QUASI-REGULAR REPRESENTATIONS AND RAPID DECAY

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Abstract. We study *property RD* in terms of rapid decay of matrix coefficients. We give new formulations of property RD in terms of a L^1-integrability condition of a Banach representation. Combining this new definition with the existence of cyclic subgroups of exponential growth in non-uniform lattices in semisimple Lie groups, we deduce that there exist matrix coefficients associated to several kinds of quasi-regular representations which satisfy a "non-RD condition" for non-uniform lattices. We obtain also that such coefficients can not satisfy *the weak inequality* of Harish-Chandra.

1. Introduction

We say that a unitary representation $\pi : G \to U(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$ of a locally compact group G with a left invariant Haar measure dg and with a length function L has property RD with respect to to L if there exist $d \geq 1$ and $C > 0$, such that π verifies for all $f \in L^1(G, dg)$,

$$\|\pi(f)\| \leq C\|f\|_{L,d}$$

where $\|\cdot\|$ denotes the operator norm, and

$$\|f\|_{L,d} = \left(\int_G |f(g)|^2(1 + L(g))^{2d} dg\right)^{\frac{1}{2}}.$$

Equivalently, a representation $\pi : G \to U(\mathcal{H})$ has property RD with respect to L if there exist $d \geq 1$ and $C > 0$ such that for each pair of unit vectors ξ, η on $\mathcal{H}$ we have

$$\int_G |\langle \pi(g)\xi, \eta\rangle|^2 \, d\mu_{L,d}(g) \leq C$$

where

$$d\mu_{d,L}(g) = \frac{dg}{(1 + L(g))^d}.$$

We say that a locally compact group has property RD with respect to L if its regular representation has property RD with respect to L.

Date: October 15, 2015.

1991 *Mathematics Subject Classification.* Primary 43; Secondary 22.

Key words and phrases. Coefficients of unitary representations, semisimple Lie groups, property RD, Furstenberg boundary.

The property of rapid decay was introduced by U. Haagerup at the end of the seventies in his work [9]. Its essence could probably be traced back to Harish-Chandra's estimates of spherical functions on semisimple Lie groups and to the work of C. Herz [10]. The terminology "property RD" was introduced later, see the work [11] of P. Jolissaint. Property RD is relevant in the context of the Baum-Connes conjecture thanks to the important work of V. Lafforgue in [13].

1.1. **Equivalent definitions of property RD, Hilbert-Schmidt and trace class operators.** Let $\pi : G \to U(\mathcal{H})$ be a unitary representation of G. For ξ and η in $\mathcal{H}$, we define the matrix coefficient associated to π, as $g \mapsto \langle \pi(g)\xi, \eta \rangle$. Let $\mathcal{L}^2(\mathcal{H})$ be the Hilbert space of Hilbert-Schmidt operators on the Hilbert space $\mathcal{H}$. Recall that the scalar product of Hilbert-Schmidt operators on $\mathcal{H}$ is defined as $\langle S, T \rangle = Tr(ST^*)$ where Tr denotes the usual trace on $B(\mathcal{H})$, and T^* denotes the adjoint operator of T. Consider the representation

$$c : G \to U(\mathcal{L}^2(\mathcal{H})),$$

defined by

$$c(g)T = \pi(g)T\pi(g^{-1}),$$

for all T in $\mathcal{L}^2(\mathcal{H})$. For S and T in $\mathcal{L}^2(\mathcal{H})$, the matrix coefficient associated to c is $g \mapsto \langle c(g)S, T \rangle$. The Hilbert space $\mathcal{L}^2(\mathcal{H})$ contains the Banach space of trace class operators on the Hilbert space $\mathcal{H}$ denoted by $\mathcal{L}^1(\mathcal{H})$. The restriction of the representation c to $\mathcal{L}^1(\mathcal{H})$ is a isometric Banach space representation for the norm $\| \cdot \|_1$ (see Subsection 2.1). In Section 2 we prove the following proposition:

Proposition 1.1. *Let G be a locally compact second countable group, and let μ be a Borel measure on G which is finite on compact subsets. Let $\pi : G \to U(\mathcal{H})$ be a unitary representation on a Hilbert space, and let c denote the corresponding Banach space representation defined above. The following conditions are equivalent.*

1. *There exists $C > 0$ such that for all S, T unit vectors in $\mathcal{L}^1(\mathcal{H})$ (unit for $\| \cdot \|_1$) we have* $\int_G |\langle c(g)S, T \rangle| \, d\mu(g) < C.$
2. *There exists $C > 0$, such that for all unit vectors ξ, η in $\mathcal{H}$ we have* $\int_G |\langle \pi(g)\xi, \eta \rangle|^2 \, d\mu(g) < C.$
3. *For all vectors S, T in $\mathcal{L}^1(\mathcal{H})$ we have* $\int_G |\langle c(g)S, T \rangle| \, d\mu(g) < \infty.$
4. *For all vectors ξ, η in $\mathcal{H}$ we have* $\int_G |\langle \pi(g)\xi, \eta \rangle|^2 \, d\mu(g) < \infty.$

Let μ be the following measure

$$d\mu(g) = \frac{dg}{(1 + L(g))^d},$$

where dg denotes a Haar measure on G, and where d is some positive real number. Applying Proposition 1.1 with this special choice of measure provides us with four equivalent definitions of property RD for unitary representations.

1.2. **A simple condition for property RD for *positive* representations.** Consider now a representation $\pi : G \to U(\mathcal{H})$ such that $\mathcal{H} \subset L^2(X, m)$ where (X, m) is a measured space i.e. a vector ξ in $\mathcal{H}$ is a complex valued function and for all ξ and η in $\mathcal{H}$ we have $\langle \xi, \eta \rangle = \int_X \xi(x)\overline{\eta(x)}dm(x)$. We say that a function ξ is positive if $\xi \geq 0$ almost everywhere with respect to m. Let

$$\mathcal{H}_+ = \{\xi \in \mathcal{H} |\, \xi \geq 0 \text{ almost everywhere with respect to } m\}$$

be the cone of positive functions of $\mathcal{H}$. In the above definition $\mathcal{H}_+$ can be $\{0\}$ but such a pathology never appears in this article. In this context, we say that π is *positive* if for all g in G, we have

$$\pi(g)\mathcal{H}_+ \subset \mathcal{H}_+.$$

Typical examples of positive representations are provided by unitary representations coming from measurable actions $G \curvearrowright (X, m)$ and "half densities" where m is a G-quasi-invariant measure (see Subsection 3.2).

The following proposition proved in Section 4 brings a simple condition for property RD for positive representations:

Proposition 1.2. *Let G be a locally compact second countable group with a length function L. Let $\pi : G \to U(\mathcal{H})$ be a unitary representation such that $\mathcal{H} = L^2(X, m)$ with (X, m) a measured space. Assume that π is positive. The following conditions are equivalent:*

(1) *π has property RD with respect to L i.e. there exist $d \geq 1$ and $C > 0$ such that for all ξ, η unit vectors in $\mathcal{H}$, we have*

$$\int_G \frac{|\langle \pi(g)\xi, \eta \rangle|^2}{(1 + L(g))^d} dg < C.$$

(2) *For each positive function $\xi \in \mathcal{H}_+$, there exists $d_\xi \geq 1$ such that*

$$\int_G \frac{\langle \pi(g)\xi, \xi \rangle^2}{(1 + L(g))^{d_\xi}} dg < \infty.$$

1.3. **Positive vectors.** We shall consider another notion of positivity. Let π be a unitary representation of a locally compact group G on a complex Hilbert space $\mathcal{H}$. Following Y. Shalom (see [20, Theorem 2.2]), we say that a nonzero vector $\xi \in \mathcal{H}$ is a *positive vector* if it satisfies

$$\langle \pi(g)\xi, \xi \rangle \geq 0, \forall g \in G.$$

This notion is particularly interesting for property RD. In fact, in order to prove property RD with respect to L for a group (i.e. for the left regular representation) it suffices to prove property RD with respect to L for a representation with a

positive vector.
Our goal is now to construct unitary representations without property RD.

1.4. **Coefficients with slow decay and lattices.** Let G be a locally compact group with a length function L and let μ be a Haar measure on G i.e. $\mu(B) = \int_G 1_B dg$, where 1_B denotes the characteristic function of a Borel subset B of G. Denote by $B_L(R)$ the ball (with respect to L) of radius R whose center is the neutral element of G. We say that G has polynomial growth with respect to L if there exists a polynomial P such that for all $R > 0$ we have

$$\mu(B_L(R)) \leq P(R).$$

It is easy to check that G has polynomial growth with respect to L if and only if there exists a positive number d such that

$$\int_G \frac{dg}{(1+L(g))^d} < \infty.$$

Cauchy-Schwarz inequality implies that a unitary representation of a group of polynomial growth with respect to L satisfies property RD with respect to L.

If a locally compact group G admits a unitary representation satisfying property RD with respect to L and with a non zero invariant vector, then G must be a group of polynomial growth with respect to L. Therefore, we are interested in representations without non zero invariant vectors.

Let G be locally compact group. Consider an action $\alpha : G \curvearrowright (X, m)$ of G on (X, m) where m is a G-quasi-invariant measure. Consider the unitary representation

$$\pi_\alpha : G \to U\left(L^2(X, m)\right)$$

associated to this action on the Hilbert space $L^2(X, m)$. Observe that π_α is a positive representation (see Subsection 3.2).
Our goal is now to construct representations without property RD.

Theorem 1.1. *Let Γ be a discrete countable group with a length function L. Consider an action $\alpha : \Gamma \curvearrowright (X, m)$ on a σ-finite measured space (X, m) with a Γ-quasi-invariant measure m. Consider $\pi_\alpha : \Gamma \to U(L^2(X, m))$ the unitary representation associated to $\alpha : \Gamma \curvearrowright (X, m)$. Assume that Γ contains a cyclic subgroup of exponential growth with respect to L. Then there exists ξ in $L^2(X, m)_+$ such that for all $d \geq 1$ we have*

$$\sum_{\gamma \in \Gamma} \frac{\langle \pi_\alpha(\gamma)\xi, \xi \rangle^2}{(1+L(\gamma))^d} = \infty.$$

The theorem is trivially true if π_α contains a non zero invariant vector which is in $L^2(X, m)_+$. For example consider the representations obtained from an action $\alpha : \Gamma \curvearrowright (X, m)$ where m is a finite Γ-invariant measure. The constant function 1_X is a non zero invariant vector which is in $L^2(X, m)_+$. Examples of representations without non zero invariant vectors are described in Subsection 4.3.

In the following, Lie groups are endowed with a length function associated to a left-invariant Riemannian metric. According to A. Lubotzky, S. Mozes and M.S Raghunatan (see [14]), any non-uniform irreducible lattice in a higher rank semisimple Lie group contains a cyclic subgroup of exponential growth.

Corollary 1.2. *Let G be a connected non-compact simple Lie groups with finite center. Let H be a closed subgroup of G. Let $\lambda_{G/H} : G \to U(L^2(G/H))$ be the corresponding quasi-regular representation. Let Γ be a non-uniform lattice in G. Then there exists ξ in $L^2(G/H)_+$ such that for all $d \geq 1$ we have*

$$\sum_{\gamma\in\Gamma} \frac{\langle \lambda_{G/H}(\gamma)\xi, \xi \rangle^2}{(1+L(\gamma))^d} = \infty.$$

In Subsection 3.3.2, we assume that G is locally compact second countable and can be written $G = KP$, where K is a compact subgroup and P is a closed subgroup which is not unimodular. Consider $\lambda_{G/P} : G \to U(L^2(G/P))$ the quasi-regular representation associated with P. The Harish-Chandra function

$$\Xi(g) = \langle \lambda_{G/P}(g)1_{G/P}, 1_{G/P} \rangle \tag{1}$$

is the diagonal coefficient of $\lambda_{G/P}$ defined by the characteristic function $1_{G/P}$ of the space G/P. Following Gangolli and Varadarajan, [8, Definition 6.1.17], we say that a function f on the group G equipped with a length function L, verifies the *weak inequality* of Harish-Chandra if there exist $C > 0$ and $d \geq 0$ such that

$$|f(g)| \leq C(1+L(g))^d\Xi(g). \tag{2}$$

We prove:

Corollary 1.3. *Let G be a non-compact semisimple Lie group with finite center. Let H be a closed subgroup of G. Let $\lambda_{G/H} : G \to U(L^2(G/H))$ be the corresponding quasi-regular representation. Then there exist ξ and η in $L^2(G/H)$ such that $g \mapsto \langle \lambda_{G/H}(g)\xi, \eta \rangle$ does not satisfy the* weak inequality *of Harish-Chandra.*

Remark 1.4. *Corollary 1.3 holds true for $H := P$ a minimal parabolic subgroup of a non-compact semisimple Lie group with finite center. Although it is known that a coefficient $g \mapsto \langle \lambda_{G/P}(g)1_{G/P}, \eta \rangle$ with $\eta \in L^2(G/P)$ not in $L^\infty(G/P)$, does not satisfy the weak inequality (it is a consequence of Fatou's Theorem for semisimple Lie groups, see* [21, Theorem 5.1]*), Corollary 1.3 gives a new proof of this result. More generally, Corollary 1.3 implies that there exist matrix coefficients associated to any quasi-regular representations that do not satisfy the weak inequality of Harish-Chandra. Although the matrix coefficient $g \mapsto \langle \lambda_{G/P}(g)1_{G/P}, \eta \rangle$ with $\eta \in L^2(G/P)$ not in $L^\infty(G/P)$ does not satisfy the weak inequality, we prove in Section 4 the following:*

Proposition 1.3. *Let G be a non-compact semisimple Lie group with finite center. Let Γ be any discrete subgroup of G. Consider P a minimal parabolic subgroup of*

G, and $\lambda_{G/P} : G \to U(L^2(G/P))$ the quasi-regular representation associated to P. Then there exist $C > 0$ and $d \geq 1$ such that for all $\eta \in L^2(G/P)$ we have

$$\sum_{\gamma \in \Gamma} \frac{\left|\left\langle \lambda_{G/P}(\gamma) 1_{G/P}, \eta \right\rangle\right|^2}{(1 + L(\gamma))^d} \leq C \|\eta\|^2$$

where $\|\cdot\|$ denotes the L^2-norm on $L^2(G/P)$.

1.5. **Acknowledgements.** I would like to thank C. Pittet for very helpful discussions and criticisms. I would like also to thank the referees for their useful remarks and comments.

2. From square integrable representations to integrable representations

2.1. **Representations.** Let G be a locally compact group. Let $\pi : G \to U(\mathcal{H})$ be a unitary representation on a Hilbert space. Consider c as in the introduction. Write $T = \sum_i \alpha_i \langle \ , \xi_i \rangle \eta_i$. We have

$$c(g)T = \sum_i \alpha_i \langle \ , \pi(g)\xi_i \rangle \pi(g)\eta_i.$$

Observe that $\|c(g)T\|_2 = \|\alpha\|_{l^2} = \|T\|_2$ where $\|T\|_2$ denotes the norm of a Hilbert-Schmidt operator T and $\|c(g)T\|_1 = \|\alpha\|_{l^1} = \|T\|_1$ where $\|T\|_1$ denotes the norm of a trace class operator T. Hence, c is a unitary representation on $\mathcal{L}^2(\mathcal{H})$ and it is an isometric Banach space representation on $\mathcal{L}^1(\mathcal{H})$. Let $\overline{\pi}$ be the conjugate unitary representation on $\overline{\mathcal{H}}$ of π. Let σ be the unitary representation:

$$\begin{aligned} \sigma : G &\to U(\overline{\mathcal{H}} \otimes \mathcal{H}) \\ g &\mapsto \overline{\pi}(g) \otimes \pi(g). \end{aligned}$$

The Banach space isomorphism Φ defined as

$$\begin{aligned} \Phi : \overline{\mathcal{H}} \otimes \mathcal{H} &\to \mathcal{L}^2(\mathcal{H}) \\ \xi \otimes \eta &\mapsto \langle \ , \xi \rangle \eta \end{aligned}$$

intertwines σ and the representation c:

$$c\Phi = \Phi\sigma,$$

and this equivalence restricts to an equivalence between Banach space representations. For more details see [19, Chap. 2, § 2.1, p. 12], [5, Chap. 9.1, from § 9.1.31 to 9.1.38] and [16, Chap. 1, § 6, p. 96] .

Lemma 2.1. *Let U, V be a pair of unit vectors in $\overline{\mathcal{H}} \widehat{\otimes} \mathcal{H}$ where $\widehat{\otimes}$ denotes the projective tensor product of Banach spaces. There exists a unique pair of unit vectors S, T in $\mathcal{L}^1(\mathcal{H})$ such that*

$$\langle \sigma(g)U, V \rangle = \langle c(g)S, T \rangle .$$

Proof. We can write $U = \Phi(S)$ and $V = \Phi(T)$ for a unique pair $S, T \in \mathcal{L}^1(\mathcal{H})$ of unit vectors because Φ is an isomorphim of Banach spaces. Furthermore, because Φ is an isomorphism of Hilbert spaces $\overline{\mathcal{H}} \otimes \mathcal{H} \supset \overline{\mathcal{H}} \widehat{\otimes} \mathcal{H}$ and $\mathcal{L}^2(\mathcal{H}) \supset \mathcal{L}^1(\mathcal{H})$ we have

$$\begin{aligned} \langle \sigma(g)U, V \rangle &= \langle \sigma(g)\Phi(S), \Phi(T) \rangle \\ &= \left\langle \Phi^{-1}\sigma(g)\Phi(S), T \right\rangle \\ &= \langle c(g)S, T \rangle . \end{aligned}$$

□

2.2. **Proof of Proposition 1.1.**

2.2.1. *An application of the Banach-Steinhaus theorem.* The next proposition is an application of the Banach-Steinhaus theorem. It enables us to prove Implication (4) ⇒ (1) from Proposition 1.1 in the next subsection.

Proposition 2.1. *Let $B : X_1 \times X_2 \times ... \times X_r \to \mathbb{C}$ be a multilinear map on a product of Banach spaces. If B is continuous on each variable, then B is continuous.*

Proof. By induction on r. See [18, p. 81, Corollary] for the case $r = 2$. □

If G is a locally compact second countable group, we denote by $d\mu(g)$ a Borel measure on G which is finite on compact subsets of G.

Proposition 2.2. *Let G be a locally compact second countable group. Let $\pi : G \to U(\mathcal{H})$ be a unitary representation on a Hilbert space and let $\sigma = \overline{\pi} \otimes \pi$. If for all $\xi, \eta \in \mathcal{H}$ we have $\int_G |\langle \pi(g)\xi, \eta \rangle|^2 d\mu(g) < \infty$, then there exists $C > 0$ such that for all unit vectors $\xi, \xi', \eta, \eta' \in \mathcal{H}$ we have $\int_G |\langle \sigma(g)\xi \otimes \eta, \xi' \otimes \eta' \rangle| d\mu(g) \le C$.*

Proof. Let $G = \cup K_n$, where K_n is an exhausting sequence of compact subsets of G. Fix $\eta, \xi', \eta' \in \mathcal{H}$, and let us define the family T_n of continuous linear operators:

$$\begin{aligned} T_n : \mathcal{H} &\to L^1(G, \mu) \\ \xi &\longmapsto \left(g \mapsto 1_{K_n}(g) \langle \sigma(g)\xi \otimes \eta, \xi' \otimes \eta' \rangle \right) \end{aligned}$$

with 1_{K_n} the characteristic function of K_n.

We have for each $\xi \in \mathcal{H}$:

$$\begin{aligned}
\sup_n \|T_n(\xi)\|_{L^1} &= \sup_n \int_{K_n} |\langle \sigma(g)\xi \otimes \eta, \xi' \otimes \eta' \rangle| \, d\mu(g) \\
&= \sup_n \int_{K_n} |\langle \pi(g)\xi, \xi' \rangle| \, |\langle \pi(g)\eta, \eta' \rangle| \, d\mu(g) \\
&\leq \sup_n \left\{ \left(\int_{K_n} |\langle \pi(g)\xi, \xi' \rangle|^2 \, d\mu(g) \right)^{\frac{1}{2}} \left(\int_{K_n} |\langle \pi(g)\eta, \eta' \rangle|^2 \, d\mu(g) \right)^{\frac{1}{2}} \right\} \\
&\leq \left(\int_G |\langle \pi(g)\xi, \xi' \rangle|^2 \, d\mu(g) \right)^{\frac{1}{2}} \left(\int_G |\langle \pi(g)\eta, \eta' \rangle|^2 \, d\mu(g) \right)^{\frac{1}{2}} \\
&< \infty,
\end{aligned}$$

by hypothesis. The Banach-Steinhaus theorem implies that $\sup_n \|T_n\| < \infty$ where $\|\cdot\|$ denotes the operator norm. Hence

$$\sup_{\{\xi, \|\xi\|=1\}} \int_G |\langle \sigma(g)\xi \otimes \eta, \xi' \otimes \eta' \rangle| \, d\mu(g) < \infty.$$

This proves that the multilinear form B :

$$\begin{aligned}
&\overline{\mathcal{H}} \times \mathcal{H} \times \mathcal{H} \times \overline{\mathcal{H}} \to \mathbb{C} \\
&\qquad (\xi, \eta, \xi', \eta') \mapsto \int_G \langle \sigma(g)\xi \otimes \eta, \xi' \otimes \eta' \rangle \, d\mu(g)
\end{aligned}$$

is continuous in ξ. Analogous arguments show that B is continuous on η, ξ' and η'. Proposition 2.1 completes the proof.

□

2.2.2. *Proof of Proposition 1.1.*

Proof. (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are obvious.
(1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are clear: take $U = \xi \otimes \xi$ and $V = \eta \otimes \eta$. Then

$$\langle \sigma(g)U, V \rangle = |\langle \pi(g)\xi, \eta \rangle|^2 .$$

Lemma 2.1 and integration complete the proof.
Let us prove (4) $\Rightarrow$ (1) :

Take U, V two unit vectors (for the projective norm) in $\overline{\mathcal{H}} \widehat{\otimes} \mathcal{H}$. Write $U = \sum_k \alpha_k \xi_k \otimes \eta_k$ and $V = \sum_l \beta_l \xi'_l \otimes \eta'_l$ with $\|\alpha\|_{l^1} = 1 = \|\beta\|_{l^1}$, where $(\xi_k)_{k \in \mathbb{N}}, (\xi'_k)_{k \in \mathbb{N}}$ and $(\eta_k)_{k \in \mathbb{N}}, (\eta'_k)_{k \in \mathbb{N}}$ are orthonormal families in $\overline{\mathcal{H}}$ and $\mathcal{H}$. We have :

$$\begin{aligned}\int_G |\langle \sigma(g)U, V\rangle|\, d\mu(g) &= \int_G \left| \left\langle \sigma(g) \sum_k \alpha_k \xi_k \otimes \eta_k, \sum_l \beta_l \xi'_l \otimes \eta'_l \right\rangle \right| d\mu(g) \\ &\leq \sum_{k,l} |\alpha_k|\, |\beta_l| \int_G |\langle \sigma(g)\xi_k \otimes \eta_k, \xi'_l \otimes \eta'_l\rangle|\, d\mu(g).\end{aligned}$$

Thanks to Proposition 2.2, there exists $C > 0$ such that for all $k, l \in \mathbb{N}$

$$\int_G |\langle \sigma(g)\xi_k \otimes \eta_k, \xi'_l \otimes \eta'_l\rangle|\, d\mu(g) \leq C.$$

Hence,

$$\begin{aligned}\int_G |\langle \sigma(g)U, V\rangle|\, d\mu(g) &\leq \sum_{k,l} |\alpha_k|\, |\beta_l|\, C \\ &= C.\end{aligned}$$

The proof of (2) $\Rightarrow$ (3) is similar and left to the reader. ☐

3. Length functions, quasi-regular representations

3.1. **Property RD.** We shall define property RD for a unitary representation of a locally compact group G.

Definition 3.1. *A length function $L : G \to \mathbb{R}_+$ on a locally compact group is a measurable function which is locally bounded (i.e. for any compact $K \subset G$, we have* $\sup\{L(g), g \in K\} < \infty$*) satisfying*

(1) $L(e) = 0$,
(2) $L(g^{-1}) = L(g)$,
(3) $L(gh) \leq L(g) + L(h)$.

Remark 3.2. *Assume that $G \curvearrowright (X, d)$ acts properly by isometries on a metric space. Fix a point $x \in X$, then $g \mapsto L(g) := d(g \cdot x, x)$ defines a length function on G.*

We will need the following:

Lemma 3.3. *Let G be a locally compact group endowed with a length function L. Let O be a compact subset in G. Then there exists a positive constant C such that*

$$1 + L(gx) \geq C(1 + L(g))$$

for all g in G and for all x in O.

Proof. Triangle inequality and the equality $L(x) = L(x^{-1})$ imply that for all $g, x \in G$

$$1 + L(g) \leq (1 + L(gx))\,(1 + L(x))\,.$$

Since L is locally bounded, we have for all x in O

$$(1 + L(g)) \leq (1 + L(gx))\,(1 + M)\,,$$

where $M = \sup\{L(x), x \in O\}$. □

Definition 3.4. *A unitary representation $\pi : G \to U(\mathcal{H})$ of a locally compact group has property RD with respect to L if there exist $C > 0$ and $d \geq 1$ such that for each pair of unit vectors $\xi, \eta \in \mathcal{H}$ we have*

$$\int_G \frac{|\langle \pi(g)\xi, \eta\rangle|^2}{(1 + L(g))^d} dg \leq C.$$

Remark 3.5. *Assume K is a symmetric relatively compact generating set of G containing the identity. Then $L(g) = \min\{n \in \mathbb{N} : \exists k_1, ..., k_n \text{ such that } g = k_1 ... k_n\}$ is a length function on G. If L' is any length function on G, then $L'(g) \leq CL(g)$, where $C = \sup\{L(k), k \in K\}$. Hence if π has rapid decay with respect to L' then it has rapid decay with respect to L.*

3.2. **Representations associated to measurable actions.** Let G be a locally compact group acting measurably on a measured space $\alpha : G \curvearrowright (X, m)$ where m is a G-quasi-invariant measure. The action is measurable in the sense that: the map

$$\alpha : (g, x) \in G \times X \mapsto \alpha(g, x) = g \cdot x \in X,$$

is measurable. For all $g \in G$ we denote by $g_* m$ the measure which verifies for all Borel subsets $A \subset X$

$$g_* m(A) = m(g^{-1}A).$$

We say that m is G-quasi-invariant if for all $g \in G$, m and $g_* m$ are in the same measure class. We say that m is G-invariant if for all $g \in G$, $g_* m = m$. We denote by

$$x \in X \mapsto \frac{dg_* m}{dm}(x)$$

the Radon-Nikodym derivative of the measure $g_* m$ with respect to m. It verifies

$$\int_X f(g \cdot x) \frac{dg_*^{-1} m}{dm}(x) dm(x) = \int_X f(x) dm(x).$$

In this situation, consider the Hilbert space $L^2(X, m)$, and define the unitary representation $\pi_\alpha : G \to U(L^2(X, m))$ as

$$\pi_\alpha(g)\xi(x) = \left(\frac{dg_* m}{dm}(x)\right)^{\frac{1}{2}} \xi(g^{-1} \cdot x). \tag{3}$$

In other words,

$$\left(\frac{dg_* m}{dm}(x)\right)^{\frac{1}{2}} dm(x)$$

can be seen as a “half density” on X.

3.3. **Quasi-regular representations.** Quasi-regular representations associated to a pair (G,H) where H is a closed subgroup of G provide examples of unitary representations associated to a measurable action. But first of all, we recall what is the measure class we consider on G/H.

3.3.1. *A measure class on G/H.* Let G be a locally compact group with a Haar measure dg, and let H be a closed subgroup of G with a Haar measure dh. The space G/H is endowed with its quotient topology. Consider its Borel σ-algebra. We shall define a Borel measure on G/H. Let Δ_G and Δ_H be the modular functions of G and H. A *rho-function* for the pair (G,H) is a continuous map $\rho : G \to \mathbb{R}_+^*$ satisfying for all $g \in G$ and for all $h \in H$

$$\rho(gh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(g).$$

It always exists, see [6, (2.54)] or [17, Chapter 8, Section 1]. Therefore, given a rho-function for the pair (G,H), there exists a quasi-invariant regular Borel measure on G/H denoted by ν such that for all $f \in C_c(G)$,

$$\int_G f(g)\rho(g)dg = \int_{G/H}\int_H f(gh)dhd\nu(gH),$$

with Radon-Nikodym derivative

$$\frac{dg_*^{-1}\nu}{d\nu}(xH) = \frac{\rho(gx)}{\rho(x)},$$

for all $g, x \in G$. See [1, Appendix B]. The quasi-regular representation associated to a pair to (G,H) is the unitary representation $\lambda_{G/H} : G \to U(L^2(G/H,\nu))$ defined as

$$\left(\lambda_{G/H}(g)\xi\right)(xH) = \left(\frac{\rho(g^{-1}x)}{\rho(x)}\right)^{\frac{1}{2}} \xi(g^{-1}xH)$$

for all $\xi \in L^2(G/H,\nu)$, for all $g \in G$ and for all $xH \in G/H$.

3.3.2. *A particular class of quasi-regular representations.* Let G be a locally compact group which is unimodular, and assume that there exists a compact subgroup K and a closed subgroup P of G such that

$$G = KP.$$

We shall define a rho-function for the pair (G,K). We denote by Δ_P the right-modular function of P. We extend to G the map Δ_P of P as $\Delta : G \to \mathbb{R}_+^*$ with $\Delta(g) = \Delta(kp) := \Delta_P(p)$. It is well defined because $K \cap P$ is compact (observe that $\Delta_P|_{K\cap P} = 1$). Notice that for all $g \in G$ and for all $p \in P$, $\Delta(gp) = \Delta(g)\Delta(p) = \Delta(g)\Delta_P(p)$. Hence the function

$$x \in G \mapsto \Delta(x) \in \mathbb{R}_+^*,$$

defines a ρ function. Observe also that

$$x \in G/P \mapsto \frac{\Delta(gx)}{\Delta(x)} \in \mathbb{R}_+^*$$

is well defined. The quotient G/P carries a unique G-quasi-invariant measure ν, such that the Radon-Nikodym derivative at $(g, x) \in G \times G/P$ denoted by $\kappa(g, x) = \frac{dg_*\nu}{d\nu}(x)$ satisfies

$$\frac{dg_*^{-1}\nu}{d\nu}(x) = \frac{\Delta(gx)}{\Delta(x)}$$

for all $g \in G$ and $x \in G/P$. Consider the quasi-regular representation $\lambda_{G/P} : G \to U(L^2(G/P))$ associated to P, defined as

$$(\lambda_{G/P}(g)\xi)(x) = \kappa(g, x)^{\frac{1}{2}}\xi(g^{-1} \cdot x).$$

We denote by $1_{G/P}$ the characteristic function of G/P.

Definition 3.6. *The* Harish-Chandra function $\Xi : G \to (0, \infty)$ *is defined as*

$$\Xi(g) := \left\langle \lambda_{G/P}(g)1_{G/P}, 1_{G/P} \right\rangle.$$

As $\lambda_{G/P}$ is a unitary representation, we have $\Xi(g) = \Xi(g^{-1})$ for all $g \in G$. Observe also that for all $k, k' \in K$ we have $\Xi(kgk') = \Xi(g)$.

3.4. **Stability of some matrix coefficients.** Let Γ be a discrete subgroup of a locally compact group G. Let $\pi : G \to U(\mathcal{H})$ be a unitary representation. A matrix coefficient, $g \mapsto \langle \pi(g)\xi, \eta\rangle$ is *stable* on G relative to Γ if there exists a relatively compact neighborhood V of the neutral element $e \in G$ and $C, c > 0$ such that

$$c\,|\langle \pi(\gamma)\xi, \eta\rangle| \leq |\langle \pi(\gamma g)\xi, \eta\rangle| \leq C\,|\langle \pi(\gamma)\xi, \eta\rangle|$$

for all $g \in V$ and for all $\gamma \in \Gamma$.

The interest of stable matrix coefficients is illustrated by the following proposition:

Lemma 3.7. *Let Γ be a discrete subgroup of a locally compact group G. Let $\pi : G \to U(\mathcal{H})$ be a unitary representation, and let L be a length function on G. Let ξ and η be in $\mathcal{H}$. If $g \mapsto \langle \pi(g)\xi, \eta\rangle$ is a stable coefficient relative to Γ, then there exists a constant $C \geq 1$ such that for all $d \geq 1$ we have:*

$$\sum_{\Gamma} \frac{|\langle \pi(\gamma)\xi, \eta\rangle|^2}{(1 + L(\gamma))^d} \leq C \int_G \frac{|\langle \pi(g)\xi, \eta\rangle|^2}{(1 + L(g))^d} dg.$$

Proof. Let V be a relatively compact neighborhood of the neutral element of G, such that $\gamma \cdot V \cap \gamma' \cdot V = \emptyset$ for all $\gamma, \gamma' \in \Gamma$ such that $\gamma \neq \gamma'$. Consider a matrix coefficient $g \mapsto \langle \pi(g)\xi, \eta \rangle$ which is stable relative to Γ. We have

$$\begin{aligned}\int_G \frac{|\langle \pi(g)\xi,\eta\rangle|^2}{(1+L(g))^d}dg &\geq \sum_\gamma \int_{\gamma.V} \frac{|\langle \pi(g)\xi,\eta\rangle|^2}{(1+L(g))^d}dg \\ &= \sum_\gamma \int_V \frac{|\langle \pi(\gamma x)\xi,\eta\rangle|^2}{(1+L(\gamma x))^d}dx \\ &\geq \sum_\gamma \int_V c^2 \frac{|\langle \pi(\gamma)\xi,\eta\rangle|^2}{(1+L(\gamma x))^d}dx \\ &\geq c' \sum_\gamma \frac{|\langle \pi(\gamma)\xi,\eta\rangle|^2}{(1+L(\gamma))^d},\end{aligned}$$

for some positive constant c' depending on V and on the constant of Lemma 3.3. □

Lemma 3.8. *Let η be in $L^2_+(G/P)$. The matrix coefficient $g \mapsto \left\langle \lambda_{G/P}(g)1_{G/P}, \eta \right\rangle$ is stable relative to every discrete subgroup of G.*

Proof. Let Γ be a discrete subgroup of G. Let V be a relatively compact neighborhood of e in G, sufficiently small so that $\gamma \cdot V \cap \gamma' \cdot V = \emptyset$ for all $\gamma \neq \gamma' \in \Gamma$. We have $\left\langle \lambda_{G/P}(\gamma g)1_{G/P}, \eta \right\rangle = \left\langle \lambda_{G/P}(g)1_{G/P}, \lambda_{G/P}(\gamma^{-1})\eta \right\rangle$. That is

$$\left\langle \lambda_{G/P}(\gamma g)1_{G/P}, \eta \right\rangle = \int_{G/P} \kappa(g,x)^{\frac{1}{2}} \kappa(\gamma^{-1},x)^{\frac{1}{2}} \eta(\gamma \cdot x) d\nu(x).$$

The function $(g,x) \in G \times G/P \mapsto \kappa(g,x)^{\frac{1}{2}}$ is a strictly positive continuous function. Therefore, as $\overline{V}$ and G/P are compact, there exist $C, c > 0$ such that for all $g \in \overline{V}$ and for all $x \in G/P$, we have

$$c \leq \kappa(g,x)^{\frac{1}{2}} \leq C.$$

Notice that $\lambda_{G/P}$ is a positive representation. Therefore, since η is in $L^2_+(G/P)$, we obtain for all $\gamma \in \Gamma$, for all $g \in V$

$$c \left\langle \lambda_{G/P}(\gamma)1, \eta \right\rangle \leq \left\langle \lambda_{G/P}(\gamma g)1_{G/P}, \eta \right\rangle \leq C \left\langle \lambda_{G/P}(\gamma)1_{G/P}, \eta \right\rangle.$$

□

We obtain immediately that the Harish-Chandra function is stable relative to every discrete subgroup of G:

Corollary 3.9. ([8, Proposition 4.6.3, p. 159].) *The Harish-Chandra function is stable relative to every discrete subgroup of G.*

Combining Corollary 3.9 with Lemma 3.7, we obtain the following:

Proposition 3.1. *Let G be a locally compact group decomposed as $G = KP$ where K is a compact subgroup and P is a closed subgroup. Let Γ be a discrete subgroup of G and let L be a length function on G. There exists a constant $C > 0$ such that for all $d \geq 1$ we have*

$$\sum_{\gamma \in \Gamma} \frac{\Xi^2(\gamma)}{(1+L(\gamma))^d} \leq C \int_G \frac{\Xi^2(g)}{(1+L(g))^d} dg.$$

The representation $\sigma : G \to U(\overline{\mathcal{H}} \otimes \mathcal{H})$ introduced in Subsection 2.1 satisfies for all $\xi, \eta \in \mathcal{H}$:

$$\langle \sigma(g)\xi \otimes \xi, \eta \otimes \eta \rangle = |\langle \pi(g)\xi, \eta \rangle|^2 .$$

The representation σ can be used to give a short and elementary proof of the following result, due to C. Herz.

Theorem 3.10. ([10], *C. Herz.*) *Let G be a connected semisimple Lie group with finite center. Then G has property RD with respect to a length function associated to a left-invariant Riemannian metric on G.*

See [10],[4],[2] for proofs.

We can now easily prove Proposition 1.3:

Proof. Consider the quasi-regular representation $\lambda_{G/P}$ associated to P a minimal parabolic subgroup of G. In [2], we prove that this representation satifies property RD with respect to L where L is associated to a left-invariant Riemannian metric. Hence, there exist $C > 0$ and $d \geq 1$ such that for all $\xi, \eta \in L^2(G/P)$ we have

$$\int_G \frac{|\langle \pi(g)\xi, \eta \rangle|^2}{(1+L(g))^d} dg \leq C\|\xi\|^2\|\eta\|^2.$$

Applying the above inequality for $\xi = 1_{G/P}$ and $\eta \in L^2_+(G/P)$, and using Lemma 3.8 and Lemma 3.7 we obtain for some $C' > 0$ and for the same $d \geq 1$, that for all $\eta \in L^2_+(G/P)$

$$\sum_{\gamma \in \Gamma} \frac{\left|\left\langle \lambda_{G/P}(\gamma) 1_{G/P}, \eta \right\rangle\right|^2}{(1+L(\gamma))^d} \leq C'\|\eta\|.$$

□

4. Proofs

4.1. **Proof of Proposition 1.2.** We start by a lemma about positive representations.

Lemma 4.1. *Let G be a locally compact second countable group with a length function L. Let $\pi : G \to U(\mathcal{H})$ be a unitary representation with $\mathcal{H} = L^2(X, m)$ where (X, m) is a measured space. Assume that π is positive (π preserves the cone of positive functions). The following assertions are equivalent.*

(1) *There exists* $d \geq 1$ *such that for all vectors* $\xi, \eta \in \mathcal{H}$ *we have*

$$\int_G \frac{|\langle \pi(g)\xi, \eta \rangle|^2}{(1+L(g))^d} dg < \infty.$$

(2) *There exists* $d \geq 1$ *such that for all* $\xi \in \mathcal{H}_+$ *we have*

$$\int_G \frac{\langle \pi(g)\xi, \xi \rangle^2}{(1+L(g))^d} dg < \infty.$$

Proof. (1) $\Rightarrow$ (2) is obvious.

Let us prove (2) $\Rightarrow$ (1). Observe first that the decomposition of a real valued function ξ into its positive and negative part satisfies $|\xi_+ - \xi_-| \leq \xi_+ + \xi_-$. By positivity of π, for all $g \in G$ we have $\pi(g)|\xi_+ - \xi_-| \leq \pi(g)\xi_+ + \pi(g)\xi_-$. Using the decomposition of a complex valued function into its real and imaginary parts, and the decomposition of a real valued function into its positive and negative parts, we obtain, according to the above observation, that $| \langle \pi(g)\xi, \eta \rangle |$ is less than or equal to a linear combination of matrix coefficients $\langle \pi(g)\xi', \eta' \rangle$ with ξ', η' positive vectors in $\mathcal{H}$. Now observe that for positive vectors,

$$\langle \pi(g)\xi', \eta' \rangle \leq \langle \pi(g)(\xi' + \eta'), (\xi' + \eta') \rangle .$$

Integration and Cauchy-Schwarz inequality complete the proof.

□

We prove Proposition 1.2:

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Thanks to the equivalence between (2) and (4) in Proposition 1.1 with $d\mu(g) = \frac{dg}{(1+L(g))^d}$ for some $d \geq 1$, it is enough to prove that there exists $d \geq 1$, such that for each pair of vectors $\xi, \eta \in \mathcal{H}$ we have

$$\int_G \frac{|\langle \pi(g)\xi, \eta \rangle|^2}{(1+L(g))^d} dg < \infty.$$

According to Lemma 4.1 it is equivalent to prove that there exists $d \geq 1$ such that for any positive function $\xi \in L^2(X, m)$ we have

$$\int_G \frac{\langle \pi(g)\xi, \xi \rangle^2}{(1+L(g))^d} dg < \infty.$$

We give a proof by contraposition.

Assume that for each $n \in \mathbb{N}$, there exists a unit positive vector ξ_n such that

$$\int_G \frac{\langle \pi(g)\xi_n, \xi_n \rangle^2}{(1+L(g))^n} dg = \infty.$$

Take a sequence $(a_n)_{n\in\mathbb{N}}$ of strictly positive real numbers such that the series $\sum_n a_n$ converges. We consider

$$\xi = \sum_n a_n \xi_n$$

which is a well defined positive element of $\mathcal{H}$. We can assume that $\xi \neq 0$. We can replace ξ by $\frac{\xi}{\|\xi\|}$ which is a unit vector, so we assume that ξ is a unit vector. Let d be a positive real number. Let n be an integer such that $n \geq d$. Notice that $\langle \pi(g)\xi_n, \xi_m \rangle \geq 0$ for all $n, m \in \mathbb{N}$ and for all g in G. Hence,

$$\begin{aligned} \int_G \frac{\langle \pi(g)\xi, \xi\rangle^2}{(1+L(g))^d} dg &\geq \int_G \frac{\langle \pi(g)\xi, \xi\rangle^2}{(1+L(g))^n} dg \\ &\geq a_n^4 \int_G \frac{\langle \pi(g)\xi_n, \xi_n\rangle^2}{(1+L(g))^n} dg = \infty. \end{aligned}$$

Finally we have found a unit positive vector ξ, such that for all $d \geq 1$ we have

$$\int_G \frac{\langle \pi(g)\xi, \xi\rangle^2}{(1+L(g))^d} dg = \infty.$$

□

4.2. **Proof of Theorem 1.1.** We state a very useful lemma due to Y. Shalom in [20, Lemma 2.3]:

Lemma 4.2. *Let $\pi : G \to U(\mathcal{H})$ be a unitary representation. Assume that there exists a non-zero positive vector ξ in the following sense:*

$$\langle \pi(g)\xi, \xi\rangle \geq 0, \forall g \in G.$$

Then for any bounded measure positive measure μ on G we have

$$\|\lambda(\mu)\|_{op} \leq \|\pi(\mu)\|_{op}.$$

We prove Theorem 1.1.

Proof. According to Proposition 1.2, it is enough to show that π_α does not satisfy property RD with respect to L. Suppose π_α has property RD with respect to L; we will find a contradiction. Let $\mathbb{Z} \subset \Gamma$ be a subgroup of exponential growth with respect to L. The restriction $\pi_{\mathbb{Z}}$ of π_α to $\mathbb{Z}$ would satisfy property RD with respect to L. Hence, according to Lemma 4.2 the left regular representation $\lambda_{\mathbb{Z}}$ would have property RD with respect to L because of Lemma 4.1. This is impossible because an amenable group can satisfy property RD with respect to L if and only if it is a group of polynomial growth with respect to L (see [11, Proposition B, (2)]).

□

4.3. **Examples of representations associated to group actions without non zero invariant vectors.**

4.3.1. *Measure preserving ergodic group actions on infinite measured space.* Let G be locally compact group group acting ergodically on a σ-finite measured space via $\alpha : G \curvearrowright (X, m)$. If $\pi_\alpha : G \to U(L^2(X, m))$ has an invariant vector ξ, there exists a measurable function $\widetilde{\xi}$ with $\int_X \widetilde{\xi}^2(x)dm(x) < \infty$, which is G-invariant and satisfying $\widetilde{\xi}(x) = \xi(x)$ almost everywhere with respect to m (see [23, 2.2.16 Lemma]). Since the action is ergodic, $\widetilde{\xi}$ is a constant function on a conull set. Hence, if m is an infinite G-invariant measure then $\pi_\alpha : G \to U(L^2(X, m))$ does not contain non zero invariant vectors.

4.3.2. *Nonsingular ergodic group actions of type III_1.* Let Γ be a discrete countable group. A nonsingular action $\alpha : \Gamma \curvearrowright (X, m)$ of Γ is an action of Γ on the measured space (X, m) such that m is a Γ-quasi-invariant measure and there is no Γ-invariant measures in the measure class of m. In the context of operator algebras we say that the equivalence relation produced by $\Gamma \curvearrowright (X, m)$ is of type III. There exist different types of equivalence relation of type III (see [22, Chapter XIII, §2]). We consider only the type III_1 case.

Consider the Maharam extension (see [15]) $\Gamma \curvearrowright (X \times \mathbb{R}, dm \otimes e^{-t}dt)$ where Γ acts by measure preserving transformations in the following way

$$\gamma \cdot (x, t) = \left(\gamma \cdot x, t + \log\left(\frac{d\gamma_*^{-1}m}{dm}(x)\right)\right).$$

We denote by

$$\rho_\alpha : \Gamma \to U(L^2(X \times \mathbb{R}, dm \otimes e^{-t}dt)),$$

the unitary representation associated to the Maharam extension. It is well known that if $\Gamma \curvearrowright (X, m)$ is of type III_1 then the action $\Gamma \curvearrowright (X \times \mathbb{R}, dm \otimes e^{-t}dt)$ is ergodic. See [12, Section 2] for a survey about the type III case. We have

Proposition 4.1. *If $\alpha : \Gamma \curvearrowright (X, m)$ is of type III_1 then π_α does not have non zero invariant vectors.*

Proof. Assume that π_α has an invariant vector. There exists a measurable function ξ such that for all γ in Γ, $\pi_\alpha(\gamma)\xi(x) = \xi(x)$ almost everywhere with respect to m. We shall prove that ξ represents the null vector. Consider the function

$$F(x, t) = \xi(x)e^{\frac{t}{2}}.$$

Observe that $\rho_\alpha(\gamma)F(x, t) = F(x, t)$ almost everywhere with respect to $dm \otimes e^{-t}dt$. There exists a measurable function $\widetilde{F}$ satisfying $\widetilde{F} = F$ almost everywhere with respect to $dm \otimes e^{-t}dt$, such that $\widetilde{F}$ is a Γ-invariant function (see [23, 2.2.16 Lemma]). Since the action $\Gamma \curvearrowright (X \times \mathbb{R}, dm \otimes e^{-t}dt)$ is ergodic, $\widetilde{F}$ is a constant function on a conull set of $X \times \mathbb{R}$. Hence ξ is a constant function on a conull set of X. Therefore $\xi = 0$ almost everywhere with respect to m (otherwise $\frac{d\gamma_* m}{dm}(x) = 1$ almost everywhere which is excluded by hypothesis). $\square$

4.3.3. *Quasi-regular representation associated to a closed subgroup of a simple Lie group.*

Proposition 4.2. *Let G be a connected non-compact simple Lie group. Let H be a closed subgroup of G such that G/H carries no finite G-invariant measure. Let Γ be a lattice of G. Then the unitary representation $\lambda_{G/H_{|\Gamma}} : \Gamma \to U\left(L^2(G/H)\right)$ does not contain non zero invariant vectors.*

Proof. Consider the quasi-regular representation $\lambda_{G/H} : G \to U\left(L^2(G/H)\right)$ associated to H. The representation $\lambda_{G/H}$ can be identified with the induced representation $Ind_H^G(1_H)$ of G associated to the trivial representation of H, see [1, Example E.1.8 (ii)]. It is well known that $Ind_H^G(1_H)$ has a non zero invariant vector if and only if G/H carries a finite invariant regular Borel measure ([1, Theorem E.3.1]). Hence $\lambda_{G/H}$ does not contain non zero invariant vectors. Combining this observation with Moore's Theorem [23, 2.2.19 Theorem] we obtain that $\lambda_{G/H_{|\Gamma}}$ does not contain non zero invariant vectors. □

4.4. **Proof of Corollary 1.2.** We give a proof of Corollary 1.2:

Proof. It is known that Γ as in the corollary contains a cyclic subgroup with exponential growth with respect to any left-invariant Riemannian metric on G. See [14, Theorem A] for the higher rank case. The rank 1 one case is well known: horospherical subgroups of Γ have exponential growth with respect to any left-invariant Riemannian metric, see [7, Chapter 3, Section 3.C] and [3, Proposition 8. 25, p. 275]. So the result follows from Theorem 1.1, Proposition 4.2 ensuring that we are not in the trivial case of Theorem 1.1. □

4.5. **Proof of Corollary 1.3.**

Proof. Let G be a non compact semisimple Lie group endowed with a length function L associated to a left-invariant Riemannian metric. Let H be a closed subgroup of G. Let $\lambda_{G/H} : G \to U\left(L^2(G/H)\right)$ be the quasi-regular representation associated to H.

Let $u \in G$ be a unipotent element of infinite order. Let $\mathbb{Z}$ be the cyclic subgroup generated by u. Observe that $\mathbb{Z}$ is discrete and is an amenable group with exponential growth with respect to L. Thus it can not satisfy property RD with respect to L by [11, Proposition B, (2)]. Since $\lambda_{G/H_{|\mathbb{Z}}}$ has a positive vector (e.g. the characteristic function of any subset of positive, finite measure) according to Lemma 4.2, the representation $\lambda_{G/H_{|\mathbb{Z}}}$ does not satisfy property RD with respect to L. Thanks to Proposition 1.2 (2) applied to $\lambda_{G/H_{|\mathbb{Z}}}$, there exists $\xi \in L^2(G/H)_+$ such that for all $d \geq 1$ we have

$$\sum_{\gamma \in \mathbb{Z}} \frac{\left\langle \lambda_{G/H}(\gamma)\xi, \xi \right\rangle^2}{(1 + L(\gamma))^d} = \infty.$$

We claim that the weak inequality fails for the coefficient $\left\langle \lambda_{G/H}(g)\xi, \xi \right\rangle$. Assume on the contrary that it holds. There would exist C_ξ and d_ξ such that for all $g \in G$ $\left\langle \lambda_{G/H}(g)\xi, \xi \right\rangle \leq C_\xi (1 + L(g))^{d_\xi} \Xi(g)$. For any $d_0 > 0$, we would have:

$$\begin{aligned} \sum_{\gamma \in \mathbb{Z}} \frac{\left\langle \lambda_{G/H}(\gamma)\xi, \xi \right\rangle^2}{(1 + L(\gamma))^{2d_\xi + d_0}} &\leq C_\xi^2 \sum_{\gamma \in \mathbb{Z}} \frac{(1 + L(\gamma))^{2d_\xi} \Xi(\gamma)^2}{(1 + L(\gamma))^{2d_\xi + d_0}} \\ &= C_\xi^2 \sum_{\gamma \in \mathbb{Z}} \frac{\Xi(\gamma)^2}{(1 + L(\gamma))^{d_0}} \\ &\leq C_\xi^2 C \int_G \frac{\Xi(g)^2}{(1 + L(g))^{d_0}} dg, \end{aligned}$$

where the last inequality follows from Proposition 3.1.

Since G is a semisimple Lie group, we can find $d_0 > 1$ such that

$$\int_G \frac{\Xi^2(g)}{(1 + L(g))^{d_0}} dg < \infty,$$

see [8, Chap 4, Theorem 4.6.4, p.161] for a reference. It would follow that

$$\sum_{\gamma \in \mathbb{Z}} \frac{\left\langle \lambda_{G/H}(\gamma)\xi, \xi \right\rangle^2}{(1 + L(\gamma))^{2d_\xi + d_0}} < \infty.$$

This is a contradiction. □

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