Lecture 3 — Groups in Fields

Last time we studied why and which fields appear definably in groups of finite Morley rank. Today we change point of view and ask which groups can be defined in fields of finite Morley rank.

The most general notion is that of an *interpretable group*; in a pure algebraically closed field, by elimination of imaginaries, this reduces to definable (equivalently, constructible) groups, and we have an answer (Weil-Hrushovski Theorem). However in an expanded field, there is no such phenomenon and even the class of definable groups is likely to be wild.

So in the expanded case we shall restrict our attention to the more reasonable class of *definably linear* groups, where linear algebra provides additional methods. In characteristic 0, simple, definably linear groups are either constructible or highly pathological (Macpherson-Pillay, Poizat; existence is open). In characteristic p, simple, definably linear groups are always algebraic but not necessarily constructible (Poizat).

1 Definable groups

1.1 Groups definable in pure fields

For this part we refer to Poizat's book [Poizat, §4], or to the relevant section in Marker's [Marker, §7.4].

Definition (algebraic group)**.** *An algebraic group is an algebraic variety (a tough notion; this is obtained by glueing Zariski-closed sets but also encompasses "projective" varieties) equipped with a compatible group structure.*

Theorem (Weil-Hrushovski: constructible groups are algebraic)**.** *Let* $\mathbb{K} \models \mathrm{ACF}$ *be a pure algebraically closed field and* G *be a definable group. Then there is a unique algebraic variety structure on* G *making it an algebraic group.*

Notice that by eliminiation of quantifiers and imaginaries, this translates immediately into the original version by Weil: every constructible group is algebraic. Hrushovski proved this using model theory; as a matter of fact he proved something much stronger and much more model-theoretic; see [Poizat, Theorem 5.23]. The proof proceeds by taking a generic "chunk" of the group, and using it to define an algebraic variety structure on G. This is explained in [Poizat, §4.5] or [Marker, §7.4].

1.2 Groups definable in expanded fields

There is little hope of saying anything sensible here.

- First return to the "exotic" field of finite Morley rank constructed by Hrushovski: a strongly minimal structure $(\mathbb{K}; \boxplus, \boxdot, \oplus, \odot)$ consisting of two different field structures (see Question (Q3) in Lecture 2). Notice that $\mathbb{K}_{\boxplus} \times \mathbb{K}^{\odot}$ is then definable in a field expansion, but one hardly sees whether the base field should be $(\mathbb{K}; \boxplus, \boxdot)$ or $(\mathbb{K}; \oplus, \odot)$.
- Now take a ranked field with non-minimal $\mathbb{K}^\times$ (these do exist in characteristic 0), say $1 < T < \mathbb{K}^\times$ is an infinite, proper, definable subgroup. Then $\mathbb{K}_+ \rtimes T$ is a definable group which can be viewed as the following linear group:

$$\left\{ \begin{pmatrix} t & a \\ & 1 \end{pmatrix} : t \in T, a \in \mathbb{K} \right\}$$

  This is *not* an algebraic group over any field.

The second example is non-abelian but *not* simple (it is 2-soluble), so it does not refute the Cherlin-Zilber conjecture. There is a reason to that.

Fact (Blossier, Martin-Pizarro, Wagner[9]). *Let* $\mathbb{K}$ *be* one of the fields we know so far *with* $\mathbb{K}^\times$ *non-minimal. Let* G *be an infinite* simple *group definable in* $\mathbb{K}$. *Then* G *is an algebraic group.*

Be very careful however. *For the moment* no field expansion we know can define a non-algebraic simple group of finite Morley rank. What about the future?

- Perhaps some day, a new field expansion will be constructed which defines a non-algebraic simple group — hence refuting the Cherlin-Zilber conjecture in a rather strong sense.
- Another possibility is that there will be non-algebraic simple groups of finite Morley rank, but none of them will be definable in a field expansion: Cherlin-Zilber would be false, but not because of fields (this might be the case should "bad groups" exist).
- And of course there is the possibility that the Cherlin-Zilber conjecture is simply true...

2 Definably linear groups in expanded fields

Since the definable class is too wild, let us be less ambitious.

2.1 Definable linearity

One special case of algebraic groups is the class of *linear algebraic groups*, for which there are three equivalent definitions:

(i). a linear algebraic group is a Zariski-closed subgroup of some $\mathrm{GL}_n(\mathbb{K})$;

(ii). a linear algebraic group is a constructible subgroup of some $\mathrm{GL}_n(\mathbb{K})$;

(iii). a linear algebraic group is an algebraic group, whose underlying set is a Zariski-closed set of some $\mathbb{K}^n$ (when using this definition, one traditionally refers to an *affine algebraic group*).

The equivalence (i)⇔(ii) is routine; the implication (ii)⇒(iii) is not hard (use the "determinant trick", i.e. view GL_n as $\{(M, \lambda) \in M_n(\mathbb{K}) \times \mathbb{K} : \det M \cdot \lambda = 1\}$ to create a Zariski-closed subset of $\mathbb{K}^{n^2+1}$). But (iii)⇒(i) is a little algebraic geometry.

Linear algebraic groups are easier to study than general algebraic groups because one can use linear algebra. It also suggests which class of groups definable in field expansions can reasonably be approached.

Definition (definably linear group). *Let* $\mathbb{K}$ *be a field structure. A group structure* G *is* definably linear *if it is a definable subgroup* $G \leq \mathrm{GL}_n(\mathbb{K})$ *(here definable is in the* full $\mathbb{K}$*-structure induced on* $\mathrm{GL}_n(\mathbb{K})$*,* not *in the pure group* $\mathrm{GL}_n(\mathbb{K})$*).*

Besides the ability to use linear algebra, another advantage of working with definably linear groups is to remove the Hrushovski-style pathology $\mathbb{K}_{\boxplus} \times \mathbb{K}^{\odot}$: it forces the structure to choose which base field it is about.

Remark. Notice that one could even study groups of the form H/N with $N \trianglelefteq H \leq \mathrm{GL}_n(\mathbb{K})$ are definable, and call them *interpretably linear*; it is not clear a priori whether H/N is then definably linear or not. This holds — and is non-trivial — if $\mathbb{K}$ is a pure algebraically closed field, i.e. in the linear algebraic case. And it is *not* mere elimination of imaginaries but something stronger. So it looks more reasonable to stick to definably linear groups.

Finally let me mention an important fact, and an open question.

Fact (Rosenlicht's Theorem; [Poizat, Theorem 4.14]). *Let* G *be a connected algebraic group. Then* $G/Z(G)$ *is a linear algebraic group.*

In particular, any simple algebraic group is actually linear, i.e., affine. In view of the Weil-Hrushovski theorem, this rephrases as: if G is a definable, connected group in a *pure* field of finite Morley rank, then $G/Z(G)$ is definably linear.

Question. *Is there anything similar for* expanded *fields of finite Morley rank?*

The question is highly non-trivial since Rosenlicht's theorem is proved by letting G act on function germs at 1; i.e., it makes essential use of the adjoint action — something completely out of grasp in the finite Morley rank setting.

[9] T. Blossier, A. Martin-Pizarro and F. O. Wagner. 'À la recherche du tore perdu'. *J. Symb. Log.* 81 (1) (2016), pp. 1–31.

2.2 Characteristic 0

We shall prove that in characteristic 0, simple, definably linear subgroups $G \leq \mathrm{GL}_n(\mathbb{K})$ are algebraic: and even more than that, that they are constructible (constructible is stronger because it says something about the inclusion map, not only up to isomorphism). We first return to our friend the Jordan decomposition.

Theorem (Poizat: definably linear groups in characteristic 0 are stable under Jordan decomposition)**.** *Let* $\mathbb{K}$ *be a field of finite Morley rank* of characteristic zero *and* $G \leq \mathrm{GL}_n(\mathbb{K})$ *be a definable subgroup. Then* G *is stable under Jordan decomposition: if* $x = s \cdot u \in G$ *with commuting semisimple* s *and unipotent* u*, then* $s, u \in G$*.*

Proof. The core idea is that the finite Morley rank setting, being close to classical algebraic geometry, should not encode field exponentials. This will make sense at some point, and certainly involves definable homomorphisms between $\mathbb{K}_+^a$ and $(\mathbb{K}^\times)^b$. They will appear in time.

Write $g = us$; we may suppose $u \neq 1$. Conjugating, we may suppose that u is strictly upper-triangular and that s is diagonal. Let $Y = \langle u \rangle_{\mathrm{def}}$ be the envelope of u and $\Theta = \langle s \rangle_{\mathrm{def}}$ be the envelope of s. We proved during the first lecture that the upper-triangular subgroup U and the diagonal subgroup T are definable: hence $Y \leq U$ and $\Theta \leq T$. We also introduce $\Gamma = \langle g \rangle_{\mathrm{def}} \leq G$ (since G is definable). Now of course $\Gamma \leq Y \times \Theta$. We shall show that Γ, as a graph, defines a map $\Theta \to Y$. Since Γ is a group, this map will be a group homomorphism.

We shall prove that $Y \simeq \mathbb{K}_+$. Consider the Lie algebra:

$$\mathfrak{u} = \left\{ \begin{pmatrix} 0 & * & * \\ & \ddots & * \\ & & 0 \end{pmatrix} \right\}$$

and the partial exponential function:

$$\begin{array}{rccc} \exp : & \mathfrak{u} & \to & U \\ & x & \mapsto & \sum_n \frac{x^n}{n!} \end{array}$$

The good thing with working with nilpotent elements is that the sum is actually finite: so $\exp : \mathfrak{u} \to U$ is definable (even constructible). It also is a bijection, so it exchanges definable subsets of $\mathfrak{u}$ with definable subsets of U. Now since Y is abelian, its preimage (intuitively, something like its Lie algebra, but Y is not yet known to be constructible) $\mathfrak{y} = \exp^{-1}(Y)$ is a definable subgroup of $\mathfrak{u}$. Applying the idea in the proof minimality of $\mathbb{K}_+$ in characteristic 0,, we see that $\mathfrak{y}$ is a vector space over $\mathbb{K}$. (This is not unexpected from an ordinary Lie algebra, but Y was not known to be topologically closed: so some model theory was involved here.) There is more: let $\ell = \exp^{-1}(u)$. Since exp exchanges definable sets, $\mathfrak{y} = \exp^{-1}(Y) = \exp^{-1}(\langle u \rangle_{\mathrm{def}}) = \langle \ell \rangle_{\mathrm{def}}$ is actually a one-dimensional vector space. As a conclusion, $Y \simeq \mathfrak{y} \simeq \mathbb{K}_+$ definably, and in particular Y is minimal and torsion-free.

With this information we finally get a map. Consider $\{y \in Y : (y, 1) \in \Gamma\}$, a definable subgroup of Y. If it equals Y, then certainly $u \in \Gamma \leq G$ and we are done. Otherwise it is trivial: this means that Γ defines a map $\Theta \to Y$, which is a group homomorphism.

We derive a contradiction. Remember that $\Theta \leq T \simeq (\mathbb{K}^\times)^n$. It suffices to show that there are no definable homorphisms from definable subgroups of $(\mathbb{K}^\times)^n$ to $\mathbb{K}_+$. This is left as an exercise. □

Exercise. Let $\mathbb{K}$ be a ranked field.

1. There are no definable group homomorphisms $\mathbb{K}_+^m \to (\mathbb{K}^\times)^n$ nor $(\mathbb{K}^\times)^n \to \mathbb{K}_+^m$.
2. If $\mathbb{K}$ has characteristic 0, then any definable group homomorphism $\mathbb{K}_+^m \to \mathbb{K}_+^n$ is $\mathbb{K}$-linear.

Corollary (Macpherson-Pillay[10]; Poizat[11]: the structure of simple, definably linear groups in char. 0)**.** *Let* $\mathbb{K}$ *be a field of finite Morley rank* of characteristic zero *and* $G \leq \mathrm{GL}_n(\mathbb{K})$ *be a quasi-simple (= simple modulo a finite centre), definable subgroup. If* G *is not constructible, then* G *contains only semisimple elements.*

[10] D. Macpherson and A. Pillay. 'Primitive permutation groups of finite Morley rank'. *Proc. London Math. Soc. (3)*, 70 (3) (1995), pp. 481–504.

[11] B. Poizat. 'Quelques modestes remarques à propos d'une conséquence inattendue d'un résultat surprenant de Monsieur Frank Olaf Wagner'. *J. Symbolic Logic*, 66 (4) (2001), pp. 1637–1646.

Proof. Suppose that G contains more than just semisimple elements; so there is a non-trivial Jordan decomposition $g = us$. We know from the previous theorem that $u, s \in G$; in particular, G contains a non-trivial unipotent element. Let $Y = \langle u \rangle_{\text{def}}$ and argue like in the previous theorem. When we proved that $\mathfrak{y}$ was a vector space over $\mathbb{K}$, we proved that it was *constructible.* Now since $\exp : \mathfrak{u} \to U$ is constructible too, so is Y.

We conclude with a classical result from algebraic geometry: a group generated by irreducible, constructible subgroups, is constructible as well (this is merely Zilber's indecomposability theorem in the constructible category). By quasi-simplicity, $G = \langle Y^g : g \in G \rangle$ is constructible, i.e. definable in the pure field. □

This was extended to the non-simple case by Mustafin[12].

Remark. It is a significant open question whether there can indeed exists a simple, definable subgroup of $\mathrm{GL}_n(\mathbb{K})$ which is not constructible. A little more is known: its connected soluble subgroups would be abelian (already in Machperson-Pillay), and it would have no involutions[13]. Quite pathological!

Some (in particular Poizat) have discussed the possibility to use a ranked field with non-minimal $\mathbb{K}^\times$ and infinite $1 < T < \mathbb{K}^\times$ to construct such a monster, where the maximal definable, connected subgroups would be conjugates of T. Apparently this was never realised.

Observe that Hrushovski amalgams mostly produced fields so far. The only group of finite Morley rank obtained by such means is the (nilpotent) Baudisch group; so presumably being able to amalgamate simple groups will require new developments in pure model theory.

2.3 Characteristic p

A useless foreword: as far as the Jordan decomposition goes, things are outrageously good.

Exercise. Let $\mathbb{K}$ be a field of finite Morley rank *of positive characteristic* and $G \leq \mathrm{GL}_n(\mathbb{K})$ be *any* (not necessarily definable) subgroup. Then G is stable under Jordan decomposition.

However the study in characteristic 0 cannot be adapted. But it can be substituted with something much harder (here again, slightly generalised by Mustafin).

Theorem (Poizat 2001; simple, definably linear groups in char.p are algebraic)**.** *Let $\mathbb{K}$ be a field of finite Morley rank* of positive characteristic *and $G \leq \mathrm{GL}_n(\mathbb{K})$ be a quasi-simple, definable subgroup. Then:*

- *G is definably isomorphic to a constructible subgroup of $\mathrm{GL}_n(\mathbb{K})$;*
- *G is definable in the* pure *field $\mathbb{K}$ augmented by a finite number of definable field automorphisms.*

Proof. The proof is rather intricate and we have no time to explain it. It uses highly non-trivial model theory and the classification of the locally finite simple groups (which relies on the classification of the finite simple groups).

The main ideas are as follows. Using saturation, we may assume that G contains an infinite, definable, connected subgroup T *consisting of diagonal matrices*, $T \leq (\mathbb{K}^\times)^n$. Using Zilber's indecomposability theorem, G is definable in $(\mathbb{K}; +, \cdot, T)$.

Observe how the Frobenius automorphism of $\mathbb{K}$ is then actually an automorphism of T. Let $\mathbb{F} = \mathbb{K} \cap \overline{\mathbb{F}_p}$, the subfield of $\mathbb{K}$ isomorphic to $\overline{\mathbb{F}_p}$ (bear in mind $\mathbb{K}$ is algebraically closed and has characteristic p).

It is a beautiful theorem by Wagner[14] that since there is a non-trivial automorphism, $(\mathbb{F}; +, \cdot, T) \preceq (\mathbb{K}; +, \cdot T)$. Now $(\mathbb{F}; +, \cdot, T)$ has the property that every definable simple group is a Chevalley group over some field; the property may be transferred back to $\mathbb{K}$ — and therefore G itself is a Chevalley group over some field $\mathbb{L}$.

Poizat works a bit more to conclude that $\mathbb{L} \simeq \mathbb{K}$ definably, and that actually a finite number of definable automorphisms of $\mathbb{K}$ are needed to retrieve the identification. □

Remark. Be careful however that G need not be constructible in $\mathrm{GL}_n(\mathbb{K})$, nor even $(\mathbb{K}; +, \cdot)$-definably isomorphic to a constructible group: twist an embedding using a non-constructible field automorphism (if there is such a thing: the question is open). This is because we apparently have no control on definable automorphisms of ranked fields in characteristic p.

[12] Y. Mustafin. 'Structure des groupes linéaires définissables dans un corps de rang de Morley fini'. *J. Algebra*, 281 (2) (2004), pp. 753–773.

[13] A. Borovik and J. Burdges. 'Definably linear groups of finite Morley rank'. Preprint. arXiv:0801.3958. 2008.

[14] F. Wagner. 'Fields of finite Morley rank'. *J. Symbolic Logic*, 66 (2) (2001), pp. 703–706.