

A little geometric algebra: Involutions in $SO_3(\mathbb{R})$

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How far off limits am I?



FORTUNA FORTES JUVAT
is a Latin saying, meaning:

Fortune favours the strong.

I have always thought so.

And now... for something completely different.

The Erlangen Programme

In his inaugural course of 1872 known as “the Erlangen Programme”, Felix Klein promoted a revolutionary idea:

*geometry is explained by algebra,
because structures control shapes.*



F. Klein

With the language of group theory Klein unified “the geometries”

(e.g. projective, orthogonal, etc.)

into the study of $GL_n(\mathbb{K})$ and subgroups/subquotients

(e.g. $PGL_n(\mathbb{K})$, $O_n(\mathbb{K})$, etc.).

So quick was the development of this new point of view that in the 30's such matrix groups were already “classical” for Hermann Weyl.

Geometric Algebra

Geometric algebra is the study of the fundamental interplay

algebraic structures $\longleftrightarrow$ geometries

both ways:

- given a geometry, understand its group of transformations
(eg. the group of transformations of Lorentz space-time is $O_{3,1}(\mathbb{R})$);
- given a group, reconstruct some geometry from it
(eg. let $\text{Alt}(5)$ act on its Sylow 5-subgroups and find an icosahedron).

Interestingly we do *not* know everything in this vein.

This is an undergraduate talk in geometric algebra.

In this talk:

I shall try to do some geometric algebra in $SO_3(\mathbb{R})$ and $PGL_2(\mathbb{C})$, and hopefully discuss basic open questions. You need to know about:

- linear algebra, groups, matrix groups;
- basic geometry, rotations.

Also, whenever I say “Fact”, it means: “Exercise”.

Outline:

- ① Introduction
- ② Geometric algebra in $SO_3(\mathbb{R})$
- ③ The groups $SO_3(\mathbb{R})$ and $PGL_2(\mathbb{C})$

Shall we?

Isometries

First I shall focus on the Euclidean vector space $\mathbb{R}^3$. It is equipped:

- with its usual $\mathbb{R}$ -linear structure: $\lambda \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 + y_1 \\ \lambda x_2 + y_2 \\ \lambda x_3 + y_3 \end{pmatrix}$;
- with its usual norm: $\left\| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Definition

An *isometry* of $\mathbb{R}^3$ is a linear map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that preserves the norm, viz. for each $\mathbf{x} \in \mathbb{R}^3$ one has $\|f(\mathbf{x})\| = \|\mathbf{x}\|$.

Fact

An isometry even preserves the dot product, viz. $\langle f(\mathbf{x}) | f(\mathbf{y}) \rangle = \langle \mathbf{x} | \mathbf{y} \rangle$. In particular it preserves perpendicularity.

The orthogonal group

Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the standard basis of $\mathbb{R}^3$.

Fact

The following are equivalent:

- f is an isometry;
- $M := \text{Mat}_{\mathcal{B}} f$ satisfies $M \cdot M^t = I_3$;
- for any orthonormal basis $\mathcal{B}'$, $M' := \text{Mat}_{\mathcal{B}'} f$ satisfies $M' \cdot M'^t = I_3$.

Definition

Let $O_3(\mathbb{R}) = \{M \in GL_3(\mathbb{R}) : M \cdot M^t = I_3\}$, called the *orthogonal group*.

Fact

It is a subgroup of $GL_3(\mathbb{R})$.

Rotations and reflections

You know what rotations and reflections are. *They are isometries.*

Fact

r is a rotation iff there is an orthonormal basis $\mathcal{B}'$ in which:

$$\text{Mat}_{\mathcal{B}'} r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \quad \text{for some } \theta.$$

Fact

s is a reflection iff there is an orthonormal basis $\mathcal{B}'$ in which:

$$\text{Mat}_{\mathcal{B}'} s = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

More on rotations and reflections

Fact

- *If r is a rotation, then $\det r = 1$;*
- *if s is a reflection, then $\det s = -1$;*
- *$SO_3(\mathbb{R}) := \{M \in O_3(\mathbb{R}) : \det M = 1\}$ is a subgroup of $O_3(\mathbb{R})$, called the special orthogonal group.*
- *Every element of $SO_3(\mathbb{R})$ is a rotation.*

Hints for the last: there is an eigenvalue (odd dimension). The orthogonal of an invariant space is invariant as well.

From now on I focus on rotations.

Let us do geometric algebra in $SO_3(\mathbb{R})$.

Involutions in $\text{SO}_3(\mathbb{R})$

Every rotation is characterised by an oriented axis and an angle.

If you forget “oriented”, one gets *two* rotations r_1, r_2 with $r_1 = r_2^{-1}$.

But if the rotation angle is π , then one need not orient the axis.

Definition

A *half-turn* is a rotation with angle π (regardless of orientation).

Hence r is a half-turn iff there is an ONB $\mathcal{B}'$ in which $\text{Mat}_{\mathcal{B}'} r = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$.

Definition

An *involution* of a group G is an element $g \in G$ with $g^2 = 1 \neq g$.

Fact

Involutions of $\text{SO}_3(\mathbb{R})$ are exactly the half-turns of the space $\mathbb{R}^3$.

The projective plane

There is a natural bijection between:

- $\{\text{involutions of } SO_3(\mathbb{R})\} = \{\text{half-turns of } \mathbb{R}^3\}$ and
- $\{\text{lines through } 0 \text{ in } \mathbb{R}^3\} = \mathbb{P}^2(\mathbb{R})$, the *projective plane*.

A geometric model for the projective plane: sphere modulo antipodation.

An algebraic model:

- $\mathbb{P}^2(\mathbb{R}) = (\mathbb{R}^3 \setminus \{0\}) / \sim$ where $\mathbf{x} \sim \mathbf{y}$ if there is $\lambda \neq 0$ with $\mathbf{y} = \lambda \mathbf{x}$;
- If P is a plane through 0, it defines a *projective line*

$$[P] = \{\text{lines contained in } P\}$$

A vector *line* becomes a projective *point*, while a vector *plane* becomes a projective *line*. For instance when:

- $\lambda = [L]$ is the projective point associated to vector line L , and
- $\pi = [P]$ is the projective line associated to vector plane P .

then: $\lambda \in \pi$ iff $L \leq P$.

Commuting involutions

Fact

Let $i \neq j \in SO_3(\mathbb{R})$ be distinct involutions. Then the following are equivalent:

- *i and j commute;*
- *$L_i \perp L_j$ (axes of the rotations i and j);*
- *$L_i \leq P_j$ (axis of i , and plane where rotation j takes place).*

Hint. If i and j commute, then i will preserve L_j setwise, so L_j is an eigenspace for i . Conversely if $L_i \perp L_j$ then construct a common eigenbasis.

As a consequence, one retrieves $(\mathbb{P}^2(\mathbb{R}), \in)$ from $(SO_3(\mathbb{R}), \cdot)$:

- the underlying set “ $\mathbb{P}^2(\mathbb{R})$ ” is the set of involutions;
- the incidence relation “ $\lambda \in \pi$ ” is given by $\lambda = [L_i]$, $\pi = [P_j]$ with $ij = ji \neq 1$.

The Hilbert-Desargues Theorem

Fact (this one is *not* an exercise, but a theorem by Hilbert)

In $(\mathbb{P}^2(\mathbb{R}), \leq)$ one can reconstruct the field $(\mathbb{R}; +, \cdot)$.

This fascinating result would deserve an entire (undergraduate) talk.
Summing up:

Using geometric algebra, in $(SO_3(\mathbb{R}), \cdot)$ one can reconstruct $(\mathbb{P}^2(\mathbb{R}), \in)$, hence $(\mathbb{R}, +, \cdot)$.

The group encodes the geometry!

An abstract point of view

Now I shall try to re-explain from a purely group-theoretic perspective.

Let $G = \mathrm{SO}_3(\mathbb{R})$. Let $i := \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$.

Fact

- *Every involution is conjugate to i (conjugate in matrix form).*
- *Every element is a product of two involutions.*
(Hence every element is inverted by some involution.)
- *Every element commutes with one involution (every rotation has an axis).*
- *No element of order > 2 commutes with two distinct involutions (only one axis).*

Centralisers of involutions in $SO_3(\mathbb{R})$

The picture is the following.

- $C := \left\{ \begin{pmatrix} 1 & & \\ & \cos \theta & -\sin \theta \\ & \sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$ is the group of all rotations with same axis as i . Clearly abelian.
- $w := \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$ inverts C .
- $N := N_G(C) = C \rtimes \langle w \rangle$ is the group of all rotations stabilising L_i .
- Conjugates of C are disjoint and cover all of G :
we have a partitioning $G = \bigsqcup_{g \in G/N} C^g$.

Now let us change group.

$\mathrm{PGL}_2(\mathbb{C})$

Let $G = \mathrm{PGL}_2(\mathbb{C}) (= \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times \mathrm{Id})$. In the quotient I use square bracket matrices.

- Let $i = \begin{bmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{bmatrix}$ and $w = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$, two involutions.
- Let $C = \left\{ \begin{bmatrix} \lambda & \\ & \mu \end{bmatrix} : \lambda, \mu \in \mathbb{C}^\times \right\} \leq C_G(i)$, inverted by w .
- Let $N = N_G(C) = C_G(i) = C \rtimes \langle w \rangle$.

Fact

- C is disjoint from its conjugates;
- $[N : C] = 2$;
- N almost covers G , meaning that $\bigsqcup_{g \in G/N} C^g$ is “large” in G .

Strikingly common features...

For the experts

Theorem (D.-Wiscons; to appear BLMS)

Let G be a connected, $U_2^\perp$ group of finite Morley rank with involutions. Let $C < G$ be definable, connected, and generically partitioning. Then $\bigsqcup_{g \in G/N} C^g$ does not contain all strongly real elements.

As instant corollaries: results by Borovik-Corredor-Nesin-Poizat, Burdges-Cherlin, Borovik-Burdges, and Altinel-Berkman-Wagner.

Conjecture

Let G be a connected, $U_2^\perp$ group of finite Morley rank with involutions. Let $C < G$ be definable, connected, and generically partitioning. Then:

- *either $N := N_G(C)$ is strongly embedded,*
- *or $G \simeq \mathrm{PGL}_2(\mathbb{K})$.*

Beyond Erlangen

Alfred Tarski, Anatoli Maltsev and Abraham Robinson brought the revolutionary idea that:

*algebra is explained by logic,
because theories control structures.*

That would take us too far. . . to “model theory”.

Actually this was a talk inspired by ongoing research in model theory.

(In particular the notion of “largeness” can be made precise in model theory.)

Want more?

I finish with recommended references.

- Emil ARTIN, *Geometric Algebra*, Wiley Classics Library, 1957.
- Lawrence GROVE, *Classical groups and geometric algebra*, Graduate Studies in Mathematics no. 39, American Mathematical Society, 2002.
- Donald TAYLOR, *The geometry of the classical groups*, Sigma Series in Pure Mathematics no. 9, Heldermann, 1992.

Thanks!