Groups with a Dimension

Adrien Deloro

NYU Shanghai

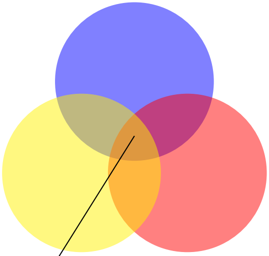
2013/10/09

Part I

What is it about?

The talk is at the intersection of three fields:

- Algebraic geometry
  (the study of polynomial equations)
- Group theory
  (more precisely, finite group theory)
- Model theory
  (a branch of mathematical logic)



Technical name: *groups of finite Morley rank*.
MSC Code: 20F11

Matrix groups

- My first point:

  every mathematician knows what a group is, because matrix groups are everywhere.

- A *matrix group* is a subgroup/subquotient of $\mathrm{GL}_n(\mathbb{K})$ "given by (polynomial) equations", where $\mathbb{K}$ is any field.

  A subquotient of G is H/K, where $K \trianglelefteq H \leq G$.

- E.g. $\mathrm{PGL}_n(\mathbb{K}) = \mathrm{GL}_n(\mathbb{K})/\{\lambda \,\mathrm{Id} : \lambda \in \mathbb{K}^\times\}$.

  E.g. $O_n(\mathbb{K}) = \{M \in \mathrm{GL}_n(\mathbb{K}) : M \cdot M^t = \mathrm{Id}\}$.

  The correct terminology would be: "groups of Lie type".

- My main point will be the following:

  there is such a thing as matrix group theory

Matrix group theory is not linear algebra

My main point:

there is such a thing as matrix group theory

I do *not* mean linear algebra.

In linear algebra one uses *much more structure* than the group structure, for instance the ambient $\mathbb{K}$-algebra structure.

Not convinced? try and give a group-theoretic description of Jordan decomposition $g = u \cdot s$ (open).

First of all I wish to explain the ubiquity of matrix groups:

matrix groups are everywhere

The Erlangen Program

One of the reasons for group theory is geometry.
Felix Klein brought the following idea (1872):

> *geometry is explained by algebra,*
> *because structures control shapes*

F. Klein

Promoting group theory Klein unified “the geometries”
(e.g. projective, orthogonal, etc.)
into the study of $\mathrm{GL}_n(\mathbb{K})$ and subgroups/subquotients
(e.g. $\mathrm{PGL}_n(\mathbb{K})$, $O_n(\mathbb{K})$, etc.).
In the 30's such matrix groups were already “classical” for H. Weyl.
Let us see them from three distinct general points of view.

Lie groups

A *Lie group* is a group equipped with a compatible (real or complex) manifold structure.
The name honours Sophus Lie who introduced abstract *Transformationsgruppen* (1870's).

S. Lie

Simple Lie groups were classified by É. Cartan (1894): it turns out that they are matrix groups (over $\mathbb{R}$ and $\mathbb{C}$).

Remark

In addition to "classical groups" new groups were discovered in the process, the "exceptional groups": E_6, E_7, E_8, F_4, G_2.

Algebraic groups

- An *algebraic group* is a group equipped with an algebraic variety structure (no details needed).
- Simple (linear) algebraic groups were classified by C. Chevalley (1955): it turns out that they are matrix groups.
- Chevalley could reach the Cartan classification of simple Lie groups but over *arbitrary* alg. closed fields.
- By doing this he opened the door to finite equivalents.

Finite simple groups

Classification of the Finite Simple Groups

A finite simple group is one of the following:

- cyclic C_p (p a prime);
- alternating Alt_n ($n \geq 5$);
- "twisted Chevalley" group (i.e. some variant of a matrix group);
- sporadic (26 exceptions).

Note on terminology

The so-called "exceptional groups" *are* groups of Lie type.
The "sporadic groups" come from no known geometry.

CFSG first announced in the 80's, completed ten years ago.
The proof is around 10,000 article pages long.

Looking for a common theorem

The last three classification theorems take the same form.

Template theorem

A simple group with nice properties is a matrix group.

Main question

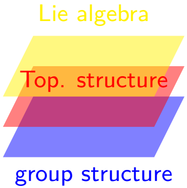
What do Lie groups, algebraic groups and finite groups have in common beyond being groups?

Time to investigate the various layers of structure.

Lie-theoretic methods

A Lie group is a group equipped with a manifold structure.

- Calculus enables one to use infinitesimals and "linearize" the group.
- Tangent space was found to bear a new kind of structure (now called Lie algebra).
- Classify finite-dimensional Lie algebras
- then basic topology finishes the work.

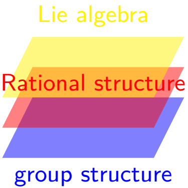


But all this is very specific to $\mathbb{R}$ and $\mathbb{C}$.

Algebraic methods

An algebraic group is a group with extra, “rational” structure.

- There is a topology and dimension (Zariski).
- Alg. groups are actually *functors*. Change base ring; go to $\mathrm{GL}_n(A)$ with $A = \mathbb{K}[\varepsilon]/(\varepsilon^2)$ This *algebraically* captures infinitesimals;
- one thus also retrieves the Lie algebra



All this is extremely far from the group structure.
So what about finite groups?

Methods in finite group theory

Finite group theory uses a variety of tools:

- Sylow theory, which relies on counting arguments;
- "character theory" (a powerful tool developed by Frobenius), but character theory is "non-effective";
- group cohomology (this is tough);
- "involutions", i.e. elements of order 2, and their centralisers;
- *lots* of patience.

Observation

The order provides inductive arguments and some vague flavour of dimension (to be precise: the non-connectedness degree).

Note. An induction-oriented proof of CFSG is *still at work.*

Main point (rephrased)

Could simple Lie groups, simple algebraic groups, and finite simple groups be explained in terms of dimension?

> *Abstract matrix group theory is the theory of groups equipped with a dimension. It captures the common core of Lie theory, alg. group theory, and finite group theory.*

To give a meaning to this I now explain *dimension without a topology*. This requires model theory.

Part II

Definable sets

Definition

Let G be a group. A subset X of G is definable if described by a logical formula involving only the group law and group elements.

Such formulas use $\exists, \forall$, negations, conjunctions, disjunctions.
The typical *basic* logical formula has the form:

$$\forall x_1 \dots \forall x_m \exists y_1 \dots \exists y_m \forall z_1 \dots \forall z_p \; w = 1$$

where w is a group word(=string), such as $x_2^{-1} \cdot y_3 \cdot x_2 \cdot y_1^{-1}$.
One can then take (finite) intersections, unions, and complements.

Limits

- No "there is a subset", no "there is a function".
- No quantifying over integers - only over the group's elements!

As a rule, if you need $\in$, then the set is likely not to be definable.

Examples

- Every finite set is definable by $x = g_1 \vee \cdots \vee x = g_n$.
- The centraliser of g in G, $C_G(g)$ is definable by $gx = xg$.
- Works with $C_G(X)$ for finite X but could a priori fail to be definable when X is infinite. If X is definable, so is $C_G(X)$.
- The normaliser of a *definable* subgroup H, $N_G(H) = \{x : xH = Hx\}$ is definable.
- The commutator subgroup G' has no reason to be definable.

The collection of definable sets

One actually needs to extend the definition a bit to get:

- Cartesian powers
- quotients

Correct definition

The definable sets of G are quotients of definable subsets of G^n by definable equivalence relations.

(where of course "definable" is the notion explained before)
If $K \trianglelefteq H$ are definable, the quotient group H/K is definable.

Note on terminology

Definable sets are a generalization of what alg. geometers call *constructible* sets.

- Notice that we have no topology.
- Model theory classifies structures according to the complexity of combinatorics of definable sets.

Dimension

A group *has a dimension function* if to any definable set can be attached an integer, which does behave like a dimension i.e.:

- $\dim A = 0$ iff A is finite;
- $\dim A \geq n + 1$ iff A contains infinitely many disjoint definable subsets of dimension $\geq n$;
- $\dim A \times B = \dim A + \dim B$;
- and so on.

Note on terminology

The official name is "group of finite Morley rank".

We shall comment on the name later.

The key example

Fact

A matrix group over an alg. closed field is a group with a dimension.

Caution!

A *real* Lie group is *not* a group with a dimension in our sense.

So I won't speak about $U_n(\mathbb{R})$ or $O_n(\mathbb{R})$. This leaves $O_n(\mathbb{C})$.
As a matter of fact the logic of the real field is not as trivial as that of the complex field. This is explained by the following:

Theorem (Macintyre)

A field with a dimension is either finite, or alg. closed.

The deep reason is that "structures with a dimension" are somehow related to alg. geometry.

Erlangen revisited

Klein's point:

> *geometry is explained by algebra,*
> *because structures control shapes*

Model theory is the following line of thought:

> *algebra is explained by logic,*
> *because combinatorics controls structures*

Group theory was a common language for classical geometries.
Similarly, model theory offers a common language for branches of mathematics: algebraic geo., semi-algebraic geo., p-adic geo.. . .
We shall focus on the algebraic geometric aspects of model theory.
The next two slides are more technical and may be slept through.

Categoricity*

- A *theory* T is a collection of logical formulas (as before).
  $\{\forall x \forall y \forall z\ x\cdot(y\cdot z) = (x\cdot y)\cdot z, \quad \forall x\ x\cdot 1 = 1\cdot x = x, \quad \forall x\ x\cdot x^{-1} = x^{-1}\cdot x = 1\}$
- A *model* of T is an object in which all axioms of T are true.
- A theory T is *κ-categorical* if any two models of T of cardinal κ are actually isomorphic.
- In other words: T is κ-categorical if it has up to isomorphism a *unique* model of cardinal κ.
  Categoricity is excellent model-theoretic behaviour.
- The theory of the field $\mathbb{C}$ is $\mathfrak{c}$-categorical ($\mathfrak{c}$=continuum).
- Likewise for any alg. closed field, and for any matrix group over an alg. closed field.
- But the theory of the field $\mathbb{R}$ is not $\mathfrak{c}$-categorical.
- It was conjectured that categoricity = alg. geometry (turned out not quite true).

Morley's Theorem*

T is κ-categorical if it has up to isom. a unique model of cardinal κ.

Theorem (Morley, 60's)

T is κ-categorical iff λ-categorical for any uncountable κ, λ.

- Starting point of pure model theory ($\neq$ univ. algebra)
- Morley introduced some form of abstract dimension.
- It was later found that any κ-categorical theory actually bears this dimension, now called "Morley rank".
- So *groups with a dimension are the invention of model-theorists doing pure model theory, who thus rediscovered algebraic geometry*.

Recap

- Lie groups, alg. groups, and finite simple groups fall into the same family.
- The only common structural aspect is the presence of a dimension function.
- Model theory provides an abstract (and relevant) notion of dimension.
- Matrix groups over alg. closed fields are groups with a dimension.

The main conjecture

Conjecture (Cherlin, Zilber)

A simple group with a dimension is finite, or a matrix group over an alg. closed field.

Note. This is an instance of the much bigger conjecture in pure model theory, since refuted, that “categoricity = alg. geometry”.

Evidence for the conjecture

- No known counterexample
- the possibility to retrieve alg. closed fields inside (most) groups with a dimension
- various classical (but basic) results on alg. groups can be adapted.

Why bother?

Conjecture (Cherlin, Zilber)

A simple group with a dimension is finite, or a matrix group over an alg. closed field.

I'm interested in this conjecture because:

- art for art's sake;
- it explains algebraic groups from an elementary point of view;
- it provides an overview of CFSG;
- it actually suggests methods for implementing chunks of CFSG into computers (GAP...)

Toolbox

Conjecture (Cherlin, Zilber)

A simple group with a dimension is finite, or a matrix group over an alg. closed field.

What could one use?

- ~~infinitesimal methods~~
- ~~algebraic-geometric functorial approach~~
- ~~cohomology~~
- ~~character theory~~
- counting arguments (very partial)
- Sylow theory (partial)

This speaks for adapting basic methods from finite group theory.

The ideological line

The key idea is to import methods from finite group theory.

Groups of finite Morley rank are:

- *historically, groups in model theory*
- *conjecturally, algebraic groups*
- *methodologically, finite groups*

CFSG relies on Sylow theory/centralisers of involutions ($x^2 = 1$).

Fact

Sylow 2-subgroups are conjugate; they have the form:
(bounded exponent$*$divisible)-by-finite.

Still open for $p > 2$.

Positive results

Theorem (Altınel, Borovik, Cherlin, 2008)

Let G be an infinite simple group with a dimension. Suppose that G contains a copy of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. Then G is a matrix group over an alg. closed field.

- The proof is 500 pages long,
- 500 « 10,000: whence a simplified, "generic" version of CFSG.
- Assumption reflects a "characteristic 2" assumption.
- What about the other cases?

Obstacles

Theorem (Feit-Thompson, 1963)

A finite simple group not C_p has a non-trivial Sylow 2-subgroup.

- Appeared in an article of 350 pages!
- Uses character theory (which we don't have).

Big open question; asked 1977

Is there a simple grp with a dimension and no element of order 2?

Worst Question

Is there a simple group of dimension 3 not isomorphic to $\mathrm{PSL}_2(\mathbb{K})$?

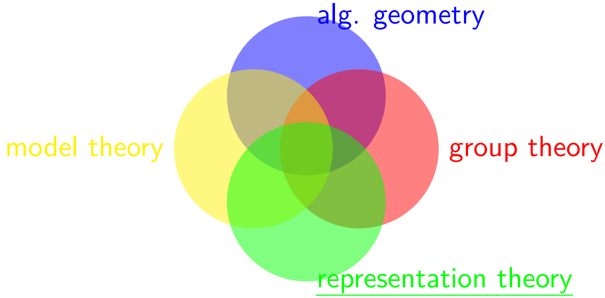
Solve this question (yes or no), you get all logicians' admiration, group-theorists' and even perhaps some physicists' interest!

Last slide!

- What I sometimes do:

  Suppose G (simple, with a dimension) has an involution. Is it a matrix group?

- At the moment: finishing the classification of minimal cases (analogue to Thompson's classical "finite" work).
- Also, and in the future:



T
H
A
N
K
S