# Lecture 2 — Fields in Groups

Today we discuss fields. We begin with the study of abstract ranked fields, which are algebraically closed (Macintyre's Theorem), but can be model-theoretically much more complicated than in geometry. Then we shall see how fields tend to spontaneously appear (definably) in ranked groups: the most famous case is that of soluble, non-nilpotent groups (Zilber's Field Theorem), but it is not the only one.

Before we start, remember that any field interpretable in a ranked group must be ranked as well. Also bear in mind that by Poizat's Theorem on rank functions, fields of finite Morley rank and ranked fields are the same.

## 1 Abstract fields of finite Morley rank

Let us begin with properties of the base field in algebraic geometry. We shall address the following questions.

**Question.**

*(Q1). First, a pure algebraically closed field has rank and degree* 1 *("strong minimality": immediate from quantifier elimination).*

*Conversely, is an (infinite) field of finite Morley rank algebraically closed? (luckily, YES).*

*(Q2). Must a field structure of finite Morley rank have rank* 1*? (unfortunately, NO).*

*(Q3). It is a non-trivial theorem by Poizat that an infinite field $\mathbb{L}$ definable in a pure algebraically closed field $\mathbb{K}$ must be isomorphic to $\mathbb{K}$ (even definably so!).*

*Is it true if $\mathbb{K}$ is simply of finite Morley rank? (unfortunately, NO).*

*(Q4). It is a non-trivial fact from algebraic geometry that if $\mathbb{K} \models \mathrm{ACF}$ is a pure algebraically closed field, then $\mathbb{K}_+$ and $\mathbb{K}^\times$ are* minimal *(i.e., contain no constructible, proper, infinite subgroups).*

*Is the same true if $\mathbb{K}$ is a field of finite Morley rank? (unfortunately, NO).*

### 1.1 Good news

We first answer Question (Q1) above.

**Theorem** (Macintyre's theorem on fields; [Poizat, Theorem 3.1])**.** *If $\mathbb{K}$ is an infinite field of finite Morley rank, then $\mathbb{K}$ is algebraically closed.*

*Proof.* This involves a bit of Galois theory. The proof is extremely interesting in its own right but not in the spirit of the lectures. It remains true in the $\omega$-stable context. □

**Exercise** (Cherlin, Shelah)**.** Any infinite (possibly non-commutative) domain of finite Morley rank is an algebraically closed field. (Here again, this generalises to $\omega$-stable.)

Hint: division ring by the DCC. Then take a minimal counter-example to commutativity and use Reineke's Theorem on minimal groups (take a definable subgroup of $\mathbb{K}^\times$) to derive a contradiction.

Good news almost stop here, save for one little bit around Question (Q4).

**Lemma** (Zilber)**.** *If $\mathbb{K}$ is a field of finite Morley rank of characteristic zero, then $\mathbb{K}_+$ is minimal (no infinite, definable, proper subgroup).*

*Proof.* Let $A < \mathbb{K}_+$ be a infinite, definable, proper subgroup; we shall prove $A = 0$. Let $N = \{x \in \mathbb{K} : xA = A\}$. Clearly $N$ is a definable subfield of $\mathbb{K}$. Since the characteristic is zero, $N$ is infinite. Since $\operatorname{rk} \mathbb{K}$ is finite, the extension $\mathbb{K}/N$ is finite. But by Macintyre's Theorem, $N$ is algebraically closed already: hence $N = \mathbb{K}$. So $A$ is a proper ideal of the field $\mathbb{K}$, forcing $A = 0$. □

## 1.2 Bad news

We turn to Question (Q3).

**Fact** (Poizat; [Poizat, Theorem 4.15])**.** *Let $\mathbb{K} \models \mathrm{ACF}$ be a pure algebraically closed field. Then any infinite field definable in $(\mathbb{K}; +, \cdot)$ is definably isomorphic to $\mathbb{K}$.*

Alas a ranked field structure can encode many fields — *even in Morley rank* 1, which makes it even worse.

**Fact** (Hrushovski[1])**.** *There exists a strongly minimal structure (i.e., of Morley rank and degree* 1*) $(\mathbb{K}; +, \cdot; \oplus, \odot)$ where $(\mathbb{K}; +, \cdot)$ and $(\mathbb{K}; \oplus, \odot)$ are fields of different characteristic.*

This was among the first so-called "amalgam" constructions. Later when such constructions were better understood, more pathologies could be constructed, yielding bad answers to Questions (Q4) and (Q2).

**Fact.**

- *There exists a field $\mathbb{K}$ of finite Morley rank of characteristic zero with $\mathbb{K}^\times$ non-minimal*[2]*.*
- *There exists a field $\mathbb{K}$ of finite Morley rank of characteristic $p > 0$ with $\mathbb{K}_+$ non-minimal*[3]*.*
- *It is open whether there exists a field $\mathbb{K}$ of finite Morley rank of characteristic $p > 0$ with $\mathbb{K}^\times$ non-minimal (but regarded as unlikely by number-theorists*[4]*).*

*In particular, there are fields of finite Morley rank $> 1$.*

Remember however that in characteristic 0, $\mathbb{K}_+$ must be minimal.

**Exercise.** Let $\mathbb{K}$ be a ranked field.

1. If $\mathbb{K}$ has characteristic 0, then the only definable field automorphism is Id.
2. If $\mathbb{K}$ has characteristic $p > 0$, then every definable set of definable field automorphisms is finite (hint: bounded implies finite; study the restriction to $\overline{\mathbb{F}_p}$). Every definable *group* of definable automorphisms is trivial.

It is a major open question whether every definable field automorphism of a ranked field of positive characteristic must be a (relative) power of the Frobenius automorphism $x \mapsto x^p$.

# 2 Presence of fields

There is one more observation from algebraic geometry and one more question.

**Question.**

*(Q5). Let $\mathbb{K} \models \mathrm{ACF}$ be a pure algebraically closed field. Then any infinite, non-abelian algebraic group over $\mathbb{K}$ interprets $\mathbb{K}$, and no other infinite field.*

*Does every infinite, non-abelian group of finite Morley rank interpret a field? A* unique *field?*

The answer is not as good as one may wish. First, because of the negative answer to Question (Q3), uniqueness should not be expected in Question (Q5). What about existence?

## 2.1 Intepretation theorem

We begin with a partial positive answer to field interpretability.

**Theorem** (field interpretation theorem)**.** *In a ranked universe, let $G$ be a definable, infinite group acting definably on a definable, abelian group $V$. Suppose the action is faithful and $V$ is $G$-*minimal*, i.e. has no definable, proper, infinite, $G$-invariant subgroup.*

*Suppose in addition one of the two:*

- *$V$ does not have bounded exponent;*
- *$G$ has an infinite centre.*

*Then there is an algebraically closed field $\mathbb{K}$ with $V \simeq \mathbb{K}_+^n$ and $G \hookrightarrow \mathrm{GL}(V)$, definably.*

[1] E. Hrushovski. 'Strongly minimal expansions of algebraically closed fields'. *Israel J. Math.* 79 (2-3) (1992), pp. 129–151.
[2] A. Baudisch et al. 'Die böse Farbe'. *J. Inst. Math. Jussieu*, 8 (3) (2009), pp. 415–443.
[3] A. Baudisch, A. Martin-Pizarro and M. Ziegler. 'Red fields'. *J. Symbolic Logic*, 72 (1) (2007), pp. 207–225.
[4] F. Wagner. 'Fields of finite Morley rank'. *J. Symbolic Logic*, 66 (2) (2001), pp. 703–706.

*Proof.* This is a definable version of Schur's Lemma. We would like to introduce the collection of covariant endomorphisms of $V$, and prove that it is a skew-field. But as we work in the definable category, it is better to restrict oneself to *definable* ones.

*Step* 1. Either $V$ has prime exponent $p > 0$, or it is divisible and torsion-free.

*Proof of Claim.* Return to Macintyre's theorem on abelian groups. $V$ decomposes as $D + B$, where both $D$ and $B$ are definable and characteristic, and consequently $G$-invariant. By $G$-minimality, either $V = B$, in which case it is easily seen to have prime exponent, or $V = D$, in which case it is divisible.

In the latter case it remains to show that $V$ is actually torsion-free. But it can be proved that for any prime $p$, $\mathrm{Tor}_p(V) = \{v \in V : \exists n : p^n v = 0\}$ (this need not be definable) is isomorphic to $\mathbb{Z}_{p^\infty}^{d_p}$ for some integer $d_p \geq 0$. Now if $\mathrm{Tor}(V) \neq 0$ then there is $p$ with $d_p > 0$. Observe how $\mathrm{Tor}_p(V)$ is then infinite countable; as it has at most $2^{\aleph_0}$ endomorphisms and $G$ is infinite, there must be $f \neq g \in G$ which coincide on $\mathrm{Tor}_p(V)$. Then $\mathrm{Tor}_p(V) \leq \ker(f - g)$ which is definable, so $\langle \mathrm{Tor}_p(V) \rangle_{\mathrm{def}} \leq \ker(f - g)$. However $\mathrm{Tor}_p(V)$ is $G$-invariant, so its envelope as well. By $G$-minimality one has $\langle \mathrm{Tor}_p(V) \rangle_{\mathrm{def}} = G$, so $f = g$ in $\mathrm{End}(V)$, against faithfulness of $G$.

We carried the proof in a sufficiently saturated model: this is allowed since by Poizat's theorem on rank functions, a ranked group always has a saturated elementary extension with the same rank function. ◇

*Notation.* Let:

$$\mathrm{DefEnd}(V) = \{\lambda : V \to V \text{ a definable endomorphism}\} \quad \text{and} \quad C = C_{\mathrm{DefEnd}(V)}(G)$$

Be very careful that $\mathrm{DefEnd}(V)$ *need not be definable* (even in the end it will not be quite clear). We aim at showing that $C$ is a definable skew-field, and then rely on our knowledge of ranked skew-fields (Macintyre's and Cherlin-Shelah's Theorems). There are three more steps.

*Step* 2. There are an integer $n$ and some $w_0 \in V$ with $V = G_n \cdot w_0$, where $G_n$ denotes the set of at most $n$ elements of $G$.

*Proof of Claim.* A little model theory. Notice that this is almost trivial if $V$ is divisible and torsion-free, since in that case $G \cdot w_0$ is infinite, and the sum has to stop by finiteness of the rank. If $V$ has exponent $p$ this uses classical techniques reminiscent of Zilber's Indecomposability Theorem, for which we unfortunately have no time. ◇

*Notation.* Let:

$$L = \bigcap_{\substack{h \in \langle G \rangle: \\ w_0 \in \ker h}} \ker h$$

For $w_1 \in L$, let:

$$\lambda_{w_0 \to w_1} : \begin{array}{ccc} V & \to & V \\ v = f(w_0) & \mapsto & f(w_1) \end{array}$$

where $f \in G_n$.

$L$ is a form of double centraliser. The map $\lambda_{w_0 \to w_1}$ is conveniently thought of as a *replacement map* insofar as it replaces the argument in the function $f$.

*Step* 3. This is well-defined; moreover $C = \{\lambda_{w_0 \to w_1} : w_1 \in L\}$.

*Proof of Claim.* Notice that since there always is such an $f$ *and since $G_n$ is a definable set*, the map is definable.

Well-definition requires a word. But if $v = f(w_0) = g(w_0)$ for $f$ and $g$ in $\langle G \rangle$, then $f - g \in \langle G \rangle$ vanishes at $w_0$. So by definition, $f - g$ vanishes at $w_1$: hence $f(w_1) = g(w_1)$ and the map is well-defined. It is left as an exercise to check that $\lambda_{w_0 \to w_1}$ is even an endomorphism.

We claim that $\lambda_{w_0 \to w_1} \in C$. For simplicity, just write $\lambda$. Now if $g \in G$ and $v = f(w_0) \in V$ with $f \in G_n$, then $g(v) = f'(w_0)$ for some other sum of at most $n$ operators $f'$. Notice that $gf - f' \in \langle G \rangle$ vanishes at $w_0$, so it must vanish at $w_1$ as well. Hence:

$$g(\lambda(v)) = g(f(w_0)) = g(f(w_1)) = f'(w_1) = \lambda(f'(w_0)) = \lambda(g(v))$$

as desired. This means $\lambda \in C$. The converse inclusion is left as an exercise. ◇

*Step* 4. $C$ is a definable, algebraically closed field; $V$ is a $C[G]$-module.

*Proof of Claim.* The previous step proves definability of $C$. It remains to show that $C$ is an infinite skew-field. If $V$ is divisible and torsion-free, $C$ contains $\mathbb{Z}\,\mathrm{Id}_V$. If $V$ has exponent $p$, $C$ contains the centre of $G$. So it remains to prove that it is a skew-field. This is a good exercise in the spirit of Schur's Lemma (hint: to kill a finite kernel $\ker \lambda > 0$, form $K = \bigcup_n \ker \lambda^n$, an infinite countable $G$-invariant subgroup of $V$, and use saturation as in the first step).

As a conclusion, $C$ is an infinite definable skew-field, hence by the Macintyre-Cherlin-Shelah theorem, an algebraically closed field. Now $V$ is a vector space over $C$, hence finite-dimensional, and the action of $G$ is linear (by definition of $C$). We are done. ◇

This concludes the proof of the theorem. □

**Question** (open, and unlikely). *Can one extend the Linearisation Theorem to the case where $V$ has exponent $p$ without assuming that $G$ has an infinite centre?*

As a corollary we retrieve the famous result which started the whole business of groups of finite Morley rank.

**Corollary** (Zilber's Field theorem; [Poizat, Theorem 3.7]). *In a ranked universe, let $G$ be a definable, infinite,* abelian *group acting definably on a definable, abelian group $V$. Suppose the action is faithful and $V$ is $G$-minimal.*

*Then there is an algebraically closed field $\mathbb{K}$ with $V \simeq \mathbb{K}_+$ and $G \hookrightarrow \mathbb{K}^\times$, definably.*

**Remarks.**

- Zilber's Field Theorem drew interest to the model-theory of $\mathbb{K}^\times$; we now know (see our "bad news" above, answering Question (Q4)) that it need not be minimal, i.e. that the embedding $G \hookrightarrow \mathbb{K}^\times$ may be proper in finite Morley rank.
- As a corollary to Zilber's Field Theorem, any soluble, connected, non-nilpotent ranked group defines an algebraically closed field.
- We should mention another interpretation result.

  **Fact** (special case of Grünenwald-Haug[5]). *Any* torsion-free*, nilpotent, non-abelian ranked group defines an algebraically closed field.*

  Torsion-freeness is essential, as we shall see in a minute.

## 2.2 Pathologies, known and potential

So far the field interpretation theorem (and Zilber's field theorem) together with the Grünenwald-Haug theorem yield partial answers to Question (Q5). One cannot go further.

**Fact** (Baudisch[6]). *There exists a nilpotent group of finite Morley rank which cannot define an infinite field.*

Of course Baudisch's group (which was obtained through a Hrushovski-style construction) is a periodic group, a group in which all elements have finite order.

Our knowledge ends here: beyond is the realm of fantasies.

**Definition** (bad group). *A* bad group *would be a non-nilpotent group of finite Morley rank all of which definable, connected, soluble subgroups would be nilpotent.*

We do not know whether there is a such a thing. Ever since Cherlin's first paper[7] on groups of finite Morley rank, they drew attention as the worse possible pathology. However one should be careful with terminology.

- First: no relationship with what was once called *bad fields*. This is just an instance of unimaginative (not to say: bad) terminology.
- Then: the very notion of a bad group has changed over the years.
- Last but not least: obsession with nilpotence of soluble subgroups tends to hide the possibility to retrieve a field using Grünenwald-Haug's method.

  Still, a bad group would likely be a negative answer to Question (Q5).

Frécon[8] made quite an announcement in 2016: there are no bad groups of Morley rank 3. This finally completed the proof of the Cherlin-Zilber conjecture in rank 3; it is still open in rank 4 as Frécon's method has not generalised yet.

[5] C. Grünenwald and F. Haug. 'On stable torsion-free nilpotent groups'. *Arch. Math. Logic*, 32 (6) (1993), pp. 451–462.

[6] A. Baudisch. 'A new uncountably categorical group'. *Trans. Amer. Math. Soc.* 348 (10) (1996), pp. 3889–3940.

[7] G. Cherlin. 'Groups of small Morley rank'. *Ann. Math. Logic*, 17 (1-2) (1979), pp. 1–28.

[8] O. Frécon. 'Bad groups in the sense of Cherlin'. Preprint. 2016.