Automorphisms of minimal simple groups of degenerate type

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Abstract

We study possible (definable) actions on minimal simple groups of degenerate type.

1 Introduction

The Cherlin-Zilber Conjecture states that simple groups of finite Morley rank are algebraic; this has been proved under a special assumption on the Sylow 2-subgroup, namely that it contains an infinite 2-group of bounded exponent. This ambitious classification program, sometimes described as a homomorphic image of the classification of the finite simple groups, is exposed in [ABC08] - a relatively short book with respect to the criteria of finite group theory.

But the question of the algebraic nature of simple groups of finite Morley rank remains open if one works without 2-unipotence; for instance, it is not known whether or not the only minimal simple group of finite Morley rank with toral-by-finite Sylow 2-subgroup is PSL_2. A painful series of analysis [CJ04, Del07, DJ08] eventually reaches three possible non-alegbraic configurations besides PSL_2, which have survived any attempt at eradication as of today.

Even worse, since there is no Feit-Thompson Theorem in the finite Morley rank category (simple groups of finite Morley rank are not known to have involutions), there might exist so-called "bad" groups, simple groups which do not interpret fields. These were first encountered thirty years ago and the bets are still open.

As the Cherlin-Zilber Conjecture has been proved in the even and mixed type cases, but no counter-example has been constructed, it is rather hard to strike the balance between what is reasonable to hope or not. The serious question is now to which extent the Conjecture is true in odd type.

The present article limits the involvement of counter-examples of the worst kind (the ones with no involutions) in less pathological contexts; that is, our aim is to study how a small simple group with no involutions can sit inside a group which has 2-torsion. The most classical result in this vein is the theorem of Delahan and Nesin on the nonexistence of involutory automorphisms of bad groups [BN94, Theorem 13.3 (iv)], generalized by Jaligot to "full Frobenius" groups [Jal01, Proposition 6.1]. Our result applies more broadly, but with a weaker conclusion.

Theorem. *Let Γ be a group of finite Morley rank with no unipotent 2-subgroup. Suppose that there is a definable, normal, minimal connected simple subgroup of degenerate type $G \lhd \Gamma$ such that $C_\Gamma(G) = 1$. Then:*

(A). $m_2(\Gamma) \leq 2$,

(B). *for any $i \in I(\Gamma^\circ)$, $C_G(i)$ is a self-normalizing Borel subgroup of G,*

(C). $m_2(\Gamma^\circ) \leq 1$.

Conclusions (A) (B) and (C) are proved in §3, 4, and 5 respectively. The methods used are similar to those of [Del07], with the difference that involutions need no longer lie inside G. The use of a so-called *concentration argument* in an abelian context is also noteworthy. The reader familiar with recent developments of the classification of small groups of finite Morley rank should find nothing striking in the proofs; they correspond to existing work, with the slight difference that involutions need no longer be inside G. Noteworthy is nonetheless, in our opinion, the use of a so-called concentration argument in an abelian configuration, 4.4. The rest is very classical.

Corollary. PSL_2 *cannot act non-trivially on a minimal simple group of finite Morley rank with no involutions.*

The next question, which seems essentially more challenging, is naturally the following.

Question. *Can* SL_2 *do so?*

To prove signficant results in odd type wihout first eliminating degenerate type, we would like, at a minimum, to eliminate actions of simple algebraic groups on simple degenerate type groups, and not just on the minimal simple groups of degenerate type. But the foregoing question is a sufficient challenge for the present.

2 Background

For general background see [Poi87], [BN94], or [ABC08]. More specific material is recalled below: unipotence theory (§2.2) and the translation of the Bender method in the finite Morley rank setting (§2.3).

2.1 Invariance

The following computation will be used throughout with no reference.

Fact 1. *Let $H \rtimes K$ be a semi-direct product with K 2-divisible. Suppose that there is an automorphism α of $H \rtimes K$ which inverts K, and centralizes or inverts H. Then $[H, K] = 1$.*

Fact 2 ([Del07, Lemme 2.2.10])**.** *Let G be a minimal connected simple group of finite Morley rank and K be a group of definable automorphisms of G. Let $H < G$ be a definable, connected, proper, K-invariant subgroup. Assume that H is non-abelian. Then there is a K-invariant Borel subgroup of G containing H.*

Fact 3 ([Bur05, Proposition 1.1]). *Let G be a connected group of finite Morley rank, let Γ be a group of finite Morley rank containing G as a definable, normal subgroup, and let S be a Sylow 2-group of Γ. Then G has an S-invariant Carter subgroup.*

not used

Fact 4 ([ABCC03]). *Let H be a solvable $p^{\perp}$-group of finite Morley rank. Let E be a non-trivial finite elementary abelian p-group acting definably on H. Then $H = \langle C_H(E_0) : E_0 \leq E, [E : E_0] = p \rangle$.*

One may drop the solvability assumption in case $p = 2$.

Fact 5 ([BBC07, Proposition 9.1]). *Let H be a group of finite Morley rank without involutions, and V a four-group acting definably on H. Then $H = \langle C_H(v) : v \in V^{\#} \rangle$.*

2.2 Unipotence

A convenient notion of unipotence in characteristic 0 has been introduced in [Bur04a]; see also [Bur04b].We shall follow the treatment of [FJ08], which introduces notation allowing finite and infinite characteristic to be treated on the same footing. This is done by using unipotence parameters, which stand for both a characteristic and a unipotence degree, if relevant.

Added some explanations. See if relevant.

A *unipotence parameter* is a pair $\tilde{q} = (p, d)$ where p is a characteristic (prime or ∞) and $d \in \mathbb{N} \cup \{\infty\}$ is such that $d = \infty$ iff $p \neq \infty$. The parameter $\tilde{q}$ is said to be trivial if $d = 0$.

It turns out that there is, for each $\tilde{q}$, notions of $\tilde{q}$-unipotence, $U_{\tilde{q}}$-radical and a $U_{\tilde{q}}$-group - interestingly enough, these notions do not in general relate to nilpotence (a $U_{\tilde{q}}$-group, for instance, need not be nilpotent).

Given a group of finite Morley rank H we shall say that a unipotence parameter $\tilde{q}$ occurs in H if $U_{\tilde{q}}(H) \neq 1$; $\tilde{q} = (p, d)$ is *maximal* for H if $\tilde{q}$ occurs in H and (p, s) does not for any $s > d$. Notice that if $\tilde{q} = (p, \infty)$ occurs in H, it is maximal for H.

The following fact tells us when there is a non-trivial (maximal) unipotence parameter. Recall that a good torus is a definable, abelian, divisible group with no torsion-free definable sections.

Fact 6 ([Bur04b, Theorem 2.19]). *Let H be a connected solvable group of finite Morley rank. If $U_{\tilde{p}}(H) = 1$ for all unipotence parameters $\tilde{p}$, then H is a good torus.*

Here is the reason why maximal unipotence parameters are so important.

Fact 7 ([FJ08, Lemma 2.11]; see [Bur04b, Theorem 2.21]). *Let G be a solvable group of finite Morley rank, and $\tilde{p}$ a maximal unipotence parameter for G. Then $U_{\tilde{q}}(G) \leq F^{\circ}(G)$.*

$U_{\tilde{p}}$-unipotence is of particular interest under a nilpotence assumption. This leads naturally to the definition of a Sylow $\tilde{p}$-subgroup as a maximal nilpotent $U_{\tilde{p}}$-group.

Still about nilpotence, there is a form of the normalizer condition.

not used (not explicitely)

Fact 8 ([Bur04b, Lemma 2.28]). *Let G be a nilpotent $U_{\tilde{r}}$-group. If $H < G$ is a definable subgroup then $U_{\tilde{r}}(N_G(H)/H) > 1$.*

Fact 9 ([Bur04a, Lemma 3.6]). *Let H be a $\tilde{q}$-unipotent $p^{\perp}$-group of finite Morley rank and P a p-group of definable automorphisms of H with bounded exponent. Then $C_H(P)$ is $\tilde{q}$-unipotent.*

Fact 10 ([Fré06, Theorem 4.11]). *Let G be a connected group of finite Morley rank. Assume that G acts definably on H, a $\tilde{r}$-unipotent group. Then $[G, H]$ is a homogeneous $\tilde{r}$-unipotent group.*

The following is an avatar of the Jaligot Lemmas. Notice that the assumption on A is always true if the unipotence parameter $\tilde{q}$ is of the form (p, ∞), or if A is normal in B.

Fact 11 ([Del07, Lemme 1.9.1]). *Let G be a minimal connected simple group of finite Morley rank. Let B be a Borel subgroup of G, not a good torus, and assume $\tilde{q}$ is a* maximal *unipotence parameter for B. Let U be a definable $\tilde{q}$-subgroup of B containing a subgroup $A \neq 1$ such that $\tilde{q}$ is a maximal unipotence parameter for $C^\circ(A)$.*

Then $U_{\tilde{q}}(B)$ is a Sylow $\tilde{q}$-subgroup of G, and is the only one that contains U. Moreover B is the only Borel subgroup with $\tilde{q}$ as a maximal unipotence parameter *containing U.*

not used

Fact 12 ([Del07, Lemme 1.10.1]). *Let G be a minimal simple group and let $B \neq B^g$ be two distinct conjugates of a Borel subgroup. If $F(B) \cap F(B^g)$ is not homogeneous, then $F^\circ(B)$ is abelian.*

2.3 Bender Compendium

We recast some highly technical but useful general facts on non-abelian intersections of Borel subgroups in a minimal simple group, following [Bur07]. The material is recorded here with no explanations, which the reader will find in [Bur07] or [Bur04b].

Fact 13 ([Bur07, Corollary 4.2]). *Let G be a minimal connected simple group of finite Morley rank. Then a definable connected non-abelian nilpotent subgroup of G is contained in exactly one Borel subgroup of G.*

Fact 14 ([Bur07, Corollary 2.2]). *Let G be a minimal connected simple group of finite Morley rank. Let B_1, B_2 be two distinct Borel subgroups of G. Then $F(B_1) \cap F(B_2)$ is torsion-free.*

Fact 15 ([Bur07, Theorem 4.3]). *Let G be a minimal connected simple group of finite Morley rank, and let B_1, B_2 be two distinct Borel subgroups of G. Suppose that $H = (B_1 \cap B_2)^\circ$ is nonabelian. Then the following are equivalent.*

1. *B_1 and B_2 are the only Borel subgroups of G containing H.*
2. *If B_3 and B_4 are distinct Borel subgroups of G containing H, then $(B_3 \cap B_4)^\circ = H$.*
3. *If $B_3 = B_1$ is a Borel subgroup containing H, then $(B_3 \cap B_1)^\circ = H$.*
4. *$C_G^\circ(H')$ is contained in B_1 or B_2.*
5. *B_1 and B_2 are not conjugate under $C_G^\circ(H')$.*
6. *$d_\infty(B1) = d_\infty(B2)$.*

Fact 16 ([Bur07]). *Let (B_h, B_ℓ) be a pair of Borel subgroups such that $H = (B_h \cap B_\ell)^\circ$ is maximal among intersections. Let $r_h = d_\infty(B_h)$ and $r_\ell = d_\infty(B_\ell)$; assume $r_h \leq r_\ell$. Also let $X = F^\circ(B_\ell) \cap F^\circ(B_h)$ and $r' = d_\infty(X)$ be the linking rank. Assume that $X \neq 1$ and fix a Carter subgroup Q of H. Then the following are true.*

1. *Description of B_ℓ:*
   (a) *$r_\ell < r_h$;*
   (b) *$F^\circ(B_\ell)$ is divisible and $\mathrm{Tor}(F^\circ(B_\ell)) \leq Z(H)$;*
   (c) *$F_r(B_\ell) \leq Z(H)$ for $r \neq r'$;*
   (d) *$F_{r'}(B_\ell) \not\leq H$; it is not abelian;*
   (e) *$U_{(0,r)}(B_\ell) \leq F_r(B_\ell)$ for all $r > r'$;*
   (f) *$U_{(0,r')}(H) = F_{r'}(H) \leq F_{r'}(B_\ell)$;*
   (g) *$N^\circ(H_{r'}) \leq B_\ell$;*
   (h) *$U_{(0,r')}(N_{F^\circ(B_\ell)}(H_{r'})) \not\leq H$.*
2. *Description of B_h:*
   (a) *$F^\circ(B_h)$ is divisible and $\mathrm{Tor}(F^\circ(B_h)) \leq Z(H)$;*
   (b) *$X = F_{r'}(B_h)$ and $B_h = N^\circ(X)$;*
   (c) *$F^\circ(B_h) \leq C^\circ(X)$;*
   (d) *$F_r(B_h) = 1$ for $r \leq r_\ell$ such that $r \neq r'$;*
   (e) *for any $r \leq r_\ell$, a $U_{(0,r)}$-Sylow subgroup of H is a a $U_{(0,r)}$-Sylow subgroup of B_h.*
3. *Carter subgroups:*
   (a) *$Q_{r'} = Z_{r'}(H)$;*
   (b) *$N(Q_{r'}) \leq B_\ell$;*
   (c) *Q is a Carter subgroup of B_h.*
4. *If in addition H is not abelian, then:*
   (a) *$N^\circ(H) = H$;*
   (b) *B_h is the only Borel subgroup containing $C^\circ(X)$;*
   (c) *If $(B_h \cap B_j)^\circ$ is another non-abelian maximal intersection such that $d_\infty(B_j) \leq r_h$, then B_j is $F^\circ(B_h)$-conjugate to B_ℓ.*

Proof. Most are in Burdges' article [Bur07] (though [Bur07, §4.5] pretends to assume non-abelianity); for instance, (1e) is [Bur07, Lemma 3.15]. However (2e) is not given explicitely there, so we mention a proof.

(2e). Let $r \leq r_\ell$. Recall that $F_r(B_h) \cdot Q_r$ is a $(0, r)$-Sylow subgroup of B_h. If $r \neq r'$, then by (2d) $F_r(B_h) = 1$, so $Q_r \leq H$ is a Sylow subgroup of B_h and the claim is proved. If $r = r'$, then $F_{r'}(B_h) = X \leq H$ by (2b), so $F_{r'}(B_h) \cdot Q_{r'} \leq H$ again and we are done. □

3 Eight-groups

We first bound the 2-rank of a group of definable automorphisms of a minimal connected simple groups of finite Morley rank of degenerate type, that is, we handle conclusion (A) of our main Theorem.

Notice that our statement is slightly more general, as we need not work in the minimal simple context.

Theorem A. *Let G be a simple group of finite Morley rank of degenerate type. Suppose that E is an elementary abelian group of order eight acting definably on G. Then $C_G^\circ(i)$ is non-solvable for some $i \in E^\#$.*

Proof. Suppose that $C_G^\circ(i)$ is always solvable. We shall use signalizer functor theory to build a proper 2-generated core. A minor hiccup is that signalizer functors are usually defined on elements of the group, not on (outer) automorphisms. We have tried to strike a balance between providing and referencing proofs.

Claim 1. G has a non-trivial nilpotent signalizer functor.

Proof of Claim: Let $\theta_0(i) = C_G^\circ(i)$; this is clearly a connected solvable E-signalizer functor, i.e.

1. $\theta_0(i)^g = \theta_0(i^g)$ for all $i \in E^*$ and $g \in G$.
2. $\theta_0(i) \cap C_G(j) \leq \theta_0(j)$ for any $i, j \in E^\#$.

Solvability is our contradictory assumption. Besides, $\theta_0(i)$ is of course infinite as otherwise G would be abelian.

If for all $i \in E^\#$, $\theta(i)$ is abelian, then we are done. Otherwise, by Fact 6, $\tilde{q} = \max(\tilde{q}_{\theta_0(j)} : j \in E^\#)$ is non-trivial. We then let $\theta(\cdot) = U_{\tilde{q}}(\theta_0(\cdot))$, which is definable, connected, nilpotent, and not identically trivial. (Notice that θ is even a characteristic subfunctor of θ_0.) $\diamondsuit$

θ is said to be *complete* if for each $i \in E^\#$,

$$\theta(i) = C_{\theta(E)}(i) \quad \text{where} \quad \theta(E) = \langle \theta(j) : j \in E^\# \rangle$$

As θ is nilpotent, the nilpotent signalizer functor theorem (any of [Bor95, BN94, Theorem B.30], [Bur04a, Theorem 6.1] or [Bur04b, p. 118], which *do not* assume that G is connected) shows that θ is complete and $\theta(E)$ is nilpotent. In particular $1 \neq \theta(E) < G$.

We proceed towards a contradiction by building a proper 2-generated core. The *2-generated p-core* $\Gamma_{E,2}(G)$ of G is the definable hull of all normalizers of p^2-groups in E:

$$\Gamma_{E,2}(G) = d(\langle N_G(U) : U \in \mathcal{E}_{p,2}(E) \rangle)$$

where $\mathcal{E}_{p,2}(E)$ denotes the collection of subgroups of order p^2 of E. (Here, of course, $p = 2$.)

Claim 2. $\Gamma_{E,2} < G$.

Proof of Claim: Consider two distinct four-groups $U \neq V \in \mathcal{E}_{2,2}(E)$. For each $u \in U^\#$,

$$\theta(u) \leq \langle C_{\theta(u)}(v) : v \in V^\# \rangle \leq \theta(V)$$

by the generation principle and the signalizer functor property. It follows $\theta(U) = \theta(V)$, and $\theta(U) = \theta(E)$. But θ clearly satisfies the stronger invariance condition that

$$\theta(i)^g = \theta(i^g) \quad \text{for all } i \in E^\# \text{ and all } g \in G \text{ for which } i^g \in E.$$

So, for any $U \in \mathcal{E}_{2,2}(E)$ and any $g \in N_G(U)$, we have

$$\theta(E)^g = \theta(U)^g = \theta(U^g) = \theta(U) = \theta(E).$$

Thus $\Gamma_{E,2}(G) \leq N_G(\theta(E)) < G$ by simplicity and because $\theta(E) < G$. $\Diamond$

We now reach our final contradiction (see [BC08a, Lemma 4.1]). Let $i \in E^{\#}$. We may write $E = \langle i \rangle \oplus V_i$ for some four-group V_i not containing i. By the generation principle (Fact 5), we have

$$C_G^\circ(i) = \langle C_G^\circ(\langle i, v \rangle) : v \in V_i^{\#} \rangle$$

But for $v \in V_i^{\#}$, one has $C_G^\circ(i, v) \leq N_G(\langle i, v \rangle) \leq \Gamma_{E,2}$. It follows $C_G^\circ(i) \leq \Gamma_{E,2}(G)$. In particular, with Fact 5 again, $G \leq \Gamma_{E,2}(G) < G$. This contradiction concludes the proof of Theorem A. $\square$

4 Toral involutions

We now attack conclusion (B) of the main Theorem.

Theorem B. *Let Γ be a connected group of finite Morley rank with no unipotent 2-subgroup. Suppose that there is a definable, normal, minimal connected simple subgroup of degenerate type $G \lhd \Gamma$ such that $C_\Gamma(G) = 1$. Let i be an involution of Γ. Then $C_G(i)$ is a self-normalizing Borel subgroup of G.*

Theorem B relies on the so-called concentration method which appeared in [Del07] to deal with minimal simple groups of odd type. As the Pruefer rank is not known to be 1, we shall work exactly as in [DJ09], which provides a consistent description of what concentration is, as opposed to [Del07] where results sometimes held by miracle, or coincidence.

Here the 2-torus is outside the smaller group; this slightly complicates matters, but the techniques may be adapted.

The proof goes as follows:

- Fixing a Borel subgroup $B \geq C_G^\circ(i)$ we will make heavy use of Nesin's $T[w]$-sets, that is $T[w] = \{b \in B : b^w = b^{-1}\}$, for a generic involution w. If $B > C_G^\circ(i)$, then $T[w]$ is (generically) large.
- The first, crucial, step is to show that B is S-invariant.
- We then show that for a generic involution w, $T[w]$ is an abelian group disjoint from $F^\circ(B)$.
- We eventually show that for generic involutions w, $T[w]$ is conjugate to $F^\circ(B)^{-i}$, and derive a contradiction.

As Γ is connected, [BC08b, Theorem 3*] implies that any involution is *toral*, i.e. lies in at least one decent torus. (Notice that our arguments will heavily rely on connectedness; it does not seem possible to show Theorem B in $G \rtimes \langle i \rangle$.)

Notation B.1. Let S be a maximal decent torus of Γ containing i.

We may assume $\Gamma = G \cdot S$. Moreover [ABC08, I, Lemma 10.5] immediately yields the connectedness of $C_G(i)$.

Notation B.2. Let $B \geq C_G(i)$ be a Borel subgroup of G, and $\tilde{q}$ a maximal unipotence parameter for B.

Lemma B.3. *If B is S-invariant, then $N_G(B) = B$.*

Proof. Suppose that S normalizes B. It then also normalizes $N = N_G(B)$, and by connectedness it centralizes the finite quotient N/B. Notice that as there are no involutions in G, this implies that any element of N inverted by i actually lies in B. Under the action of i, we may write $N = N^+ \cdot N^-$ with $N^+ = C_N(i)$ and $N^- = \{n \in N : n^i = n^{-1}\}$ (see [ABC08, I, Lemma 10.4] for instance). Both N^+ and N^- are subsets of B, whence $N = B$. $\square$

Until the end of §4 we assume $B > C_G(i)$.

Of course once this is refuted, we shall know that $B = C_G(i)$ is S-invariant, whence self-normalizing by Lemma B.3, and Theorem B will be proved.

If B were a good torus, then a conjugate of B would be in $S \leq C^\circ(i)$. In particular one would have $B = C_G^\circ(i)$, a contradiction. Fact 6 thus implies that $\tilde{q}$ is non-trivial.

4.1 $T[w]$ sets

Let us begin with remarks on torsion.

Lemma B.4.

1. *No element $a \in G^\#$ is G-conjugate to its inverse a^{-1}.*
2. *No toral element of G is inverted by an involution of Γ.*

Proof.

1. Suppose that $a^h = a^{-1}$ for some $a, h \in G$. As a is not an involution, we have $h \in N(d(a)) \setminus C(a)$. But clearly h has order two modulo $C(a)$. So lifting torsion, G contains an involution by [BN94, Ex. 11 p. 93], a contradiction.
2. Assume t is a toral element inverted by an involution k. Then $t \in \hat{T}$ for some maximal decent torus $\hat{T}$ of Γ. By conjugacy of maximal decent tori, $\hat{T}$ contains a conjugate S^g of S. So $S^g \leq C^\circ(t)$ which is k-invariant, and by connectedness of the Sylow 2-subgroup, it follows that $k \in S^g$, so k centralizes t. Then $t^2 = 1$ and $t = 1$. $\square$

Lemma B.5. *Suppose $H < G$ is a definable, connected, S-invariant subgroup, and let $k \in N(H)$ be an involution. Then $H^{-k} \subseteq F^\circ(H)$.*

Proof. Considering $N(H)$ which contains S and k, we may by connectedness of the Sylow 2-subgroup assume that $k \in S$. Let $\hat{H} = H \cdot S$, a definable *connected* solvable group. Then $H^{-k} \subseteq [H, S] \leq (H \cap \hat{H}')^\circ \leq (H \cap F^\circ(\hat{H}))^\circ \leq F^\circ(H)$. $\square$

We now introduce Nesin's $T[w]$ sets.

Notation B.6. For an involution w, set $T[w] := \{x \in B \mid x^w = x^{-1}\}$.

Fact B.7 ([Del07, Lemme 3.1.28]). *The following set is generic in i^G:*

$$I^* := \{w \in i^G \setminus N(B) \mid \mathrm{rk}(T[w]) \geq \mathrm{rk}(i^B)\}$$

Proof of Claim: Consider the group $\hat{G} = G \rtimes \langle i \rangle$ (notice that $i^G = i^{\hat{G}}$) and the function

$$\begin{array}{cccc} \pi: & i^G \setminus N(B) & \to & \hat{G}/B \\ & w & \mapsto & wB \end{array}$$

We argue classically as follows. Added proof.

Notice that the intersection $i^G \cap N_{\hat{G}}(B)$ is not generic in i^G (this is an immediate analog of for instance [CJ04, Fact 2.36], which still holds although $i \notin G$). It follows that the domain of π has rank $\mathrm{rk}\, G - \mathrm{rk}\, C_G(i)$ and degree 1.

Now using freely the conjugacy of involutions in $\hat{G}$, one sees that the fiber over some coset wB is exactly the set $wT[w]$, which is in definable bijection with $T[w]$. Let k be such that for w generic in $i^G \setminus N(B)$, the fiber has rank k ("generic" is not ambiguous, as the degree is 1). One has

$$\mathrm{rk}\, G - \mathrm{rk}\, C_G(i) - k \leq \mathrm{rk}\, G - \mathrm{rk}\, B$$

whence $\mathrm{rk}\, k \geq \mathrm{rk}\, B - \mathrm{rk}\, C_G(i) = \mathrm{rk}\, B - \mathrm{rk}\, C_B(i) = \mathrm{rk}\, i^B$. ♢

Notation B.8. Let w be an involution from I^*.

The involution w normalizes $d(T[w])$, which will actually turn out to be equal to $T[w]$. As in previous work [CJ04, Del07], it is the key to understanding B. We shall prove that $C^\circ(i)$ is not "light" in B, meaning that it meets interestingly the unipotent radical of B (Lemma B.10.1). This will yield uniqueness of B (Proposition B.10). Afterwards we shall show that $T[w]$ is a definable group disjoint from $F^\circ(B)$ (Proposition B.17), and then use concentration methods in §4.4.

But let us start with a basic remark on $T[w]$.

Lemma B.9. *For all $t \in T[w]$, $d(t)$ is torsion-free. In particular the group $d(T[w]) = d^\circ(T[w])$ has no unipotent torsion.*

Proof. Let $t \in T[w]$ have order p for some prime number p. By Lemma B.4 (2), t is not toral in B. It follows $U_p(B) \neq 1$ and $C^\circ_{U_p(B)}(t) \neq 1$, so by Fact 11 B is the only Borel subgroup of G containing $C^\circ(t)$. This implies $w \in N(B)$, a contradiction.

Hence no element of $T[w]$ is torsion. As $T[w]$ is closed under taking definable hulls of elements, this means that for all $t \in T[w]$, $d(t)$ is torsion-free, thus connected. In particular $T[w] \leq d^\circ(T[w])$, and $d(T[w])$ is connected.

Let p be a prime number; suppose $U_p(d(T[w])) \neq 1$. By Fact 11 B is the only Borel subgroup containing $U_p(d(T[w]))$, hence is w-invariant, a contradiction to $w \notin N(B)$. □

4.2 An invariant Borel subgroup

In this section we show the following.

Proposition B.10. *B is the only Borel subgroup containing $C_G(i)$. In particular, B is S-invariant and self-normalizing in G.*

This will rely on a technical analysis of unipotence parameters occuring in $C_G(i)$, Lemma B.10.1 below.

4.2.1 The reduced rank of $C_G(i)$

Recall that $\tilde{q}$ denotes a maximal unipotence parameter of B (Notation B.2).

Lemma B.10.1. $U_{\tilde{q}}(C_G(i)) \neq 1$.

As an appetizer and before we move to a deeper proof, here is the argument in finite characteristic.

Proof of Lemma B.10.1 in finite characteristic. Suppose that $\tilde{q} = (p, \infty)$ for a prime number p; that is, suppose $U_p(B) \neq 1$ but $U_p(C(i)) = 1$. A fairly standard use of Fact 11 shows that $B \cap B^w$ is abelian; it follows that $T[w]$ is abelian, and in particular $T[w] = d(T[w])$.

Let $t \in T[w]^\#$ and consider $X = C_{U_p(B)}(t)$. If $X \neq 1$, then by Fact 11 B is the only Borel subgroup containing $C^\circ(t)$, which forces $w \in N(B)$, a contradiction. Therefore $X = 1$ and $T[w]$ acts freely on $U_p(B)$.

Notice further that by assumption, $C_{U_p(B)}(i)$ is finite; this shows $\operatorname{rk} U_p(B) \leq \operatorname{rk} i^B$. Let $A \leq U_p(B)$ be any $T[w]$-minimal subgroup; clearly $\operatorname{rk} A \leq \operatorname{rk} i^B \leq \operatorname{rk} T[w]$. In view of these rank inequalities, Zilber's field Theorem implies that there is a field structure $\mathbb{K}$ with $A \simeq \mathbb{K}_+$ and $T[w] \simeq \mathbb{K}^\times$. In particular $T[w]$ acts transitively on $A^\#$, and this violates Lemma B.4 (1). □

4.2.2 The reduced rank of $C_G(i)$, continued

With characteristic zero unipotence the matters are of course slightly more subtle, but the argument is essentially the same.

Proof of Lemma B.10.1, general case. Let $\tilde{q}_i$ be a maximal unipotence parameter of $C_G(i)$ and suppose towards a contradiction that $\tilde{q}_i \prec \tilde{q}$.

Claim 1. $U_{\tilde{q}}(C^\circ_B(a)) = 1$ for $a \in B$ such that $a^w = a^{\pm 1}$.

Proof of Claim: Suppose towards a contradiction that $X := U_{\tilde{q}}(C^\circ_B(a)) \neq 1$. Let $L = C^\circ_G(a)$. Let U_1 be a Sylow $\tilde{q}$-subgroup of L containing X. As w normalizes L, there is an L-conjugate w' of w normalizing U_1. By our assumption that $\tilde{q} \succ \tilde{q}_i$ and $w \in i^G$, w' must invert $U_1 \geq X$. So w' inverts X. As $\tilde{q}_L \succeq \tilde{q}$, w' inverts $U_{\tilde{q}_L}(L)$ too.

Let $N = N^\circ_G(X) \geq U_{\tilde{q}}(Z(F^\circ(B)))$, and let U_2 be a Sylow $\tilde{q}$-subgroup of N containing $U_{\tilde{q}}(Z(F^\circ(B)))$. As w' normalizes N, there is an N-conjugate w'' of w' normalizing $U_{\tilde{q}}(B)$. Fact 11 now forces $w'' \in N(B)$. Still by our assumption that $\tilde{q} \succ \tilde{q}_i$, w'' must invert $U_{\tilde{q}}(B)$; in particular $U_{\tilde{q}}(B) = U_{\tilde{q}}(Z(F^\circ(B))) \leq N$. But w'' also inverts $U_{\tilde{q}_N}(N)$. In particular

$[U_{\tilde{q}}(B), U_{\tilde{q}_N}(N)] = 1$ and $U_{\tilde{q}_N}(N) \leq B$; as $U_{\tilde{q}}(B) \leq N$ it follows $\tilde{q}_N = \tilde{q}$ and $U_{\tilde{q}}(B) = U_{\tilde{q}}(N)$. Moreover $N \leq N_G^\circ(U_{\tilde{q}}(N)) \leq B$.

Now $w' \in w''^N \subseteq N(B)$ inverts $U_{\tilde{q}_L}(L)$ and $U_{\tilde{q}}(B)$; it follows that $[U_{\tilde{q}_L}(L), U_{\tilde{q}}(B)] = 1$. In particular $U_{\tilde{q}_L}(L) \leq B$ and $\tilde{q}_L = \tilde{q}$. Therefore $X = U_{\tilde{q}}(L)$. The latter is w-invariant, so w normalizes N and $U_{\tilde{q}}(N) = U_{\tilde{q}}(B)$. Thus w normalizes B, against its definition. $\Diamond$

By Claim 1, $U := [T[w], U_{\tilde{q}}(B)]$ is nontrivial; moreover it is a homogeneous $\tilde{q}$-unipotent subgroup by Fact 10.

Claim 2. $d(T[w]) = T[w]$ is abelian and acts freely on $U^\#$.

Proof of Claim: For the first part it suffices clearly to check that $d(T[w])$ is abelian. So suppose that $d(T[w])' \neq 1$. As $d(T[w])'$ is w-invariant, one has $d(T[w])' = C_{d(T[w])'}(w) \cdot (d(T[w])')^{-w}$. In particular there is $a \in d(T[w])'$ such that $a^w = a^{\pm 1}$. But as $d(T[w])' \leq F^\circ(B)$, one has $U_{\tilde{q}}(Z(F^\circ(B))) \leq C_B^\circ(a)$, contradicting Claim 1.

The second statement is a clear consequence of Claim 1, in view of the $\tilde{q}$-homogeneity of U. $\Diamond$

We now finish the proof of Lemma B.10.1. As $\tilde{q}_i \prec \tilde{q}$ and U is $\tilde{q}$-homogeneous, one has $C_U(i) = 1$. In particular, $\operatorname{rk} U \leq \operatorname{rk} i^B$. Let $A \leq U$ be $T[w]$-minimal. One has $\operatorname{rk} A \leq \operatorname{rk} U \leq \operatorname{rk} i^B \leq \operatorname{rk} T[w]$. Moreover, $T[w]$ acts faithfully on A by Claim 2. By Zilber's field theorem, $T[w]$ acts transitively on $A^\#$, and this contradicts Lemma B.4 (1). Here ends the proof of Lemma B.10.1. $\square$

4.2.3 Uniqueness of B

With Lemma B.10.1 at hand, we can finally prove Proposition B.10.

Proof of Proposition B.10. Suppose that $C_G(i)$ lies in two distinct Borel subgroups B_1 and B_2; fix maximal unipotence parameters $\tilde{q}_1$ and $\tilde{q}_2$ for B_1 and B_2. By Lemma B.10.1, $\tilde{q}_1 = \tilde{q} = \tilde{q}_2$.

We may assume that $H := (B_1 \cap B_2)^\circ$ is maximal among intersections of distinct Borel subgroups contaning $C(i)$. Notice that $U_{\tilde{q}}(H) \neq 1$ by Lemma B.10.1. If H were nonabelian, then $\tilde{q}_1 \neq \tilde{q}_2$ by Fact 15, a contradiction. Shorter proof.

So $H \geq C(i)$ is abelian. In particular, $C_G^\circ(U_{\tilde{q}}(H)) \geq C_G^\circ(i)$; by Lemma B.10.1 again, $\tilde{q}$ is a maximal unipotence parameter for $C_G^\circ(U_{\tilde{q}}(H))$. By Fact 11, it follows that B_1 is the only Borel subgroup of G with maximal unipotence parameter $\tilde{q}$ which contains $U_{\tilde{q}}(H)$. The same applies to B_2: a contradiction.

It follows that B is unique, whence S-invariant; by Lemma B.3 it is self-normalizing in G. $\square$

We introduce further notation for the set of involutions which behave like i, as far as B is concerned.

Notation B.11. Let $I_B = \{j \in i^\Gamma \cap N(B) : C_G^\circ(j) \leq B\}$.

Remark B.12. $I_B = i^B$.

Indeed, one inclusion is clear. Now let $j \in I_B$; as $\Gamma = G \cdot S$, there is $g \in G$ such that $j = i^g$. By Proposition B.10, B is the only Borel subgroup containing $C_G^\circ(j) = C_G^\circ(i^g)$; it follows $g \in N_G(B) = B$ from Proposition B.10.

4.3 Intersections of Fitting subgroups

In Proposition B.10 we have shown that there is a unique Borel subgroup B containing $C_G(i)$; in particular B is S-invariant.

Recall that $\tilde{q}$ denotes a maximal unipotence parameter for B (Notation B.2). Recall further that w is a generic involution in i^G (Notation B.8).

We show that $F(B) \cap F(B)^w$ is trivial (Proposition B.17 below). This will have interesting consequences on $T[w]$ (Corollary B.18).

4.3.1 Two adequate subgroups

Though this is not our concern, it could be shown that i centralizes $U_{\tilde{q}}(Z(F^\circ(B)))$.

Lemma B.13. *There is a non-trivial, definable, $\tilde{q}$-subgroup Y of B characteristic in B, central in $F^\circ(B)$, and such that any involution $k \in i^\Gamma \cap N(B)$ with $C_G^\circ(k) \not\leq B$ inverts Y.*

Proof. If $U_{\tilde{q}}(Z(F^\circ(B))) \leq Z(B)$, let $Y = U_{\tilde{q}}(Z(F^\circ(B)))$; otherwise let $Y = [U_{\tilde{q}}(Z(F^\circ(B))), B]$ (in which case one will bear Fact 10 in mind); only the last property need be proved.

So let $k \in i^\Gamma \cap N(B)$ be such that $C_G^\circ(k) \not\leq B$. Suppose that $Y_0 = C_Y(k)$ is non-trivial. By Fact 9, Y_0 is a $\tilde{q}$-group again. Clearly $C_B(k)$ normalizes Y_0. By Lemma B.5 and Proposition B.10, $B^{-k} \subseteq F^\circ(B)$ also normalizes Y_0. Hence Y_0 is normal in B.

It follows from Fact 11 that B is the only Borel subgroup of parameter $\tilde{q}$ containing Y_0. However $Y_0 \leq C_G^\circ(k)$ which is contained in a Borel subgroup of unipotence parameter $\tilde{q}_k = \tilde{q}_i$ (since i and k are conjugate), whence $C_G^\circ(k) \leq B$, against the assumption. $\square$

Lemma B.14. *There is a unipotence parameter $\tilde{s} \prec \tilde{q}$ and a non-trivial $\tilde{s}$-group $T_{\tilde{s}} \subseteq T[w]$ such that $T_{\tilde{s}}$ does not centralize $U_{\tilde{q}}(Z(F^\circ(B)))$.*

Proof. We show that the set $T[w]$ does not centralize $U_{\tilde{q}}(Z(F^\circ(B)))$. Suppose it does, and let $C = C^\circ(T[w])$. Clearly $\tilde{q}_C \succ \tilde{q}$. Let $U \leq C$ be a Sylow $\tilde{q}$-subgroup of C containing $U_{\tilde{q}}(Z(F^\circ(B)))$. As w normalizes C, there is a C-conjugate w' of w which normalizes U. It follows $w' \in N(B)$. By Lemma B.5, one has $B^{-w'} = (F^\circ(B))^{-w'}$.

Now w' inverts $T[w]$, and this implies $T[w] \subseteq F^\circ(B)$. Therefore $d(T[w])$ is a definable, connected (Lemma B.9), nilpotent subgroup of B and B^w. By Fact 13, $d(T[w])$ is abelian, and it coincides with $T[w]$. Therefore $T[w] \leq F^\circ(B)$. We reach a distinct contradiction in two cases, depending on whether $C_G^\circ(w') \leq B$ or not.

- Suppose $C^\circ_G(w') \leq B$. Then $C^\circ_G(w') = C^\circ_B(w')$ has the same rank as $C^\circ_G(i) = C^\circ_B(i)$. We may write $B = B^{+w'} \cdot B^{-w'}$, and in particular $\operatorname{rk} T[w] \leq \operatorname{rk} B^{-w'} = \operatorname{rk} B^{-i} \leq \operatorname{rk} T[w]$. Hence $T[w]$ is the only subgroup contained in $(F^\circ(B))^{-w'}$ and generic in it. In particular $C_B(w')$ normalizes $T[w]$. As there are no involutions in G, $T[w] \cdot C_B(w') = T[w] \rtimes C_B(w')$ has rank $\operatorname{rk} T[w] + \operatorname{rk} C_B(w') = \operatorname{rk} T[w] + \operatorname{rk} C_B(i) \geq \operatorname{rk} B$. Thus $T[w] \rtimes C_B(w')$ is a generic subgroup of B, so $B = T[w] \rtimes C_B(i)$ normalizes $T[w]$; B is w-invariant, a contradiction.
- Now suppose that $C^\circ_G(w') \not\leq B$, and consider the group Y given by Lemma B.13. In particular, w' inverts $U_{\tilde{q}_C}(C)$ and also $Y \leq Z^\circ(F^\circ(B)) \leq C^\circ_G(T[w]) = C$. So $U_{\tilde{q}_C}(C) \leq C^\circ_G(Y) \leq B$, against $U_{\tilde{q}_C}(C) \succ \tilde{q}$. This is a contradiction again.

Hence $T[w]$ does not centralize $U_{\tilde{q}}(Z(F^\circ(B)))$: there is an element $t \in T[w]$ not centralizing $U_{\tilde{q}}(Z(F^\circ(B)))$; we take any indecomposable (in the sense of Burdges: [Bur04b, Chapter II]) group appearing in $d(t) \subseteq T[w]$. Notice that $\tilde{s} \prec \tilde{q}$, as otherwise $t \in U_{\tilde{q}}(B) \leq F^\circ(B)$. □

Notation B.15.

1. Let $\tilde{s}$ be a unipotence parameter and $T_{\tilde{s}}$ be a non-trivial $\tilde{s}$-group contained in the set $T[w]$ and not centralizing $U_{\tilde{q}}(Z(F^\circ(B)))$.
   (Clearly $T_{\tilde{s}} \not\leq F^\circ(B)$.)
2. Let $X = [T_{\tilde{s}}, U_{\tilde{q}}(Z(F^\circ(B)))]$.
   (X is a non-trivial, $\tilde{q}$-homogeneous subgroup by Fact 10.)

Though X need not be normal in B, we shall make good use of it.

Lemma B.16. *No involution of I_B centralizes $T_{\tilde{s}}$.*

Proof. Suppose $j \in I_B$ centralizes $T_{\tilde{s}}$ (I_B was introduced in Notation **??**). Then $j \in C_\Gamma(T_{\tilde{s}})$ which is normalized by w; by downwards invariance, there is a $C_\Gamma(T_{\tilde{s}})$-conjugate w' of w which normalizes a Sylow 2-subgroup containing j. As Sylow 2-subgroups of Γ are connected, w' centralizes j. In particular, w' normalizes $C^\circ_G(j)$, and also B, which is the only Borel subgroup of G containing it. Hence $w' \in N(B)$. As w' inverts $T_{\tilde{s}}$, it now follows from Lemma B.5 that $T_{\tilde{s}} \leq F^\circ(B)$, a contradiction to Notation B.15 (1). □

4.3.2 Controlling the intersection

Proposition B.17. $F(B) \cap F(B)^w = 1$.

Proof.

Notation 1. Let $L = F(B) \cap F(B)^w$.
Assume $L \neq 1$ towards a contradiction. L is torsion-free by Fact 14.

Notation 2. Let $N = N^\circ_G(L)$ and $H_0 = N^\circ_B(L)$.

This is where X (Notation B.15 (2)) plays its crucial role.

Claim 3. $H_0' \geq X$.

Proof of Claim: Notice that $T_{\tilde{s}} \leq N$ and $T_{\tilde{s}} \leq H_0$. But one also clearly has $U_{\tilde{q}}(Z(F^\circ(B))) \leq H_0$. So $X \leq H_0'$. ◊

In particular $H_0' \neq 1$.

Claim 4. There is a w-invariant Borel subgroup containing N.

Proof of Claim: Clearly $U_{\tilde{q}}(Z(F^\circ(B))) \leq N$. If N is abelian, then $U_{\tilde{q}}(N)$ is a nilpotent $\tilde{q}$-group which contains $U_{\tilde{q}}(Z(F^\circ(B)))$. The Jaligot Lemma (Fact 11) then forces that $U_{\tilde{q}}(B)$ is the only Sylow $\tilde{q}$-subgroup of G containing $U_{\tilde{q}}(N)$; this implies $w \in N(B)$, a contradiction. So N is non-abelian, and Fact 2 concludes the proof. ◊

Notation 5. Let $B_M \geq N$ be a w-invariant Borel subgroup and $H = (B \cap B_M)^\circ$. We let $\tilde{q}_M$ denote a maximal unipotence parameter for B_M.

$H \geq H_0$ is non-abelian, as H_0 isn't (Claim 3).

Claim 6. $\tilde{q}_M \succ \tilde{q}$.

Proof of Claim: $U_{\tilde{q}}(Z(F^\circ(B))) \leq B_M$ so $\tilde{q}_M \succeq \tilde{q}$. If equality holds, then the Jaligot Lemma (Fact 11) implies that w normalizes $U_{\tilde{q}}(B)$, a contradiction. So $\tilde{q}_M \succ \tilde{q}$. ◊

Claim 7. (B, B_M) is a maximal pair with non-abelian intersection H, and B_M is the heavy one. The linking parameter is $\tilde{q}$.

Proof of Claim: As H is non-abelian and asymmetry holds, it is a maximal intersection ([Bur07, Theorem 4.3]). Now $H' \geq H_0' \geq X$ (Claim 3) which is $\tilde{q}$-homogeneous, so the linking parameter is $\tilde{q}$. ◊

Recall that I_B (Notation **??**) is the set of involutions that could freely be exchanged for i.

Claim 8. There is $j \in I_B$ normalizing B_M and H.

Proof of Claim: $U_{\tilde{q}}(Z(F^\circ(B))) \leq N \leq B_M$. Let $U \leq B_M$ be a Sylow $\tilde{q}$-subgroup of B_M containing $U_{\tilde{q}}(Z(F^\circ(B)))$. As w normalizes B_M, there is by downwards invariance a B_M-conjugate j of w normalizing U.

The Jaligot Lemma (Fact 11) implies that $U_{\tilde{q}}(B)$ is the only Sylow $\tilde{q}$-subgroup of G containing U; this forces $j \in N(B)$. As j clearly normalizes B_M, it also normalizes H.

If $C_G^\circ(j) \not\leq B$, then j inverts the group Y given by Lemma B.13. So j inverts $U_{\tilde{q}_M}(B_M)$ (because $\tilde{q}_M \succ \tilde{q}$), and $Y \leq U_{\tilde{q}}(Z(F^\circ(B))) \leq B_M$, so $U_{\tilde{q}_M}(B_M) \leq C_G^\circ(Y) \leq B$, a contradiction. Hence $C_G^\circ(j) \leq B$, which means $j \in I_B$. ◊

Claim 9. $H^{-j} \subseteq F_{\tilde{q}}(B_M)$.

Proof of Claim: It is unfortunately not possible to quickly argue $H^{-j} \subseteq F(B) \cap F(B_M)$: B_M is j-invariant (Claim 8) and S is connected, but it is not sure that S itself normalizes B_M, so Lemma B.5 is not available now.

We shall instead use the Fitting subgroup theorem [BN94, Corollary 7.5], carefully making H^{-j} centralize all of $F(B_M)$. Recall that $F_{\tilde{r}}(B_M) = 1$ for $\tilde{r} \prec \tilde{q}$ by Fact 16 (2d), bearing in mind that the "linking parameter" of the configuration, that is the unipotence parameter of H', is $\tilde{q}$ (Claim 7).

Let $x \in H^{-j}$ and $\tilde{r} \preceq \tilde{q}_M$. If $\tilde{r} \succ \tilde{q}$, then j inverts $F_{\tilde{r}}(B_M)$, so x centralizes $F_{\tilde{r}}(B_M)$. Hence $H^{-j} \subseteq C(F_{\tilde{r}}(B_M))$. We now consider the case $\tilde{r} = \tilde{q}$. Then $F_{\tilde{q}}(B_M) \leq F^\circ(H)$ which is abelian as a connected nilpotent subgroup of two distinct Borel subgroups (Fact 13), so $F_{\tilde{q}}(B_M) \leq Z(F(H))$. But by Lemma B.5 applied to B, $H^{-j} \subseteq B^{-j} \subseteq F^\circ(B)$. Therefore $H^{-j} \subseteq H \cap F^\circ(B) \subseteq F(H) \leq C(F_{\tilde{q}}(B_M))$.

Therefore H^{-j} commutes with all terms in Burdges' decomposition of $F(B_M)$ along unipotent subgroups; it follows $H^{-j} \subseteq C(F(B_M)) \leq F(B_M)$ by [BN94, Corollary 7.5]. Then $H^{-j} \subseteq F(B) \cap F(B_M) = F_{\tilde{q}}(B_M)$ by Fact 16 (2b), and we are done. $\Diamond$

We now derive a contradiction. Consider the group $T_{\tilde{s}}$ introduced in Notation B.15 (1). Let $V \leq H$ be a Sylow $\tilde{s}$-subgroup of H containing $T_{\tilde{s}}$. Notice that by Fact 13, V is abelian.

As j normalizes H (Claim 8), there is an H-conjugate k of j normalizing V. Notice that I_B is closed under conjugation by $H \subseteq B$, so $k \in I_B$. Moreover k normalizes H, so it also normalizes B_M. Hence we may assume that $k = j$.

It follows from Lemma B.16 that j cannot centralize U. Hence V^{-j} is non-trivial. But this is an $\tilde{s}$-group (as a quotient of V), and also $V^{-j} \leq F_{\tilde{q}}(B_M)$, which is $\tilde{q}$-homogeneous by Fact 16 (2b). It follows $\tilde{s} = \tilde{q}$, against Notation 1. This concludes the proof of Proposition B.17. $\square$

4.3.3 Consequences

Corollary B.18. *$T[w]$ is a torsion-free, abelian group; it is disjoint from $F^\circ(B)$, and from $C^\circ_G(j)$ for any $j \in I_B$. B has no p-unipotence.*

Proof. From Proposition B.17 we deduce that $(B \cap B^w)^\circ$ is abelian. By Lemma B.9, $d(T[w]) = d^\circ(T[w]) \leq (B \cap B^w)^\circ$ is an abelian group. It is in particular equal to $T[w]$ and torsion-free. Of course $T[w] \cap F^\circ(B) \leq F^\circ(B) \cap F^\circ(B)^w = 1$, by Proposition B.17 again.

Now let $x \in T[w]^\#$ be centralized by $j \in I_B$. Then $j \in C_\Gamma(x)$ which is w-invariant. By downwards invariance, there is a $C_\Gamma(x)$-conjugate w' of j which normalizes a Sylow 2-subgroup containing j. By connectedness of the Sylow 2-subgroup of Γ, it follows $w' \in C(j)$. In particular, w' normalizes $C^\circ_G(j)$, and the only Borel subgroup of G containing it, namely B (as $j \in I_B$). So $w' \in N(B)$; and also, w' inverts x. It follows from Lemma B.5 that $x \in F^\circ(B)$, a contradiction to Proposition B.17.

Finally, suppose $U_p(B) \neq 1$ for some p, and fix a $T[w]$-minimal subgroup A of $U_p(B)$. If $T[w]$ centralizes A, then $A \leq C^\circ(T[w])$ which is therefore contained in a unique Borel subgroup (Jaligot Lemma, Fact 11), namely B. This is a contradiction as B is not w-invariant. Hence $T[w]$ does not centralize A, and by Wagner's Theorem $T[w]$ contains a non-trivial decent torus, a contradiction to absence of torsion in $T[w]$. $\square$

4.4 Concentration Methods

We now move to proving Theorem B; this will require the concentration machinery, which was first introduced in [Del07]. A more complete exposition is in [DJ09].

We shall study the group $N_B^\circ(T_{\tilde{s}})$. There are three possibilities: it can be a subgroup of a maximal non-abelian intersection, B being the "heavy" or "light" Borel (with respect to unipotence degrees), or no such non-abelian intersection can exist. This trichotomy will not seem extravagant to the reader familiar with [Del07, §3.3.4]; concentration appeared there, in order to deal with the "non-abelian, B heavy" configuration. Interestingly we shall also use it in the abelian scenario. There remains the third case, which we immediately deal with in a very standard way.

4.4.1 A case of no interest

We get rid of a non-abelian intersection with B light by a quick argument. B_M is another Borel subgroup, and $\tilde{q}_M$ a maximal unipotence parameter for B_M.

Lemma B.19. *There is no non-abelian intersection $H = (B \cap B_M)^\circ$ containing $N_B^\circ(T_{\tilde{s}})$ such that $\tilde{q} \prec \tilde{q}_M$.*

Proof. Suppose it is the case.

Notation 1. Let B_M be a Borel subgroup containing $N_B^\circ(T_{\tilde{s}})$ such that $H = (B \cap B_M)^\circ$ is a maximal, *non-abelian* intersection, and $\tilde{q} \prec \tilde{q}_M$.

Notation 2. Let Q be a Carter subgroup of H.

By assumption $\tilde{q} \prec \tilde{q}_M$. In particular, Q is a Carter subgroup of B_M by Fact 16 (3c).

Claim 3. The linking parameter is $\tilde{q}$.

Proof of Claim: Suppose not. Then $U_{\tilde{q}}(Z(F^\circ(B))) \leq Z(H) \leq C(T[w])$, a contradiction to Lemma B.14. $\diamondsuit$

Claim 4. Q is not a Carter subgroup of $C_G^\circ(T_{\tilde{s}})$.

Proof of Claim: Suppose it is. Notice that $T_{\tilde{s}} \leq Q$. By downwards invariance, there is a $C_G^\circ(T_{\tilde{s}})$-conjugate w' of w normalizing Q. Notice that w' cannot normalize B, as otherwise $T_{\tilde{s}} \leq F^\circ(B)^{-w'}$, and $T_{\tilde{s}}$ commutes with $U_{\tilde{q}}(Z(F^\circ(B)))$, against its definition (Notation B.15 (1)).

Bear in mind that the linking parameter is $\tilde{q}$ (Claim 3). In view of Fact 16 (3a), consider the group $N = N_G^\circ(Q_{\tilde{q}}) \geq H$; N is w'-invariant. If $N = H$, then w' normalizes the only two Borel subgroups containing H; in particular it normalizes B, a contradiction. If B is the only Borel subgroup containing N, same contradiction. Hence B_M is the only Borel subgroup containing N.

In particular, $N_G^\circ(Q) \leq N_G^\circ(Q_{\tilde{q}}) \leq B_M$, and $N_G^\circ(Q) \leq N_{B_M}^\circ(Q) = Q$. So Q is a Carter subgroup of G. It is therefore also a Carter subgroup of B; by downwards invariance we may assume that S normalizes Q.

As B_M is unique over N, one finds that S normalizes B_M; so does w'. In particular, applying Lemma B.5, one sees $T_{\tilde{s}} \leq F^\circ(B_M)$. But $\tilde{s} \prec \tilde{q}$ by construction (Notation B.15), so by Claim 3 and Fact 16 (2d), it follows $T_{\tilde{s}} = 1$, a contradiction. $\diamondsuit$

We notice that $N_G^\circ(Q_{\tilde{q}}) \leq B$. Otherwise, $N_G^\circ(Q_{\tilde{q}}) \leq B_M$, so $N_G^\circ(Q) \leq B_M$, and Q is a Carter subgroup of G, a contradiction to Claim 4.

Now $N^\circ_{C_G^\circ(T_{\tilde{s}})}(Q) \leq B$, so $N_{C_G^\circ(T_{\tilde{s}})}(Q) \leq C_B^\circ(T_{\tilde{s}}) \leq H$. It follows that Q is a Carter subgroup of $C_G^\circ(T_{\tilde{s}})$, a contradiction to Claim 4 again. $\square$

4.4.2 Preparing conjugacy

We now prepare for concentration. This requires conjugating unipotent elements to semi-simple elements, in a very loose way which is explained in [DJ09].

Proposition B.20. *There are two definable, abelian, connected subgroups $K \leq \Sigma \leq B$ with the following properties:*

Conj1. *$K = (F^\circ(B))^{-i}$ and $B = K \rtimes C_G^\circ(i)$*

Conj2. *for any $k \in K^\#$, B is the only Borel subgroup containing $C_G^\circ(k)$*

Conj3. *$T[w] \leq \Sigma$ has the same rank as K*

Conj4. *Σ is a Carter subgroup, or a Sylow $\tilde{s}$-subgroup, in $C_G^\circ(T[w])$*

Conj5. *K and Σ are S-invariant*

Concentration matters start here. As we are dealing simultaneously with two distinct cases, the proof will be two-fold.

Lemma B.20.1. Proposition B.20 holds if there is a non-abelian intersection $H = (B \cap B_M)^\circ$ containing $N_B^\circ(T_{\tilde{s}})$ with $\tilde{q} \succ \tilde{q}_M$.

Proof. We shall follow the original [Del07, §3.3.5 et 3.3.6]

Notation 1. Let $\tilde{r}'$ denote the linking parameter (see Fact 16).

Claim 2. We may assume that S normalizes B_M.

Proof of Claim: $H_{\tilde{r}'}$ is a Sylow $\tilde{r}'$-subgroup of B by Fact 16 (2e). So up to B-conjugacy we may assume that S normalizes $H_{\tilde{r}'}$. As B_M is the only Borel subgroup containing $N^\circ(H_{\tilde{r}'})$ by Fact 16 (1h), we are done. $\diamondsuit$

Notation 3 (see [Del07, Notations 3.3.31 et 3.3.33])**.** Let $\Sigma = U_{\tilde{r}'}(H)$ and $K = \Sigma^{-i}$.

Notice that conclusion Conj2e is satisfied.

Claim 4 (see [Del07, Lemme 3.3.32])**.** Σ is $\tilde{r}'$-homogeneous and abelian; $T[w] \leq \Sigma$.

Proof of Claim: Everything as in [Del07, Lemme 3.3.32], except perhaps the homogeneity of $T[w]$, which relied on the presence of involutions in Carter subgroups. We have to handle this question in another way.

Let $U = T_\ell[w]$ for some $\ell \neq \tilde{r}'$: we show $U = 1$. Let $V \geq U$ be a Sylow ℓ-subgroup of H; as $\ell \neq \tilde{r}'$, there is a Carter subgroup Q of H containing V. As S normalizes B and B_M (Claim 6), we may assume that S normalizes Q. Now $[V, i]$ lies in $F^\circ(B) \cap F^\circ(B_M) = F_{\tilde{r}'}(B)$ by Fact 16 (2b), so $[V, i] \leq \Sigma$. On the other hand $[V, i] \leq [V, S]$ is ℓ-homogeneous by Fact 10. As $\ell \neq \tilde{r}'$, it follows $[V, i] = 1$. So i centralizes U, and in view of Corollary B.18 this implies $U = 1$. Hence $T[w]$ is $\tilde{r}'$-homogeneous. $\diamondsuit$

Claim 5 (see [Del07, Lemme 3.3.34])**.** $K = (F^\circ(B))^{-i}$ and $B = K \rtimes C_G(i)$. B is the only Borel subgroup containing $C^\circ_G(K)$. Moreover, K and $T[w]$ have the same rank.

Points Conj2a, Conj2b, Conj2c follow.

Claim 6 (see [Del07, Lemme 3.3.37])**.** $C^\circ_{F_{\tilde{r}'}(B)}(T[w]) = \Sigma$.

In [Del07, Lemme 1.10.4], the following was proved under certain assumptions on the behaviour of the Carter subgroup of H.

Claim 7. $U_{\tilde{r}'}(B_M) = F_{\tilde{r}'}(B_M)$.

Proof of Claim: Let U be any Sylow-$\tilde{r}'$-subgroup of B_M. As S normalizes B_M (Claim 6), we may assume that S normalizes U. We know $U = U^{+i} \cdot U^{-i}$ (the latter is a set). One has $U^{+i} \leq B$, whence $U^{+i} \leq H$; by Fact 9 it is a $\tilde{r}'$-group, so $U^{+i} \leq \Sigma \leq F^\circ(B)$. On the other hand $U^{-i} \subseteq F^\circ(B_M)$ by Lemma B.5. Hence $U \leq F^\circ(B)$ and $U \leq F_{\tilde{r}'}(B_M)$. $\diamondsuit$

Claim 8 (see [Del07, Proposition 3.3.38])**.** Σ is a Sylow $\tilde{r}'$-subgroup of $C^\circ_G(T[w])$.

Proof of Claim: By Claim 11 and [Del07, Lemme 1.10.3], $F_{\tilde{r}'}(B_M)$ is the only Sylow $\tilde{r}'$-subgroup of G containing Σ. Let $U \geq \Sigma$ be a Sylow-$\tilde{r}'$-subgroup of $C^\circ_G(T[w])$. One has $U \leq C^\circ_{F_{\tilde{r}'}(B)}(T[w]) = \Sigma$ by Claim 10. $\diamondsuit$

Requirement Conj2d is met too. This concludes the proof of Lemma B.20.1. $\square$

Lemma B.20.2. Proposition B.20 holds if there is *no* non-abelian intersection $H = (B \cap B_M)^\circ$ containing $N^\circ_G(T_{\tilde{s}})$.

This case is a little more interesting; we at first did not expect to prove anything similar in this configuration.

Proof. The current assumption means that for any Borel subgroup $B_M \geq N^\circ_B(T_{\tilde{s}})$ such that $B_M \neq B$, $(B \cap B_M)^\circ$ is abelian.

Notation 1. Let $\Sigma = N^\circ_B(T_{\tilde{s}})$.

Clearly $T[w] \leq \Sigma$.

Claim 2. Σ is abelian.

Proof of Claim: Let $B_M \geq N_G^\circ(T_{\tilde{s}})$; up to conjugating by w, we may assume $B_M \neq B$. Then $\Sigma \leq (B \cap B_M)^\circ$ which is abelian by assumption; so is Σ. ◊

Claim 3. It is not the case that B is the only Borel subgroup of G containing $N_G^\circ(\Sigma)$.

Proof of Claim: Assume it is. Let $B_M \geq N_G^\circ(T_{\tilde{s}})$ such that $B_M \neq B$; then $N_{B_M}^\circ(\Sigma) \leq (B \cap B_M)^\circ$ which is abelian, whence a subgroup of $C_B^\circ(T_{\tilde{s}}) \leq N_B^\circ(T_{\tilde{s}}) = \Sigma$. So Σ is a Carter subgroup of B_M; it is therefore also a Carter subgroup of $C_G^\circ(T_{\tilde{s}})$.

As the latter is w-invariant, there is a $C_G^\circ(T_{\tilde{s}})$-conjugate w' of w normalizing Σ. Since B is the only Borel subgroup of G containing Σ, w' normalizes B. It follows by Lemma B.5 that $T_{\tilde{s}} \subseteq B^{-w'} \subseteq F^\circ(B)$, a contradiction to its definition (Notation B.15 (1)). ◊

Claim 4. Σ is an abelian Carter subgroup of B; we may assume that S normalizes Σ.

Proof of Claim: Suppose that Σ is not a Carter subgroup of B and let $B_1 \geq N_G^\circ(\Sigma)$ be a Borel subgroup. We suppose that $B_1 \neq B$. Then $(B \cap B_1)^\circ \geq N_B^\circ(\Sigma) > \Sigma$. But on the other hand, $B_1 \geq \Sigma = N_B^\circ(T_{\tilde{s}})$, so $(B \cap B_1)^\circ$ is abelian, whence in $C_B^\circ(T_{\tilde{s}}) \leq N_B^\circ(T_{\tilde{s}}) = \Sigma$, a contradiction. If follows that B is the only Borel subgroup of G containing $N_G^\circ(\Sigma)$, against Claim 15. Hence Σ is a Carter subgroup of B; by Fact 3 we may assume that S normalizes Σ. ◊

This proves clauses Conj2d and Conj2e. Now bear in mind that $X = [U_{\tilde{q}}(Z(F^\circ(B))), T_{\tilde{s}}]$ (Notation B.15 (2)).

Claim 5. X is normal in B.

Proof of Claim: By Claim 16, Σ is a Carter subgroup of B; it follows $B = F^\circ(B) \cdot \Sigma$. Now $X \leq Z(F^\circ(B))$ and $\Sigma = N_B^\circ(T_{\tilde{s}}) \leq N_B(X)$, so $X \lhd B$. ◊

Notation 6. Let $K = \Sigma^{-i}$.

Claim 7. $K = (F^\circ(B))^{-i}$ and $B = K \rtimes C(i)$. Moreover K and $T[w]$ have the same rank.

Proof of Claim: Clearly $K \subseteq (F^\circ(B))^{-i}$; K is a group as Σ is abelian (Claim 16). Moreover, i normalizes $\Sigma = N_B^\circ(T_{\tilde{s}})$, which contains $T[w]$. Of course $T[w] \cap \Sigma^{+i} = 1$ by Corollary B.18, so $\operatorname{rk}\Sigma \geq \operatorname{rk}\Sigma^{+i} + \operatorname{rk} T[w]$; it follows $\operatorname{rk} K \geq \operatorname{rk} T[w] \geq \operatorname{rk}(F^\circ(B))^{-i} \geq \operatorname{rk} K$. Equality follows.

Now K is generic in $(F^\circ(B))^{-i}$, so it is the only definable group generic in $(F^\circ(B))^{-i}$. It follows that $C_B(i)$ normalizes K. In particular the group $K \rtimes C_G(i)$ has rank exactly $\operatorname{rk}(F^\circ(B))^{-i} + \operatorname{rk} C(i) = \operatorname{rk} B$, and therefore $B = K \rtimes C(i)$.

By definition of w (Notation B.8), we know $\operatorname{rk} T[w] \geq \operatorname{rk} K$. On the other hand, $T[w]$ is disjoint from $C_\Sigma(i)$ (Corollary B.18), so $\operatorname{rk} K \geq \operatorname{rk} T[w]$. ◊

In particular, Conj2a and Conj2c hold.

Claim 8. *B is the only Borel subgroup of G containing $C_G^\circ(K)$.*

Proof of Claim: Let $\beta \geq C_G^\circ(K)$ be a Borel subgroup of G. One has $\beta \geq U_{\tilde{q}}(Z(F^\circ(B)))$ and $\beta \geq T_{\tilde{s}}$, so $X \leq \beta'$. It follows $U_{\tilde{q}_\beta}(Z(F^\circ(\beta))) \leq C_G^\circ(X) \leq N_G^\circ(X) = B$, forcing $\tilde{q}_\beta = \tilde{q}$. Fact 11 yields the conclusion. $\Diamond$

Conj2b is an immediate consequence.

Claim 9. *Σ is a Carter subgroup of $C_G^\circ(T_{\tilde{s}})$.*

Proof of Claim: Suppose not. Then $(N_G^\circ(\Sigma) \cap C_G^\circ(T_{\tilde{s}}))^\circ > \Sigma$. As $N_G^\circ(\Sigma)$ is S-invariant and $C_{N_G^\circ(\Sigma)}(i) \leq N_B(\Sigma) = \Sigma$, we may assume that there is $z \in N_{C_G^\circ(T_{\tilde{s}})}^\circ(\Sigma) \setminus \Sigma$ which is inverted by i.

We shall need the order of z to be infinite; suppose it is p. By Lemma B.4 (2), z cannot be toral in $N_G^\circ(\Sigma)$. It follows that the group $U = U_p(N_G^\circ(\Sigma))$ is non-trivial. Now $U \cdot \Sigma$ is nilpotent, and in particular $U_0 = C_U^\circ(\Sigma) \neq 1$. Now $U_0 \leq C_G^\circ(\Sigma) \leq C_G^\circ(K) \leq B$ by Claim 20, whence B contains p-unipotence, against Corollary B.18. Hence z has infinite order, and up to taking a power we assume $d(z) = d^\circ(z)$.

Let $t \in T_{\tilde{s}}$. Recall that $[t, i] \neq 1$ by Proposition B.17. Now z and $z^{-1} = z^i$ commute with t, so z commutes with t and t^i. In particular z commutes with $[t, i] \in K^\#$. Hence z normalizes B; as $d(z) = d^\circ(z)$ it follows $z \in N_G^\circ(B) = B$, so $z \in N_B(\Sigma) = \Sigma$, a contradiction. $\Diamond$

As $C_G^\circ(T[w]) \leq C_G^\circ(T_{\tilde{s}})$, condition Conj2d is satisfied. All axioms of Proposition B.20 are proved. $\square$

This ends the proof of Proposition B.20.

4.4.3 Conjugating and counting

From now on the involution w is free to vary. Recall from Notation B.8 that w was arbitrary in the set I^* of involutions of $i^G \setminus N_\Gamma(B)$ with large $T[w]$-set; in particular, Corollary B.18 and Proposition B.20 hold for any such w.

Lemma B.21. *$T[w]$ and K are G-conjugate.*

Proof. Σ is a Carter subgroup, or a Sylow subgroup, in the w-invariant solvable group $C_G^\circ(T[w])$ by Conj2d. In particular, by Fact 3, there is a $C_G^\circ(T[w])$-conjugate w' of w normalizing Σ. But S also normalizes Σ (Conj2e), so there are $n \in N_\Gamma(\Sigma)$ and $j \in S$ such that $w' = j^n$.

Now w', like w, inverts $T[w]$; if w' normalizes B, then by Lemma B.5, $T[w] \leq F^\circ(B)$, against Corollary B.18. Hence $w' \in I^*$, and $T[w] \leq T[w']$ are connected groups of equal rank; it follows $T[w] = T[w']$. So $T[w] = T[w'] = \Sigma^{-w'} = (\Sigma^{-j})^n$.

As $j \in S$, $j \in C(i)$ must normalize B; it follows $j \in N(B)$. As $j \in i^\Gamma$, there is a unique Borel subgroup of G (which is Γ-conjugate to B) containing $C_G^\circ(j)$; let us call it B_j.

Suppose that $B_j \neq B$, and consider $L = C_K^\circ(j)$. If $L \neq 1$, then $L \leq B_j$ which is $S \leq C(j)$-invariant; by Lemma B.5 one has $L \leq B_j^{-i} \leq F^\circ(B_j)$.

In particular, $U_{\tilde{q}}(Z(F^\circ(B_j))) \leq C^\circ_G(L) \leq B$ by Conj2b, and Fact 11 forces $B_j = B$, a contradiction. So if $B_j \neq B$, one has $L = 1$, whence j inverts K. In particular, $K \leq \Sigma^{-j}$, and $K^n \leq T[w]$; equality follows from rank equality.

Now suppose that $B_j = B$; by definition (Notation **??**), $j \in I_B$. By Remark **??**, $j = i^b$ for some $b \in B$, and it follows $\Sigma^{-j} = K$. In this case again, $T[w]$ and K are conjugate.

So far conjugation takes place in $\Gamma = G \cdot S$; it suffices to notice that $S \leq N_\Gamma(K)$ to have G-conjugacy. □

Lemma B.22. $\mathrm{rk}\{T[w] : w \in I^*\} = \mathrm{rk}\, G - \mathrm{rk}\, B$.

Proof. Let $w_1, w_2 \in I^*$ be such that $T[w_1] \cap T[w_2] \neq 1$. By Lemma B.21 there are $g_1, g_2 \in G$ such that $T[w_i] = K^{g_i}$. By Conj2b, one has $g_1 g_2^{-1} \in N_G(B) = B \leq N(K)$, so $T[w_1] = T[w_2]$. Moreover, w_1 and w_2 lie in $N(B^{g_1})$. We now drop the subscript and write g for g_1.

The product $w_1 i^g$ lies in G as $i^G \subseteq iG$, so $w_1 i^g \in N_G(B^g) = B^g$; moreover $w_1 i^g \in (B^g)^{-i^g} = K^g$. So $w_1 w_2 \in K^g$ which is 2-divisible: w_1 and w_2 are K^g-conjugate.

The family of $T[w]$'s has therefore rank $\mathrm{rk}\, I^* - \mathrm{rk}\, K = \mathrm{rk}\, G - \mathrm{rk}\, B$. □

Fact B.23. *Let G be a simple group of finite Morley rank, and $M < G$ be a definable, proper subgroup. Suppose that there is a definable subset $K \neq \{1\}$ of M such that the generic element of K lies in only finitely many conjugates of K. Let $\Theta = \{\theta[\lambda] : \lambda \in \Lambda\}$ be a uniformly definable family of subsets of B such that $\mathrm{rk}\,\Theta = \mathrm{rk}\, G - \mathrm{rk}\, N_G(K)$. Then for generic $\lambda \in \Lambda$, $\theta[\lambda]$ is* not *G-conjugate to K.*

It suffices to apply Fact B.23 with $M = B$, $\theta[\lambda] = T[w]$, to get a contradiction. This ends the proof of Theorem B.

5 Finer study of the 2-rank

We eventually handle conclusion (C) of the theorem.

Theorem C. *Let Γ be a connected group of finite Morley rank with no unipotent 2-subgroup. Suppose that there is a definable, normal, minimal connected simple subgroup of degenerate type $G \lhd \Gamma$ such that $C_\Gamma(G) = 1$. Then $m_2(\Gamma) \leq 1$.*

The proof relies on the two following propositions, which correspond respectively to [DJ09, §14.2] and [DJ09, §15.3].

Proposition C.1. *Let Γ be a connected group of finite Morley rank with no unipotent 2-subgroup. Suppose that there is a definable, normal, minimal connected simple subgroup of degenerate type $G \lhd \Gamma$ such that $C_\Gamma(G) = 1$. Then $Pr_2(\Gamma) \leq 1$.*

Proposition C.2. *Let Γ be a connected group of finite Morley rank with no unipotent 2-subgroup. Suppose that there is a definable, normal, minimal connected simple subgroup of degenerate type $G \lhd \Gamma$ such that $C_\Gamma(G) = 1$. Then $m_2(\Gamma) \leq 1$.*

5.1 The Pruefer 2-rank

In this subsection we show Proposition C.1.

Suppose that $Pr_2(\Gamma) \geq 2$; by Theorem A, the Pruefer 2-rank is exactly 2. Let $S \leq \Gamma$ be a maximal decent torus; we may assume that $\Gamma = G \cdot S$. In particular, the Sylow 2-subgroup is connected. Notice that since S is abelian, G controls Γ-conjugacy; for any involution $i \in \Gamma$, one has $i^\Gamma = i^G$. We shall show that involutions in Γ are conjugate, which will collapse the Pruefer 2-rank to 1. If they are not, then by connectedness of the Sylow 2-subgroup, there are exactly three distinct Γ-classes, which are also G-classes, and no two conjugate involutions commute.

Notation C.1.1. For $i \in I(\Gamma)$, let $B_i = C_G(i)$ and $\tilde{q}_i$ a maximal unipotence parameter of B_i.

Recall from Theorem B that B_i is a Borel subgroup of G. We shall introduce crossed $T[w]$-sets, a technique directly borrowed from [Del07].

Notation C.1.2. For $i, j \in I(\Gamma)$, let $T_i[j] = \{b \in B_i : b^j = b^{-1}\}$.

Lemma C.1.3 (Devil's Ladder)**.** Let $X \subseteq B_i$ be a non-trivial, definable subset of B_i inverted by an involution $j \in I(\Gamma)$. Then $U_{\tilde{q}_i}(Z(F^\circ(B_i))) \leq C^\circ_G(X) \leq B_i$; in particular $N_\Gamma(X) \leq N_\Gamma(B_i)$.

Proof. Notice that by Fact 11, it suffices to prove the first claim, and the second will follow.

By assumption, $i \in C_\Gamma(X)$ which is j-invariant; by downwards invariance, there is a $C_\Gamma(X)$-conjugate j' of j which normalizes a Sylow 2-subgroup containing i. By connectedness of the Sylow 2-subgroup of Γ, $j' \in C_\Gamma(i)$; as it still inverts X, we may assume $j \in C_\Gamma(i)$. Let $V = \langle i, j\rangle$, a four-group.

By connectedness of the Sylow 2-subgroup, there is a decent torus containing V; it must centralize i, whence normalize B_i; it follows $X \subseteq F^\circ(B_i)$ (the reader not convinced may want to look at Lemma B.5); in particular, $U_{\tilde{q}_i}(Z(F^\circ(B_i))) \leq C^\circ_G(X)$. Notice that it now suffices to show $C^\circ_G(X) \leq B_i$.

If $C^\circ_G(X)$ is abelian, then $C^\circ_G(X) \leq N^\circ_G(U_{\tilde{q}_i}(Z(F^\circ(B_i))) = B_i$, and we are done. So we assume that $C^\circ_G(X)$ is not abelian.

By Fact 2, there is a V-invariant Borel subgroup B_α of G containing $C^\circ_G(X) \geq U_{\tilde{q}_i}(Z(F^\circ(B_i)))$; let $\tilde{q}_\alpha$ be a corresponding maximal unipotence parameter. If $\tilde{q}_\alpha = \tilde{q}_i$, then by Fact 11, one has $B_\alpha = B_i$ and we are done. So we may suppose that $\tilde{q}_\alpha \succ \tilde{q}_i$.

If $U_{\tilde{q}_\alpha}(Z(F^\circ(B_\alpha)))$ is central in B_α, then its centralizer in G lies in $C^\circ_G(U_{\tilde{q}_i}(Z(F^\circ(B_i)))) \leq B_i$; against $\tilde{q}_i \prec \tilde{q}_\alpha$. It follows that $Y_\alpha = [U_{\tilde{q}_\alpha}(Z(F^\circ(B_\alpha))), B_\alpha] \neq 1$; it is a non-trivial, definable, connected, $\tilde{q}_\alpha$-homogeneous (Fact 10), characteristic subgroup of B_α; and therefore V-invariant. As $\tilde{q}_\alpha \succ \tilde{q}_i$, it is clear that i inverts Y_α.

In particular, j and ij cannot both centralize Y_α. It follows that there is $k \in \{j, ij\}$ such that $Z = Y_\alpha^{-k}$ is non-trivial. Notice that Z is still a $\tilde{q}_\alpha$-group, by homogeneity of Y_α, and that it remains V-invariant. Also notice that $Y_\alpha \leq C^\circ_G(Z)$.

Claim. $C^\circ_G(Z) \leq B_\alpha$.

Proof of Claim: If $C^\circ_G(Z)$ is abelian, then clearly $C^\circ_G(Z) \leq N^\circ_G(Y_\alpha) = B_\alpha$. So suppose that $C^\circ_G(Z)$ is not abelian; by Fact 2, there is a V-invariant Borel subgroup B_β such that $Y_\alpha \leq C^\circ_G(Z) \leq B_\beta$. If $\tilde{q}_\beta = \tilde{q}_\alpha$, then by Fact 11 our claim is proved. So we suppose that $\tilde{q}_\beta \succ \tilde{q}_\alpha$.

If $U_{\tilde{q}_\beta}(Z(F^\circ(B_\beta)))$ is central in B_β, then its centralizer in G lies in $C^\circ_G(Y_\alpha) \leq B_\alpha$, against $\tilde{q}_\beta \succ \tilde{q}_\alpha$. So $Y_\beta = [U_{\tilde{q}_\beta}(Z(F^\circ(B_\beta))), B_\beta] \neq 1$; it is a non-trivial, definable, connected, $\tilde{q}_\beta$-homogeneous subgroup characteristic in B_β; in particular it is V-invariant. As $\tilde{q}_\beta \succ \tilde{q}_i$, it is clear that i inverts Y_β. But i also inverts $Y_\alpha \leq B_\beta$; it follows $Y_\beta \leq C^\circ_G(Y_\alpha) \leq B_\alpha$, against $\tilde{q}_\beta \succ \tilde{q}_\alpha$. $\Diamond$

Now Z is centralized by $\ell = ik \in \{j, ij\}$. In particular $Z \leq B_\ell$. We claim that $B_\ell = B_\alpha$. Indeed, B_ℓ is clearly V-invariant, and $\tilde{q}_\ell \succeq \tilde{q}_\alpha \succ \tilde{q}_i$, so i must invert $U_{\tilde{q}_\ell}(Z(F^\circ(B_\ell)))$; as i also inverts Z, it follows $U_{\tilde{q}_\ell}(Z(F^\circ(B_\ell))) \leq C^\circ_G(Z) \leq B_\alpha$, whence $\tilde{q}_\ell = \tilde{q}_\alpha$ and as $Z \leq B_\ell$, the conclusion follows from Fact 11.

Hence $X \subseteq C^\circ_G(X) \leq B_\alpha = B_\ell = C_G(\ell)$, whence X is centralized by $\ell \in \{j, ij\}$. As it is also centralized by i, j cannot invert it: a contradiction. $\square$

Notation C.1.4. For non-conjugate involutions $i, j \in I(\Gamma)$ we let $\pi_{i,j}$ be the definable map

$$\pi_{i,j}: \begin{array}{ccc} j^G & \to & \Gamma/C_\Gamma(i) \\ j' & \mapsto & j'C_\Gamma(i) \end{array}$$

Lemma C.1.5. $\pi_{i,j}$ has generically finite fibers.

Proof. Suppose that generically, the fiber over $jC_\Gamma(i)$ is infinite; we may assume $j \notin N_\Gamma(B_i)$. Taking products, we find an infinite, definable subset X of $C_\Gamma(i)$ inverted by j. Let $x \in X$; write $x = jk$ for another involution k. Modulo G, j both inverts and centralizes x; it follows $x^2 \in G$, whence $x^2 \in C_G(i) = B_i$. If $x^2 \neq 1$, as j inverts x, it follows by Lemma C.1.3 that $j \in N_\Gamma(B_i)$, a contradiction: hence X is a set of involutions, all in $C_\Gamma(i, j)$.

As there are only three G-classes of involutions, we may assume that there are $k, \ell \in X$ which are G-conjugate. In particular, $x = k\ell \in G$ and $1 \neq x \in C_G(i) = B_i$ is inverted by k, whence $j \in N_\Gamma(d(x)) \leq N_\Gamma(B_i)$ by Lemma C.1.3: a contradiction. $\square$

Proof of Proposition C.1. Recall that by connectedness of the Sylow 2-subgroup, there must be exactly three classes of involutions.

Let i, j be non-conjugate. Using functions $\pi_{i,j}$ and $\pi_{j,i}$, we see that $\operatorname{rk}\Gamma - \operatorname{rk} C_\Gamma(j) \leq \operatorname{rk}\Gamma - \operatorname{rk} C_\Gamma(i)$ and symmetrically. It follows that $\operatorname{rk} C_\Gamma(i) = \operatorname{rk} C_\Gamma(j)$ and that $\pi_{i,j}$ and $\pi_{j,i}$ have generic images.

So the generic coset of $C_\Gamma(i)$ contains a representative of the other two conjugacy classes, say j and k. As j and k are not conjugate, their product contains an involution $\ell \in C_\Gamma(i)$, which by structure of the Sylow 2-subgroup can be but conjugate to i. As the only Γ-conjugate of i in $C_\Gamma(i)$ is i, it follows $\ell = i$, whence $j \in C_\Gamma(i)$, and the coset collapses into G. $\square$

5.2 The 2-rank

Proposition C.2. *Let Γ be a connected group of finite Morley rank with no unipotent 2-subgroup. Suppose that there is a definable, normal, minimal connected simple subgroup of degenerate type $G \lhd \Gamma$ such that $C_\Gamma(G) = 1$. Then $m_2(\Gamma) \leq 1$.*

We now attack Proposition C.2. Let S be a maximal decent torus of Γ; it has Pruefer rank 1. We suppose that the 2-rank is ≥ 2 and shall prove a contradiction.

Remark C.2.1. The Sylow 2-subgroup of Γ is isomorphic to that of PSL_2 and all involutions are Γ-conjugate.

Notice that Γ has exactly three G-conjugacy classes of involutions.

Notation C.2.2. Let $j \in C(i) \setminus \{i\}$ be an involution and $V = \langle i, j \rangle$ (a four-group). Fix decent tori S_i and S_j containing i (resp. j) and normalized by j (resp. i).

Instead of extending G by a decent torus (as we did in §4 and §5.1), we need to be a little more careful in order to capture the Sylow 2-subgroup.

Notation C.2.3. Let $\Delta = \langle S_i, S_j \rangle$.

Δ is a definable, connected subgroup of Γ; it has same Sylow 2-subgroup as Γ (and as PSL_2), and acts faithfully on G: we may assume that $\Gamma = G \cdot \Delta$.

Notation C.2.4. For $i \in I(\Gamma)$, let $B_i = C_G(i)$.

By Theorem B, B_i is a Borel subgroup of $C_G(i)$.

Lemma C.2.5.

1. For distinct involutions $k \neq \ell \in \Delta$, $F^\circ(B_k) \cap F^\circ(B_\ell) = 1$.
2. For distinct, *commuting* involutions $k \neq \ell \in \Delta$, $F^\circ(B_k) = B_k^{-\ell}$.

Proof. Let us focus on i and j for the moment.

Notation 1. Let $\tilde{q}$ be a maximal unipotence parameter for B_i.

By Remark C.2.1, $\tilde{q}$ is a maximal unipotence parameter for any B_k.

Claim 2. Let $K < G$ be a definable, connected, V-invariant subgroup. If $U_{\tilde{q}}(Z(F^\circ(B_i))) \leq K$, then $K \leq B_i$.

Proof of Claim: If K is abelian, this is obvious by Fact 11. Otherwise, there is a V-invariant Borel subgroup B_α of G containing K. If $\tilde{q}_\alpha \preceq \tilde{q}$ then by Fact 11 $B_\alpha = B_i$, we are done. So we suppose $\tilde{q}_\alpha \succ \tilde{q}$. In particular, all involutions in V invert $U_{\tilde{q}_\alpha}(B_\alpha)$: a contradiction. $\Diamond$

Notation 3. Let $H = (B_i \cap B_j)^\circ$.

Claim 4. H is proper in B_j and abelian; moreover $N_G^\circ(H) = C_G^\circ(H)$.

Proof of Claim: If H is trivial, everything is proved. If on the other hand $H = B_j$, then i centralizes B_j; hence $C_G^\circ(V) = B_j$ and $G = B_j$ by Fact 5, a contradiction.

We now suppose H non-abelian and consider $N = N_G^\circ(H')$. As $H' \leq F^\circ(B_i) \cap F^\circ(B_j)$, the V-invariant group N is in B_i; by Fact 11, B_i is the only Borel subgroup of unipotence parameter $\tilde{q}$ which contains N. The same applies to B_j, proving $B_i = B_j$, a contradiction. So H is abelian.

Let $N = N_G^\circ(H)$; this group is V-invariant, and one write $N = N^{+i} \cdot N^{-i}$. Notice that as i centralizes H, one has $N^{-i} \subseteq C_G(H)$. We focus on $L = N^{+i}$, which is j-invariant; we may write $L = L^{+j} \cdot L^{-j}$. Notice that $L^{+j} \leq H \leq C_G(H)$, and as above $L^{-j} \subseteq C_G(H)$. Hence $N \subseteq C_G(H)$, and the result follows from connectedness of N. $\Diamond$

Claim 5. We may suppose that j inverts $F^\circ(B_i)$.

Proof of Claim: Supppose that $X_i = C_{F^\circ(B_i)}(j)$ and $X_j = C_{F^\circ(B_j)}(i)$ are both non-trivial. Notice that they are subgroups of H.

By Claim 2, one finds $C_G^\circ(X_i) \leq B_i$, and $C_G^\circ(X_j) \leq B_j$. In particular, $C_G^\circ(H) \leq H$, and by Claim 4, H is a Carter subgroup of G. By downwards invariance, there is a Sylow 2-subgroup of Δ normalizing H. As $V \leq N_\Gamma(H)$, at least one involution of V must be in $N_\Gamma^\circ(H)$; so we may assume that $S_i \cdot \langle j \rangle$ normalizes H. Now j inverts S_i and centralizes H, whence S_i centralizes H. In particular it centralizes X_j, and normalizes the only Borel subgroup of parameter $\tilde{q}$ containing its centralizer. So S_i normalizes B_j, and Δ normalizes B_j.

The involution j centralizes B_j; by conjugacy in Δ, so does i. This forces $B_j = H$, against Claim 4. Hence we may assume that j inverts $F^\circ(B_i)$. $\Diamond$

Claim 6. $B_i^{-j} = F^\circ(B_i)$ and $B_j^{-i} = F^\circ(B_j)$.

Fixed the proof. Please check.

Proof of Claim: By Claim 5, we may assume that j inverts $F^\circ(B_i)$. In particular, $F^\circ(B_i)$ is abelian. As $B_i' \leq F^\circ(B_i) \leq C_{B_i}(F^\circ(B_i))$, it follows that $C_{B_i}(F^\circ(B_i))$ is normal in B_i. It is clearly nilpotent, whence $C_{B_i}(F^\circ(B_i)) \leq F(B_i)$.

Now pick any $x \in B_i^{-j}$. Then x must normalize $F^\circ(B_i)$ which is inverted by j too, and it follows $x \in C_{B_i}(F^\circ(B_i)) \leq F(B_i)$. In particular, $B_i^{-j} \subseteq F(B_i)$. But as x centralizes $F^\circ(B_i)$, one also has that the whole coset $xF^\circ(B_i)$ is contained in B_i^{-j}. Hence B_i^{-j} is a union of $F^\circ(B_i)$-cosets in $F(B_i)$.

Write $B_i = H \cdot B_i^{-j}$ under the action of j and notice that B_i^{-j} has degree 1. It follows that there is only one coset in our union!

Hence $B_i^{-j} = F^\circ(B_i)$ under the assumption given by Lemma 5. As $F^\circ(B_i)$ is abelian, any involution of jB_i has the same action; as i centralizes B_i, any involution $k \in C(i)\setminus\{i\}$ satisfies $B_i^{-k} = F^\circ(B_i)$. Conjugating j to i (in Δ), $F^\circ(B_j) = B_j^{-i}$. $\Diamond$

We now prove Lemma C.2.5.

1. Let $k \neq \ell$ be distinct involutions, and suppose that $X = F^\circ(B_k) \cap F^\circ(B_\ell) \neq 1$. We may assume that $k = i$; in particular, as j inverts $F^\circ(B_i)$, X is V-invariant.

   By Claim 2, $U_{\tilde{q}}(Z(F^\circ(B_i))) \leq C^\circ_G(X) \leq B_i$, and B_i is the only Borel subgroup of G containing $C^\circ_G(X)$ of parameter $\tilde{q}$. It follows that ℓ normalizes B_i. So we may assume that $\ell = j$, and j inverts X by Claim 6, a contradiction.

2. Now suppose that k and ℓ commute; we may suppose $k = i$ and $\ell = j$, and this is Claim 6. $\square$

Proof of Proposition C.2. We are done with tori (which were crucial in the first place, to have Borel subgroups, and to prove Claim 5 of Lemma C.2.5), and let $\hat{G} = G \rtimes V$, which has exactly three conjugacy classes of involutions. Notice however that $C_G(i)$ remains a Borel subgroup, and that its rank does not depend on the involution i.

Claim. $\operatorname{rk} G^{-i} = 2 \operatorname{rk} F^\circ(B_i)$.

Proof of Claim: For non G-conjugate involutions $i, j \in \hat{G}$, the definable hull $d(ij)$ must contain an involution $\ell_{i,j}$, which by structure of the Sylow 2-subgroup represents the third G-class.

Consider the definable function

$$\begin{array}{rccc} \varphi: & i^G \times j^G & \to & k^G \\ & (i_1, j_1) & \mapsto & \ell_{i_1,j_1} \end{array}$$

The map φ is clearly onto. If (i_1, j_1) and (i_2, j_2) lie above the same involution k, then $i_1, i_2 \in C_{\hat{G}}(k)$; as $i_1 i_2 \in G$, one has $i_1 i_2 \in B_k$; as $i_1 \in C(k)$, one even has $i_1 i_2 \in F^\circ(B_k)$ (Lemma C.2.5 (2b)). Conversely, letting $x, y \in F^\circ(B_k) = B_k^{-i_1}$ and bearing in mind that G has no 2-torsion, one finds that $(x i_1, y j_1)$ lies above k.

In particular the fibers of φ have rank exactly $2 \operatorname{rk} F^\circ(B_i)$. It follows $2(\operatorname{rk} G - \operatorname{rk} B_i) - 2 \operatorname{rk} F^\circ(B_i) = \operatorname{rk} G - \operatorname{rk} B_i$, that is $\operatorname{rk} G = \operatorname{rk} B_i + 2 \operatorname{rk} F^\circ(B_i)$. The decomposition of G under the action of i yields the conclusion. $\Diamond$

By Lemma C.2.5 (2a),

$$\operatorname{rk} F^\circ(B_j)^{F^\circ(B_i)} = 2 \operatorname{rk} F^\circ(B_i)$$

Notice however that as $F^o(B_j)$ is contained in G^{-i} and $F^o(B_j)$ is contained in $C(i)$, $F^o(B_j)^{F^\circ(B_i)}$ is contained in G^{-i}.

The same is true of $F^\circ(B_k)^{F^\circ(B_i)}$ for another $k \in C(i) \setminus \{i\}$. In particular, as G^{-i} has degree one, they must intersect; meaning that there are $x_1, x_2 \in F^\circ(B_i)$ such that $F^\circ(B_{j^{x_1}}) \cap F^\circ(B_{k^{x_2}}) \neq 1$. By Lemma C.2.5 (2a) again, it follows $j^{x_1} = k^{x_2}$, meaning that G conjugates involutions in $C(i) \setminus \{i\}$, against the fact that there are three classes. $\square$

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