$*$-Locally$^\circ_\circ$ Soluble Groups of Finite Morley Rank

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Part I

Generix rises

From Gregory's talk

- Groups of finite Morley rank are groups with a dimension on definable sets

  for pure model-theorists: definable = in *eq*, with parameters
- Infinite simple groups of finite Morley rank were conjectured to be algebraic (Cherlin, Zilber)
- Strategy: do finite group theory (Borovik)
- Partial answer under technical assumption ("even type")
- Talk is in "odd type"

G will always stand for a group of finite MR.

Groups of odd type

- G has odd type if its Sylow 2-subgroups S are divisible-by-finite, $S^\circ \simeq \mathbb{Z}_{2^\infty}^d$
  $d = \mathrm{Pr}_2(G)$ is called the Pruefer rank
- Real-life fact: simple alg. group has odd type iff base field has char. $\neq 2$
- Open problem: classify simple groups of odd type
  too complicated as "bad" sections could appear
- Observation: d measures the size in a loose sense
- Real-life fact: only simple alg. group with $\mathrm{Pr}_2(G) = 1$ is $\mathrm{PSL}_2(\mathbb{K})$.
- Burdges et al. have results for large values of $\mathrm{Pr}_2(G)$.

An extreme form of smallness

When does a simple group of odd type have small $\mathrm{Pr}_2(G)$?
When is it $\mathrm{PSL}_2(\mathbb{K})$?

Definition

G is minimal simple (aka "FT" in Jaligot's first preprints) if it is simple but every proper, definable, connected subgroup is soluble.

- Real-life fact: only minimal simple algebraic group is $\mathrm{PSL}_2(\mathbb{K})$
- Hopeless problem: classify minimal simple groups
  (bad groups would be min. simple)
- More realistic: classify min. simple groups of odd type.
- We now review Jaligot's work in odd type.

2000: "FT-Groupes"

FT-groupes

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1 Introduction.

Selon la conjecture de Cherlin-Zil'ber, un groupe simple infini de rang de Morley fini devrait être isomorphe, comme groupe abstrait, à un groupe algébrique sur un corps algébriquement clos. Ce préprint est dédié à l'étude d'une classe restreinte de groupes de rang de Morley fini qu'il convient de classifier :

Définition 1.1. – *On appelle FT-groupe tout groupe de rang de Morley fini connexe, non résoluble et dont les sous-groupes propres, définissables et connexes sont résolubles.*

Un FT-groupe simple G est un groupe de Frobenius complément-rempli avec pour complément de Frobenius un sous-groupe de Borel de G, ou bien a deux sous-groupes de Borel distincts avec une intersection infinie ([4], théorème 5.2). Devant la consistance et les propriétés algébriques bien déterminées du premier cas, les arguments d'analyse locale effectués ici ne concernent que le deuxième cas. Ce préprint contient un résultat très partiel, dans le but d'obtenir un jour une preuve d'une conjecture plus faible que la conjecture de Cherlin-Zil'ber :

Conjecture - *Un FT-groupe simple G est soit un groupe de Frobenius complément-rempli, avec pour complément de Frobenius un sous-groupe de*

1

Theorem (Jaligot, 2000 preprint)

Let G be a minimal connected simple group of odd type and Pruefer rank 1. Let S be a Sylow 2-subgroup and $i \in S^\circ$.
Suppose that there exist a Borel subgroup $B \geq C_G^\circ(i)$ and some B-minimal subgroup $U \leq B$ such that $C^\circ_{C^\circ_G(i)}(U) = 1$.
Then $G \simeq \mathrm{PSL}_2(\mathbb{K})$ with $\mathrm{char}\,\mathbb{K} \neq 2$.

- Real life: $B = \{$upper-triangular matrices$\}$, $U = \{$strictly upper-triangular matrices$\}$
- Proof by rank computations à la Borovik-Nesin (CIT-groups)
- Jaligot retrieves information on the action on G/B "Zassenhaus group" and then applies Nesin's general tools
- First result *specifically in odd type*

A few details on "rank computations"

Consider $\Gamma = \mathrm{PSL}_2(\mathbb{K})$ and $\iota = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

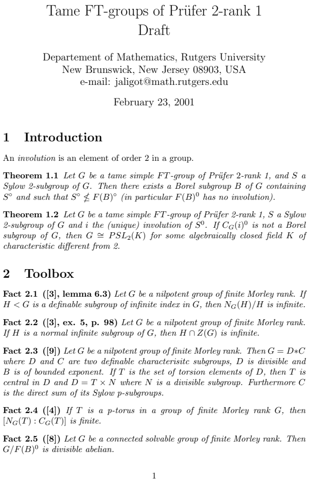
Then $C^\circ_\Gamma(\iota) = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\} < \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\} = B$ (Borel)

Goal: find some rank estimate on Γ, equiv. $C^\circ_\Gamma(\iota)$.

Let $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, a conjugate of ι.

- *Key observation:* $\{\beta \in B : \beta^\omega = \beta^{-1}\} = C^\circ_\Gamma(\iota)$.
- *Key idea:* use $\{\beta \in B : \beta^\omega = \beta^{-1}\}$ throughout.
  For generic ω, this set is large.
- First issue: not a subgroup!

2001: "Tame FT-Groups"



Tame FT-groups of Prüfer 2-rank 1
Draft

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February 23, 2001

1 Introduction

An *involution* is an element of order 2 in a group.

Theorem 1.1 *Let G be a tame simple FT-group of Prüfer 2-rank 1, and S a Sylow 2-subgroup of G. Then there exists a Borel subgroup B of G containing S° and such that $S^\circ \not\leq F(B)^\circ$ (in particular $F(B)^0$ has no involution).*

Theorem 1.2 *Let G be a tame simple FT-group of Prüfer 2-rank 1, S a Sylow 2-subgroup of G and i the (unique) involution of S^0. If $C_G(i)^0$ is not a Borel subgroup of G, then $G \cong PSL_2(K)$ for some algebraically closed field K of characteristic different from 2.*

2 Toolbox

Fact 2.1 ([3], lemma 6.3) *Let G be a nilpotent group of finite Morley rank. If $H < G$ is a definable subgroup of infinite index in G, then $N_G(H)/H$ is infinite.*

Fact 2.2 ([3], ex. 5, p. 98) *Let G be a nilpotent group of finite Morley rank. If H is a normal infinite subgroup of G, then $H \cap Z(G)$ is infinite.*

Fact 2.3 ([9]) *Let G be a nilpotent group of finite Morley rank. Then $G = D{*}C$ where D and C are two definable characterisitc subgroups, D is divisible and B is of bounded exponent. If T is the set of torsion elements of D, then T is central in D and $D = T \times N$ where N is a divisible subgroup. Furthermore C is the direct sum of its Sylow p-subgroups.*

Fact 2.4 ([4]) *If T is a p-torus in a group of finite Morley rank G, then $[N_G(T) : C_G(T)]$ is finite.*

Fact 2.5 ([8]) *Let G be a connected solvable group of finite Morley rank. Then $G/F(B)^0$ is divisible abelian.*

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Theorem (Jaligot, 2001 preprint)

*Let G be a **tame** minimal connected simple group of odd type and Pruefer rank 1. Let S be a Sylow 2-subgroup and $i \in S^\circ$.*
Suppose that $C^\circ_G(i)$ is not a Borel subgroup.
Then $G \simeq \mathrm{PSL}_2(\mathbb{K})$ with $\mathrm{char}\,\mathbb{K} \neq 2$.

- Real life: $C^\circ_G(i) = \{\text{diagonal matrices}\}$ is not a Borel subgroup.
- Tameness="no bad fields appear"
  Used throughout; every time in bold, upper case.
- Jaligot aims a controlling intersections of Borel subgroups
- + same methods as for J00.

A few details

The key idea:

Fact

In $\mathrm{PSL}_2(\mathbb{K})$*, Borels meet over tori, no more.*

"Jaligot Lemma" template

In minimal simple groups, Borels do not share unipotence.

Key problem: (as of 2001), no good fMR notion of unipotence.

2004: "Tame Minimal Connected Simple Groups"

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Tame minimal simple groups of finite Morley rank

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Abstract

We consider tame minimal simple groups of finite Morley rank and of odd type. We show that the Prüfer 2-rank of such a group is bounded by 2. We also find all potential nonalgebraic configurations; there are essentially four of them, and we delineate them with some precision.

1. Introduction

The role of groups of finite Morley rank in model theory was first seen in the work of Zilber on $\aleph_1$-categorical theories ([33], cf. [35]). Motivated by a sense that most interesting structures occur "in nature," Cherlin and Zilber independently proposed:

Classification Conjecture. *A simple infinite group of finite Morley rank is isomorphic as an abstract group to an algebraic group over an algebraically closed field.*

To date there have been three fruitful lines of attack on this problem. First of all, one may simply attempt to mimic the theory of algebraic groups. The second line of attack is to embed the problem in model theory proper. The third line, taken here and in numerous related recent articles, is to see what can be done by the methods of finite group theory, consisting of local geometrical analysis and some considerations involving involutions

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Theorem (Cherlin-Jaligot, 2004 paper)

*Let G be a **tame** minimal simple group of odd type. Then $\mathrm{Pr}_2(G) \leq 2$. Lots of information on non-algebraic configurations.*

- Mixes many methods (local analysis, genericity arguments, number theory)
- Critical use of tameness (yields some form of unipotence)
- Intersection control made systematic
- The Burdges-Cherlin-Jaligot 2007 "milkshake" completes the picture.

The Monsters

CJ04 reaches three potential non-algebraic configurations.

Theorem (CJ04+BCJ07 milkshake)

Let G be a tame minimal simple group of odd type with $G \not\simeq \mathrm{PSL}_2(\mathbb{K})$. Then:

- $\mathrm{Pr}_2(G) \leq 2$ *and involutions are conjugate;*
- *for* $i \in I(G)$, $C_G^\circ(i)$ *is a Borel;*
- *depending on* S, *three possibilites:*
  1. $S = \mathbb{Z}_{2^\infty}$, *not much to say;*
  2. $S = \mathbb{Z}_{2^\infty} \rtimes \mathbb{Z}/2\mathbb{Z}$, *Weyl involutions invert* $C_G^\circ(i)$ *(cf.* $\mathrm{SO}_3(\mathbb{R})$*);*
  3. $S = \mathbb{Z}_{2^\infty}^2$; *there is* w *of order* 3 *permuting involutions in* S°.

No progress towards removing these has been made since.

Part II

The Mentor

from milkshakes to Laphroaig

First step

Theorem (D. 2007)

Let G be a minimal simple group of odd type. Then, (essentially) same conclusions as CJ04.

Critical tool: Burdges' unipotence theory to control intersections.
Applications:

- a "Jaligot Lemma" in characteristic 0
- systematic use of Burdges' "Bender method"

Question

And what about larger groups?

$*$-Local$^\circ_\circ$ solubility

Groups of Finite Morley Rank with Solvable Local Subgroups

Eric Jaligot
(joint work with Adrien Deloro)

In the Classification of the Finite Simple Groups, the study of finite simple groups whose proper subgroups are all solvable, the *minimal simple* groups, has been a cornerstone in the whole process. The *local analysis* of these groups, done by J. Thompson originally for the Odd Order Theorem, has been used to get a classification in presence of involutions. This has then been slightly generalized, with only very few additional groups, to a classification of nonsolvable finite groups in which normalizers of nontrivial abelian subgroups are all solvable. This full classification appeared in a series of papers starting with [Tho68]

The work presented here is an analog of this final transfer in the context of groups of finite Morley rank. Indeed, a large body of results have been obtained in the last few years concerning *solvable* and *minimal connected simple* groups of finite Morley rank, i.e. connected simple groups of finite Morley rank in which proper definable connected subgroups are all solvable. The present work can be considered as a "collapse" on these two classes of the more general class of groups of finite Morley rank naturally defined by mimicing the finite case. More precisely, and as we prefer to work with connected groups throughout, we say that a group of finite Morley rank is

- *locally solvable* if $N(A)$ is solvable for any nontrivial definable abelian subgroup A.
- *locally*$^\circ$ *solvable* if $N(A)$ is solvable for any nontrivial definable connected abelian subgroup A.
- *locally solvable*$^\circ$ if $N^\circ(A)$ is solvable for any nontrivial definable abelian subgroup A.
- *locally*$^\circ$ *solvable*$^\circ$ if $N^\circ(A)$ is solvable for any nontrivial definable connected abelian subgroup A.

Definition

G is $*$-locally$^\circ_\circ$ soluble if: $A \neq 1$, definable, connected, soluble $\Rightarrow N_G^\circ(A)$ soluble

- Project announced at Oberwolfach, January 2007.
- Results as well but under much nicer assumption (close to minimal simple).
- Talk is about the project.

First results

$*$-Locally$^\circ_\circ$ soluble project resulted in 3 papers + 1 preprint.

1. general setting and rewriting of Burdges' theory
2. $*$-locally$^\circ_\circ$ soluble groups and involutions
3. measuring Pruefer p-ranks in $*$-locally$^\circ_\circ$ soluble groups
4. DJprep was meant to deal with $*$-locally$^\circ_\circ$ soluble groups of odd type (and generalise CJ04).

A parallel

Meanwhile, the American Connection (Burdges-Cherlin-D) worked on toral automorphisms of min. simple groups with no involutions.

Theorem (BCD, 2009 preprint)

Let G be a minimal simple group with no involutions. Then we understand definable actions of $\hat{T} \simeq \mathbb{Z}_{2^\infty}^d$ on G:

- $\Pr_2(\hat{T}) = 1$;
- $C_G(\hat{i})$ *is a Borel for* $\hat{i} \in I(\hat{T})$;
- *no dihedral group allowed.*

Important corollary: PSL_2 cannot act on G.

Metamorphosen

paper	size	tame?	type	involutions
J00	min. simple	N	odd	inner
J01	min. simple	Y	odd	inner
CJ04	min. simple	Y	odd	inner
D07	min. simple	N	odd	inner
DJprep	Loc$^\circ_\circ$ Sol.	N	odd	inner
BCD09	min. simple	N	deg.	outer

Can one put this all together?

Part III

De même qu'autrefois nous partions pour la Chine

(Shanghai Express)

The Shanghai Gesture

Theorem (DJ4)

Let $G \trianglelefteq \hat{G}$ be definable, connected groups of odd type. Suppose G is $*$-locally$^\circ_\circ$ soluble group. Suppose that for $\hat{i} \in I(\hat{G})$, $C^\circ_G(\hat{i})$ is soluble. Then, (essentially) the same conclusions as D07, ie. if $G \not\simeq \mathrm{PSL}_2(\mathbb{K})$ then:

- $\mathrm{Pr}_2(\hat{G}) \leq 2$ *and involutions are conjugate;*
- *for $\hat{i} \in I(\hat{G})$, $C^\circ_G(\hat{i})$ a Borel;*
- *The three monsters + an extra BCD09 monster.*

Portions of the proof

(For the experts):

- Study of the Sylow 2-subgroup à la Cotorix
- Rank computations in the tradition of J00 and J01.
- Algebraic identification now relies on BN pairs (Wiscons)
- Concentration argument in an inductive setting
  The technical part, also the one that underwent deepest changes.
  Claim: if Sylow 2-subgroup is connected then $C_G^\circ(\hat{\imath})$ is a Borel
  No prior knowledge of $\mathrm{Pr}_2(\hat{G})$ required!

  "Concentration" for both non-abelian and abelian intersections
- Strong embedding: copy-paste of the core of the milkshake.

  Rest of BCJ07 plays no role: do not consider non-central involutions since all involutions are central in Borels.

Last slide

Result will be used to study modules of MR 3 (j.w. Borovik).

Question

Can one classify simple groups no simple sections of which have a connected Sylow 2-subgroup?

"To Be Continued"

DJ4: Modnet preprint #736