t))$. For any $g \in G_K$ and a formal series $\sum c_n t^n$, $g(\sum c_n t^n) = \sum g(c_n)\chi(g)^n t^n$. If $\sum c_n t^n$ is G-invariant, then $c_n \in C(n)^{G_K}$ for all $n \in \mathbb{Z}$. By Tate-Sen theorem, $c_n = 0$ for $n \neq 0$ and $c_0 \in K$. So $C((t))^{G_K} = K$. $Frac(B_{HT})^{G_K}$ has to be K.

(2)Assume $0 \neq b = \sum c_i t^i \in B_{HT}$ such that

$$g(b) = \eta(g)b, \eta(g) \in \mathbb{Q}_p, \text{ for all } g \in G_K$$

Then $g(c_i)\chi^i(g) = \eta(g)c_i$ for all $i \in \mathbb{Z}$ and $\gamma \in G_K$. Hence

$$g(c_i) = (\eta\chi^{-i})(g)c_i.$$

For all i such that $c_i \neq 0$, then $\mathbb{Q}_p c_i$ is a 1-dimensional sub-$\mathbb{Q}_p$-vector space of C stable under G_K. Thus the 1-dimensional representation associated to the character $\eta\chi^{-i}$ is C-admissible. By Sen-Tate's theorem, for all i such that $c_i \neq 0$ the action of I_K through $\eta\chi^{-i}$ is finite, which can be true for at most one i. Thus there exists $i_0 \in \mathbb{Z}$ such that $b = c_{i_0} t^{i_0}$ with $c_{i_0} \neq 0$. So b is invertible in B_{HT}.

□

Then we can define the associated functor D_{HT}.

Definition 2.14. *The covariant functor $D : Rep_C(G_K) \to Vec_K$ is*

$$D_{HT}(V) = (B_{HT} \otimes_C V)^{G_K}.$$

By the general formalism in Section 1, we have the following G_K-equivariant map.

$$\alpha_V : B_{HT} \otimes_K D_{HT}(V) \to B_{HT} \otimes_{\mathbb{Q}_p} V.$$

Definition 2.15. *We say a p-adic representation V of G_K is Hodge-Tate if $C \otimes V$ is B_{HT}-admissible, i.e.* $\dim D_{HT}(C \otimes V) = \dim_{\mathbb{Q}_p} V$.

We then also have the result by Thoerem 1.3 that the functor $D_{HT} : Rep_{\mathbb{Q}_p}(G_K) \to Vec_K$ is exact and faithful and any subrepresentation or quotient of a Hodge-Tate representation is also a Hodge-Tate representation.

The difference from the general case is that the Hodge-Tate ring B_{HT} has a direct sum decomposition. So $D_{HT}(V)$ is not only just a C-vector space. Indeed, $D_{HT}(V) = (B_{HT} \otimes_C V)^{G_K} = \bigoplus_{q\in\mathbb{Z}}(C(q)) \otimes_C V)^{G_K}$. It is a graded vector space.

Definition 2.16. *A graded vector space over a field F is a F-vector space D which has a direct sum decomposition $\bigoplus D_q$ for F-subspaces $D_q \subset D$ and we define the q-th graded piece of D to be $gr^q(D) := D_q$. The morphisms between two such obejects $T : D \to D'$ should resprect the grading, i.e. $T(D_q) \subseteq D'_q$ for all q. We denote the category by Gr_F. Also write $Gr_{F,f}$ to be the full subcategory of objects of finite dimension.*

Now since $D_{HT}(V)$ has a graded structure, we may try to construct a functor from $Gr_{K,f}$ to $Rep_C(G_K)$. For any $D \in Gr_{K,f}$, we define

$$V(D) := gr^0(B_{HT} \otimes_K D) = \bigoplus_q C(q) \otimes_K D_{n-q}.$$

Since D is of finite dimension, then $V(D) \in Rep_C(G_K)$. Indeed, $V(D)$ is a Hodge-Tate representation.

Theorem 2.17. *The covariant functors D_{HT} and V between the categories of Hodge-Tate representations in $Rep_C(G_K)$ and finite dimensional vector spaces in $Gr_{K,f}$ are actually quasi-inverse equivalent.*

Theorem 2.18. *For any $W \in Rep_C(G_K)$, the natural map*

$$K' \otimes_K D_K(W) \to D_{K'}(W)$$

is an isomorphism for all finite extension K'/K contained in $\bar{K} \subseteq C$. Also, if we replace K' by $\widehat{K^{ur}}$, it is still an isomorphism. In particular, for any finite extension K'/K, an object $W \in Rep_C(G_K)$ is Hodge-Tate iff it is Hodge-Tate in $Rep_C(G_{K'})$ iff it is Hodge-Tate in $Rep_C(G_{\widehat{K^{ur}}})$.

The proof of Theorem 2.17 is easy to deduce. But the proof of Theorem 2.18 is more difficult. See [Bri08, Thm 2.46].

Remark 2.19. It is good, according to the theorem, being Hodge-Tate is insensitive to restrcition to inertia group. But it is bad that the Hodge-Tate property is also insensitive to finite extensions. This shows Hodge-Tate property is not sufficiently fine.

3. The field B_{dR} and de Rham representations

We know the ring $B_{HT} = \bigoplus_q C_K(q)$ is a graded C_K-algebra and has a semilinear G_K-action. This ring can be used to classify certain representations called Hodge-Tate representations, but the class of representations we are interested in are smaller then Hodge-Tate representations. So we need to find another period ring to construct a smaller good class of representations and it might be great if it could recover the ring B_{HT}.

In this section, we will construct such a period ring B_{dR} refining B_{HT} which is $(\mathbb{Q}_p, G_K)$-regular and prove if $V \in Rep_{\mathbb{Q}_p}(G_K)$ is a de Rham representation then it is also Hodge-Tate. In fact, B_{dR} is a field and complete with respect to a discrte valuation. Its residue field is just $C = \widehat{\overline{K}}$. Moreover, the valuation defines a filtration on B_{dR} and its associated graded ring is B_{HT}. We may expect more interesting things except for the general results.

3.1. **Motivation for constructing B_{dR}.** The motivation for constructing B_{dR} might be seen from geometry. We know B_{HT} can be used to construct a comparison theorem between p-adic étale cohomology and Hodge cohomology. And algebraic de Rham cohomology is equipped with a filtration whose graded piece is a subquotient of Hodge cohomology. This leads us to consider another structure: filtration.

We can see a motivating example.

Example 3.1. Let R be a discrete valuation ring with maximal ideal m and residue field k. Let K be the fraction field of R. Then we have a natural filtration structure on K, i.e. $A_i = m^i$ for $i \in \mathbb{Z}$. Then the associated graded ring of K is non-canonically isomophic to $k[t, 1/t]$. Let $\widehat{K}$ be the fraction field of the m-adic completion of R, then $gr^{\bullet}(K) = gr^{\bullet}(\widehat{K})$.

Note that $B_{HT} \simeq C[T, T^{-1}]$. Inspired by the above example, we want to find a complete discrete valuation ring B_{dR}^+ over K. It has a G_K-action such that the residue field is G_K-equivariantly isomorphic to C. Moreover, its Zariski cotangent space is isomorphic to $C(1)$ in the category $Rep_C(G_K)$.Then we could have an isomorphism $gr^{\bullet}(B_{dR}) \simeq B_{HT}$ as graded C_K-algebras with G_K-action, in which B_{dR} is the fraction field of B_{dR}^+.

In order to construct B_{dR}^+, we notice that if k is a perfect field with characteristic $p > 0$, there is a complete discrete valuation ring $W(k)$ (the Witt ring of k, [Ser13]) which has k as residue field . The problem is that C has characteristic 0. So we can not use the construction of Witt ring directly. Instead, we might look at $\mathcal{O}_C/(p)$ and try to get from $\mathcal{O}_C/(p)$ to $\mathcal{O}_C$ then to C. If we follow this idea, the key point lies in that $\mathcal{O}_C/(p)$ is not perfect. Indeed, the p-th power is surjective but not injective, $a^p = 0$ in $\mathcal{O}_C/(p)$ for all $v(a) > 1/p$.

3.2. **Ring $\mathcal{O}^\flat$.** In this subsection, we will solve the problem caused by that $\mathcal{O}_C/(p)$ is not perfect. Let A be a ring (not necessarily perfect) with characteristic $p > 0$, we construct an associated perfect ring in the following way

$$A^{perf} = \varprojlim_{x \to x^p} A = \{(x_0, x_1, \cdots) \in \prod_{n \geq 0} A | x_{i+1}^p = x_i, \forall i\}$$

This $\mathbb{F}_p$-algebra has product ring structure and it is perfect. In fact, A^{perf} has a universal property in the sense that the projection map $pr_0 : A^{perf} \to A$ is final among all maps to A from perfect $\mathbb{F}_p$-algebras. Note that this construction is functorial.

Now we apply this construction to the ring $\mathcal{O}_{\overline{K}}/(p)$, define

$$\mathcal{O}^\flat := \varprojlim_{x \to x^p} \mathcal{O}_{\overline{K}}/(p) = \varprojlim_{x \to x^p} \mathcal{O}_C/(p)$$

The ring $\mathcal{O}^\flat$ is endowed with natural G_K-action via functoriality. Also $\mathcal{O}_{\overline{K}}/(p)$ is a $\overline{k}$-algebra, by functoriality we have a ring map

$$\overline{k} = \varprojlim_{x \to x^p} (\bar{k}) \to \mathcal{O}^\flat$$

explicitly,

$$c \to (j(c), j(c^{1/p}), j(c^{1/p^2}), \cdots)$$

where $j : \bar{k} \to \mathcal{O}_{\overline{K}}/(p)$ is the unique k-algebra section to the reduction map $\mathcal{O}_{\overline{K}}/(p) \to \bar{k}$.

Now we have a perfect ring $\mathcal{O}^\flat$ with charactristic $p > 0$. Then we follow the idea lifting $\mathcal{O}^\flat$ to the ring $\mathcal{O}_C$. The following proposition provides such a way.

Proposition 3.2. *The map*

$$\varprojlim_{x \to x^p} \mathcal{O}_C \to \mathcal{O}^\flat$$

defined by $(x^{(n)})_{n \geq 0} \to (x^{(n)} \mod p)_{n \geq 0}$ *is a bijection.*

Proof. Let $x = (x_n)_{n \in \mathbb{N}} \in \mathcal{O}^\flat$. Fix $(\widehat{x}_n)_{n \in \mathbb{N}}$, $(\widetilde{x}_n)_{n \in \mathbb{N}}$ to be two liftings of x in $\varprojlim_{x \to x^p} \mathcal{O}_C$. As $\widehat{x}_{n+m} \equiv \widetilde{x}_{n+m}$ *mod* p, we get $\widehat{x}_{n+m}^{p^m} \equiv \widetilde{x}_{n+m}^{p^m}$ *mod* p^{m+1}. Taking $\widetilde{x}_n = \widehat{x}_{n+1}^p$, we get $\widehat{x}_{n+m+1}^{p^{m+1}} \equiv \widehat{x}_{n+m}^{p^m}$ *mod* p^{m+1}. The sequence $(\widehat{x}_{n+m}^{p^m})_{m \in \mathbb{N}}$ is Cauchy, hence converge in $\mathcal{O}_C$. Moreover its limit $x^{(n)} \in \mathcal{O}_C$ does not depend on the choice of the lifting $(\widehat{x}_n)_{n \in \mathbb{N}}$. The sequence $(x^{(n)})_{n \in \mathbb{N}}$ is also a lifting of x. In particular, we have

$$(x^{(n+1)})^p = \lim_{m \to \infty} x^{(n+1+m)p^{m+1}} = x^{(n)}$$

which shows that $(x^{(n)})_{n \in \mathbb{N}}$ is a preimage of x, which is the inverse map. □

Remark 3.3. Using this bijective map, we can transfer the natural F_p-algebra structure on $\mathcal{O}^\flat$ over to such a structure on the inverse limit set $\varprojlim_{x \to x^p} \mathcal{O}_C$. The multiplicative structure translates through the bijiection $(xy)^{(n)} = x^{(n)} y^{(n)}$ and this implies that $\mathcal{O}^\flat$ is a domain. The additive structure can be seen from

$$(x+y)^{(n)} = \lim_{m \to \infty} (x^{(n+m)} + y^{(n+m)})^{p^m}$$

Let $|\cdot|_p : C \twoheadrightarrow p^{\mathbb{Q}} \cap \{0\}$ be the normalized absolute value, i.e. $|p|_p = 1$. We define a map $|\cdot|_{\mathcal{O}^\flat}$ on $\mathcal{O}^\flat$ by $|x|_{\mathcal{O}^\flat} = |x^{(0)}|_p$.

Lemma 3.4. *The map* $|\cdot|_{\mathcal{O}^\flat}$ *on* $\mathcal{O}^\flat$ *is a* G_K*-equivariant absolute value on* $\mathcal{O}^\flat$*. In particular,* $\mathcal{O}^\flat$ *is* $v_{\mathcal{O}^\flat}$*-adically separated and complete. The subfield* $\bar{k}$ *of* $\mathcal{O}^\flat$ *maps isomorphically onto the residue field of* $\mathcal{O}^\flat$*.*

Proof. To prove $|\cdot|_{\mathcal{O}^\flat}$ is an absolute value, we follow its definition. $x^{(0)} = 0$ iff $x = 0$ and $|xy|_{\mathcal{O}^\flat} = |x|_{\mathcal{O}^\flat} |y|_{\mathcal{O}^\flat}$ since $(xy)^{(0)} = x^{(0)} y^{(0)}$. To show $|x+y|_{\mathcal{O}^\flat} \leq max(|x|_{\mathcal{O}^\flat}, |y|_{\mathcal{O}^\flat})$ for all $x, y \in \mathcal{O}^\flat$, suppose $x, y \neq 0$, $x^{(0)}, y^{(0)} \neq 0$. So we may assume $|x^{(0)}|_p \leq |x^{(0)}|_p$, then for all $n \geq 0$

$$|x^{(n)}|_p = |x^{(0)}|_p^{p^{-n}} \leq |y^{(0)}|_p^{p^{-n}} = |y^{(n)}|_p$$

The ratio $x^{(n)}/y^{(n)}$ lies in $\mathcal{O}_C$ for all $n \geq 0$ and form a p-power compatible sequence. So it corresponds to an element z in $\mathcal{O}^\flat$ and $xy = z$ in $\mathcal{O}^\flat$, this means $y|x$ in $\mathcal{O}^\flat$.Therefore

$$|x+y|_{\mathcal{O}^\flat} = |y(z+1)|_{\mathcal{O}^\flat} = |y|_{\mathcal{O}^\flat} |z=1|_{\mathcal{O}^\flat} \leq |y|_{\mathcal{O}^\flat} \leq \max(|x|_{\mathcal{O}^\flat}, |y|_{\mathcal{O}^\flat}).$$

So we have proved $|\cdot|_{\mathcal{O}^\flat}$ is an absolute value. It is easy to show $\mathcal{O}^\flat$ is the valuation ring.

To prove the completeness, write $v = -\log_p |\cdot|_p$ on C, then $v_{\mathcal{O}^\flat}(x) = p^n v(x^{(n)})$. Thus, $v_{\mathcal{O}^\flat}(x) \geq p^n$ iff $v(x^{(n)}) \geq 1$ iff $x^{(n)} \mod p = 0$. So if we write $\theta_n : \mathcal{O}^\flat \to \mathcal{O}_C/(p)$ denote the n-th projection, then $\operatorname{Ker}\theta = \{x \in \mathcal{O}^\flat | v_{\mathcal{O}^\flat}(x) \geq p^n\}$

Since $x_n = 0$ implies $x_m = 0$ for all $m \leq n$, the $v_{\mathcal{O}^\flat}$-adic topology is the same with the subsapce topology within $\prod_{m\geq 0}(\mathcal{O}_C/(p)$ where the factors are given discrete topology. Since $\mathcal{O}^\flat$ is closed in $\prod_{m\geq 0}(\mathcal{O}_C/(p))$, so it is complete.

The map $\theta_0 : \mathcal{O}^\flat \to \mathcal{O}_C/(p)$ is a $\bar{k}$-algebra morphism, since it is local, so we have ,

$$\bar{k} \to \mathcal{O}^\flat/m \hookrightarrow \bar{k}$$

So the subfield $\bar{k}$ of $\mathcal{O}^\flat$ maps isomorphically onto its residue field.

□

Example 3.5. We now introduce an important element in $\mathcal{O}^\flat$,

$$\varepsilon = (\varepsilon^{(n)})_{n\geq 0} = (1, \zeta_p, \zeta_{p^2}, \cdots)$$

with $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)}$ is a primitive p-th root of unity and hence $\varepsilon^{(n)}$ is a primitive p^n-th root of unity for all $n \geq 0$. Any two such elements differ by a $\mathbb{Z}_p^\times$ power. We can calculate

$$v_{\mathcal{O}^\flat}(\varepsilon - 1) = \frac{p}{p-1}$$

We will need the next theorem later in Kisin's theory of $\mathfrak{S}$-module.

Theorem 3.6. *The field* $\operatorname{Frac}(\mathcal{O}^\flat)$ *of char* p *is algebraically closed.*

Proof. We have to show any irreducible $P(X) = X^N + a_1 x^{(N-1)} = \cdots a_N \in \operatorname{Frac}(\mathcal{O}^\flat)[X]$ has a roott in $\operatorname{Frac}(R)$. $\operatorname{Frac}(\mathcal{O}^\flat)$ is perfect, so $P(X)$ is separable. We may assume $P(X)$ is in $\mathcal{O}^\flat[X]$.As $P[X]$ is separable, so there exist $U_0(X), V_0(X) \in \operatorname{Frac}(\mathcal{O}^\flat)[X]$ such that $P[X]U_0(X) + P'(X)V_0(X) = 1$. Let $m_0 \in \mathbb{N}$ such that $U(X) = \widetilde{p}^{m_0} U_0(X) \in \mathcal{O}^\flat[X]$ and $V(X) = \widetilde{p}^{m_0} V_0(X) \in \mathcal{O}^\flat[X]$, then we have $P[X]U(X) + P'(X)V(X) = \widetilde{p}^{m_0}$, $\widetilde{p} = (p, p^{1/p}, \cdots) \in \mathcal{O}^\flat$. Choose $m \in \mathbb{N}$ with $n > 2m_0$. The polynomial $P^{(m)}(X) := X^N + a_1^{(m)} X^{(N-1)} + \cdots a_N^{(m)} \in \mathcal{O}_C[X]$ has a root in $\mathcal{O}_C$ The map $\mathcal{O}^\flat \to \mathcal{O}_C; x \to x^{(m)}$ is surjective. There exists $\alpha \in \mathcal{O}^\flat$ whose m-th component $\alpha^{(m)}$ is such a root. Reducing modulo $p\mathcal{O}_C$, we have $pr_m(P(\alpha)) = \alpha_m^N + a_{1,m}\alpha_m^{N-1} + \cdots + a_{N,m} = 0$ hence $v_{\mathcal{O}^\flat}(P(\alpha)) \geq p^m$. This implies $v_{\mathcal{O}^\flat}(P(\alpha)U(\alpha)) \geq m > m_0$,so $v_{\mathcal{O}^\flat}(P'(\alpha)V(\alpha)) = v_{\mathcal{O}^\flat}(p^{m_0} - P(\alpha)U(\alpha)) = m_0$. Thus $v_{\mathcal{O}^\flat}(P'(\alpha)) \leq m_0$. As $v_{\mathcal{O}^\flat}(P(\alpha)) = p^m \geq m > 2m_0 \geq 2v_{\mathcal{O}^\flat}(P'(\alpha))$. By Hensel's lemma, $P(X)$ has a zero $\equiv \alpha \mod \widetilde{p}^{m-m_0}\mathcal{O}^\flat$.

□

Note that $\mathbb{A}_{inf} := W(\mathcal{O}^\flat) \subset W(\operatorname{Frac}(\mathcal{O}^\flat))$, then $\mathbb{A}_{inf}$ is a domain with characteristic 0.

3.3. **The construction of B_{dR}.**

We now come to the construction of B_{dR}. The first step is to lift the map $\theta_0 : \mathcal{O}^\flat \to \mathcal{O}_C/(p)$ (the projection to the 0th component) to a map $\theta : \mathbb{A}_{inf} \to \mathcal{O}_C$.

Lemma 3.7. *Suppose* k *is a perfect ring with characteristic* $p > 0$, *there is a unique and multiplicative section* $r : k \to W(k)$. *Moreover, every element in* $W(k)$ *can be written uniquely in the form* $\sum_{n\in\mathbb{N}}[a_n]p^n$, *in which* $[a_n]$ *is the Teichmuller lift of* a_n, *i.e.* $[a_n] = (a_n, 0, 0\cdots) \in W(k)$.

Proof. For any $x \in k$, we can find a compatible sequence $(x_n)_{n\in\mathbb{N}}$ in k, $x_n^{p^n} = x_0 = x$ and $x_{n+1}^p = x_n$. For each x_n, we choose a lifting $\widehat{x}_n \in W(k)$. Then $\widehat{x}_{n+1}^p \equiv \widehat{x}_n \mod p$ implies $\widehat{x}_{n+1}^{p^{n+1}} \equiv \widehat{x}_n^{p^n} \mod p^{n+1}$. Therefor $r(x) := \lim_{n\to\infty} \widehat{x}_n^{p^n}$ exists. If we choose another lifting sequence $(\widehat{a}_n)_{n\in\mathbb{N}}$. Then $x_n \equiv a_n \ mod\ p$, and $x_n^{p^n} \equiv a_n^{p^n} \ mod\ p^{n+1}$. Hence, the limits coincide in $W(k)$. This means $r(x)$ is independent of the choice of liftings. It is easy to see r is a section. For the multiplicity, we can choose the lifting of xy to be the multiple of their own lifting.

Moreover if t is another section, we can always choose $\widehat{x}_n = t(x_n)$, then

$$r(x) = \lim_{n\to\infty} \widehat{x}_n^{p^n} = \lim_{n\to\infty} t(x_n)^{p^n} = t(x)$$

So the uniqueness follows.

To prove the last statement, we choose $a \in W(k)$. Considering the projection $pr : W(k) \to k$, we have $a = [pr(a)] + pa'$ for some $a' \in W(k)$. Let $a_0 = pr(a)$ and then do the same thing for a'. Let $a_1 = pr(a')$ and continue this process. We then get a unique sequence $(a_n)_{n\in\mathbb{N}}$ and $a = \sum_{n\in\mathbb{N}}[a_n]p^n$. □

Now, we can construct the map θ in a concrete form:

$$\theta(\sum [c_n]p^n) = \sum c_n^{(0)} p^n.$$

There is another formula of θ

$$\theta : (r_0, r_1, \cdots) \to \sum r_n^{(n)} p^n.$$

This map is G_K-equivariant, we want to prove it is actually a ring homomorphism.

Lemma 3.8. *The map $\theta : \mathbb{A}_{inf} \to \mathcal{O}_C$ is a ring homomorphism.*

Proof. It suffices to prove that

$$\theta_n = \theta \mod p^n : W_n(\mathcal{O}^\flat) = W(\mathcal{O}^\flat)/(p^n) \to \mathcal{O}_{C_K}/(p^n) = \mathcal{O}_{\overline{K}}/(p^n)$$

is a ring map for all $n \geq 1$. First, we check θ_n is additive.

Write $x = (x_0, \cdots, x_{n-1})$ with $x_i \in \mathcal{O}^\flat$, so

$$\theta_n(x) = \sum_{i=0}^{n-1} x_i^{(i)} p^i = \sum_{i=0}^{n-1} p^i (x_i^{(n)})^{p^{n-i}}$$

Recall that the $(n-1)$-th phantom component

$$\Phi_{n-1} : W_n(\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\bar{K}}) \to \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$$

$$(x_0, x_1, \cdots, x_{n-1}) \to \sum_{i=0}^{n-1} p^i x_i^{p^{n-i-1}}$$

is a ring homomorphism. If $x_i \equiv y_i \mod p\mathcal{O}_{\overline{K}}$ then $x_i^{p^{n-i-1}} \equiv y_i^{p^{n-i-1}} \mod p^{n-i}\mathcal{O}_{\overline{K}}$, so $\sum_{i=0}^{n-1} p^i x_i^{p^{n-i-1}} \equiv \sum_{i=0}^{n-1} p^i y_i^{p^{n-i-1}} \mod p^{n-i}\mathcal{O}_{\overline{K}}$. Hence $\Phi_{n-1} = \overline{\Phi}_{n-1} \circ \pi_n$ where $\pi_n : W_n(\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}) \twoheadrightarrow W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ is the natural quotient map and $\overline{\Phi}_{n-1} W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}} \to \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$.

Since π_n is surjective and additive amnd Φ_{n-1} is additive, $\overline{\Phi}_{n-1}$ is also additive. Let $f_n : \mathcal{O}^\flat \to \mathcal{O}_C/p\mathcal{O}_C$ denotes the projection $r \to r^{(n)} \mod p$, we have

$$\theta_n = \overline{\Phi}_{n-1} \circ W_n(f_n).$$

The map $W_n(f_n)$ is additive since f_n is a ring homomorphism and additive structure on W_n is functorial in ring homomorphims. So θ_n is additive.

Then come to the multiplicity, this can be reduced to the case $x = [r_1]$ and $x' = [r_2]$

$$\theta([r_1][r_2]) = \theta([r_1r_2]) = (r_1r_2)^{(0)} = r_1^{(0)}r_2^{(0)} = \theta([r_1])\theta([r_2]).$$

□

Proposition 3.9. *The homomorphism θ is surjective.*

Proof. For any $a \in \mathcal{O}_C$, there exists $x \in \mathcal{O}^\flat$ such that $x^{(0)} = a$, then $\theta[x] = x^{(0)} = a$. □

Write $\widetilde{p} = (p, p^{1/p}, \cdots) \in \mathcal{O}^\flat$, $v_{\mathcal{O}^\flat}(\widetilde{p}) = 1$. Let $\xi = [\widetilde{p}] - p \in \mathbb{A}_{inf}$. We now state the most important property of the map θ.

Proposition 3.10. *The ideal* $\operatorname{Ker}(\theta)$ *is a principal ideal generated by ξ. Moreover, an element $w = (r_0, r_1, \ldots,) \in \operatorname{Ker}(\theta)$ is a generator if and only if r_1 is a unit in $\mathcal{O}^\flat$.*

Proof. $\theta(\xi) = \theta([\widetilde{p}]) - p = \widetilde{p}^{(0)} - p = 0$, so ξ is in $\operatorname{Ker}(\theta)$. Note that $v_{\mathcal{O}^\flat}(\widetilde{p}) = 1$. Choose any elment $y \in \operatorname{Ker}(\theta)$. We can write $y = \sum_{n\in\mathbb{N}}[y_n]p^n$. Then $\theta(y) = \sum y_n^{(0)}p^n = 0$. So $y_0^{(0)} \in p\mathcal{O}_C$, i.e. $v_{\mathcal{O}^\flat}(y_0) \geq 1$. This means there exists an element $z_0 \in \mathcal{O}^\flat$ such that $y_0 = \bar{\xi}z_0$, which implies that $y - \xi[z_0] \in \operatorname{Ker}(\theta)$ has reduction 0 modulo p. We can write $y - \xi[z_0] = py'$. Then y' is also in $\operatorname{Ker}(\theta)$. By induction, we can construct a sequence $(z_n)_{n\in\mathbb{N}}$ such that $y - \xi\sum_{i=0}^{i}[z_i] \in p^{n+1}\operatorname{Ker}(\theta)$. Since $\mathbb{A}_{inf}$ is p-adically separated and complete, $y = \xi\sum_{i=0}^{\infty}p^i[z_i] \in \xi\mathbb{A}_{inf}$.

Assume $w = (r_0, r_1, \cdots) \in \operatorname{Ker}(\theta)$, then we have

$$w = \xi \cdot (s_0, s_1, \cdots) = (\widetilde{p}, -1, \cdots)(s_0, s_1, \cdots) = (\widetilde{s}_0, \widetilde{p}^p s_1 - s_0^p, \cdots),$$

So $r_1 = \widetilde{p}^p s_1 - s_0^p$. Therefore, r_1 is a unit if and only if s_0 is a unit, which can be seen via the valuation on $\mathcal{O}^\flat$. That s_0 is a unit is equivalent to that $(s_0, s_1 \cdots)$ is a unit. Then it is equivalent to w is a generator of $\operatorname{Ker}(\theta)$. □

According to the proof above, we can get that any element x in $\operatorname{Ker}(\theta)$ whose reduction $\bar{x}$ satisfies $v_{\mathcal{O}^\flat}(\bar{x}) = 1$ is a generator of $\operatorname{Ker}(\theta)$. Define $\varpi = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} = 1 + [\varepsilon^{1/p}] + [\varepsilon^{1/p}]^2 + \cdots + [\varepsilon^{1/p}]^{p-1} \in W(R)$. It is a generator of $\operatorname{Ker}\theta$ by using this criterion. In fact, one has $\bar{\varpi} = \frac{\varepsilon-1}{\varepsilon^{1/p}-1} = (\varepsilon^{1/p} - 1)^{p-1}$. So $v_{\mathcal{O}^\flat}(\bar{\varpi}) = (p-1)v_{\mathcal{O}^\flat}(\varepsilon^{1/p} - 1) = \frac{p-1}{p}v_{\mathcal{O}^\flat}(\varepsilon - 1) = 1$.

Now we extend the map θ to C which is a G_K-equivariant surjective ring homomorphism.

$$\theta_{\mathbb{Q}} : \mathbb{A}_{inf}[1/p] \twoheadrightarrow \mathcal{O}_C[1/p] = C.$$

Corollary 3.11. *For all $j \geq 1$*

$$\mathbb{A}_{inf} \cap (\operatorname{Ker}\theta_{\mathbb{Q}})^j = (\operatorname{Ker}\theta)^j$$

Also, $\cap(\operatorname{Ker}\theta)^j = \cap(\operatorname{Ker}\theta_{\mathbb{Q}})^j = 0.$

Let B_{dR}^+ be the completion of $\mathbb{A}_{inf}[1/p]$ for the $\operatorname{Ker}(\theta_{\mathbb{Q}})$-adic topology, i.e.,

$$B_{dR}^+ := \varprojlim_j \mathbb{A}_{inf}[1/p]/(\operatorname{Ker}(\theta_{\mathbb{Q}})^j).$$

The transition maps are G_K-equivariant, so B_{dR}^+ is also G_K-equivariant which is compatible with action on its subring $\mathbb{A}_{inf}[1/p]$. The map $\theta_{\mathbb{Q}}$ can also extend to a surjective ring homomorphism $\theta : B_{dR}^+ \to C$.

Proposition 3.12. *The ring B_{dR}^+ is a complete discrete valuation ring with residue field C_K and any generator of $\operatorname{Ker}\theta_{\mathbb{Q}}$ is a uniformizer of B_{dR}^+.*

Remark 3.13. There is a natural Frobenius map φ on $\mathbb{A}_{inf}[1/p]$, but it does not extend to B_{dR}^+ since it does not preserve $\operatorname{Ker}\theta_{\mathbb{Q}}$, $\varphi(\xi) = [\widetilde{p}^p] - p \notin \operatorname{Ker}\theta_{\mathbb{Q}}$. This impels us to find a finer ring with a Frobenius map compatible with which on $\mathbb{A}_{inf}[1/p]$.

Definition 3.14. *The de Rham period ring is $B_{dR} := \operatorname{Frac}(B_{dR}^+)$, which has a natural G_K-action and G_K-stable filtration, i.e. $Fil^i(B_{dR}) = (\operatorname{Ker}\theta)^i$, ($\operatorname{Ker}\theta$ is the maximal ideal of B_{dR}^+).*

Next, we will prove B_{dR} is really a refinement of B_{HT}. The fact is that the associated graded algebra of B_{dR} is G_K-equivariantly isomorphic to B_{HT}.

The strategy is to prove that B_{dR}^+ admits a uniformizer t, on which G_K acts by p-adic cyclotomic charcter. To construct t, recall that $[\varepsilon] - 1 \in \operatorname{Ker}\theta$, so the logarithm makes sense,

$$t := \log([\varepsilon]) = \log(1 + ([\varepsilon] - 1)) = \sum_{n\geq 1}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n} \in B_{dR}^+.$$

The G_K-action on t is

$$g(t) = \log(g([\varepsilon])) = \log([\varepsilon]^{\chi(g)}) = \chi(g)t.$$

This means $\mathbb{Z}_p t$ is a canonical copy of $\mathbb{Z}_p(1)$ as a G_K-stable line in B_{dR}^+.

Proposition 3.15. *The element $t \in Fil^1 B_{dR}$ and $t \notin Fil^2 B_{dR}$, i.e. t generates the maximal ideal of B_{dR}^+.*

Proof. According to the definition of t, we have $t \equiv [\varepsilon] - 1 \mod Fil^2 B_{dR}^+$. $[\varepsilon] - 1 = \varpi([\varepsilon^{1/p}]-1)$. As $\theta([\varepsilon^{1/p}]-1) \neq 0$, we have $[\varepsilon^{1/p}]-1 \in B_{dR}^{+\times}$. So $[\varepsilon]-1 \in Fil^1 B_{dR}^+ \backslash Fil^2 B_{dR}^+$. Then so does t. This means t is a uniformizer of B_{dR}^+. □

Then we have the corollary.

Corollary 3.16. *$gr^i B_{dR} = C(i)$ for all $i \in \mathbb{Z}$ and $gr^\bullet B_{dR} \simeq B_{HT}$.*

Remark 3.17. The construction of B_{dR}^+ only involves the field C_K. If $K' \subseteq C_K$ is a complete discrete value field (eg, finite extension or $\widehat{K^{ur}}$, we get the same field whether we use K or K'. The group G_K, $G_{K'}$ can both be seen as the subgroup of isometric automorphism group of C_K.

3.4. **De Rham representation.** In this subsection, we will put the de Rham period ring into the general formalism of admissible representations. Since B_{dR} is a field, it is trivially that it is $(\mathbb{Q}_p, G_K)$-regular. We will also explore the results brought by the filtration structure on B_{dR}.

Theorem 3.18. *$B_{dR}^{G_K} = K$.*

Proof. $K \subset B_{dR}^{G_K}$ follows the Proposition 3.16. Since the G_K-action respect the filtration, the field extension $B_{dR}^{G_K}$ of K with the subspace filtration has associated graded K-algebra that injects into $(gr^\bullet(B_{dR}))^{G_K} = B_{HT}^{G_K}$. By Tate-Sen theorem, $B_{HT}^{G_K} = K$. So $B_{dR}^{G_K} = K$. □

Now we have already see B_{dR} is $(\mathbb{Q}_p, G_K)$-regular and $B_{dR}^{G_K} = K$. By the general formalism of admissible representation, we define the covariant functor $D_{dR} : Rep_{\mathbb{Q}_p}(G_K) \to Vec_K$ by

$$D_{dR}(V) = (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

For any $V \in Rep_{\mathbb{Q}_p}(G_K)$, we have an injective G_K-equivariant comparision map

$$\alpha_V : B_{dR} \otimes_K D_{dR}(V) \to B_{dR} \otimes_{\mathbb{Q}_p} V.$$

The general result tells us that $\dim_K(D_{dR}(V)) \leq \dim_{\mathbb{Q}_p}(V)$.

Definition 3.19. *$V \in Rep_{\mathbb{Q}_p}(G_K)$ is called de Rham representation or B_{dR}-admissible if $\dim_K D_{dR}(V) = \dim_{\mathbb{Q}_p}(V)$.*

The map α_V is an isomorphism if and only if V is a de Rham representation. We use $Rep^{dR}_{G_K} \subseteq Rep_{\mathbb{Q}_p}(G_K)$ to denote the subcategory of de Rham representations. Note that $D_{dR} : Rep^{dR}_{G_K} \to Vec_K$ is a faithful, exact functor and any subrepresentation or quotient of a de Rham representation is also a de Rham representation.

Now we focus on the target category of the functor D_{dR}. Since B_{dR} has a natural filtration, this induces that $D_{dR}(V) = (B_{dR} \otimes V)^{G_K}$ also has a filtration.

$$Fil^i(D_{dR}) := (Fil^i B_{dR} \otimes V)^{G_K}.$$

Now we define a new category.

Definition 3.20. *A filtered K-module is a finite dimensional K-vector space D endowed with a decreasing, separated and exhaustive filtration $(Fil^i D)_{i \in \mathbb{Z}}$, resepectively which means $Fil^{i+1} D \subset Fil^i D$ for all $i \in \mathbb{Z}$, $Fil^i D = 0$ for $i \gg 0$ and $Fil^i D = D$ for $i \ll 0$. A morphism $(D_1, Fil^\bullet D_1) \to (D_2, Fil^\bullet D_2)$ is a K-linear map $f : D_1 \to D_2$ which satisfies $f(Fil^i(D_1) \subset Fil^i(D_2)$ for all $i \in \mathbb{Z}$. We denote by MF_K the category of filtered K-modules.*

Remark 3.21. In the category MF_K, there are good notions of kernel and cokernel of a map $T : D_1 \to D_2$. The subspace filtration

$$Fil^i(\operatorname{Ker} T) := \operatorname{Ker}(T) \cap Fil^i(D_1)$$

and the quotient filtration

$$Fil^i(\operatorname{Coker} T) := (Fil^i D_2 + T(D_1))/T(D_1)$$

Though MF_K has kernel and cokernel, it is not an abelian category. For example, we choose $D_1 = D_2 = D$ and $T = id$, the filtration on D_1 is $Fil^i D_1 = D$ for $i \leq 0$ and $Fil^i D_1 = 0$ if $i > 0$. The filtration on D_2 is $Fil^i D_2 = D$ for $i \leq 1$ and $Fil^i D_2 = 0$ if $i > 1$. Then T is not an isomorphism in MF_K. There is another construction of new filtration associated with tensor product.

$$Fil^n(D_1 \otimes D_2) = \bigoplus Fil^i(D_1) \otimes Fil^{n-i}(D_2)$$

Proposition 3.22. *For $V \in Rep^{dR}_{G_K}$, the G_K-equivariant comparison isomorphism*

$$\alpha_V : B_{dR} \otimes_K D_{dR}(V) \to B_{dR} \otimes V$$

respects the filtrations and the inverse does too.

Proposition 3.23. *For any complete discretely-valued extension K'/K inside C_K and any $V \in Rep_{\mathbb{Q}_p}(G_K)$, the natural map $K' \otimes_K D_{dR,K}(V) \to D_{dR,K'}(V)$ is an isomorphism in $Fil_{K'}$. In particular, V is de Rham as G_K-representation if and only if it is de Rham as $G_{K'}$-representation.*

At the end of this section, we state a proposition relating de Rham representations and Hodge-Tate representations.

Proposition 3.24. *If V is de Rham then V is Hodge-Tate and $gr^{\bullet}(D_{dR}(V)) = D_{HT}(V)$ as graded vector spcaes. In general, there is an injection $gr^{\bullet}(D_{dR}(V)) \hookrightarrow D_{HT}(V)$.*

Proof. We have an exact sequence

$$0 \to Fil^{i+1}B_{dR} \to Fil^{i}B_{dR} \to C(i) \to 0$$

This induces a left exact sequence

$$0 \to Fil^{i+1}D_{dR}(V) \to Fil^{i}D_{dR}(V) \to (C(i) \otimes V)^{G_K}$$

so we have $gr^{i}D_{dR}(V) \hookrightarrow (C(i) \otimes V)^{G_K}$ for all $i \in \mathbb{Z}$ and then $gr^{\bullet}D_{dR}(V) \hookrightarrow D_{HT}(V)$. If V is de Rham, then

$$\dim_{\mathbb{Q}_p} V = \dim_K(D_{dR}(V)) = \dim_K(gr^{\bullet}D_{dR}(V)) \leq \dim_K(D_{HT}(V)) \leq \dim_{\mathbb{Q}_p}(V)$$

So V is Hodge-Tate. □

4. A finer period ring: B_{cris}

We have seen that being a de Rham representation is not sensitive to complete finitely ramified extensions, which means that it can not distinguish good reduction and potentially good reduction in geometry. A drawback of B_{dR}^{+} is that the Frobenius automorphism of $\mathbb{A}_{inf}[1/p]$ does not preserve $\operatorname{Ker}\theta_{\mathbb{Q}}$, so there is no natural Frobenius endomorphism of $B_{dR} = \operatorname{Frac}(B_{dR}^{+}) = B_{dR}^{+}[1/t]$.

The fact that B_{dR} refines B_{HT} inspires us to find a period ring with a finer structure than B_{dR}. This impels us to construct a Frobenius-stable ring which is also a subring of B_{dR}. At the same time, we need to find a new target category with richer algebraic structures. In fact, there exists such a period ring $B_{cris} \subset B_{dR}$ which is also $(\mathbb{Q}_p, G_K)$-regular. The target category of the functor D_{cris} is MF_K^{φ}, the category of filtered φ-modules. The ring B_{cris} has three structures: G_K-action, Frobenius operator and filtration. The amazing thing is that we can recover the original p-adic representaion by using the three structures.

4.1. Motivation for MF_K^{φ}.

In this subsection, we will find some motivation form geometry generalizing the category MF_K to classify good p-adic representations.

For an abelian variety over a p-adic field K, a good reduction means it is the K-fiber of an abelian scheme. An abelian scheme is a smooth proper scheme over $\operatorname{Spec}\mathcal{O}_K$ with connected geometric fibers.

If $\mathcal{A}$ is an abelian scheme over $\mathcal{O}_K$, let A_0 be its special fiber and A_K be its generic fiber. The theory of crystalline cohomology over k provides a finitely generated $W(k)$-module $H^i_{cris}(A_0/W(k))$ equipped with a Frobenius semilinear endomorphism φ such that the induced endomorphism of the K_0-vector space $H^i_{cris}(A_0/W(k)[1/p]$ is bijiective. This impels the following definition.

Definition 4.1. *An isocrystal over K_0 is a finite dimensional K_0-vector space D equppied with a bijective Frobenius semilinear endomorphism $\varphi_D : D \to D$.*

Moreover, there is a comparison isomorphism between crystalline and de Rham cohomology

$$H^i_{dR}(A_K/K) \simeq K \otimes_{K_0} H^i_{cris}(A_0/W(k))[1/p]$$

Thus, $D = H^i_{cris}(A_0/W(k))[1/p]$ is an isocrystal over K_0 for which the scalar extension $D_K = D \otimes_{K_0} K$ is equipped with a filtration due to the filtration of $H^i_{dR}(A_K/K)$. These considerations lead us to the new concept which refines MF_K.

Definition 4.2. *Let K be a p-adic field. A filtered φ-module over K is a triple $(D, \varphi, Fil^\bullet)$ where (D, φ_D) is an isocrystal over K_0 and $(D_K, Fil^\bullet)$ is an object in MF_K. A morphism $D_1 \to D_2$ between two filtered φ-modules is a K_0-linear map $D_1 \to D_2$ that is compatible with both φ_{D_1} anad φ_{D_2} and has a K-linear extension $D_{1K} \to D_{2K}$ that is a morphism in MF_K. The category is denoted by MF^φ_K.*

We will see that this category is the correct one which $D_{cris}(V)$ lies in.

4.2. **Divided power structure.** In this subsection, we shall introduce some notions needed in the construction of B_{cris}. The references for this part are [Ber06] and [BO15].

Definition 4.3. *Let A be a ring, I be an ideal of A. A collection $\gamma_n : I \to I, n > 0$ is called a divided power structure (or PD-structure) on I if for all $n \neq 0$, $m > 0$, $x, y \in I$, and $a \in A$, we have*

(i) $\gamma_1(x) = x$, $\gamma_0(x) = 1$,
(ii) $\gamma_n(x)\gamma_m(x) = \binom{n+m}{m}\gamma_{n+m}(x)$,
(iii) $\gamma_n(ax) = a^n\gamma_n(x)$,
(iv) $\gamma_n(x + y) = \sum_{i=0}^{n} \gamma_i(x)\gamma_{n-i}(y)$,
(v) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$.

We say (A, I, γ) is a divided power ring. A homomorphism of divided power rings $f : (A, I, \gamma) \to (B, J, \delta)$ is a ring homomorphism $f : A \to B$ such that $f(I) \subset J$ and such that $\delta_n(f(x)) = f(\gamma_n(x))$ for all $x \in I$ and $n > 0$.

Definition 4.4. *Let (A, I, γ) be a divided power ring and $f : A \to B$ be a ring homomorphism.*

(i) *We say γ extends to B if there exists a PD-structure $\bar{\gamma}$ on B such that $f : (A, I, \gamma) \to (B, IB, \bar{\gamma})$ is a PD-morphism.*
(ii) *Let (B, J, δ) be a divided power ring. We say that γ and δ are compatible if γ extends to B and $\bar{\gamma}$ and δ agrees on $IB \cap J$.*

Definition 4.5. *If A is a ring, B is an A-algebra and (B, J, δ) is a divided power ring, then we call B a divided power A-algebra.*

Given a divided power ring (A, I, γ), let D be the category of divided power A-algebras that are compatible with γ. Morphisms in D are just divided power A-algebra homomorphisms. Let D' be the category of pairs (B, J) of an A-algebra B and an arbitrary ideal $J \subset B$. Morphisms are A-algebra homomorphism which induces A-linear morphism between the given ideals. We have the forgetful functor

$$\omega : D \to D', \quad (B, J, \delta) \to (B, J).$$

Theorem 4.6. *The functor ω admits a left adjoint functor D_γ. That is to say*

$$Hom_D(D_\gamma(B,J),(C,K,\varepsilon)) \cong Hom_{D'}((B,J),(C,K)).$$

Proof. See [BO15, 3.19] or [Ber06, 2.3.1]. □

We call the divided power ring $D_\gamma(B,J)$ the divided power envelope of J in B relative (A,I,γ).

4.3. **Properties of the ring A_{cris}.** Let A^0_{cris} be the divided power envelope of $\mathbb{A}_{inf}$ with respect to $\operatorname{Ker}\theta$, in concrete terms which is the G_K-stable $\mathbb{A}_{inf}$-subalgebra

$$\mathbb{A}_{inf}[\alpha^{[m]}]_{m\geq 1,\alpha\in \operatorname{Ker}\theta} = \mathbb{A}_{inf}[\xi^{[m]}]_{m\geq 1}$$

in $\mathbb{A}_{inf}[1/p]$, $\xi^{[m]} = \xi^m/m!$. In fact, A^0_{cris} is a $\mathbb{Z}$-flat domain since it is a subalgebra of $\mathbb{A}_{inf}[1/p]$ and a $\mathbb{Z}$-module is flat if and only if it is torsion free over $\mathbb{Z}$. We define

$$A_{cris} := \varprojlim A^0_{cris}/(p^n)$$

to be the p-adic completion of A^0_{cris}. Then A_{cris} is p-adically separated and complete.

We have already a map $\theta : \mathbb{A}_{inf} \to \mathcal{O}_C$. By mapping $x^n/n!$ to 0 for all $x \in \operatorname{Ker}(\theta)$ and $n \in \mathbb{N}_{>0}$, we get the map $\theta : A^0_{cris} \to \mathcal{O}_C$. Since $\mathcal{O}_C$ is p-adically complete, we can extend to the map $\theta : A_{cris} \to \mathcal{O}_C$ (from now on, in this subsection θ will always denote the map $A_{cris} \to \mathcal{O}_C$. Define $Fil^iA^0_{cris}$ to be the ideal of A^0_{cris} generated by $\{\xi^{[n]}\}_{n\geq i}$. We also define Fil^iA_{cris} to be the p-adic closure of the ideal of A_{cris} generated by $\{\xi^{[n]}\}_{n\geq i}$. In particular, $Fil^1A_{cris} = \operatorname{Ker}(\theta)$. To see this, suppose there exists $a \in A_{cris}$ such that $\theta(a) = 0$ and a is not in the Fil^1A_{cris}. Then there exists a sequence $(a_n)_{n\in\mathbb{N}} \in A^0_{cris}$ such that $a = \lim a_n$. In fact, we can choose the sequence such that for every a_n, there exists $b_n \in Fil^1A_{cris}$ and c_n such that $\theta(a_n) = p^n\theta(c_n)$ and $a_n = b_n + p^nc_n$. Hence, $a = \lim b_n$ which means a is in Fil^1A_{cris}. For the similar reason, one can prove $Fil^iA_{cris} = \mathbb{A}_{inf}\xi^{[i]} + Fil^{i+1}A_{cris}$. The map θ induces an isomorphism

$$gr^iA_{cris} \simeq \mathcal{O}_C\xi^{[i]}$$

Proposition 4.7. *Let γ be the map $x \to \frac{x^p}{p}$. We have an isomorphism*

$$\mathbb{A}_{inf}[\delta_m]_{m\in\mathbb{N}}/(p\delta_0 - \xi^p, p\delta_{m+1} - \delta_m^p)_{m\in\mathbb{N}_{>0}} \simeq A^0_{cris}.$$

in which δ_m is sent to the image of $\gamma^{m+1}(\xi)$.

Corollary 4.8. *There is an isomorphism*

$$A_{cris}/pA_{cris} \simeq (\mathcal{O}^\flat/\tilde{p}^p\mathcal{O}^\flat)[\delta_m]_{m\in\mathbb{N}}/(\delta_m^p)_{m\in\mathbb{N}}.$$

The proof of the proposition and the corollary can be seen in [Bri08, 6.1.2]. Note that we can see A^0_{cris} is p-adically separated from the proposition 4.7 and the map $A^0_{cris} \to A_{cris}$ is injective. Now we want to construct a map $A_{cris} \to B^+_{dR}$ and use the corollary to prove the map is actually injective.

Let $m \in \mathbb{N}$. We have a natural map $\mathbb{A}_{inf} \hookrightarrow B^+_{dR} \to B_{m+1} := B^+_{dR}/\operatorname{Ker}(\theta)^{m+1}$. Since p is invertible in B^+_{dR} then in B_{m+1}, this map can extend to a map $A^0_{cris} \to B_{m+1}$. As $A^0_{cris} = \sum_{n=0}^m \xi^{[n]}\mathbb{A}_{inf} + Fil^{m+1}A^0_{cris} \subset \frac{1}{m!}\mathbb{A}_{inf} + Fil^{m+1}A^0_{cris}$ and the image of $Fil^{m+1}A^0_{cris}$ in B_{m+1} is zero, the image of A^0_{cris} in B_{m+1} is contained in the image of $\frac{1}{m!}\mathbb{A}_{inf}$, i.e. $A^0_{cris} \to \frac{1}{m!}\mathbb{A}_{inf}/\operatorname{Ker}(\theta)^{m+1} \hookrightarrow B_{m+1}$. Since $\mathbb{A}_{inf}$ is p-adically complete, we can extend the map $A^0_{cris} \to B_{m+1}$ to a map $A_{cris} \to B_{m+1}$. Then passing to the projective limit, it gives rise to a map

$$f : A_{cris} \to B_{dR}.$$

Proposition 4.9. *The map $f : A_{cris} \to B_{dR}^+$ is injective.*

Proof. Consider the map $A_{cris} \to B_i$. Then $Fil^i A_{cris}$ is a subset of the kernel of this map, which means $f(Fil^i A_{cris}) \subset Fil^i B_{dR}^+$. We can prove by induction that $f^{-1}(Fil^i B_{dR}^+) = Fil^i A_{cris}$. When $i = 1$, the map $f_1 : A_{cris} \to B_1$ is just the map $\theta : A_{cris} \to \mathcal{O}_C$, by definition $Fil^1 A_{cris} = \mathrm{Ker}(\theta)$, we have $f^{-1}(Fil^1 B_{dR}^+) = Fil^1 A_{cris}$. Suppose $f^{-1}(Fil^{n-1} B_{dR}^+) = Fil^{n-1} A_{cris}$ and $x \in f^{-1}(Fil^n B_{dR}^+)$. Then $x = \xi^{[n-1]} + x_1$ with $x_0 \in \mathbb{A}_{inf}$ and $x_i \in Fil^n A_{cris}$. We then have $f(\xi^{[n-1]}x_0) \in Fil^n B_{dR}^+$, i.e. $x_0 \in \xi\mathbb{A}_{inf}$. So $x \in \xi\xi^{[n-1]}\mathbb{A}_{inf} + Fil^n A_{cris} = Fil^n A_{cris}$.

So $\mathrm{Ker}(f) = \cap Fil^i A_{cris}$. Let $x \in \mathrm{Ker}(f)$. The image of $Fil^{p^n} A_{cris}$ in A_{cris}/pA_{cris} is the ideal generated by $\{\delta_m\}_{m \geq n-1}$. By the preceeding lemma, the intersection of latter is $\{0\}$. So $x \in pA_{cris}$. We then can write $x = py$. $pf(y) = 0$ implies $y \in \mathrm{Ker}(f)$. By induction, this means $x \in \cap p^n A_{cris} = \{0\}$ since A_{cris} is p-adcially seperated. $\square$

With this proposition, we can regrad A_{cris} as a subring of B_{dR}^+ and it is actually an integral domain.

In fact, by [Fon17, 4.1.2], the image of A_{cris} in B_{dR}^+ has a concrete form

$$\{\sum_{n\geq 0} a_n \frac{\xi^n}{n!} \mid a_n \in W(R), a_n \to 0 \text{ for the } p\text{-adic topology}\}$$

This infinite sums are taken with respect to the discretely-valued topology of B_{dR}^+. It converges since ξ is in the maximal ideal of B_{dR}^+.

Proposition 4.10. *A_{cris} is G_K-stable in B_{dR}^+.*

Proof. Note that $\theta : \mathbb{A}_{inf} \to \mathcal{O}_C$ is G_K-equivariant. The ideal $\mathrm{Ker}(\theta)$ is G_K-stable. Then the action of G_K extends to A_{cris}^0. $A_{cris} = \varprojlim A_{cris}^0/(p^n)$, then G_k acts on each component $A_{cris}^0/(p^n)$ for all $n \in \mathbb{N}$. The transition map $A_{cris}^0/(p^{n+1}) \to A_{cris}^0/(p^n)$ is also G_K-equivarinat, so A_{cris} is G_K-stable. $\square$

4.4. **Construction of the period ring B_{cris}.** Define $B_{cris}^+ := A_{cris}[1/p] \subseteq B_{dR}^+$ and it is G_K-stable. Recall the element $t = \log[\varepsilon] = \sum_{n\geq 1}(-1)^{n+1}([\varepsilon]-1)^n/n$.

Proposition 4.11. (i) *$t \in A_{cris}$ and $t^{p-1} \in pA_{cris}$, so $t^p/p! \in A_{cris}$.*
(ii) *For all $a \in \mathrm{Ker}(A_{cris} \twoheadrightarrow \mathcal{O}_{C_K})$ we have $a^m/m! \in A_{cris}$, for all $m \geq 1$.*

Proof. (1) Since $[\varepsilon]-1 = b\xi$ for some $b \in \mathbb{A}_{inf}$, $\frac{([\varepsilon]-1)^n}{n} = (n-1)!b^n\gamma_n(\xi)$ and $(n_1)!b^n$ tends to 0 as $n \to \infty$ with respect to the p-adic topoloy. So $t \in A_{cris}$. To show $t^{p-1} \in pA_{cris}$, we just need to show that $([\varepsilon]-1)^{p-1} \in pA_{cris}$. Note that $[\varepsilon]-1 = (\varepsilon - 1, \cdots)$ and $(\varepsilon-1)^{(n)} = \lim_{m\to+\infty}(\zeta_{p^{n+m}}-1)^{p^m}$ where $\zeta_{p^n} = \varepsilon^(n)$ is a primitive n-th root of unity. Then $v((\varepsilon-1)^{(n)}) = 1/(p^{n-1}(p-1))$ and

$$(\varepsilon-1)^{p-1} = u_1(p^p, 1, \cdots) = u_2\varpi^p.$$

where u_1 and u_2 are units. Then

$$([\varepsilon]-1)^{p-1} \equiv [\varpi^p]\cdot(\cdots) = (\xi-p)^p\cdot(\cdots) \equiv \xi^p\cdot(\cdots) \mod pA_{cris}.$$

But $\xi^p = p(p-1)!\gamma_p(\xi) \in pA_{cris}$, then we get the result.

(2)if $a = \sum a_n\gamma_n(\xi) \in A_{cris}^0$, then

$$\frac{a^m}{m!} = \sum_{\sum i_n = m}\prod_n a_n \frac{\xi^{ni_n}}{(n!)^{i_n}(i_n)!}.$$

We have $\frac{(ni)!}{(n!)^i i!} \in \mathbb{N}$ for $n \geq 1$ for $n \geq 1$ and $i \in \mathbb{N}$. If $i > 0$, $\frac{(ni)!}{(n!)^i i!}$ can be interpreted combinatorialy as the number of choices to put ni balls into i unlabeled boxes. Thus

$$\frac{a^m}{m!} = \sum_{\sum i_n = m} \prod_n a_n \frac{(ni_n)!}{(n!)^{i_n}(i_n)!} \gamma_{ni_n}(\xi) \in A^0_{cris}.$$

and $\theta(\frac{a^m}{m!}) = 0$.

The case for $a \in A_{cris}$ can be proved by continuity.

□

Definition 4.12. *The crystalline period ring is* $B_{cris} = B^+_{cris}[1/t] = A_{cris}[1/t] \subset B^+_{dR}[1/t] = B_{dR}$.

Analogus to the construction of B_{dR}, the construction of B^+_{cris} and B_{cris} with their Frobenius and Galois structures only depend on the field C, which means if we replace K with any complete discretely valued field $K' \subset C$ we get the same B^+_{cris} and B_{cris}. In particular, the Galois group can be regared as a subgroup of the isometric automorphism group of C. Since $W(k) \subseteq \mathbb{A}_{inf} \subseteq A_{cris}$, we have $K_0 = W(k)[1/p] \subseteq B_{cris}$, so $K_0 \subseteq B^{G_K}_{cris} \subseteq B^{G_K}_{dR} = K$. Our aim is to prove B_{cris} is $(\mathbb{Q}_p, G_K)$-regular.

Theorem 4.13. *The natural* G_K*-equivariant map* $K \otimes_{K_0} B_{cris} \to B_{dR}$ *is injective.*

Proof. We give the sketch of the proof. It suffices to prove that $\mathcal{O}_K \otimes_{W(k)} A_{cris} \to B^+_{dR}$. We now introduce some auxiliary rings. Define A_{max} to be the p-adic completion of $\mathbb{A}_{inf}[\frac{\xi}{p}]$. Similarly, let $\xi_\pi = [\widetilde{\pi}] - 1 \in \mathcal{O}_K \otimes_{W(k)} \mathbb{A}_{inf}$ (π is a uniformizer of K) and $A_{max,K}$ the p-adic completion of $(\mathcal{O}_K \otimes_{W(k)} \mathbb{A}_{inf})[\frac{\xi_\pi}{\pi}]$. Note that $\mathrm{Ker}(1 \otimes \theta : \mathcal{O}_K \otimes_{W(k)} A_{cris} \to \mathcal{O}_C$ is the principle ideal genrated by ξ_π. The ring A_{max} (resp. $A_{max,K}$) is filtered by $Fil^i A_{max} = (\frac{\xi}{p})^i A_{max}$ (resp. $Fil^i A_{max,K} = (\frac{\xi_\pi}{p})^i A_{max}$). Then

$$A_{max}/(p) \simeq \mathbb{A}_{inf}[T]/(pT - \xi, p) \simeq (\mathcal{O}^\flat / \widetilde{p}\mathcal{O}^\flat)[T]$$

and similarly,

$$A_{max,K}/(\pi) \simeq (\mathcal{O}_K \otimes \mathbb{A}_{inf})[T]/(\pi T - \xi_\pi, \pi) \simeq (\mathcal{O}^\flat / \widetilde{\pi}\mathcal{O}^\flat)[T].$$

The image of $Fil^i A_{max}$ (resp. $Fil^i A_{max,K}$) in $A_{max}/(p)$ (resp. $A_{max,K}/(\pi)$) is the ideal generated by T^i. One can use the same methods which prove $A_{cris} \to B^+_{dR}$ is injective to see the natural maps $A_{max} \to B^+_{dR}$ and $A_{max,K} \to B^+_{dR}$ are injective.

If $n \in N$, one has $\xi^{[n]} \in A_{max}$. So $A_{cris} \subset A_{max}$ in B^+_{dR}. There is a map $\mathcal{O}_K \otimes_{W(k)} \mathbb{A}_{inf} \to A_{max,K}$. Let e be the ramification index of K. We can deduce $(\frac{[\widetilde{\pi}]}{\pi})^e = \alpha \frac{[\widetilde{p}]}{p}$ where $\alpha = \frac{[\widetilde{u}]}{u} \in (\mathcal{O}_K \otimes_{W(k)} \mathbb{A}_{inf})^\times$. This shows that $(\mathcal{O}_K \otimes_{W(k)} \mathbb{A}_{inf})[\frac{\xi}{p}] \subset (\mathcal{O}_K \otimes_{W(k)} \mathbb{A}_{inf})[\frac{\xi_\pi}{\pi}]$. Moreover, for $n \in \mathbb{N}$, we have

$$(\frac{[\widetilde{\pi}]}{\pi})^n \in \frac{1}{p}(\mathcal{O}_K \otimes_{W(k)} \mathbb{A}_{inf})[\frac{\xi}{p}].$$

Thus we have the inclusion $(\mathcal{O}_K \otimes_{W(k)} \mathbb{A}_{inf})[\frac{\xi}{p}] \subset (\mathcal{O}_K \otimes_{W(k)} \mathbb{A}_{inf})[\frac{\xi_\pi}{\pi}] \subset \frac{1}{p}(\mathcal{O}_K \otimes_{W(k)} \mathbb{A}_{inf})[\frac{\xi}{p}]$. Taking p-adic completions, we get the inclusions

$$\mathcal{O}_K \otimes_{W(k)} A_{max} \subset A_{max,K} \subset \frac{1}{p}\mathcal{O}_K \otimes A_{max}.$$

So the map $\mathcal{O}_K \otimes A_{cris} \to B^+_{dR}$ is the composite of

$$\mathcal{O}_K \otimes A_{cris} \to \mathcal{O}_K \otimes A_{max} \subset B^+_{dR}.$$

Since $\mathcal{O}_K$ is $W(k)$-flat, this is injective.

$\square$

With B_{dR} a field, we get an injective map $K \otimes_{K_0} Frac(B_{cris}) \to B_{dR}$, put G_K action on this map , we have $\mathrm{Frac}(B_{cris})^{G_K} = K_0$.

Theorem 4.14. *The domain B_{cris} is $(\mathbb{Q}_p, G_K)$-regular.*

Proof. It remains to verify that if $b \in B_{cris}$ is nonzero and $\mathbb{Q}_p b$ is G_K-stable then b is a unit. We first reuce to the case that b is in B_{dR}^+ and not in the maximal ideal. Indeed, if b is the in i-th filtration of B_{dR}, then we replace b with $t^{-i}b$.

Let $\eta : G_K \to \mathbb{Q}_p^\times$ be the character of group action on $\mathbb{Q}_p b$. Then the residue class $\bar{b}$ in C spans a $\mathbb{Q}_p$-line in C with G_K-action by η. So η must be continuous and the image of η are in $\mathbb{Z}_p^\times$ (since every continuous $\mathbb{Q}_p$-representation has a G_K-stablle $\mathbb{Z}_p$-lattice). Moreover $C(\eta^{-1})^{G_K} \neq 0$, where the underlying space of $C(\eta^{-1})$ is C and the G_K-action is given by $g.c = \eta^(-1)(c)g(c) = (\eta(c))^{-1}g(c)$. By Tate-Sen theorem, we know $\eta(I_K)$ is finite, where $I_K = Gal(\overline{K}/K^{ur})$. Since $I_K = G_{\widehat{K^{ur}}}$ and the constrcution of B_{cris} will not be influenced by replacing K with $\widehat{K^{ur}}$, we get that $\bar{b}$ is algebraic over $\widehat{K^{ur}} = W(\bar{k})[1/p] \subset B_{dR}^+$.

As we know, B_{dR} is a complete discretely valued field. By using Hensel's lemma, there exists a unique element $c \in B_{dR}^+$ such that $\bar{c} = \bar{b}$ and c is algebraic over $\widehat{K^{un}}$ as$\bar{b}$ is algebraic over $\widehat{K^{ur}}$. So $b - c \in tB_{dR}^+$. Moreover, because of the uniqueness of the lifting c, the G_K-action on B_{dR}^+ restricted to c is given by η. So if $b - c \neq 0$, $b - c$ spans a G_K-stable $\mathbb{Q}_p$-line in tB_{dR}^+ with character η. And by passing to the quotient by $t^2 B_{dR}^+$, the image of $b - c$ is a nonzero element of $C(1)$ on which G_K acts by η. In other words, $C(\chi \cdot \eta)$ has a nonzero G_K-invariant element, where χ is the p-adic cyclotomic character. By Tate-Sen theorem, $\chi\eta(I_K)$ is finite which contradicts to the fact that $\eta(I_K)$ is finite. Hence $b = c$.

By replacing K with K_0, $L := \widehat{K_0^{ur}}(b) \subset B_{cris}$ is a finite extension of $\widehat{K_0^{ur}}$. Let L_0 be the maximal unramified subfield of $\widehat{K_0^{ur}}$, i.e. $L_0 = \widehat{K_0^{ur}}$. Then there is an injective map $L \otimes_{L_0} B_{cris} \to B_{dR}$. Hence the subring $L \otimes_{L_0} L$ is a domain. So $L = L_0$ and therefore $b \in L_0^\times \subset B_{cris}^\times$. $\square$

By the general formalism of admissible representation in Section 1, we consider the fucntor $D_{cris} : Rep_{\mathbb{Q}_p}(G_K) \to Vec_{K_0}$ defined by

$$V \to (B_{cris} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

We then have the comparison morphism

$$\alpha_V : B_{cris} \otimes_K D_{cris}(V) \to B_{cris} \otimes_{\mathbb{Q}_p} V$$

The category of crystalline representation is denoted by $Rep_{G_K}^{cris}$. By Theorem 1.3, α_V is injective and $\dim_{K_0} D_{cris}(V) \leq \dim_{\mathbb{Q}_p}(V)$. We say V is crystalline if it is B_{cris}-admissible, i.e. $\dim_{K_0}(D_{cris}(V)) = \dim_{\mathbb{Q}_p}(V)$ (or α_V is an isomorphism). By the Theorem 4.13, we then add a subspace filtration on $K \otimes_{K_0} D_{cris}(V)$ via its injection into $D_{dR}(V)$. In fact, we have any crystalline representation is also a de Rham representation.

Proposition 4.15. *Let V be in $Rep_{G_K}^{cris}$, then the natural map $K \otimes_{K_0} D_{cris}(V) \to D_{dR}(V)$ in Fil_K is an isomorphism. In particular, crystalline representations are de Rham.*

Proof. According to the definition of filtration structure on $K \otimes_{K_0} D_{cris}(V)$, the natural map is a subobject inclusion in Fil_K. So we just need to compare the K-dimensions. Since V is a crystalline representation, $\dim_{K_0} D_{cris}(V) = \dim_{\mathbb{Q}_p}(V)$ and since $\dim_K D_{dR}(V) \leq \dim_{\mathbb{Q}_p}(V)$, we must have $\dim_K D_{dR}(V) = \ dim_{\mathbb{Q}_p}(V)$, so V is a de Rham representation. □

4.5. **Structures of B_{cris}.** In this subsection, we will introduce the Frobenius operator on B_{cris}. Recall that $\widetilde{p} = (p, p^1/p, \cdots) \in \mathcal{O}^\flat$, $\xi = \widetilde{p} - p \in \operatorname{Ker}\theta$. We first see how Frobenius map on $\mathbb{A}_{inf}[1/p]$ acts on A^0_{cris}.

Lemma 4.16. *The $\mathbb{A}_{inf}$-algebra $A^0_{cris} \subseteq \mathbb{A}_{inf}[1/p]$ is $\varphi_{\mathcal{O}^\flat}$-stable.*

Proof. We can compue $\varphi_{\mathcal{O}^\flat}(\xi) = [\widetilde{p}^p] - p = [\widetilde{p}]^p - p = (\xi + p)^p - p = \xi^p + pw$ for some $w \in \mathbb{A}_{inf}$. Hence, $\varphi_{\mathcal{O}^\flat}(\xi) = p\cdot(w+(p-1)!\cdot(\xi^p/p!))$, so $\varphi_{\mathcal{O}^\flat}(\xi^m) = p^m\cdot(w+(p-1)!\cdot(\xi^p/p!))^m$ for all $m \geq 1$. Note that $p^m/m! \in \mathbb{Z}_p$ for all $m \geq 1$, so $\varphi_{\mathcal{O}^\flat}(\xi^m/m!) \in A^0_{cris}$ for all $m \geq 1$. □

Since A^0_{cris} is $\varphi_{\mathcal{O}^\flat}$-stable, by continuity, we can extend uniquely to a Frobenius map on A_{cris}, and hence a map φ on $B^+_{cris} = A_{cris}[1/p]$. What's more ,we have $\varphi(t) = pt$ with $p \in (B^+_{cris})^\times$, so φ uniquely extends to an endomorphism of B_{cris}.

Theorem 4.17. *The Frobenius endomorphism $\varphi : A_{cris} \to A_{cris}$ is injective. In particular, the induced Frobenius endomorphism of $B_{cris} = A_{cris}[1/t]$ is injective.*

Proof. Recall in Theorem 4.13 A_{max} is the p-adic completion of $\mathbb{A}_{inf}[\delta]$ ($\delta = \frac{\xi}{p}$). We have $A_{cris} \subset A_{max}$. As $\varphi(\xi) \cong \xi^p \ mod\ p\mathbb{A}_{inf}$ i.e. $\varphi(\xi/p) \in \xi^{p-1}\delta + \mathbb{A}_{inf}$, so φ extends into a φ-linear map $\varphi : A_{max} \to A_{max}$. So if we can prove the injectivity for A_{max}, the injectivity for A_{cris} follows.

Again recall in Theorem 4.13 $A_{max}/pA_{max} = (\mathcal{O}^\flat/\bar{\xi}\mathcal{O}^\flat)[\bar{\delta}]$ (where $\bar{x}$ is the image of x mod p). If $a \in A_{max}$ such that $\varphi(a) = 0$, then $\bar{a}^p = 0$ in A_{max}/pA_{max}, i.e. $a \in (\varphi^{-1}(\xi), p)A_{max}$. Let $i \in \{1, \cdots, p-1\}$ be such that $a \in (\varphi^{-1}(\xi), p)^i A_{max}$. Then we can write $a = \sum_{j=0} ip^{i-j}\varphi^{-1}(\xi)^j a_{i,j}$ with $a_{i,j} \in A^{i+1}_{max}$. We get $0 = \varphi(a) = \sum_{j=0} ip^{i-j}\xi^j\varphi(a_{i,j})$, i.e. $p^i \sum_{j=0} i\varphi(a_{i,j})\delta^i = 0$. As A_{max} has no p-torsion, $\sum_{j=0} i\varphi(a_{i,j})\delta^i = 0$. Hence $\sum_{j=0} i(\bar{a}_{i,j})^p\bar{\delta}^i = 0$ in A_{max}/pA_{max}.

But $\bar{a}^p_{i,j} \in (\mathcal{O}^\flat/\bar{\xi}\mathcal{O}^\flat)[\bar{\delta}^p]$ and $(1, \bar{\delta}, \cdots, \bar{\delta}^i)$ is linearly independent over $(\mathcal{O}^\flat/\bar{\xi}\mathcal{O}^\flat)[\bar{\delta}^p]$. This imples $\bar{a}^p_{i,j} = 0$, i.e. $\bar{a}_{i,j} \in \bar{\xi}^{1/p}(A_{max}/pA_{max})$. So $a_{i,j} \in (\varphi^{-1}(\xi), p)A_{max}$ for all $j \in \{0, \cdots, i\}$. It follows that $a \in (\varphi^{-1}(\xi), p)^{i+1}A_{max}$. By induction, we deduce that $a \in (\varphi^{-1}(\xi), p)^p A_{max}$. As $(\varphi(\xi)^{-1})^p \cong \xi$ mod pA_{max} and $\xi = p\delta \in pA_{max}$, we have $(\varphi(\xi)^{-1})^p \in pA_{max}$. So $(\varphi^{-1}(\xi), p)^p A_{max} \subset pA_{max}$, then $a \in A_{max}$. As $\varphi(p) = p$ and A_{max} has no p-torsion, we have $a \in p^n A_{max}$ for all $n \in \mathbb{N}$. So $a = 0$. □

Note that Frobenius map on K_0 is an automorphism, we then can prove any Frobenius semilinear injection $D \to D$ for a finite-dimensional K_0-vector space D is automatically bijective. So the functor D_{cris} actually takes values in the category MF^{φ}_K.

We have a natural map

$$B_{cris} \subseteq K \otimes_{K_0} B_{cris} \subseteq B_{dR}$$

so we can give B_{cris} a subsapce filtration,i.e,

$$Fil^i B_{cris} := B_{cris} \cap Fil^i B_{dR}.$$

Be aware that the filtration is not φ-stable.

Theorem 4.18. *The space* $(Fil^0 B_{cris})^{\varphi=1}$ *is equal to* $\mathbb{Q}_p$.

The proof is very complicated. We will not state it here. The references are [Fon17, 5.3.7].

The key piont of the whole story is that the three structures : G_K-action, Frobenius operator and the filtration can help us recover V from $D_{cris}(V)$ when V is crystalline. Actually we can define a functor on the essential image of D_{cris} in MF_K^{φ},

$$V_{cris}(D_{cris}(V)) = Fil^0(B_{cris} \otimes_{K_0} D_{cris}(V))^{\varphi=1} \simeq V.$$

Remark 4.19. If V is an arbitrary element in MF_K^{φ}, $V_{cris}(V)$ may not be isomorphic to V. In fact, in Section 6, after we introduce the notion of admissiblity and weak admissibility, we can introduce the category of weakly admissible filtered φ-modules and this category is equivalent to $Rep_{G_K}^{cris}$. The quasi-inverse functors are given by D_{cris} and V_{cris}.

In the end, as a response to the fact, mentioned in the begining of this section, that being de Rham is not sensitive to finite ramified extension, we give the following proposition. The proof can be seen in [BC09, 9.3].

Proposition 4.20. *If* K'/K *is a finite ramified extension, it is not true that* V *is in* $Rep_{G_K}^{cris}$ *if it is in* $Rep_{G_{K'}}^{cris}$. *If* $K' = \widehat{K^{ur}}$, *then* V *is crystalline as a* G_K*-representation if and only if it is crystalline as a* $G_{K'}$*-representation.*

5. Étale φ-module

Different from the previous sections, we will describe p-adic representations of $G_E = Gal(E_s/E)$ for arbitrary fields E of characteristic $p > 0$ and a fixed separable closure E_s in this section. We will construct equivalence between various representations of G_E and some categories consisting of what we call étale φ-modules. These are modules equipped with a φ-semilinear endomorphism, in which φ is the Frobenius operator on the coefficient rings.

The aim of introducing this section is make some preparations for Section 6 where we will adapt the theory of étale modules to study $\mathfrak{S}$-modules and find an analogous way to describe the essential image of $Rep_{G_K}^{cris}$ in $Rep_{G_{K_\infty}}$ in terms of $\mathfrak{S}$-modules. The references are [Fon07] and [BC09].

We follow the strategy that we study $\mathbb{F}_p$-representations firstly, then $\mathbb{Z}_p$ representations, and finally pass to $\mathbb{Q}_p$ representations.

5.1. p-torsion representations.

In this subsection, we consider the category $Rep_{\mathbb{F}_P}(G_E)$ of continuous finite-dimensional $\mathbb{F}_p$ representations of G_E and show the equivalence of this category and the category of étale φ-modules over E. We see that there are two structures on E_s, one is G_E-action and the other is the Frobenius map $\varphi_{E_s} : E_s \to E_s$ which sends x to x^p. Correspondly, there are two identities $E_s^{G_E} = E$ and $E_s^{\varphi_{E_s}=1} = \mathbb{F}_p$. We will use these two structures to define the functors between these two categories.

Firstly, we introduce an important semilinear algebra object.

Definition 5.1. *An* φ*-module over* E *is a pair* (M_0, φ_{M_0}), *where* M_0 *is a* E*-vector space and* φ_{M_0} *is a* φ_E*-semilinear endomorphism (*φ_E *is the Frobenius map on* E*), i.e,*

$$\varphi_{M_0}(x + y) = \varphi_{M_0}(x) + \varphi_{M_0}(y), \forall x, y \in M_0$$

$$\varphi_{M_0}(\lambda x) = \varphi_E(\lambda)\varphi_{M_0}(x) = \lambda^p \varphi_{M_0}(x), \forall \lambda \in E, x \in M_0$$

Definition 5.2. *A φ-module is called étale if the E-linearization $\varphi_E^*(M_0) \to M_0$ (i.e.,the E-linear map $c \otimes m \to c\varphi_{M_0}(m)$) is an isomorphism, in which $\varphi_E^*(M_0) = E \otimes_{\varphi_E, E} M_0$. The category of étale φ-module over E is denoted $\Phi M_E^{\acute{e}t}$.*

As usual, we can construct new objects from the old ones. Tensor product is trivial. We introduce the notion of duality in $\Phi M_E^{\acute{e}t}$. The point lies in the semilinear endormorphism. Let $M \in \Phi M_E^{\acute{e}t}$, its dual has the underlying E-vector space the usual one of M. $\varphi_{M^\vee} : M^\vee \to M^\vee$ sends an E-linear functorial $l : M \to E$ to the composite of E-linear pullback functional $\varphi_E^*(l) : \varphi_E^*(M) \to E$ (i.e. $c \otimes m \to c \cdot l(m)^p = c \cdot \varphi_E(l(m))$ and the inverse $M \simeq \varphi_E^*(M)$.

The following is a fundamental result.

Lemma 5.3. *The category $\Phi M_E^{\acute{e}t}$ is an abelian category.*

Proof. If $f : M \to L$ is a morpism, let K(resp. N) be its kernel (resp. cokernel) in Vec_E and K'(resp. N') be the kernel (resp. cokernel) of $1 \otimes f : \varphi^*(M) \to \varphi^*(L)$ in Vec_E. Since $\varphi_E : E \to E$ is flat, we have $\varphi^*(K) = K'$ and $\varphi^*(N) = N'$.

Consider the commutative diagram, in which the bottom row is exact

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \varphi^*(N) & \longrightarrow & \varphi^*(M) & \longrightarrow & \varphi^*(L) & \longrightarrow & \varphi^*(K) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & L & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

By the five lemma, we have $\varphi^*(N) \simeq N$ and $\varphi^*(K) \simeq K$. So they are both in $\Phi M_E^{\acute{e}t}$. ∎

As we mentioned, the separable closure E_s of E has two compatible structure: G_E-action and φ_E-semilinear endomorphism φ_{E_S}. Thus, we can define two functors.

Definition 5.4. *For any $V \in Rep_{\mathbb{F}_p}(G_E)$, we define $D_E(V) = (E_s \otimes_{\mathbb{F}_p} V)^{G_E}$ as an E-vector space equipped with the φ_E-semilinear endomorphism $\varphi_{D_E V}$ induced by $\varphi_{E_S} \otimes 1$. For any $M \in \Phi M_E^{\acute{e}t}$, define $V_E(M)$ to be $(E_s \otimes M)^{\varphi=1}$ with its evident G_E-action induced by the G_E-action on E_s, here $\varphi = \varphi_{E_s} \otimes \varphi_M$.*

Remark 5.5. It is sometimes convenient to use contravariant versions D_E^* and V_E^* (Actually when we want to study $\mathfrak{S}$-modules, contravariant versions are more convenient). They are defined via

$$D_E^*(V) = D_E(V^\vee), V_E^*(M) = V_E(M^\vee)$$

In another formulation

$$D_E^*(V) \simeq Hom_{\mathbb{F}_p(G_E)}(V, E_s), V_E^*(M) = Hom_{E,\varphi}(M, E_s)$$

Lemma 5.6. *For any $V \in Rep_{\mathbb{F}_p}(G_E)$, the E-vector space $D_E(V)$ is finite-dimensional with dimension equal to $\dim_{\mathbb{F}_p}(V)$, and the E-linealization of $\varphi_{D_E(V)}$ is an isomorphism. In particular, $D_E(V_0)$ lies in $\Phi M_E^{\acute{e}t}$ with E-rank equal to the $\mathbb{F}_p$-rank of V_0.*

For any $M \in \Phi M_E^{\acute{e}t}$, the $\mathbb{F}_p$-vector space $V_E(M)$ is finite-dimensional with dimension at most $\dim_E(M)$. In particular, $V_E(M_0)$ lies in $Rep_{\mathbb{F}_p}(G_E)$ with $\mathbb{F}_p$-rank at most $\dim_E(M_0)$.

Proof. For the detailed proof, see [Fon07, A1.2]. Observe that $E_s \otimes_{\mathbb{F}_p} V$ equipped with its diagonal G_E-action is a finite dimensional E_s-vector space equipped with a continuous semilinear G_E-action for the discrete topology. So the natural E_S-linear and G_E-equivariant map

$$E_s \otimes_E (D_E(V)) = E_s \otimes (E_s \otimes_E V)^{G_E} \to E_s \otimes_{\mathbb{F}_p} V$$

is an isomorphism, thus $\dim_E D_E(V) = \dim_{\mathbb{F}_p}(V)$. Furthermore, this map is Frobenius compatible So to prove $\varphi_{D_E(V)}$ is an isomorphism, we just need to prove for any finite-dimensional $\mathbb{F}_p$-vector space V the E_S-linearization of $\varphi_{E_S} \otimes 1$ of $E_s \otimes_{\mathbb{F}_p} V$ is an isomorphism.

For proving $V_E(M)$ has dimension at most $\dim_E M$, we consider the map

$$E_s \otimes_{\mathbb{F}_p} (V_E(V)) = E_s \otimes_{\mathbb{F}_p} (E_S \otimes_E M)^{\varphi=1} \to E_s \otimes_E M$$

it is injective. Since any element in $E_S \otimes_{\mathbb{F}_p} (V_E(V))$ is a finite sum of elementary tensors, it suffices to prove that if $v_1, \cdots, v_r \in V_E(M_0)$ are $\mathbb{F}_p$-linearly independent then in $E_s \otimes_E M_0$ they are E_s-linearly independent. If not, $\sum a_i v_i = 0$ with not all a_i zero. By minimality, we have $a_i \neq 0$ for all i. We may assume $a_1 = 1$. Then

$$v_1 = -\sum_{i>1} \varphi_{E_s}(a_i)\varphi(v_i) = -\sum_{i>1} \varphi_{E_s}(a_i)v_i$$

Hence $\sum_{i>1}(a_i - \varphi_{E_s}(a_i))v_i = 0$. Then we must have $a_i = \varphi_{E_s}(a_i)$ fro all $i > 1$. So $a_i \in \mathbb{F}_p$ for all $i > 1$. This contradicts our assumption. □

By Lemma 5.6, we have two functors $D_E : Rep_{\mathbb{F}_p}(G_E) \to \Phi M_E^{ét}$ and $V_E : \Phi M_E^{ét} \to Rep_{\mathbb{F}_p}(G_E)$. Note that

$$(E_s \otimes_{\mathbb{F}_p} V)^{\varphi=1} = (E_s)^{\varphi=1} = V, \; (E_s \otimes_E M)^{G_E} = (E_s)^{G_E} \otimes_E M = M$$

these define an isomorphism $V_E(D_E(V)) \to V$ and an injection $D_E(V_E(M)) \to M$.

Theorem 5.7. *Via the natural isomorphism $V_E \circ D_E \simeq id$ and the inclusion $D_E \circ V_E \to id$, the covariant functors D_E and V_E are exact rank-preserving quasi-inverse equivalences of categories.*

Proof. see [Fon07, A1.2] or [BC09, 3.1.8]. □

5.2. **Étale φ-modules over $\mathcal{O}_{\mathcal{E}}$.** In this subsection, we will study the category $Rep_{\mathbb{Z}_p}(G_E)$ of continuous G_E-representations on finitely generated $\mathbb{Z}_p$-modules. The idea is to firstly handle torsion objects and then pass to the inverse limit to handle general objects. We will also see this kind of idea later dealing with finite free $\mathfrak{S}$-modules.

Now from $\mathbb{F}_p$ to $\mathbb{Z}_p$, the key point is to find the corresponding semilinear algebra objects over some coefficient ring. More concretely, we need some ring of characteristic 0 and admits an endomorphism lifting φ_E. Since in general we do not demand that E is a perfect field, the construction of Witt vectors can not be used. But actually, such rings exist. We quote two theorems without proof. For details, one can refer to [Mat89, 29.1,29.2].

Theorem 5.8. *Let $(A, \pi A, k)$ be a discrete valuation ring and K an extension of k; then there exists a discrete valuation ring $(B, \pi B, K)$ containing A.*

Theorem 5.9. *Let* (A, m_A, k_A) *be a complete local ring, and* (R, m_R, k_R) *be an unramified discrete valuation ring of characteristic* $(0, p)$*. Then for every homomorphism* $h : k_R \to k_A$*, there exists a local homomorphism* $g : R \to A$ *which induces* h *on the residue fields.*

Applying $A = \mathbb{Z}_p$ to the above theorems, then if K is an extension of $\mathbb{F}_p$, there exists a unique (up to isomorphism) unramified discrete valuation ring R of characteristic 0 with residue field K. Moreover, there exists an endomorphism lifting the Frobenius map φ_K.

Now we apply the theorems to the field E, we can get a pair $(\mathcal{O}_{\mathcal{E}}, \varphi)$, in which $\mathcal{O}_{\mathcal{E}}$ is an unramified discrete valuation ring of characteristic 0 with residue field E and φ lifts the Frobenius map φ_E on E. We give an explicit example of such a ring.

Example 5.10. Assume $E = k((u))$ with k perfect of characteristic $p > 0$. Let $W(k)$ denotes the Witt vectors of k. The explicit pair $(\mathcal{O}_{\mathcal{E}}, \varphi)$ can be constructed as follows.

Let $\mathfrak{S} = W(k)[[u]]$, this is a 2-dimensional regular local ring and (p) is a prime ideal at which the residue field is $k((u)) = E$. Then the localization $\mathfrak{S}_(p)$ at prime ideal (p) is 1-dimensional regular local ring, it is a discrete valuaion ring with uniformizer p. Since u is not in (p), u is a unit in $\mathfrak{S}_(p)$. This means $\mathfrak{S}_{(p)}$ is the localizaion of Dedekind domain $\mathfrak{S}[1/u]$ at (p).

Thus, $\mathfrak{S}^{\wedge}_{(p)}$ is the p-adic completion of $\mathfrak{S}[1/u]$ over $W(k)$. Explicitly,

$$\mathfrak{S}^{\wedge}_{(p)} \simeq \{\sum_{n \in \boldsymbol{Z}} a_n u^n | a_n \in W(k) \text{ } and \text{ } a_n \to 0 \text{ } as \text{ } n \to -\infty\}$$

we define $\mathcal{O}_{\mathcal{E}} = \mathfrak{S}^{\wedge}_{(p)}$. The endomorphism $\sum a_n u^n \to \sum \sigma(a_n) u^{np}$ extends to a local endomorphism of the completion $\mathcal{O}_{\mathcal{E}}$.

For the rest of the section, we fix a choice of $(\mathcal{O}_{\mathcal{E}}, \varphi)$ as above.

Definition 5.11. *The category* $\Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$ *of étale* φ*-modules over* $\mathcal{O}_{\mathcal{E}}$ *consists of pairs* $(\mathcal{M}, \varphi_{\mathcal{M}})$ *where* $\mathcal{M}$ *is a finitely generated* $\mathcal{O}_{\mathcal{E}}$*-module and* $\varphi_{\mathcal{M}}$ *is a* φ*-semilinear endomorphism of* $\mathcal{M}$ *whose* $\mathcal{O}_{\mathcal{E}}$*-linearization is an isomorphism.*

Remark 5.12. Since we have a canonical map $\mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}/(p) = E$, we can consider $\Phi M^{ét}_E$ as the full subcategory of p-torsion objects in $\Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$.

The lifting φ fixes p, so $\varphi^*(\mathcal{M})$ and $\mathcal{M}$ have the same $\mathcal{O}_{\mathcal{E}}$-rank and invariant factors. To prove $\mathcal{O}_{\mathcal{E}}$-linearization of $\varphi_{\mathcal{M}}$ is an isomorphism, we only need to prove it is surjective. Furthermore, surjectivity can be verified modulo p also because p is fixed by φ, so the étaleness property can be checked by working with finite-dimensional $\mathcal{O}_{\mathcal{E}}/(p) = E$-vector space $\mathcal{M}/p\mathcal{M}$.

Lemma 5.13. *The category* $\Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$ *is an abelian categroy.*

The proof is similar to the proof of Lemma 5.3.

Note that $\mathcal{O}_{\mathcal{E}}$ is a complete discrete valuation ring with residue field E. We fix a separable closure E_s of E, then the maximal unramified extension $\mathcal{O}_{\mathcal{E}^{ur}}$ of $\mathcal{O}_{\mathcal{E}}$ with residue field E_s is unique up to isomorphism. It is strictly henselian (generally not complete) discrete valuation ring with uniformizer p, so its fraction field $\mathcal{E}^{ur} = \mathcal{O}_{\mathcal{E}^{ur}}[1/p]$.

By the universal property of maximal unramified extension, if $f : \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$ is a local ring map , $\bar{f} : E \to E$ is its reduction which has a lifting $\bar{f}' : E_s \to E_s$, then there is a unique local ring endomorphism $f' : \mathcal{O}_{\mathcal{E}^{ur}} \to \mathcal{O}_{\mathcal{E}^{ur}}$ over f lifting $\bar{f}'$.

If we take f to be φ and $\bar{f}'$ to be φ_{E_s}. Then we get a unique local endomorphism of $\mathcal{O}_{\mathcal{E}^{ur}}$ denoted φ again. If we take f to be identity map, and $\bar{f}' \in Gal(E_s)$, we get an induced G_E-action on $\mathcal{O}_{\mathcal{E}^{ur}}$. Moreover, this action is continuous and commutes with φ on $\mathcal{O}_{\mathcal{E}^{ur}}$ because of the uniqueness of f'. In particular, the induced G_E-action on $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ is continuous and commutes with the induced Frobenius endomorphism.

We have the following important facts:

Lemma 5.14. (i) $\mathcal{O}_{\mathcal{E}} = \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{G_E}$, $\mathcal{E} = \widehat{\mathcal{E}^{ur}}{}^{G_E}$.
(ii) $\mathbb{Z}_p = \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{\varphi=1}$,$\mathbb{Q}_p = (\widehat{\mathcal{E}^{ur}})^{\varphi=1}$.

Proof. Since G_E and φ fix p, and $\widehat{\mathcal{E}^{ur}} = \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}[1/p]$, the integral claims imply the field claims. So we focus on the integral cases. Note that the inclusions $\mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{G_E}$ and $\mathbb{Z}_p \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{\varphi=1}$ are local maps between p-adically separated and complete rings. So it suffices to prove the maps $\mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{G_E}/(p^n)$ and $\mathbb{Z}_p \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{\varphi=1}/(p^n)$ are surjective for all $n \geq 1$. We prove this by induction on n.

We first consider the case $n = 1$. There is an exact sequence

$$0 \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \xrightarrow{p} \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \to E_s \to 0.$$

By adding a G_E-action, we get a left exact sequence

$$0 \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{G_E} \xrightarrow{p} \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{G_E} \to E_s^{G_E}$$

then this gives rise to a linear injection $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{G_E}/(p) \hookrightarrow E_s^{G_E} = E$ of modules over $\mathcal{O}_{\mathcal{E}}/(p) = E$, so this injection is actually an isomorphism. So the natural mao $\mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{G_E}/(p)$ is surjective. Similarly, we also have a left exact sequence

$$0 \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{\varphi=1} \xrightarrow{p} \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{\varphi=1} \to E_s^{\varphi=1}$$

Since $E_s^{\varphi=1} = \mathbb{F}_p = \mathbb{Z}_p/(p)$, we see that $\mathbb{Z}_p \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{\varphi=1}/(p)$ is surjective. This proves the case $n = 1$.

Now we assume $\mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{G_E}/(p^{n-1})$ is surjective. For any $x \in \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{G_E}$, we want to seek $y \in \mathcal{O}_{\mathcal{E}}$ such that $x \equiv y \mod p^n$. We choose $z \in \mathcal{O}_{\mathcal{E}}$ such that $x \equiv z \mod p^{n-1}$, so $x - z = p^{n-1}x'$ for some $x' \in \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{G_E}$. There exists $z' \in \mathcal{O}_{\mathcal{E}}$ such that $x' \equiv z' \mod p$, so $x \equiv z + p^{n-1}z' \mod p^n$ with $z + p^{n-1}z' \in \mathcal{O}_{\mathcal{E}}$.

For the φ-invariant case, choose any $x \in \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{\varphi=1}$. We want to seek $y \in \mathbb{Z}_p$ such that $x \equiv y \mod p^n$. Choose $y' \in \mathbb{Z}_p$ such that $x \equiv z \mod p^{n-1}$. then $x = y' + p^{n-1}z$ for some $z \in \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}{}^{\varphi=1}$. Also, there exists $z' \in \mathbb{Z}_p$ such that $z \equiv z' \mod p$. So $x \equiv y' + p^{n-1}z' \mod p^n$ with $y' + p^{n-1}z' \in \mathbb{Z}_p$. Hence we have proved the lemma.

□

Theorem 5.15 (Fontaine)**.** *There are covariant naturally quasi-inverse equivalences of abelain categories*

$$D_{\mathcal{E}} : Rep_{\mathbb{Z}_p}(G_E) \to \Phi M^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}, V_{\mathcal{E}} : \Phi M^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}} \to Rep_{\mathbb{Z}_p}(G_E).$$

defined by

$$D_{\mathcal{E}}(V) = (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathbb{Z}_p} V)^{G_E}, V_{\mathcal{E}}(M) = (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1}.$$

These functors preserve rank and invariant factors over $\mathcal{O}_{\mathcal{E}}$ and $\mathbb{Z}_p$. In particular, they are length-preserving over $\mathcal{O}_{\mathcal{E}}$ and $\mathbb{Z}_p$ for tosion objects and preserve being finite free over $\mathcal{O}_{\mathcal{E}}$ and $\mathbb{Z}_p$.

The functors $D_{\mathcal{E}}$ and $V_{\mathcal{E}}$ are compatible with the formation of duality functors $Hom_{\mathcal{O}_{\mathcal{E}}}(\cdot, \mathcal{E}/\mathcal{O}_{\mathcal{E}})$ and $Hom_{\mathbb{Z}_p}(\cdot, \mathbb{Q}_p/\mathbb{Z}_p)$ on torsion objects and the formation of duality functors $Hom_{\mathcal{O}_{\mathcal{E}}}(\cdot, \mathcal{O}_{\mathcal{E}})$ and $Hom_{\mathbb{Z}_p}(\cdot, \mathbb{Z}_p)$ on finite free module objects.

Proof. See [Fon07, 1.2.6]. □

Remark 5.16. There are contravariant version of this theorem. See [Fon07, 1.2.7]. The functors are defined by

$$D^*_{\mathcal{E}}(V) = Hom_{G_E}(V, \mathcal{E}^{ur}/\mathcal{O}_{\mathcal{E}^{ur}})$$

and

$$V^*_{\mathcal{E}}(M) = Hom_{\Phi M^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}}(M, \mathcal{E}^{ur}/\mathcal{O}_{\mathcal{E}^{ur}}).$$

on torsion objects. For the finite free objects, they are defined by

$$D^*_{\mathcal{E}}(V) = Hom_{G_E}(V, \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}).$$

and

$$V^*_{\mathcal{E}}(M) = Hom_{\Phi M^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}}(M, \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}).$$

5.3. **p-adic representations of G_E.** We have discussed $\mathbb{Z}_p$-representations of G_E. Now we want to use these results to discuss the category $Rep_{\mathbb{Q}_p}(G_E)$ in a similar way.

Lemma 5.17. *For $V \in Rep_{\mathbb{Q}_p}(G)$, there exists a G-stable $\mathbb{Z}_p$-lattice $\Lambda \subseteq V$.*

Proof. Let $\rho : G \to Aut_{\mathbb{Q}_p}(V)$ be the continuous action map. Choose a $\mathbb{Z}_p$-lattice $A_0 \subseteq V$ and we have $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A_0$. So we have $Aut_{\mathbb{Z}_p} A_0 \subseteq Aut_{\mathbb{Q}_p}(V)$ and this is an open subgroup since $\mathbb{Z}_p$ is an open subgroup of $\mathbb{Q}_p$. So the preimage $G_0 = \rho^{-1}(Aut_{\mathbb{Z}_p}(A_0))$ of this subgroup is open in G. G_0 has finite index since G is compact. So we can choose finite coset representatives $\{a_i\}$. Thus the finite sum $A = \sum_i \rho(a_i)A_0$ is a $\mathbb{Z}_p$-lattice in V and it is G-stable since A_0 is G_0 stable and $G = \amalg a_i A_0$. □

For $V \in Rep_{\mathbb{Q}_p}(G_E)$, we define the $\mathcal{E}$-vector space

$$D_{\mathcal{E}}(V) = (\widehat{\mathcal{E}^{ur}} \otimes_{\mathbb{Q}_p} V)^{G_E}$$

equipped with $\varphi_{\mathcal{E}}$-semilinear endomorphism $\varphi_{D_{\mathcal{E}}(V)}$ induced by the Frobenius endomorphism on $\widehat{\mathcal{E}^{ur}}$.

Proposition 5.18. *For $V \in Rep_{\mathbb{Q}_p}(G_E)$,$D := D_{\mathcal{E}}(V)$ has finite $\mathcal{E}$-dimension $\dim_{\mathcal{E}}(D) = \dim_{\mathbb{Q}_p}(V)$, and the $\mathcal{E}$-linearizaion of φ_D is an isomorphism. Moreover, there is a φ_D-stable lattice $L \subseteq D$ such that the $\mathcal{E}$-linearization $\varphi_{\mathcal{O}_{\mathcal{E}}}(L) \to L$ is an isomorphism.*

Proof. By Lemma 5.17, we have $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda$ for $\Lambda \in Rep_{\mathbb{Z}_p}(G_E)$ that is finite free as a $\mathbb{Z}_p$-module. Then we have

$$D_{\mathcal{E}}(V) = D_{\mathcal{O}_{\mathcal{E}}}(\Lambda)[1/p] \simeq \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{E}}} \Lambda$$

as $\mathcal{E}$-vector space with a $\varphi_{\mathcal{E}}$ a-semilinear endomorphism. Since $\mathcal{O}_{\mathcal{E}}(\Lambda) \in \Phi M^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}$ and it is finite free as an $\mathcal{O}_{\mathcal{E}}$-module with rank equal to $\dim_{Q_p}(V)$, the proof is done. □

Motivated by the proposition, we have the following definition

Definition 5.19. *An étale φ-module over $\mathcal{E}$ is a finite dimensional $\mathcal{E}$-vector space D equipped with a $\varphi_{\mathcal{E}}$-semilinear endomorphism φ_D whose linearization is an isomorphoism and which admits a φ_D-stable $\mathcal{O}_{\mathcal{E}}$-lattice $L \subseteq D$ such that $(L, \varphi_D|_L) \in \Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$. The category is denoted by $\Phi M^{ét}_{\mathcal{E}}$.*

Theorem 5.20. *The fuinctors $D_{\mathcal{E}}(V) := (\widehat{\mathcal{E}^{ur}} \otimes_{\mathbb{Q}_p} V)^{G_E}$ and $V_{\mathcal{E}}(V) := (\widehat{\mathcal{E}^{ur}} \otimes_{\mathcal{E}} V)^{\varphi=1}$ are rank-preserving exact quasi-inverse equivalences between $Rep_{\mathbb{Q}_p}(G_E)$ and étale$\varphi -$ modules over $\mathcal{E}$.*

Proof. It is easy to see $D_{\mathcal{E}}(V) = D_{\mathcal{E}}(\Lambda)[1/p]$ and $V_{\mathcal{E}}(V) = D_{\mathcal{E}}(L)[1/p]$ for a G_E-stable $\mathbb{Z}_p$-lattice Λ in V and an *é*tale φ-module L. Then the theorem can be proved by p-localization on Theorem 5.15 comparing $Rep_{\mathbb{Z}_p}(G_E)$ and $\Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$. □

6. Kisin's theory

For many purposes, such as the theory of Galois deformation with artinian coefficients, it will be useful if we have a finer theory in which p-adic vector spaces are replaced with lattices or torsion modules. Fontaine and Laffaille ([FL82]) imposed stringent restrictions on the Hodge-Tate weights and absolute ramification in K to get such a theory.

In this subsection, we will introduce the work of Kisin which gives a new classification of crystalline representations and some important results about the lattices in crystalline representations in terms of $\mathfrak{S}$-modules.

Throughout this secton, we fix a p-adic field K and a unifomizer π. k is its redisue field which is a perfect field with char $p > 0$. $W = W(k)$ is its ring of Witt vectors. $K_0 = W(k)[1/p]$, then K/K_0 is a totally ramified field extension. We denote $E(u) \in K_0[u]$ the Eisenstein polynomial of π, which is irreducible. Moreover, we fix an agebraic closure $\overline{K}$ of K, and a sequence of elements $\pi_n \in \overline{K}$, for nonnegative integer n, satisfying $\pi_0 = \pi$ and $\pi_n = \pi_{n+1}^p$. We write $K_{n+1} = K(\pi_n)$.

6.1. Geometric interpretation of $MF_K^{\varphi,N,Fil\geq 0}$.

In this subsection, we try to give the relation between the category $MF_K^{\varphi,N}$ of filtered (φ, N)-modules and the category of certain vector bundles over rigid-analytic open unit disc over K_0, following the idea of Berger.

Here are some notations. We denote the rigid-analytic open unit disc over K_0 by $D([0, 1))$. The points of $D([0, 1))$ can be identified with the orbits of $Gal(\bar{K}/K_0)$ acting on the set $\{x \in \bar{K} | |x| < 1\}$. We choose a coordinate u. $I \subset [0, 1)$ is a subinterval, then we write $D(I) \subset D([0, 1))$ for the admissible open subsapce whose $\bar{K}$-points correspond to $x \in \bar{K}$ with $|x| \in I$. We define

$$\mathcal{O}_I := \Gamma(D(I), \mathcal{O}_{D(I)})$$

to be the rings of rigid analytic functions. In particular, $\mathcal{O} = \mathcal{O}_{[0,1)}$. If $I = (a, b)$, we write $D(a, b)$ directly instead of $D((a, b))$. Concretely, the ring $\mathcal{O}$ is of the form

$$\{\sum_{n=0}^{\infty} a_n u^n \in K_0[[u]] | \lim_{n\to\infty} |a_n| r^n = 0, \text{for } \forall r < 1\}.$$

Obviously, it is a K_0-subalgebra of $K_0[[u]]$.

Remark 6.1. For $0 < r < 1$ which also lies in the value group $\bar{K}^{\times}$, the ring $\mathcal{O}_{[0,r]}$ of power series converging on the rigid-analytic closed disc $D_{[0,r]}$ has a norm defined by

$$|f|_r := \sup_{x \in D_{[0,r]}} |f(x)| < \infty$$

These norms make $\mathcal{O}$ into a Fréchet space as projective limit of topological spaces $\mathcal{O}_{[0,r]}$.

Let $\mathfrak{S} = W[[u]]$ and $\widehat{\mathfrak{S}_n} = K(\pi_n)[[u - \pi_n]]$. The ring $\widehat{\mathfrak{S}}_n$ is equipped with $(u - \pi_n)$-adic filtration. Then $\widehat{\mathfrak{S}}_n[1/(u - \pi_n)]$ has a natural filtration.

We have the following natural inclusions by sending u to u,

$$\mathfrak{S} = W[[u]] \hookrightarrow \mathfrak{S}[1/p] \hookrightarrow \mathcal{O} \hookrightarrow \widehat{\mathfrak{S}}_n.$$

6.2. **Frobenius map and monodromy operator on $\mathcal{O}$.** We now define the Frobenius map on the ring $\mathcal{O}$. Note that on W, we have a natural Frobenius map φ. We define two maps on $\mathfrak{S}$, φ_W and $\varphi_{\mathfrak{S}/W}$. $\varphi_W : \mathfrak{S} \to \mathfrak{S}$ is a $\mathbb{Z}_p[[u]]$-linear map which acts on W via the Frobenius. And $\varphi_{\mathfrak{S}/W} : \mathfrak{S} \to \mathfrak{S}$ is a W-linear map sending u to u^p.

So we can induce maps on $\mathcal{O}_I$ for $I \subset [0, 1)$:

$$\varphi_W : \mathcal{O}_I \to \mathcal{O}_I, \quad \varphi_{\mathfrak{S}/W} : \mathcal{O}_I \to \mathcal{O}_{p^{-1}I}.$$

Finally, we define φ as their composites:

$$\varphi := \varphi_W \circ \varphi_{\mathfrak{S}/W} : \mathcal{O}_I \to \mathcal{O}_{p^{-1}I}.$$

In particular, on the ring $\mathcal{O}$, there is a Frobenius map $\varphi : \mathcal{O} \to \mathcal{O}$, in concrete form,

$$\varphi\left(\sum_{n\geq 0} a_n u^n\right) = \sum_{n\geq 0} \varphi(a_n) u^{np}.$$

We define the infinite product

$$\lambda := \prod_{n\geq 0} \varphi^n(E(u)/E(0))$$

λ converges on $D([0, 1))$,which means it is in $\mathcal{O}$ (In fact, if $s(u) \in \mathfrak{S}[1/p] \subset \mathcal{O}$ has constant term 1, then the product $\prod_{n\geq 0} \varphi^n(s)$ converges in $\mathcal{O}$ [Ked04a, Rem 4.5]). Note that λ depends on the choice of uniformizer π.

Besides the Frobenius map, we now define a derivation on $\mathcal{O}$.

$$N_\nabla := -u\lambda \frac{d}{du} : \mathcal{O} \to \mathcal{O}.$$

We also use N_∇ to denote the induced derivation on $\mathcal{O}_I$.

Now we adjoin a formal variable l_u to $\mathcal{O}$ which acts formally like $\log(u)$. We then extend the natural maps $\mathcal{O} \to \widehat{\mathfrak{S}}_n$ to $\mathcal{O}[l_u]$ by sending l_u to

$$\log\left[\left(\frac{u - \pi_n}{\pi_n}\right) + 1\right] := \sum_{i=1}^{\infty} (-1)^{i-1} i^{-1} \left(\frac{u - \pi_n}{\pi_n}\right)^i \in \widehat{\mathfrak{S}}_n.$$

we can also extend φ to $\mathcal{O}[l_u]$ by setting $\varphi_W(l_u) = l_u$ and $\varphi_{\mathfrak{S}/W}(l_u) = pl_u$, then $\varphi(l_u) = pl_u$, and we extend N_∇ to a derivation on $\mathcal{O}[l_u]$ by setting $N_\nabla(l_u) = -\lambda$. We also define $N = \frac{d}{dl_u}$ which acts as differentiation of the formal variable l_u.We then have:

- $N(\varphi(l_u^a)) = N(p^a l_u^a) = ap^a l_u^{a-1}$, so that $N\varphi = p\varphi N$.

- $N_\nabla\varphi = p\frac{E(u)}{E(0)}\varphi N_\nabla$, because

$$\begin{aligned} N_\nabla\varphi(u^a l_u^b) &= N_\nabla(u^{pa}(pl_u)^b) \\ &= p^b(-u\lambda l_u^b pau^{pa-1} - \lambda u^{pa} bl_u^{p-1}) \\ &= p\frac{E(u)}{E(0)}\varphi(\lambda)(au^{pa}(pl_u)^b + bu^{pa}(pl_u)^{b-1}) \\ &= p\frac{E(u)}{E(0)}\varphi(N_\nabla(u^a l_u^b)) \end{aligned}$$

- $NN_\nabla = N_\nabla N$.

Definition 6.2. *A φ-module is a finite dimensional K_0-vector space D together with a bijective Frobenius semilinear map $\varphi : D \to D$. A (φ, N)-module is a φ-module D, euqipped with a K_0-linear map $N : D \to D$ which satisfies $N\varphi = p\varphi N$.*

Remark 6.3. The monodromy operator N is nilpotent. Indeed, if v is an eigenvector $Nv = \lambda v$, then $N\varphi(v) = p\varphi(\lambda v) = p\varphi(\lambda)\varphi(v)$. But $p\varphi(\lambda)$ and λ have different valuations unless $\lambda = 0$. There are finitely many eigenvalues. So all eigenvalues are 0. This means N has to be nilpotent.

Definition 6.4. *A filtered (φ, N)-module (resp. φ-module) is a (φ, N)-module (resp. φ-module) D equipped with a decreasing separated and exhaustive filtration on $D_K = D \otimes_{K_0} K$.*

Definition 6.5. *Newton number and Hodge number.*

If D is a one-dimensional (φ, N)-module, $v \in D$ is a basis vector, then $\varphi(v) = av$ for some $a \in K_0$. Then we define $t_N(D) = v_p(a)$ the p-adic valuation of a. If D has dimension n and a basis $(a_i)_{1\le i\le n}$, then we define $t_N(D) = t_N(\bigwedge^n D)$, where $\bigwedge^n D$ has the underlying vector space generated by $a_1 \wedge a_2 \cdots \wedge a_n$ and $\varphi(a_1 \wedge a_2 \cdots \wedge a_n) = \varphi(a_1) \wedge \varphi(a_2) \cdots \wedge \varphi(a_n)$, $N(a_1 \wedge a_2 \cdots \wedge a_n) = N(a_1) \wedge N(a_2) \cdots \wedge N(a_n)$.

For a one-dimensional filtered (φ, N)-module D, we denote $t_H(D)$ the unique integer i such that $gr^i(D_K)$ non-zero. If D has dimension n, define $t_H(D) = t_H(\bigwedge^n D)$.

Definition 6.6. *A filtered (φ, N)-module D is called weakly admissible if $t_H(D) = t_N(D)$ and for any (φ, N)-submodule $D' \subset D$, $t_H(D') = t_N(D')$, where $D'_K \subset D_K$ is equipped with the subspace filtration. We call a filtered (φ, N)-module effective if $Fil^0 D = D$.*

Now we introduce an important theorem in [CF00b] about crystalline representations and weakly admissible filtered φ-module.

Theorem 6.7 (Colmez-Fontaine)**.** *The functor D_{cris} defined by $D_{cris}(V) = (B_{cris} \otimes V)^{G_K}$ induces an equivalence of categories*

$$Rep^{cris}_{G_K} \to MF^{\varphi,w.a}_K$$

where $MF^{\varphi,w.a}_K$ is the category of weakly admissible filtered φ-modules. The inverse is given by $V_{cris}(D) = Fil^0(B_{cris} \otimes D_K)^{\varphi=1}$.

With this theorem, dealing with a crystalline representation is equivalent to dealing with its associated weakly admissible module.

Definition 6.8. *A φ-module over $\mathcal{O}$ is a finite free $\mathcal{O}$-module $\mathfrak{M}$ with a Frobenius semi-linear, injective map $\varphi : \mathfrak{M} \to \mathfrak{M}$. A (φ, N_∇)-module over $\mathcal{O}$ is a φ-module $\mathfrak{M}$ over $\mathcal{O}$, together with a differential operator $N_\nabla^{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that*

$$N_\nabla^{\mathfrak{M}}(fm) = N_\nabla(f)m + fN_\nabla^{\mathfrak{M}}(m)$$

and

$$N_\nabla^{\mathfrak{M}}\varphi = (p/E(0))E(u)\varphi N_\nabla^{\mathfrak{M}}$$

We will write N_∇ for $N_\nabla^{\mathfrak{M}}$.

We now introduce a useful lemma.

Lemma 6.9. *Let $I \subset [0, 1)$ be an interval, $\mathfrak{M}$ a finite free $\mathcal{O}_I$-module, and $\mathfrak{N} \subset \mathfrak{M}$ an $\mathcal{O}_I$-sumodule. Then the following conditions are equivalent*

(i) *$\mathfrak{N} \subset \mathfrak{M}$ is closed.*
(ii) *$\mathfrak{N}$ is finitely generated.*
(iii) *$\mathfrak{N}$ is finite free.*

Proof. See [Kis, 1.1.4]. □

Remark 6.10. The ring $\mathcal{O}$ is a Bézout domain, i.e. every finitely generated ideal is principle. In general, $\mathcal{O}$ is not noetherian.

Definition 6.11. *A φ-module over $\mathcal{O}$ is of finite E-height, if the cokernel of the $\mathcal{O}$-linear map $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by a power $E(u)^h$ for some integer $h \geq 0$. A (φ, N_∇)-module over $\mathcal{O}$ is of finite height if it is of finite E-height as a φ-module. We use $Mod_{/\mathcal{O}}^{\varphi,N_\nabla}$ (resp. $Mod_{/\mathcal{O}}^{\varphi}$) to denote the category of (φ, N_∇)-modules(resp. φ-modules) over $\mathcal{O}$ of finite height.*

The least integer h in the definition is called the E-height of $\mathfrak{M}$. In the definition, the condition that Coker$(1 \otimes \varphi)$ killed by $E(u)^h$ means that this cokernel is supported on $x_0 \in D([0, 1))$.

Our main goal in this section is to show

Theorem 6.12. *There are exact tensor-compatible functors*

$$\mathcal{M} : MF^{\varphi,N,Fil_{\geq 0}} \rightleftharpoons Mod_{/\mathcal{O}}^{\varphi,N_\nabla} : D$$

and natural isomorphisms of functors :

$$\mathcal{M} \circ D \simeq id, D \circ \mathcal{M} \simeq id$$

$Fil_{\geq 0}$ means being effective.

We first define the functor $\mathcal{M}$. Let D be an effective filtered (φ, N)-module.We define a map

$$\mathcal{O}[l_u] \otimes_{K_0} D \xrightarrow{\varphi_W^{-n} \otimes \varphi^{-n}} \mathcal{O}[l_u] \otimes_{K_0} D \to \widehat{\mathfrak{S}}_n \otimes_{K_0} D = \widehat{\mathfrak{S}}_n \otimes_K D_K.$$

which the second map is deduced from the map $\mathcal{O}[l_u] \to \widehat{\mathfrak{S}}_n$ we defined just before Definition 6.2. Now we look at the map $\varphi_W^{-n} \otimes \varphi^{-n}$. Note that φ_W is bijective. We can see that φ_W^{-n} has a simple zero at each zero of $\varphi_W^{-n} \circ \varphi^n(E(u)/E(0)) = E(u^{p^n})/E(0)$ in $\bar{K}$. So as a function on $D([0, 1))$ it has a simple zero at $x_n \in D([0, 1))$ corresponding to the $Gal(\bar{K}/K_0)$-conjugacy class of a $\pi_n \in \bar{K}$. Since x_n is a simple zero, the element

$\varphi_W^{-n}(\lambda) \in \mathcal{O}$ is mapped to a uniformizer under the map $\mathcal{O}[l_u] \to \widehat{\mathfrak{S}}_n$. So we can extend to the map

$$\iota_n : \mathcal{O}[l_u, 1/\lambda] \otimes_{K_0} D \to \widehat{\mathfrak{S}}_n[1/(u-\pi_n)] \otimes_K D_K.$$

Set

$$\mathcal{M}(D) = \left\{ x \in (\mathcal{O}[l_u, 1/\lambda] \otimes_{K_0} D)^{N=0} : \iota_n(x) \in Fil^0(\widehat{\mathfrak{S}}[1/(u-\pi_n)] \otimes_K D_K), n \geq 0 \right\}$$

Note that N acts on $\mathcal{O}[l_u, 1/\lambda] \otimes_{K_0} D$ by

$$N(a \otimes b) = N(a) \otimes b + a \otimes N(b)$$

Note that $(\mathcal{O}[l_u, 1/\lambda] \otimes_{K_0} D)^{N=0}$ is an $\mathcal{O}$-module. Since $\varphi(1/\lambda) = \frac{E(u)/E(0)}{\lambda}$, φ naturally acts on the ring $\mathcal{O}[1/\lambda]$. So we can give $(\mathcal{O}[l_u, 1/\lambda] \otimes_{K_0} D)$ a Frobenius map by those on D and $\mathcal{O}[l_u, 1/\lambda]$. It is also equipped with a differential operator N_∇, induecd by the operator $N_\nabla \otimes 1$ on $\mathcal{O}[l_u, 1/\lambda] \otimes_{K_0} D$.

Lemma 6.13. *Suppose that D is effective. Then the operators φ and N_∇ on $(\mathcal{O}[l_u, 1/\lambda] \otimes_{K_0} D)^{N=0}$ induce on $\mathcal{M}(D)$ the structure of a (φ, N_∇)-module over $\mathcal{O}$.Moreover, the $\mathcal{O}$-linear map*

$$1 \otimes \varphi : \varphi^*\mathcal{M}(D) \to \mathcal{M}(D)$$

is injective, and has cokernel isomorphic to

$$\oplus_{i \geq 0} (\mathcal{O}/E(u)^i)^{h_i}$$

where $h_i = \dim_K gr^i D_K$.

Proof. see [Kis, 1.2.2]. ☐

In the rest of this section, we define the fucntor $D : Mod_{/\mathcal{O}}^{\varphi, N_\nabla} \to MF_K^{\varphi, N, Fil_{\geq 0}}$ by sending a (φ, N_∇)-module to its fiber at the origin of the disc

$$D(\mathcal{M}) := \mathcal{M}/u\mathcal{M}$$

We equip $D(\mathcal{M})$ with Frobenius and monodromy operators

$$\varphi := \varphi_{\mathcal{M}} \bmod u \text{ and } N := N_\nabla^{\mathcal{M}} \bmod u$$

It is easy to see $D(\mathcal{M})$ is a finite dimensional K_0-vecor space and $N\varphi = p\varphi N$. Another point is φ is bijective. Indeed, $\varphi^*(\mathcal{M}) \to \mathcal{M}$ is injective and its cokernel is killed by some power of $E(u)$, then after modulo u, its cokernel is killed by some power of p. However, p is a unit in K_0. So after modulo u, φ is bijective .

In order to show that $D(\mathcal{M})$ is an object in the category $MF_K^{\varphi, N, Fil_{\geq 0}}$, we need to equip $D(\mathcal{M})_K$ with an effective filtration.

To construct the filtration on $D(\mathcal{M})_K$, we adopt the following notations: if $J \subset I \subset [0,1)$ are intervals, and $\mathcal{M}$ is a finite $\mathcal{O}_I$-module, we write $\mathcal{M}_J = \mathcal{M} \otimes_{\mathcal{O}_I} \mathcal{O}_J$. If we think of $\mathcal{M}$ as a coherent sheaf on $D(I)$, then $\mathcal{M}_J$ corresponds to the restriction of $\mathcal{M}$ to $D(J)$.

Lemma 6.14. *Let $\mathcal{M}$ be a φ-module over $\mathcal{O}$ of finite E-height. There is a unique $\mathcal{O}$-linear, φ-equivariant morphism*

$$\xi : D(\mathcal{M}) \otimes_{K_0} \mathcal{O} \to \mathcal{M}$$

whose reduction modulo u induces the identity on $D(\mathcal{M})$. ξ is injective, and its cokernel is killed by a finie power of λ. If $\pi \in (|\pi|, |\pi|^{1/p})$, then the image of the map $\xi_{[0,r)}$ induced by ξ coincides with the image of $1 \otimes \varphi : (\varphi^*\mathcal{M})_{[0,r)} \to \mathcal{M}_{[0,r)}$.

Proof. Recall that $\mathcal{O}$ is a Fréchet space, with its topology defined by the morms $|\cdot|_r$ for $r \in (0,1)$, given by $|f|_r = sup_{x\in D([0,r])}|f(x)|$. We identity $\mathcal{M}$ with $\mathcal{O}^d$ and define $|\cdot|_r$ on $\mathcal{M}$ by taking the maximum of $|\cdot|$ applied to the coordinates of elements. For a subset $\Sigma \subset \mathcal{M}$, define $|\Sigma|_r = \sup_{x\in\Sigma}|x|_r$.

For the existence of ξ, note that the data of ξ is equivalent to a k_0-linear section $D(\mathcal{M}) \to \mathcal{M}$ to the natural surjection such that it is compatible with Frobenius map. Now choose any K_0-linear section $s_0 : D(\mathcal{M}) \to \mathcal{M}$ of natural map $\mathcal{M} \to D(\mathcal{M})$. We define a new map $s : D(\mathcal{M}) \to \mathcal{M}$ by

$$s = \lim_{n\to\infty} \varphi^n s_o \varphi^{-n} = s_0 + \sum_{i=1}^{\infty}(\varphi^i \circ s_0 \circ \varphi^{-1} - \varphi^{i-1} \circ s_0 \circ \varphi^{1-i})$$

We need to check this map is well defiend. To see the limit indeed converges pointwise, one works on a fixed $r \in (0,1)$and $L \subset D(\mathcal{M})$ a $\mathcal{O}_{K_0}$-lattice. Then $\varphi^{-1}(L) \subset p^{-j}L$ for some non-negative integer j. After increasing j, we may also assume $|\varphi(m)|_r \le |p^{-j}m|_r$ for all $m \in \mathcal{M}$. Since $\varphi\circ s_0\circ\varphi^{-1}-s_0 \in u\mathcal{M}$, we have $\widetilde{L} := u^{-1}(\varphi\circ s_0\circ\varphi^{-1}-s_0)(L) \subset \mathcal{M}$ so that

$$|(\varphi^{i+1} \circ s_0 \circ \varphi^{-i-1} - \varphi^i \circ s_0 \circ \varphi^{-i}(L)|_r \le |p^{-ij}u^{p^i}\varphi^i(\widetilde{L})|_r \le p^{2ij}r^{p^i}|\widetilde{L}|_r.$$

Since $|\widetilde{L}|_r$ is finite, and $p^{2ij}r^{p^i} \to 0$ as $i \to \infty$, for any $j \ge 0$ and $r \in (0,1)$, so the map s is well defined. By construction, s is φ-compatible, so ξ exists.

To prove the uniqueness, given any other such map s', the difference s'_s sends $D(\mathcal{M})$ into $u\mathcal{M}$. But since φ is a bijiection on $D(\mathcal{M})$, and $\varphi^j \circ (s'_s) = (s-s')\circ\varphi^j$, for $j \ge 1$, this means $(s-s')(D(\mathcal{M})) \subset u^{p^j}\mathcal{M}$. Since j is an arbitrary positive integer, $s - s' = 0$. It follows that s is the unique such map. Extending s to $D(\mathcal{M}) \otimes_{K_0} \mathcal{O}$ by $\mathcal{O}$-linearity, yields the required map ξ.

Now we prove the rest of the lemma. Since ξ is an isomorphism modulo u, it follows that for some sufficiently large positive integer i, $\xi_{[0,r)}$ is an isomorphism. Since ξ commutes with φ, we have a commutative diagram

$$\begin{array}{ccc} \varphi^*(D(\mathcal{M})\otimes_{K_0}\mathcal{O}) & \xrightarrow{\varphi^*\xi} & \varphi^*\mathcal{M} \\ \sim\downarrow & & \downarrow 1\otimes\varphi \\ D(\mathcal{M})\otimes_{K_0}\mathcal{O} & \xrightarrow{\xi} & \mathcal{M} \end{array}$$

Since the cokernel of the right vertical mao $1\otimes\varphi$ is killed by E^h, we see that this map is an isomorphism away from the point $\pi \in D([0,1))$; in particular it is an isomorphism over $D([0,r^p))$. Since $\varphi^{-1}(D([0,r^{p^i}))) = D([0,r^{p^{i-1}}))$ and ξ is an isomorphism over $D([0,r^{p^i}))$, we deduce that the top arrow $\varphi^*\xi$ is an isomorphism over $D([0,r^{p^i-1}))$. The right vertical map is also an isomorphism over $D([0,r^{p^{i-1}}))$, so the bottom arrow ξ must be an isomorphism over $D([0,r^{p^i-1}))$. It follows by induction that ξ is an isomorphism over $D([0,r^p))$, hence $\varphi^*\xi$ is an isomorphism over $D([0,r))$, then $\xi_{D([0,r))}$ is injective, so ξ is injective. From the diagram, we can also see that $\xi_{D([0,r))}$ and $1\otimes\varphi_{D([0,1))}$ have the same image. Finally, coker$\xi_{D([0,r))}$ is killed by E^h, it follows from the same commutative diagram that ξ is killed by λ^h.

□

Now we are ready to define the filtration on $D(\mathcal{M})$. We first define a decreasing filtration on $\varphi^*\mathcal{M}$ by

$$Fil^i\varphi^*\mathcal{M} = \{x \in \varphi^*\mathcal{M} : 1 \otimes \varphi(x) \in E(u)^i\mathcal{M}\}.$$

This is a filtration on $\varphi^*\mathcal{M}$ by finite free $\mathcal{O}$-modules since closed $\mathcal{O}$-submodule is finite free by Lemma 6.9. By transport of structure, this defines a filtration on $(1\otimes\varphi)(\varphi^*\mathcal{M})$, hence on $(1 \otimes \varphi)(\varphi^*\mathcal{M})_{[0,r)}$. Using the map $\xi_{[0,r)}$, we obtain a filtration on $(D(\mathcal{M}) \otimes_{K_0} \mathcal{O})_{[0,r)}$. Via the composite map

$$(D(\mathcal{M}) \otimes_{K_0} \mathcal{O})_{[0,r)} \to D(\mathcal{M}) \otimes_{K_0} \mathcal{O}/E(u)\mathcal{O} \xrightarrow{\sim} D(\mathcal{M}) \otimes K_0K = D(\mathcal{M})_K.$$

we get a filtration on $D(\mathcal{M})_K$.

So if $\mathcal{M}$ is a (φ, N_∇)-module over $\mathcal{O}$ of finite E-height, the above discussion gives $D(\mathcal{M})$ a structure of a filtered (φ, N)-module. What left is to show the equivalence of the categories.

Proposition 6.15. *Let D be an effective filtered (φ, N)-module. There is a natural isomorphism of filtered (φ, N)-modules $D(\mathcal{M}(D)) \xrightarrow{\sim} D$.*

Proof. We give a sketch of the proof. The details can be seen in [Kis, Prop 1.2.8]. Set $D_0 = (\mathcal{O}[l_u] \otimes_{K_0} D)^{N=0}$, D_0 is isomorphic to $\mathcal{M}(D)$ at $u = 0$, so that

$$D(\mathcal{M}(D)) = \mathcal{M}(D) \otimes_{\mathcal{O}} \mathcal{O}/u\mathcal{O} \xrightarrow{\sim} (K_0[l_u] \otimes_{K_0} D)^{N=0}.$$

The composite map

$$\eta : (K_0[l_u] \otimes_{K_0} D)^{N=0} \subset K_0[l_u] \otimes_{K_0} D \xrightarrow{l_u \to 0} D.$$

is an isomorphism of filtered (φ, N)-modules, where on the right hand N acts by $-N \otimes 1$. η is an injection between K_0-vector space with same dimension, so it is bijective. Simple calculations show η is also φ-compatible.

It remains to show it is strictly compatible with filtrations. The idea is to prove firstly the filtration on $D(\mathcal{M}(D))_K$ is identified with the given filtration on D_K, using the isomorphisms

$$\gamma : D(\mathcal{M}(D))_K = (K_0[l_u] \otimes_{K_0} D)^{N=0} \otimes_{K_0} \mathcal{O}/E(u)\mathcal{O} = D_0/E(u)D_0$$

$$\xrightarrow{\sim} \widehat{\mathfrak{S}}_0 \otimes_K D_K/(u - \pi_n)\widehat{\mathfrak{S}}_0 \otimes_K D_K = D_K.$$

Then prove the following composite is a natural inclusion

$$D \xrightarrow{\eta^{-1}} D(\mathcal{M}(D)) \hookrightarrow D(\mathcal{M}(D))_K \xrightarrow{\gamma} D_K.$$

□

The inverse statement is also true, see [Kis, Prop 1.2.13].

Proposition 6.16. *Given $\mathcal{M} \in Mod_{/\mathcal{O}}^{\varphi,N_\nabla}$, there is a canonical isomorphism*

$$\mathcal{M}(D(\mathcal{M})) \xrightarrow{\sim} \mathcal{M}.$$

Combining Proposition 6.15 and 6.16 proves Theorem 6.12.

6.3. **Slope and weak admissiblity.** In this section, we will recall Kedlaya's theory of slopes and apply it to translate the condition of weak admissibility. We will not focus on the proofs of theorems. One can refer to [Ked04b] and [Ked05] for more details.

Recall some definitions of ring of functions

Definition 6.17. (i) *The Robba ring is defined by*

$$\mathcal{R} := \varinjlim_{r \to 1^-} \mathcal{O}_{(r,1)}$$

$\mathcal{R}$ is equipped with a Frobenius endomorphism φ induced by the maps $\varphi : \mathcal{O}_{(r,1)} \to \mathcal{O}(r^{1/p}, 1)$.

(ii) *The bounded Robba ring is defined by*

$$\mathcal{R}^b := \varinjlim_{r \to 1^-} \mathcal{O}^b_{(r,1)}$$

where $\mathcal{O}^b_{(r,1)} \subset \mathcal{O}_{(r,1)}$ is the set of bounded functions on $D((r,1))$. The Frobenius on $\mathcal{R}$ induces a Frobenius on R^b.

We also define

$$\mathcal{R}^{int} := \left\{ \sum_{n \in \mathbb{Z}} a_n u^n \in \mathcal{R} \middle| a_n \in W(k), \forall n \in \mathbb{Z} \right\};$$

This is a henselian discrete valuation ring with uniformizer p ([Ked04b, lemma 3.9]). In particular, $R^b = Frac(\mathcal{R}^{int})$. So $\mathcal{R}^b$ is a discrete valuation field. Moreover, the nonzero elements of $\mathcal{R}^b$ are precisely the units of $\mathcal{R}$.

Remark 6.18. As $E(u)$ is a polynomial with $W(k)$-coefficients, so $E(u) \in \mathcal{R}^{int} \subseteq \mathcal{R}^b$. Also, $E(u)/p$ is in $\mathcal{R}^b$ but not in $\mathcal{R}^{int}$, so $E(u) \in (R^{int})^{\times}$.

Definition 6.19. *We denote by $Mod^{\varphi}_{/\mathcal{R}^b}$ the catgeory of finite dimensional $\mathcal{R}^b$-vector space $\mathcal{M}$ equipped with an isomorphism $\varphi^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$. Morphisms in $Mod^{\varphi}_{/\mathcal{R}^b}$ are φ-compatible morphisms of $\mathcal{R}^b$-modules.*

Theorem 6.20 (Kedlaya)**.** *There exists an $\mathcal{R}$-algebra $\mathcal{R}^{alg}$, which contains a copy of $W(\bar{k})$, where $\bar{k}$ denotes an algebraic closure of k. $\mathcal{R}^{alg}$ is equpped with a lifting φ of Frobenius on $\mathcal{R}$. For any $\mathcal{M} \in Mod^{\varphi}_{/\mathcal{R}}$, there exists a finite extension E of $W(\bar{k})[1/p]$ such that $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}^{alg} \otimes_{W(\bar{k})[1/p]} E$ admits a basis of φ-eigenvectors $v_1, \cdots, v_n$ such that $\varphi(v_i) = \alpha_i(v_i)$ for some $\alpha_i \in E$. The set of p-adic vaulations of $\alpha_1, \cdots, \alpha_n$ is uniquely determined by $\mathcal{M}$ called the set of slopes of $\mathcal{M}$. If all p-adic valuations of α_i are equal to some $s \in \mathbb{Q}$, then $\mathcal{M}$ is called pure of slope s.*

We denote by $Mod^{\varphi,s}_{/\mathcal{R}^b}$ the full subcategory of $Mod^{\varphi}_{/\mathcal{R}^b}$ which consists of elements of pure slope s. The following important theorem in[Ked06, 1.7.1] elucidates the struture of $\mathcal{R}$-modules.

Theorem 6.21. (i) *The functor $\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{R}^b} \mathcal{R}$ induces an equivalence*

$$Mod^{\varphi,s}_{/\mathcal{R}^b} \to Mod^{\varphi,s}_{/\mathcal{R}}.$$

(ii) *For any $\mathcal{M}$ in $Mod^{\varphi,s}_{/\mathcal{R}}$, there exists a canonical filtration called the slope filtration*

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_r = \mathcal{M}$$

by φ-stable submodules such that $\mathcal{M}_i/\mathcal{M}_{i+1}$ is finite free over $\mathcal{R}$ and pure of slope s_i, and $s_1 < s_2 < \cdots s_r$.

Remark 6.22. If $\mathcal{M} = \mathcal{R} \cdot x \in Mod^{\varphi}_{\mathcal{R}}$, $\varphi(x) = \alpha x$, then $\mathcal{M}$ is pure of slope $v_p(\alpha)$.

Given a nonzero object in $Mod^{\varphi,N_\nabla}_{/\mathcal{O}}$, then $(\mathcal{M}_{\mathcal{R}}, \varphi)$ is a nonzero object in $Mod^{\varphi}_{/\mathcal{R}}$. The question here is if the slope filtration on $(\mathcal{M}_{\mathcal{R}}, \varphi)$ can be induced by a filtration on $\mathcal{M}$.

Definition 6.23. *Let $\mathcal{M}$ be a finite free $\mathcal{R}$-module (resp. $\mathcal{O}_I$-module for some interval $I \subset [0, 1)$) , an $\mathcal{R}$(resp.$\mathcal{O}_I$) submodule $\mathcal{N} \subset \mathcal{M}$ is saturated if it is finitely generated and if $\mathcal{M}/\mathcal{N}$ is torsion free, euqivalently, free over $\mathcal{R}$ (resp. $\mathcal{O}_I$).The saturation of $\mathcal{N}$ is the smallest saturated submodule containing $\mathcal{N}$.*

Proposition 6.24. *Let $\mathcal{M}$ be in $Mod^{\varphi,N_\nabla}_{/\mathcal{O}}$ and set $\mathcal{M}_{\mathcal{R}} = \mathcal{M} \otimes \mathcal{R}$. If*

$$0 = \mathcal{M}_{0,\mathcal{R}} \subset \mathcal{M}_{1,\mathcal{R}} \subset \cdots \mathcal{M}_{r,\mathcal{R}} = \mathcal{M}_{\mathcal{R}}$$

denotes the slope filtration of $\mathcal{M}_{\mathcal{R}}$ then for $i = 0, 1, \cdots, r$, $\mathcal{M}_{i,\mathcal{R}}$ extends uniquely to a saturated $\mathcal{O}$-submodule $\mathcal{M}_i \subset \mathcal{M}$ which is stable by φ and N_∇.

We are now able to give a translation of the condition of weak admissibility.

Theorem 6.25. *Let D be an effective filtered (φ, N)-module. Then D is weakly admissible if and only if $\mathcal{M}(D)$ is pure of slope 0.*

Proof. We first deal with the case that D has rank 1. Choose a basis $e \in D$ and $\varphi(e) = ae$ for some $a \in K_0$. Then we have $\mathcal{M}(D) = \lambda^{-t_H(D)}(\mathcal{O}[l_u] \otimes_{K_0} D)^{N=0}$, so that

$$\varphi(\lambda^{t_H(D)} e) = (E(u)/E(0))^{t_H(D)} a \lambda^{-t_H(D)} e.$$

Since $E(0)$ has p-adic valuation 1 and $E(u)$ is a unit in $\mathcal{R}^{int}$, $\mathcal{M}(D)$ has slope $t_N(D) - t_H(D)$. Then this proves the theorem for the case that D is of rank 1.

Now we consider the general case. Suppose D is weakly admissible. Then by Prop. 6.25, the slope filtration on $(D)_{\mathcal{R}}$ is induced by a filtration of $\mathcal{M}(D)$ by (φ, N_∇) -submodules

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \cdots \subset \mathcal{M}_r = \mathcal{M}(D).$$

$\mathcal{M}_{i,\mathcal{R}}/\mathcal{M}_{i+1,\mathcal{M}}$ has the slope s_i and $\mathcal{R}$-rank d_i. So $\bigwedge^{d_1} \mathcal{M}_1$ has slope $d_1 s_1$ by [Ked04b, Prop.5.13]. By Prop 6.15, there exsits a filtered (φ, N_∇) -submodule such that $\mathcal{M}(D_1) = \mathcal{M}_1$. Since $\mathcal{M}(\bigwedge^{d_1} D_1) \simeq \bigwedge^{d_1} \mathcal{M}(D_1)$, we have $s_1 d_1 = t_N(D_1) - t_H(D_1)$. Then the weak admissibility shows that $s_1 \geq 0$. So all $s_i \geq 0$. Note that $det(\mathcal{M}(D)) \simeq \bigwedge(det(\mathcal{M}_i/\mathcal{M}_{i+1})$ then $det(\mathcal{M}(D))$ has slope $\sum s_i d_i$. Then $\sum s_i d_i = t_N(D) - t_H(D) = 0$, so that $r = 1$ and $s_1 = 0$.

Conversely, if $\mathcal{M}(D)$ is pure of slope 0, then $t_N(D) = t_H(D)$. If $D' \subset D$ is a (φ, N)-submodule, then $\mathcal{M}(D') \subset \mathcal{M}(D)$ has all slopes ≥ 0 by [Ked04b, Prop.4.4]. Then the slope of the determinant of $\mathcal{M}(D')$ is ≥ 0. Hence $t_N(D') - t_H(D') \geq 0$. So D is weak admissible.

□

6.4. **The final fully faithful functor.** In this subsection, we give descriptions about (φ, N)-modules over $\mathfrak{S}$ of finite E-height and construct the fully faithful functor from $MF_K^{\varphi,N,Fil\geq 0,w.a}$ to $Mod_{/\mathfrak{S}}^{\varphi,N} \otimes \mathbb{Q}_p$.

Definition 6.26. *Let $Mod_{/\mathcal{O}}^{\varphi,N}$ be the category whose objects are φ-modules $\mathcal{M}$ with a K_0-linear map $N : \mathcal{M}/u\mathcal{M} \to \mathcal{M}/u\mathcal{M}$ which satisfies $N\varphi = p\varphi N$, and also it is of finite E-height as a φ-module over $\mathcal{O}$. Let $Mod_{/\mathcal{O}}^{\varphi,N,0}$ be the full subcategory consisting of modules pure of slope 0.*

Define a functor from $Mod_{/\mathcal{O}}^{\varphi,N_\nabla} \to Mod_{/\mathcal{O}}^{\varphi,N}$ by sending $\mathcal{M}$ to $\widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}}$ has underlying module $\mathcal{M}$ equipped with the operator φ and N the reduction of N_∇ modulo u.

Lemma 6.27. *The functor defined above is fully faithful.*

By now, we have the composite functor

$$MF_K^{\varphi,N,Fil\geq 0,w.a} \xrightarrow{\mathcal{M}} Mod_{/\mathcal{O}}^{\varphi,N_\nabla,0} \hookrightarrow Mod_{/\mathcal{O}}^{\varphi,N,0}$$

This is a fully faithful functor because of Theorem 6.12 and Proposition 6.27.

Definition 6.28. *Define a (φ, N)-module over $\mathfrak{S}$ to be a finite free $\mathfrak{S}$-module $\mathfrak{M}$ equipped with*

(i) *a φ-semi-linear Frobenius $\varphi : \mathfrak{M} \to \mathfrak{M}$*
(ii) *a linear endomorphism $N : \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ such that $N\varphi = p\varphi N$ on $\mathfrak{M}/u\mathfrak{M}$.*

We say that $\mathfrak{M}$ is of finite E-height if the cokernel of $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by some power of E. We denote by $Mod_{/\mathfrak{S}}^{\varphi,N}$ the category of (φ, N)-modules over $\mathfrak{S}$ of finite E-height.

Lemma 6.29. *The functor*

$$\Theta : Mod_{/\mathfrak{S}}^{\varphi,N} \otimes \mathbb{Q}_p \to Mod_{/\mathcal{O}}^{\varphi,N,0} \qquad \mathfrak{M} \to \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$$

is an equivalence of categories.

Proof. Let $\mathcal{M}$ be in $Mod_{/\mathcal{O}}^{\varphi,N,0}$, then $\mathcal{M}_{\mathcal{R}} := \mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}$ is in $Mod_{/\mathcal{R}}^{\varphi,0}$. Then there exists some $\mathcal{M}_{\mathcal{R}^b}$ in $Mod_{/\mathcal{R}^b}^{\varphi,0}$ such that $\mathcal{M} = \mathcal{M}_{\mathcal{R}^b} \otimes_{\mathcal{R}^b} \mathcal{R}$. So we have isomorphisms

$$\mathcal{M}_{\mathcal{R}^b} \otimes_{\mathcal{R}^b} \mathcal{R} \xrightarrow{\sim} \mathcal{M}_{\mathcal{R}} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}.$$

By [Ked04b, Prop.6.5], after choosing suitable $\mathcal{R}^b$-basis for $\mathcal{M}_{\mathcal{R}^b}$ and $\mathcal{O}$-basis for $\mathcal{M}$, the corresponding matrix of the composite isomorphism is the identity. This means as subsets of $\mathcal{M}_{\mathcal{R}}$, $\mathcal{M}_{\mathcal{R}^b}$ and $\mathcal{M}$ are spanned by a common basis. Note that $\mathfrak{S}[1/p] = \mathcal{O} \cap \mathcal{R}^b \subset \mathcal{R}$, let $\mathcal{M}^b$ be the $\mathfrak{S}[1/p]$ -span of the basis, then we have $\mathcal{M}^b = \mathcal{M}_{\mathcal{R}^b} \cap \mathcal{M} \subset \mathcal{M}_{\mathcal{R}}$. So $\mathcal{M}^b$ is φ-stable and of finite E-height, since $\mathcal{M}$ is. Now given any $\mathcal{N}$ in $Mod_{/\mathfrak{S}}^{\varphi,N}$, $\mathcal{N} \otimes \mathbb{Q}_p$ can be recovered as $\Theta(\mathcal{N})_{\mathcal{R}^b} \cap \Theta(\mathcal{N})$. This means we can recover $\mathcal{N}$ up to p-isogeny from $\Theta(\mathcal{N})$ (which gives a quasi-inverse functor on the essential image of Θ). So Θ is in fact fully faithful.

To prove Θ is an equivalence, we need to prove it is essentially surjective, i.e $\mathcal{M}^b$ arises from an object of $Mod_{/\mathfrak{S}}^{\varphi,N}$. We will give the corresponding construction, for the details of the proof, see [Kis, Lemma 1.3.13].

Since $\mathcal{M}_{\mathcal{R}^b}$ is pure of slope 0, there exists a φ-stable $\mathcal{R}^{int}$-lattice $\mathcal{L}$ in $\mathcal{M}_{\mathcal{R}^b}$. Let $\mathfrak{M}' = \mathcal{M}^b \cap \mathcal{L}$, and set

$$\mathfrak{M} = \mathcal{R}^{int} \otimes_{\mathfrak{S}} \mathfrak{M}' \cap \mathfrak{M}'[1/p] \subset \mathcal{M}_{\mathcal{R}^b}.$$

Then $\mathcal{M}^b = \mathfrak{M}[1/p]$.

□

Corollary 6.30. *There exists a fully faithful functor from the category $MF_K^{\varphi,N,Fil\geq 0,w.a}$ to the category $Mod_{/\mathfrak{S}}^{\varphi,N} \otimes \mathbb{Q}_p$.*

Proof. The functor is the composite

$$MF_K^{\varphi,N,Fil\geq 0,w.a} \xrightarrow{\mathcal{M}} Mod_{/\mathcal{O}}^{\varphi,N_\nabla,0} \xrightarrow{Prop\ 6.27} Mod_{/\mathcal{O}}^{\varphi,N,0} \xrightarrow{lemma\ 6.29} Mod_{/\mathfrak{S}}^{\varphi,N}.$$

□

6.5. **$\mathfrak{S}$-modules and crystalline representations.** Recall from Section 3.2 that for a p-adic field K with residue field k, there is a perfect ring $\mathcal{O}^\flat$ of characteristic p and its Witt ring is denoted by $\mathbb{A}_{inf} = W(\mathcal{O}^\flat)$. There is a surjective ring homomorphism $\theta : \mathbb{A}_{inf} \to \mathcal{O}_C$ where $C = \widehat{\overline{K}}$. Write $\underline{\pi} = (\pi_n)_{n\geq 0} \in \mathcal{O}^\flat$. Let $[\underline{\pi}] \in \mathbb{A}_{inf}$ be the Teichmuller lift of $\underline{\pi}$. By sending u to $[\underline{\pi}]$, we have an embedding $W[u] \hookrightarrow \mathbb{A}_{inf}$. Indeed, if the image of $f(u) = \sum_{i=0}^m a_i u^i$ is 0, i.e. $\sum_{i=0}^m a_i [\underline{\pi}]^n = 0$, noting that $\theta([\underline{\pi}] = \pi)$, we have $\sum_{i=0}^m a_i \pi^i = 0 \in \mathcal{O}_C$. Then $f(u) = E(u)F(u) \in W[u]$ since $E(u)$ is the Eisenstein polynomial of π. In particular, we may assume $f(u)$ is of the minimal degree in the kernel. But $E([\underline{\pi}]) \neq 0 \in \mathbb{A}_{inf}$. Since $\mathbb{A}_{inf}$ is an integral domain, $F(u)$ is also in the kernel but with smaller degree. This contradicts to our assumption. So $W[u] \to \mathbb{A}_{inf}$ is indeed an embedding. Furthermore, we use the product of valuation topology of $\mathcal{O}^\flat$ on $\mathbb{A}_{inf}$. Then this embedding can be extended to $\mathfrak{S} \to \mathbb{A}_{inf}$. In fact $[\underline{\pi}]^n = [\underline{\pi}^n]$ and $(\underline{\pi}^n)^{(0)} = \pi^n$. So we have a natural map

$$\mathfrak{S} = W[[u]] \hookrightarrow \mathbb{A}_{inf}$$
$$\sum_{n\geq 0} a_n u^n \to \sum_{n\geq 0} a_n [\underline{\pi}]^n.$$

$\theta_{\mathfrak{S}}$ is the map $\mathfrak{S} \to \mathcal{O}_K$ sending u to π. This embedding is compatible with Frobenius endomorphism. Since $\underline{\pi} \in \operatorname{Frac} \mathcal{O}^\flat$ is nonzero, $[\underline{\pi}] \in W(\operatorname{Frac} \mathcal{O}^\flat)^\times$. So we naturally have an embedding

$$\mathfrak{S}[1/u] \hookrightarrow W(\operatorname{Frac}(\mathcal{O}^\flat)).$$

Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$ just as we did in Example 5.11 and write $\mathcal{E} = \operatorname{Frac}(\mathcal{O}_{\mathcal{E}})$. $\mathcal{O}_{\mathcal{E}}$ is a complete discrete valuation ring with residue field $k((u))$ and p is a uniformizer. It has a Frobenius endomorphism induced the $\varphi_{\mathfrak{S}}$ on $\mathfrak{S}$. In fact, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{E}} & \longrightarrow & W(\operatorname{Frac}(\mathcal{O}^\flat)) \\ \downarrow & & \downarrow \\ k((u)) & \longrightarrow & \operatorname{Frac}(\mathcal{O}^\flat) \end{array}$$

Recall Theorem 3.7 that $\operatorname{Frac}(\mathcal{O}^\flat)$ is algebraically closed. So there is a maximal unramifed extenison $\mathcal{O}_{\mathcal{E}^{ur}}/\mathcal{O}_{\mathcal{E}}$ with respect to a separable closure of $k((u))$ in $\operatorname{Frac}(\mathcal{O}^\flat)$. Write $\mathcal{E}^{ur} = \operatorname{Frac}(\mathcal{O}_{\mathcal{E}^{ur}}) \subset W(\operatorname{Frac}(\mathcal{O}^\flat))[1/p]$. We also denote by $\widehat{\mathcal{E}^{ur}}$ the p-adic clsoure of $\mathcal{E}^{ur}$ in $W(\operatorname{Frac}(\mathcal{O}^\flat))[1/p]$ and by $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ its ring of integers.

Write $G_{K_\infty} = Gal(\bar{K}/K_\infty)$. Note that G_{K_∞} preserves the separable closure of $k((u))$ in Frac($\mathcal{O}^\flat$) (see [BC09, P56]), then via the universal property of maximal unramified extension, G_{K_∞} acts on $\mathcal{O}_{\mathcal{E}^{ur}}$. There is an important theorem allowing us to study G_{K_∞} more conveniently.

Theorem 6.31. *The action of G_{K_∞} on $\mathcal{O}_{\mathcal{E}^{ur}}$ induces an isomorphism of topological groups*

$$G_{K_\infty} \xrightarrow{\simeq} Aut(\mathcal{O}_{\mathcal{E}^{ur}}/\mathcal{O}_{\mathcal{E}}) \simeq G_{k((u))}.$$

Recall the theory of étale φ-modules over $\mathcal{O}_{\mathcal{E}}$ in Section 5, we have an equivalence between the category of $Rep_{\mathbb{Z}_p}(G_{K_\infty})$ and the catgeory $\Phi M^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}$. In particular, the full subcategory of finite free $\mathbb{Z}_p$ representations and the full category of finite free *é*tale φ-modules are equivalent and so are the full subcategories of torison objects. We want to adapt the theory of *é*tale φ-modules to study $\mathfrak{S}$-modules.

Define $\mathfrak{S}^{ur} := \mathcal{O}_{\mathcal{E}^{ur}} \cap \mathbb{A}_{inf} \subset W(Frac(\mathcal{O}^\flat))$ and $\widehat{\mathfrak{S}^{ur}}$ to be its p-adic completion. Since G_{K_∞} acts on $\mathcal{O}_{\mathcal{E}^{ur}}$, it acts on $\mathfrak{S}^{ur}$ and $\mathcal{E}^{ur}$.

Lemma 6.32. *Let $\mathfrak{M}$ be a fnitely generated $\mathfrak{S}$-module equipped with a $\mathfrak{S}$-linear endomorphism $\varphi^*\mathfrak{M} \to \mathfrak{M}$. Suppose $\mathfrak{M}$ is isomorphic as a $\mathfrak{S}$-module to a finite direct sum $\oplus_{i\in I}\mathfrak{S}/p^{n_i}\mathfrak{S}$ where $n_i \in \mathbb{N}^+$ and that* Coker$(1\otimes\varphi)$ *is killed by some power of $E(u)$. Then*

(i) *The association $\mathfrak{M} \to Hom_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{ur}[1/p]/\mathfrak{S}^{ur})$ is an exact functor in $\mathfrak{M}$.*
(ii) *The natural map*

$$Hom_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{ur}[1/p]/\mathfrak{S}^{ur}) \to Hom_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathscr{E}^{ur}/\mathscr{O}_{\mathscr{E}^{ur}})$$

is an isomorphism, and both sides are isomorphic to $\oplus_{i\in I}\mathbb{Z}_p/p^{n_i}\mathbb{Z}_p$ as $\mathbb{Z}_p$-modules.

Proof. See [Fon07, A.1.2 and B.1.8.4]. □

Definition 6.33. *We denote by $Mod^{\varphi}_{/\mathfrak{S}}$ the category of finite free $\mathfrak{S}$-modules equipped with a $\mathfrak{S}$-linear map $1\otimes\varphi : \varphi^*\mathfrak{M} \to \mathfrak{M}$ whose cokernel is killed by some power of $E(u)$.*

By passing to inverse limit, we have the corollary

Corollary 6.34. *Let $\mathfrak{M}$ be in $Mod^{\varphi}_{/\mathfrak{S}}$. Then*

$$V_{\mathfrak{S}}(\mathfrak{M}) := Hom_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}^{ur}})$$

is a free $\mathbb{Z}_p$-module of rank $rk_{\mathfrak{S}}\mathfrak{M}$, and the functor $\mathfrak{M} \to V_{\mathfrak{S}}(\mathfrak{M})$ is an exact functor in $\mathfrak{M}$. Moreover, the natural map

$$Hom_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}^{ur}}) \to Hom_{\mathcal{O}_{\mathcal{E}},\varphi}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}, \mathcal{O}_{\widehat{\mathcal{E}^{ur}}})$$

is a bijection.

Since $E(u) \cong u^e \mod p$, $e = [K : K_0]$, $E(u)$ is a unit in $\mathcal{O}_{\mathcal{E}}$ which is a discrete valuation ring with uniformizer p. Hence we have a functor

$$Mod^{\varphi}_{/\mathfrak{S}} \to Mod^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$$
$$\mathfrak{M} \to \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$$

where $Mod^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$ is the full subcategory of $\Phi M^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}$ consisting of finite free $\mathcal{O}_{\mathcal{E}}$-modules. Note that the natural map $\mathfrak{S} \to \mathcal{O}_{\mathcal{E}}$ factors through the map $\mathfrak{S}_{(p)} \to \mathcal{O}_{\mathcal{E}}$. The latter map is a local extension of discrete valuation ring, so it is faithfully flat. As the localization map is flat, $\mathfrak{S} \to \mathcal{O}_{\mathcal{E}}$ is also flat. So the functor $- \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is actually exact. In fact, we want to prove this functor is fully faitful. But we need some lemmas before that.

Lemma 6.35. *Let $h : \mathfrak{M} \to \mathfrak{M}'$ be a morphism in $Mod^{\varphi}_{/\mathfrak{S}}$ which becomes an isomorphism after tensoring by $\mathcal{O}_{\mathcal{E}}$, then h is an isomorphism.*

Proof. Since h is an isomorphism after tensoring $\mathcal{O}_{\mathcal{E}}$, $\mathfrak{M}$ and $\mathfrak{M}'$ have the same rank as $\mathfrak{S}$-modules. So h is an isomorphism if and only if its determinant is an isomorphism. Now it suffices to prove the result for the rank 1 case.

Let $\mathcal{M} = \mathfrak{M} \otimes \mathcal{O}$ and $\mathcal{M}' = \mathfrak{M}' \otimes \mathcal{O}$. By [Kis, 1.3.10 and 1.3.13] $\mathcal{M}$ and $\mathcal{M}'$ are both in the category $Mod^{\varphi, N_{\nabla}}_{/\mathcal{O}}$. Let $D = D(\mathcal{M})$ and $D' = D(\mathcal{M}')$, since we have the equivalence of categories by Theorem 6.12

$$MF_K^{\varphi, N, Fil \geq 0} \xrightarrow{\simeq} Mod^{\varphi, N_{\nabla}}_{/\mathcal{O}}$$

we have a nonzero map $h' : D \to D'$. Since they are of dimension 1, h' is an isomorphism. By the fully faithful functor in Corollary 6.30

$$MF_K^{\varphi, N, Fil \geq 0, w.a} \to Mod^{\varphi, N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$$

We see that h becomes an isomorphim after inverting p. So h has to be the multiplication of p^r for some non-negative integer r and a unit. As h becomes an isomorphism after tensoring by $\mathcal{O}_{\mathcal{E}}$, r must be 0. Then h is an isomorphism. □

Lemma 6.36. *Let $\mathfrak{M}$ be in $Mod^{\varphi}_{/\mathfrak{S}}$ with $\mathfrak{S}$-rank d, define*

$$\mathfrak{M}' := Hom_{\mathbb{Z}_p[G_{K_\infty}]}(V_{\mathfrak{S}}(\mathfrak{M}), \widehat{\mathfrak{S}^{ur}})$$

then $\mathfrak{M}'$ is a finite free $\mathfrak{S}$-module of rank d and the natural map $\mathfrak{M} \to \mathfrak{M}'$ is injective.

Proposition 6.37. *The functor $- \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} : Mod^{\varphi}_{/\mathfrak{S}} \to Mod^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$ is fully faithful.*

Proof. We first give an important construction. Let $\mathcal{M}$ be in $Mod^{\varphi}_{\mathcal{O}_{\mathcal{E}}}$. If $\mathfrak{M} \subset \mathcal{M}$ is a finitely generated $\mathfrak{S}$-module which is φ-stable and such that $\mathfrak{M}/\varphi^*\mathfrak{M}$ is killed by some power of $E(u)$. Define $F(\mathfrak{M}) := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$. According to [Fon07, B1.2.4] $F(\mathfrak{M})$ is a finite free $\mathfrak{S}$-module and is naturally a submodule of $\mathcal{M}$, which contains $\mathfrak{M}$. In particular, $F(\mathfrak{M})$ is in the category $Mod^{\varphi}_{\mathfrak{S}}$.

We need to prove for any objects $\mathfrak{M}_1, \mathfrak{M}_2$ in $Mod^{\varphi}_{\mathfrak{S}}$, $\mathcal{M}_1 = \mathfrak{M}_1 \otimes \mathcal{O}_{\mathcal{E}}$, $\mathcal{M}_2 = \mathfrak{M}_2 \otimes \mathcal{O}_{\mathcal{E}}$

$$Hom_{\mathfrak{S}, \varphi}(\mathfrak{M}_1, \mathfrak{M}_2) \to Hom_{\mathcal{O}_{\mathcal{E}}, \varphi}(\mathcal{M}_1, \mathcal{M}_2)$$

is bijective.

Suppose $h : \mathcal{M}_1 \to \mathcal{M}_2$ is a morphism in $Mod^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$. We have to show this induces a map $\mathfrak{M}_1 \to \mathfrak{M}_2$.

We first consider the case that h is the identity map. Then $V_{\mathfrak{S}}(\mathfrak{M}_1) = V_{\mathfrak{S}}(\mathfrak{M}_2)$, so $\mathfrak{M}_1, \mathfrak{M}_2$ are both contained in $Hom_{\mathbb{Z}_p[G_{K_\infty}]}(V_{\mathfrak{S}}(\mathfrak{M}_1), \widehat{\mathfrak{S}^{ur}})$ which is a finite $\mathfrak{S}$-module of rank $d = rk_{\mathcal{O}_{\mathcal{E}}}(\mathcal{M}_1)$. We write $\mathfrak{M}_3 := \mathfrak{M}_1 + \mathfrak{M}_2$. Then $\mathfrak{M}_3$ is a finite $\mathfrak{S}$-module of rank d and it is also φ-stable and of finite E-height since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are. Since $F(\mathfrak{M}_3) \otimes \mathcal{O}_{\mathcal{E}} \simeq \mathcal{M}_1$, the morphisms $\mathfrak{M}_1 \to F(\mathfrak{M}_3)$ and $\mathfrak{M}_2 \to F(\mathfrak{M}_3)$ are both isomorphisms via the lemma.

In the general case, let $\mathcal{M}_3 = h(\mathfrak{M}_1)$ and $\mathfrak{M}_3 = h(\mathfrak{M}_1)$ (note that the natural map $\mathfrak{M}_1 \to \mathcal{M}_1$ is injective). Write $\mathfrak{M}_3' = \mathcal{M}_3 \cap \mathfrak{M}_2$. $\mathfrak{M}_3$ and $\mathfrak{M}_3'$ are φ-stable. Since $\mathfrak{M}_3 \subset \mathcal{M}_2$, $\mathfrak{M}_3$ is of finite E-height. As for $\mathfrak{M}_3'$, consider the exact sequence

$$0 \to \mathfrak{M}_3' \to \mathcal{M}_3 \oplus \mathfrak{M}_2 \to \mathcal{M}_2$$

the second map is defined by $m \to m \oplus m$ and the third map is defined by $m_1 \oplus m_2 \to m_1 - m_2$. Note that $1 \otimes \varphi$ is injective on each term of this sequence and $\mathcal{M}_3$ is in the category of $Mod^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$. The cokernel of $1 \otimes \varphi$ on $\mathfrak{M}_3'$ may be considered as a $\mathfrak{S}$-submodule of the cokernel of $1 \otimes \varphi$ on $\mathfrak{M}_2$. We have $F(\mathfrak{M}_3) = F(\mathfrak{M}_3')$. Then we define the composite map

$$\mathfrak{M}_1 \to F(\mathfrak{M}_3) = F(\mathfrak{M}_3') \to F(\mathfrak{M}_2) = \mathfrak{M}_2.$$

□

Proposition 6.38. *Let D be an effective, weakly admissible filtered φ-module, and $\mathfrak{M}$ in $Mod^{\varphi}_{/\mathfrak{S}}$ a module whose image in $Mod^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$ is equal to the image of D under the fully faithful functor $Mod^{\varphi, Fil \geq 0.w.a}_{\mathfrak{S}} \to Mod^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$. Then there exists a canonical bijection*

$$V_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\simeq} V^*_{cris}(D) = Hom_{Fil,\varphi}(D, B_{cris}).$$

which is compatible with the action of G_{K_∞} on the two sides.

Proof. See [BC09, Prop11.3.3], for a more general case, see [Kis, Prop2.1.5]. □

Now we can prove the conjecture given by Breuil in [Bre02]

Corollary 6.39. *The functor $Rep^{cris}_{G_K} \to Rep_{G_{K_\infty}}$ defined by restricting the action of a G_K-representation to G_{K_∞} is fully faithful.*

Proof. It suffices to prove the corollary for the full subcategory $Rep^{cris,+}_{G_K}$ consisting of crystalline representations with non-negative Hodge-Tate weights (otherwise we can twist by $\mathbb{Q}_p(n)$ for large enough n.

Consider the following diagram

$$\begin{array}{ccc} Rep^{cris,+}_{G_K} & \longrightarrow & Rep_{G_{K_\infty}} \\ \downarrow & & \uparrow \simeq \\ Mod^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p & \longrightarrow & Mod^{\varphi}_{/\mathcal{O}_{\mathcal{E}}} \otimes \mathbb{Q}_p \end{array}$$

The left side functor is the composite $Rep^{cris,+}_{G_K} \xrightarrow{\simeq} MF^{\varphi,Fil\geq 0,w.a}_{K} \xrightarrow{fully\ faithful} Mod^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$. So it is also fully faithful. The functor in the bottom is fully faithful via the Lemma 6.37. The functor on the right sends $\mathcal{M}$ in $Mod^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$ to $Hom_{\mathcal{O}_{\mathcal{E}}}(\mathcal{M}, \widehat{\mathcal{E}^{ur}})$ and it is an equivalence according to the Theorem 5.15. So it rests to prove this diagram is commutative. In fact, by Proposition 6.39 and Corollary 6.35, the following diagram

$$\begin{array}{ccccc} MF^{\varphi,w.a,Fil\geq 0}_{K} & \xrightarrow{V^*_{cris}} & Rep^{cris}_{G_K} & \longrightarrow & Rep_{G_{K_\infty}} \\ \downarrow & & & & \uparrow \\ Mod^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p & & \longrightarrow & & Mod^{\varphi}_{\mathcal{O}_{\mathcal{E}}} \otimes \mathbb{Q}_p \end{array}$$

is commutative. The commutativity of the first diagram follows.

□

Now by using $\mathfrak{S}$-modules, we can describe G_{K_∞}-stable $\mathbb{Z}_p$-lattices in p-adic G_{K_∞}-representations.

Lemma 6.40. *Let* $\mathfrak{M}$ *be in* $Mod^{\varphi}_{/\mathfrak{S}}$ *and* $V = V_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. *Set* $\mathcal{M} = \mathcal{E} \otimes \mathfrak{M}$. *Then the map*

$$\mathfrak{N} \to V_{\mathfrak{S}}(\mathfrak{N}) = Hom_{\mathfrak{S},\varphi}(\mathfrak{N}, \widehat{\mathfrak{S}^{ur}})$$

is a bijection (up to isomorphism) between finite free φ*-stable* $\mathfrak{S}$*-submodules* $\mathfrak{N} \subset \mathcal{M}$ *such that* $\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{N} \simeq \mathcal{M}$ *and* $\mathfrak{N}/\varphi^*(\mathfrak{N})$ *is killed by some power of* $E(u)$*, and* G_{K_∞}*-stable* $\mathbb{Z}_p$*-lattices* $L \subset V$*.*

Proof. By Theorem 5.15, the set of G_{K_∞}-stable lattices $L \subset V$ is in bijiection (up to isomorphism) with the set of finite free, φ-stable $\mathcal{O}_{\mathcal{E}}$-lattices $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{N} \in \Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$.

To prove the injection, choose any $\mathfrak{N}$, then $V_{\mathfrak{S}}(\mathfrak{N})$ is a G_{K_∞}-lattices in V by Proposition 6.39. If there exists $\mathfrak{N}'$ such that $V_{\mathfrak{S}}(\mathfrak{N}) = V_{\mathfrak{S}}(\mathfrak{N}')$, then $\mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \simeq \mathfrak{N}' \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. By Lemma 6.36, we have $\mathfrak{N} \simeq \mathfrak{N}'$. So the map of this lemma is injective.

To prove the surjectivity, given a G_{K_∞}-stable lattice $L \subset V$, let $\mathcal{N} = V^*_{\mathcal{E}}(L) = Hom_{\mathbb{Z}_p[G_{K_\infty}]}(L, \mathcal{O}_{\widehat{\mathcal{E}^{ur}}})$ be the corresponding finite free $\mathcal{O}_{\mathcal{E}}$-module. Let $\mathfrak{N} = \mathcal{N} \cap \mathfrak{M}[1/p]$, then $\mathfrak{N}$ is a finite free $\mathfrak{S}$-module such that $\mathfrak{n}/\varphi^*(\mathfrak{N})$ is killed by some power of $E(u)$ and $\mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \mathcal{N}$ (see the proof of [Kis, Lemma1.3.13]). Then $V_{\mathfrak{S}}(\mathfrak{N}) \simeq V^*_{\mathcal{E}}(\mathcal{N}) \simeq L$. So the map of the lemma is surjective. □

Denote by $Rep^{cris\circ}_{G_K}$ the category of G_K-stable $\mathbb{Z}_p$-lattices spanning representations in $Rep^{cris}_{G_K}$, the category of crystalline representations. To give an important statement concerning the category $Rep^{cris\circ}_{G_K}$, we set a new category $Mod^{\varphi,+}_{/\mathfrak{S}}$, which is the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a Frobenius semi-linear isomorphism

$$1 \otimes \varphi : \varphi^*(\mathfrak{M})[1/E(u)] \xrightarrow{\simeq} \mathfrak{M}[1/E(u)].$$

This definition is slightly different from $Mod^{\varphi}_{/\mathfrak{S}}$. Indeed, $Mod^{\varphi}_{/\mathfrak{S}}$ is a full subcategory of $Mod^{\varphi,+}_{/\mathfrak{S}}$. As we will see, the aim of constructing this new catgeory is to consider crystalline representations with arbitrary Hodge-Tate weights.

Theorem 6.41. *There exists a fully faithful functor*

$$\mathfrak{M} : Rep^{cris\circ}_{G_K} \to Mod^{\varphi,+}_{/\mathfrak{S}}.$$

Proof. Denote by $Rep^{\circ}_{G_{K_\infty}}$ the category of continuous representations of G_{K_∞} on finite free $\mathbb{Z}_p$-modules. Then we have functors

$$Mod^{\varphi}_{/\mathfrak{S}} \xrightarrow{\mathfrak{M} \to \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}} Mod^{\varphi}_{/\mathcal{O}_{\mathcal{E}}} \xrightarrow{\simeq} Rep^{\circ}_{G_{K_\infty}}.$$

This composite functor is fully faithful. We denote by $Rep^{fh,\circ}_{G_{K_\infty}}$ the essential image of this functor.

Let $Rep^{-,\circ}_{cris}$ be the category of G_K-stable $\mathbb{Z}_p$-lattices in crystalline representations with Hodge-Tate weights ≤ 0. Then the functor

$$Rep^{-,\circ}_{cris} \to Rep^{\circ}_{G_{K_\infty}}; \quad L \to L|_{G_{K_\infty}}.$$

is fully faithful (Note that in the proof of the Corollary 6.40 , we restrict the functor to the full subcategory $Rep^{cris,+}_{G_K}$ and use the contravariant functor $D^*_{cris} : Rep^{cris,+}_{G_K} \to Mod^{\varphi,Fil\geq 0,w.a}_{K}$. Now we restrict to the full subcategory consisting of those representations with non-positive Hodge-Tate weights, then we need to use the covariant functor D_{cris} correspondently).

If L is in $Rep_{cris}^{-,\circ}$, then by the Proposition 6.39, we have $L[1/p]$ has a G_{K_∞}-stable lattice in $Rep_{G_{K_\infty}}^{fh,\circ}$. Then by the Corollary 6.35 and the bijection in the Lemma 6.41, $L|_{G_{K_\infty}}$ is also in $Rep_{G_{K_\infty}}^{fh,\circ}$.

We now can define the functor $\mathfrak{M}$ as the composite

$$\mathfrak{M} : Rep_{cris}^{-,\circ} \to Rep_{G_{K_\infty}}^{fh,\circ} \xrightarrow{\simeq} Mod_{/\mathfrak{S}}^{\varphi}.$$

To extend this functor to one on $Rep_{G_K}^{cris\circ}$, let $\mathfrak{S}(1)$(resp. $\mathfrak{S}(-1)$) be the object with the underlying $\mathfrak{S}$-module equal to $\mathfrak{S}$ and φ given by sending 1 to $E(0)/pE(u)$ (resp. $pE(u)/E(0)$). We can see that in $\mathfrak{S}(-1)$ and $\mathfrak{S}(1)$ are both in $Mod_{/\mathfrak{S}}^{\varphi,+}$. Now for any L in Rep_{cris}°, we set $\mathfrak{M}(L) = \mathfrak{M}(L(-m)) \otimes \mathfrak{S}(i)^{\otimes m}$ for m a positive integer such that $L(-m)[1/p]$ has all Hodge-Tate weights ≤ 0 ($L(-m)$ is the Tate-twist of L).

□

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