Groups of finite Morley rank and their representations

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Abstract

Notes for a mini-course given at Universidad de los Andes in May 2017. There were four lectures of 105 minutes each, although 2 hours might have been more reasonable.

Contents

Introduction

The following inclusions sum up our setting and goals:

- {algebraic geometry} $\subseteq$ {model theory};
- {constructible sets} $\subseteq$ {definable sets};
- {linear algebraic groups} $\subseteq$ {groups of finite Morley rank}
  (equality, in the simple case, is called the Cherlin-Zilber conjecture);
- for a group $G = \mathbb{G}_{\mathbb{K}}$, where $\mathbb{G}$ is an algebraic group and $\mathbb{K}$ a field of finite Morley rank,
  {algebraic representations of G} $\subseteq$ {modules of finite Morley rank for G}.

That is, we use groups of finite Morley rank to describe groups of $\mathbb{K}$-points (for $\mathbb{K}$ an algebraically closed field). Naturally, much is said about fields of finite Morley rank, taking into account both their abstract properties and their interplay with groups. We emphasise the topics of field interpretation, and definably linear groups. This will lead us to exploring modules of finite Morley rank and definable representations of algebraic groups, where active research is ongoing.

The course is *not* about the Cherlin-Zilber conjecture; the final open questions do *not* directly relate to the classification programme of infinite simple groups of finite Morley rank.

These lecture notes may be a tad less fully self-contained than those for another course I gave in Los Andes, on "Groups of small Morley rank" (see the bibliography). The overlap between both courses, hence between both sets of notes, is actually small; I will occasionally direct there. In particular the complete beginner is advised to start with the other course.

Lecture 1 – Rank and Groups

In this lecture. We quickly introduce groups of finite Morley rank; some prior knowledge of model theory will help find the pace reasonable. Morley rank and the Borovik-Poizat axioms are presented; then Macintyre's theorem on abelian groups is proved.

The versed reader may however find some interest in the (very classical) definability of the Jordan decomposition inside $\mathrm{GL}_n(\mathbb{K})$.

References:

- An extremely well-explained reference is [**Borovik-Nesin**, §4–5].
- More details on model theory in [**Poizat**, Introduction–§1].
- Even more model theory in [**Marker**, §3.2, §6.2].
- As a general reference for linear algebraic group theory, [**Humphreys**].

Let us first describe the setting for our lectures: universes of finite Morley rank. This requires a minimal model-theoretic framework which is better understood in view of the theory of linear algebraic groups; knowledge of the latter helps, but is not a formal prerequisite. Conversely the non-logician lost in Morley rank should focus on the Borovik-Poizat axioms, which are quite natural from the point of view of algebraic geometry.

1 Morley rank

1.1 Definable and interpretable sets

Definition (structure)**.** *A* structure *is a set M equipped with relations. Each relation R_i is a subset of M^{n_i} for some n_i which can depend on R_i. Using the graph trick we may also allow functions $f_i : M^{n_i} \to M$. We write $\mathcal{M} = (M; \{R_i\})$ to denote the structure.*

A group structure *is a structure $(G; \cdot, =, \dots)$, with possibly more relations than just the group law, i.e. it can be an expansion of the group language. Likewise a* field structure *is possibly an expansion of* $(\mathbb{K}; +, \cdot, =, \dots)$. *The phrase* pure group *(or* pure field*) helps emphasise the other case.*

Definition (definable and interpretable sets)**.** Definable sets *are the members of the* definable class*, which is the smallest collection:*

- *containing all singletons, all Cartesian powers M^n, and all relations R_i;*
- *stable under Boolean combinations (viz. under finite intersections, finite unions, and taking complements; but infinitary combinations are* not *allowed);*
- *stable under projection (viz. if $A \subseteq M^{n+1}$ is definable, so is $\pi(A) = \{\overline{x} \in M^n : \exists y \in M, (\overline{x}, y) \in A\}$; and likewise for the other projections).*

Likewise the interpretable class *is the smallest collection:*

- *containing all definable sets;*
- *stable under taking quotients (viz. if $A \subseteq M^n$ is interpretable and $E \subseteq A^2$ is an interpretable equivalence relation on A, then A/E is interpretable).*

There is a difference; however we shall soon abuse terminology. The reason comes from geometry and requires one more definition.

[**Borovik-Nesin**]: Alexandre Borovik and Ali Nesin. *Groups of finite Morley rank.* Vol. 26. Oxford Logic Guides. The Clarendon Press — Oxford University Press, New York, 1994, pp. xviii+409

[**Poizat**]: Bruno Poizat. *Stable Groups.* Vol. 87. Mathematical Surveys and Monographs. American Mathematical Society, Providence, 2001, pp. xiii+129

[**Marker**]: David Marker. *Model theory: An Introduction.* Vol. 217. Graduate Texts in Mathematics. Springer-Verlag, New York, 2002, pp. viii+342

[**Humphreys**]: James Humphreys. *Linear algebraic groups.* Vol. 21. Graduate Texts in Mathematics. New York: Springer-Verlag, 1975, pp. xiv+247

Definition (Zariski-closed and constructible sets)**.** *Let* $\mathbb{K} \models \mathrm{ACF}$*, i.e. let* $\mathbb{K}$ *be a pure algebraically closed field.*

- *A subset* $A \subseteq \mathbb{K}^n$ *is* Zariski-closed *if it is defined by a system of polynomial equations.*

  To a model-theorist, it means that A *is defined by a conjunction of atomic formulas in the pure field language (with parameters).*

- *Now* $A \subseteq \mathbb{K}^n$ *is* constructible *if it is in the Boolean algebra generated by Zariski-closed sets.*

  To a model-theorist, it means that A is defined by a quantifier-free formula (always with parameters).

Fact (Chevalley's Constructible Theorem [Marker, Corollary 3.2.8])**.** *If* $\mathbb{K}$ *is an algebraically closed field, then the constructible class is stable under projections.*

Model-theoretically, it just means that the constructible class and the definable class coincide. This is a straightforward consequence of quantifier elimination.

Fact (Tarski's Elimination Theorem [Marker, Theorem 3.2.2])**.** ACF *eliminates quantifiers.*

Later on we shall consider algebraically closed fields with extra structure; of course Tarski's Theorem will no longer hold. We now turn to the interpretable class in a model of ACF.

Fact (Poizat; first appeared in [**Poi83**], alternative proof in [**Hol93**]; see [Marker, Theorem 3.2.20])**.** ACF *eliminates imaginaries, i.e. if* $A \subseteq \mathbb{K}^n$ *is definable in* $\mathbb{K} \models \mathrm{ACF}$ *and* $E \subseteq A^2$ *is a definable equivalence relation, then there are definable* $B \subseteq \mathbb{K}^m$ *and* $f : A \twoheadrightarrow B$ *such that on* A*, one has* $\underline{x} E \underline{y}$ *iff* $f(\underline{x}) = f(\underline{y})$*.*

Hence A/E *can be safely parametrised by the* definable *set* B*.*

As a consequence, if $\mathbb{K} \models \mathrm{ACF}$ is a pure algebraically closed field, then the interpretable class can be considered to be no larger than the definable class (geometers apparently have no name for this phenomenon).

Later, when working with expansions of algebraically closed fields, interpretability will no longer collapse to ordinary, "Cartesian" definability. So there will be two options:

- work with definable sets mostly;
- also allow interpretable sets.

Now our topic is group theory, and group theorists are extremely keen on taking quotients. So the most convenient (algebraically speaking) choice is to allow interpretable sets.

Definition (universe)**.** *The* universe *of a structure* $\mathcal{M}$ *is the class of its interpretable sets (with parameters).*

Remark. Be extremely careful when reading most sources on groups of finite Morley rank (such as [Borovik-Nesin]) that *definable* is systematically used to mean *interpretable*. The versed model-theorist will use the phrase *definable in* $\mathcal{M}^{\mathrm{eq}}$; I won't.

1.2 Definability of the Jordan decomposition

Here the model-theorist may pause and learn something in group theory.

Lemma. *Let* $\mathbb{K}$ *be an algebraically closed field and* $G = \mathrm{GL}_n(\mathbb{K})$ *as a group. Then the sets of semisimple (i.e. diagonalisable), resp. unipotent (i.e.* $\mathrm{Id} +$*nilpotent) elements, are definable.*

Proof. It is *not* a good idea to think in terms of the spectrum (set of eigenvalues), as G is an abstract group and we have lost the action on $\mathbb{K}^n$. We need something more intrinsincally group-theoretic.

Take a diagonal matrix d with distinct entries. Then $C(d)$ is exactly the diagonal subgroup, and $\bigcup_{g \in G} C(d^g)$ is the set of semisimple elements, which is therefore definable.

Being unipotent in characteristic $p > 0$ could easily be defined by: having order dividing p^n. Nothing similar works in characteristic 0. But in either case, being unipotent means

[Poi83]: Bruno Poizat. 'Une théorie de Galois imaginaire'. *J. Symbolic Logic* 48(4) (1983), 1151–1170 (1984)
[Hol93]: Jan Holly. 'Definable operations on sets and elimination of imaginaries'. *Proc. Amer. Math. Soc.* 117(4) (1993), pp. 1149–1157

being conjugate to the strictly upper-triangular group:

$$U := \left\{\begin{pmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{pmatrix}\right\}$$

To prove definability of $\bigcup_{g\in G} U^g$ it suffices to prove that of U.

The associative $\mathbb{K}$-algebra $M_n(\mathbb{K})$ is around; the group does not know this but we do. Observe that $C_{M_n(\mathbb{K})}(X)$ is always a $\mathbb{K}$-vector space. By the descending chain condition on vector subspaces (which relates to dimension theory and we shall return to this shortly), $C_{M_n(\mathbb{K})}(U)$ is a *finite* intersection $C_{M_n(\mathbb{K})}(u_1, \dots, u_k)$ for some tuple $(u_1, \dots, u_k) \in U^k$. In particular $C_G(U) = G \cap C_{M_n(\mathbb{K})}(U) = C_G(u_1, \dots, u_k)$ is definable. A computation shows that:

$$\hat{U}_{1,n} := C_G(U) = \{\lambda I + \mu E_{1,n} : (\lambda, \mu) \in \mathbb{K}^\times \times \mathbb{K}_+\}$$

where $E_{i,j}$ is the matrix with only one non-zero entry, in cell (i, j).

Now it is a fact that for any $i \neq j$, the group $\hat{U}_{i,j} := \{\lambda I + \mu E_{i,j} : (\lambda, \mu) \in \mathbb{K}^\times \times \mathbb{K}_+\}$ is G-conjugate to $\hat{U}_{1,n}$, hence definable as well (take a “permutation matrix”). Consider the group they generate when $i < j$:

$$\hat{U} := \left\langle \hat{U}_{i,j} : i < j \right\rangle = \left\{\begin{pmatrix} \lambda & * & * \\ & \ddots & * \\ & & \lambda \end{pmatrix} : \lambda \in \mathbb{K}^\times\right\}$$

One should be careful in general with generation but here again, by dimension theory, finite products will do: so $\hat{U}$ is definable. Therefore so is its normaliser:

$$B := N_G(\hat{U}) = \left\{\begin{pmatrix} * & * & * \\ & \ddots & * \\ & & * \end{pmatrix}\right\}$$

It so happens that $B' = U$. One should be careful in general with commutator subgroups, but here again finite products of commutators do (actually U is even the *set* of commutators of B). Hence U is definable. □

Remark. The question would have been trivial if we had worked in the *ring* $M_n(\mathbb{K})$.

The Jordan decomposition will reappear in the third and last lectures. To some extent it is central in this class.

1.3 Morley rank

One more notion from model theory.

Definition (Morley rank)**.** *Let $\mathcal{M}$ be an ω-saturated structure. The Morley rank of definable $X \subseteq \mathcal{M}^n$ is given by the following induction, where α is an ordinal:*

- $\mathrm{MR}(X) \geq 0$ *iff* $X \neq \emptyset$;
- $\mathrm{MR}(X) \geq \alpha + 1$ *iff there are definable* $(Y_i)_{i\in\omega}$ *disjoint, all contained in* X*, and with* $\mathrm{MR}(Y_i) \geq \alpha$;
- $\mathrm{MR}(X) \geq \alpha$ *for limit* α *iff* $\mathrm{MR}(X) \geq \beta$ *for all* $\beta < \alpha$.

Remarks.

- The Morley rank of $X = \varphi(\mathcal{M})$ does not increase if we go to $\mathcal{N} \succeq \mathcal{M}$, precisely because $\mathcal{M}$ is ω-saturated. So it is a “strong” notion, a property of the theory.
- In general the induction need not terminate. Two special cases are worth highlighting:
  - if $\mathrm{MR}(M)$ is an ordinal, the structure (its complete theory) is said to be *totally transcendental* (this is equivalent to being ω-*stable* if the language is countable);
  - if $\mathrm{MR}(M)$ is a *finite* ordinal, we say that it has *finite Morley rank.*

Examples.

- A pure algebraically closed field $\mathbb{K} \models \text{ACF}$ has Morley rank 1 (by quantifier elimination). A definable subset $A \subseteq \mathbb{K}^n$ has Morley rank equal to its *Zariski dimension*, which may be defined topologically, or equivalently through commutative algebra, or equivalently geometrically. The model-theoretic intuition is certainly a decent point of view as well: Zariski dimension is a special case of Morley rank [Poizat, §4.2], [Marker, pp. 226–227]; cf [Humphreys, §3].
- Any divisible torsion-free group, seen as a pure group, has Morley rank 1 (QE again).
- Later we shall see that infinite fields of finite Morley rank are algebraically closed (a theorem by Macintyre). Which infinite groups have finite Morley rank is a good question (the group-theoretically simple case is the famous Cherlin-Zilber conjecture, which stands open; exotic groups of finite Morley rank alien to algebraic geometry are already known to exist, though *none so far is simple*).

Remark. Having finite Morley rank is inherited by definable sets (and definable algebraic structures). Hence a group or field which is definable in a structure of finite Morley rank, has finite Morley rank too.

Definition (Morley degree). *If $X \subseteq \mathcal{M}^n$ has ordinal* MR*, then it also has an integer* Morley degree $\deg X$*, which is the maximal number of Y_i one can simultaneously find in the definition.*

2 Groups of finite Morley rank

2.1 Borovik-Poizat axioms

It so happens that for our mostly algebraic purposes, a less model-theoretic and more naive setting is enough. In the early 80's Borovik first suggested an axiomatic framework for what he considered worth of interest. Poizat later added the fourth (missing) axiom and was able to prove "completeness" of the resulting definition, in the sense of his Equivalence Theorem below.

Remark. Bear in mind that we call *definable* the elements of the universe of a structure (instead of *interpretable*).

Definition (rank function; Borovik-Poizat axioms)**.** *Let $\mathcal{M}$ be a structure and $\mathcal{U}$ be its universe. A rank function is a finite-valued map $\text{rk} : \mathcal{U} \setminus \{\emptyset\} \to \omega$ with the following properties, in which A, B stand for definable sets and $f : A \twoheadrightarrow B$ is a definable surjection:*

- $\text{rk}\, A \geq n+1$ *iff there are infinitely many disjoint definable $B_i \subseteq A$ with* $\text{rk}\, B_i \geq n$ *("monotonocity");*
- *for every integer k, the set $F_k := \{b \in B : \text{rk}\, f^{-1}(b) = k\}$ is definable ("definability");*
- *if $B = F_k$ for some k then* $\text{rk}\, A = k + \text{rk}\, B$ *("additivity");*
- *there is an integer ℓ such that for each $b \in B$, either $f^{-1}(b)$ has at most ℓ elements or is infinite ("elimination of infinity").*

As a consequence of the first axiom, $\text{rk}\, A = 0$ iff A is finite.

Theorem (Poizat's Equivalence Theorem; [Poizat, §2.4])**.** *Let G be a group structure. Then G has finite Morley rank iff the universe of G has a rank function; in which case* $\text{MR} = \text{rk}$ *everywhere.*

This also applies to expansions of groups: for instance to field structures as well.

Poizat's Equivalence Theorem is non-trivial for two reasons:

- it is not clear whether in a group of finite Morley rank, MR extends to *interpretable* sets;
- it is not even clear in the first place whether a ranked group is a group of finite Morley rank, as the Borovik-Poizat axioms do *not* require ω-saturation.

The proof involves a thorough analysis of groups with a suitable dimension on definable sets (something common to both groups of finite Morley rank and ranked groups). We can but direct to Poizat's book for details.

As a corollary to Poizat's theorem, both points of view (orthodox model-theoretic, revisionist algebraic) can be adopted in the study of groups of finite Morley rank/ranked groups. *Both phrases will be used indifferently.* And since MR must then be the only rank function, it is safe to say "rank" and "degree" with no reference to Michael Morley.

Conjecture (Cherlin-Zilber Algebraicity Conjecture, [**Che79**], [**Zil77**])**.** *Let G be a simple, infinite, ranked group. Then G is the group of points of an algebraic group, viz. there are an algebraically closed field $\mathbb{K}$ and an algebraic group $\mathbb{G}$ such that $G \simeq \mathbb{G}(\mathbb{K})$ (as an abstract group).*

Despite its significance in model theory, as the last standing conjecture "à la Zilber", its central role in the historical development of the topic of groups of finite Morley rank, its tight relations with other parts of mathematics, its beauty... this conjecture hardly matters here.

2.2 Basic properties

Our first results are classical: the descending chain condition, existence of a definable connected component.

Lemma (descending chain condition; [Borovik-Nesin, §5.1], [Poizat, §1.3], [Marker, §7.1])**.** *Let G be a ranked group. Then every descending sequence of definable subgroups terminates.*

Proof. Let $(H_i)_{i \in I}$ be such a sequence. As rank and degree are ordinals, at some point the sequences $\operatorname{rk} H_i$ and $\deg H_i$ must become stationary.

So it remains to show that whenever $K \leq H$ are definable groups with same rank and degree, equality holds. This is because H is a disjoint union of translates of K, which all have same rank and degree as K: so there is only one such, i.e. $H = K$. □

Lemma (and definition: connected component; [Borovik-Nesin, §5.2], [Poizat, §1.4], [Marker, §7.1])**.** *Let G be a group of finite Morley rank. Then there is a smallest definable subgroup of finite index. It is definably characteristic in G and called the* connected component *of G, denoted G°.*

Remark. Be very careful that there is no general notion of connected components for definable sets: there is no topology here.

Proof. Intersect all definable subgroups of finite index: by the DCC, we get a definable subgroup, of finite index, and clearly minimal as such. It is easily seen that G° is definably characteristic in G. □

Corollary (and definition: definable envelope)**.** *Let $X \subseteq G$ be any subset. Then there is a smallest definable subgroup containing X, called the* (definable) envelope *of X and denoted by $\langle X \rangle_{\text{def}} \leq G$.*

Remark. Here again there is no general analogue of the Zariski closure, i.e. no general definition of a "definable closure" in our sense (the phrase exists in model theory but means something else). This can only produce definable *groups*—because the DCC works only for definable *subgroups.*

2.3 Abelian groups

We finish with a nice application of the DCC.

Theorem (Macintyre's Theorem on abelian groups [**Mac71**a]; see [Borovik-Nesin, Theorem 6.7])**.** *Let A be an abelian group of finite Morley rank. Then there are definable, characteristic subgroups $D, B \leq A$ such that:*

- *D is divisible and B has bounded exponent;*
- *$A = D + B$ and $D \cap B$ is finite.*

Proof. We use additive notation. Consider the following chain of subgroups:

$$A \geq 2A \geq \dots \geq n!A \geq \dots$$

[Che79]: Gregory Cherlin. 'Groups of small Morley rank'. *Ann. Math. Logic* 17(1-2) (1979), pp. 1–28
[Zil77]: Boris Iossifovitch Zilber. 'Groups and rings whose theory is categorical'. Russian. *Fund. Math.* 95(3) (1977), pp. 173–188
[Mac71a]: Angus Macintyre. 'On ω_1-categorical theories of abelian groups'. *Fund. Math.* 70(3) (1971), pp. 253–270

By the descending chain condition, it must become stationary at say $n_0!A = D$, which is a definable, characteristic subgroup. We claim that D is divisible. This is by stationarity: $D = n_0!A = (n_0 + n)!A \leq n \cdot n_0!A = nD$, which is n-divisible.

Now let $B = \{a \in A : n_0!a = 0\}$ be the subgroup of elements of order dividing $n_0!$. Clearly B is a definable and characteristic subgroup of bounded exponent. We claim that $A = D + B$. For if $a \in A$, then $n_0!a \in D$ and since D is divisible there is $d \in D$ with $n_0!d = n_0!a$. So by construction $b = a - d \in B$, and $a = d + b$.

It remains to show that the intersection $I = D \cap B$ must be finite. Consider the definable homomorphism $f : D \to D$ with $f(d) = n_0!d$. The kernel of f is I. So all fibres of f have rank equal to $\operatorname{rk} I$. But D as we know is divisible, so f is onto. All this shows $\operatorname{rk} D = \operatorname{rk} D + \operatorname{rk} I$, and therefore by additivity $\operatorname{rk} I = 0$: so $I = D \cap B$ is finite. □

More can be said, for which you need to know the *quasi-cyclic Prüfer p-group* $\mathbb{Z}/p^\infty\mathbb{Z} = \bigcup_n \mathbb{Z}/p^n\mathbb{Z} \simeq \{z \in \mathbb{C}^\times : \exists n \in \mathbb{N} : z^{p^n} = 1\}$. See the exercises.

Final notes and exercises

Elimination theorems

It is well-known that an elimination theorem is both useful and expensive.

- Other instances of elimination of quantifiers: the theory of real closed fields RCF in the language of *ordered* rings (Tarski-Seidenberg Theorem, see [Marker, Corollary 3.3.18]), the theory of p-adically closed fields in Macintyre's language [**Mac76**].
- Any (expanded) *field* of finite Morley rank eliminates imaginaries [**Wag01**, Corollary 6]. In an arbitrary theory of finite Morley rank EI has no reason to hold; but the whole universe approach has the effect of carefully eluding the problem of interpretability vs. definability, and people in groups of finite Morley rank never distinguish the two notions.
- In the o-minimal case there are various results covering all reasonable cases, with the proviso that *an o-minimal theory need not eliminate imaginaries* [**Pil86**]. Apart from that, the theory of an o-minimal group does eliminate imaginaries [**Edm03**], and in an o-minimal theory any interpretable group is definable [**EPR14**].
  I have no idea what happens with the p-adics.

Definability of the Jordan decomposition

- Our proof that $\hat{U}$ is definable relies of course on the Chevalley-Zilber generation lemma ([Humphreys, Proposition 7.5]; in model theory it is sometimes referred to as "Zilber's Indecomposability Theorem" [Zil77, Theorem 3.3]; see [Borovik-Nesin, §5.4], [Poizat, §2.2], [Marker, §7.3]).
  Although the lemma appears repeatedly in this class, we neither prove nor state it. It is typical of $\aleph_1$-categorical behaviour and its generalisations are not as nice.
- It is an excellent question whether given $G = \mathbb{G}_\mathbb{K}$ the group of $\mathbb{K}$-points of an algebraic group $\mathbb{G}$ defined in an algebraically closed field $\mathbb{K}$, the two following universes coincide:
  - the universe of the pure group $(G; \cdot)$;
  - the trace on G of the universe of $\mathbb{K}$, i.e. sets of form A/E with $A \subseteq G^n$ $\mathbb{K}$-definable and $E \subseteq A^2$ a $\mathbb{K}$-definable equivalence relation.

  It is a non-trivial observation by Poizat that in case $\mathbb{K} \models \mathrm{ACF}$ is a pure algebraically closed field *and* $\mathbb{G}$ is a simple algebraic group (typically SL_n), then both universes do agree [**Poi88**], [Poizat, Corollary 4.16].
  There are counterexamples without simplicity; see [**Bal89**] or [**Fré10**]. For which affine groups $\mathbb{G}$ it holds is open.

[Mac76]: Angus Macintyre. 'On definable subsets of p-adic fields'. *J. Symbolic Logic* 41(3) (1976), pp. 605–610
[Wag01]: Frank Wagner. 'Fields of finite Morley rank'. *J. Symbolic Logic* 66(2) (2001), pp. 703–706
[Pil86]: Anand Pillay. 'Some remarks on definable equivalence relations in o-minimal structures'. *J. Symbolic Logic* 51(3) (1986), pp. 709–714
[Edm03]: Mário Edmundo. 'Solvable groups definable in o-minimal structures'. *J. Pure Appl. Algebra* 185(1-3) (2003), pp. 103–145
[EPR14]: Pantelis E. Eleftheriou, Ya'acov Peterzil and Janak Ramakrishnan. 'Interpretable groups are definable'. *J. Math. Log.* 14(1) (2014), pp. 1450002.1–1450002.47
[Poi88]: Bruno Poizat. 'MM. Borel, Tits, Zilber et le général nonsense'. *J. Symbolic Logic* 53(1) (1988), pp. 124–131
[Bal89]: John T. Baldwin. 'Some notes on stable groups'. In: *The model theory of groups (Notre Dame, IN, 1985–1987).* Vol. 11. Notre Dame Mathematical Lectures. Univ. Notre Dame Press, Notre Dame, IN, 1989, pp. 100–116
[Fré10]: Olivier Frécon. 'Splitting in solvable groups of finite Morley rank'. *J. Log. Anal.* 2 (2010), pp. 1–15

Morley rank

If we drop the ω-saturation clause on $\mathcal{M}$ then we compute its so-called *Cantor rank* (which can increase if we go up to an elementary extension). The topic of groups of finite Cantor rank seems untractable if not artificial; Poizat however did something there [**Poi10**].

The Cherlin-Zilber conjecture

Immensely more could be said as in the past four decades most of the attention devoted to groups of finite Morley rank has focused on the question of algebraicity of abstract, simple groups. Under the influence of Borovik, methods were borrowed from the classification of the finite simple groups, making the theory of groups of finite Morley rank resemble finite group theory more than algebraic group theory or model theory.

One should however not forget that there are other legitimate approaches (though none proved as fruitful as Borovik's) *and other legitimate open problems.* The present course precisely tries to introduce questions of a different nature, where there is more algebraic group theory, at times more model theory, and remarkably less finite group theory. I believe that there is something significant to be done at the intersection between model theory and representation theory, and the $\aleph_1$-categorical setting is a good laboratory to run the experiment.

For a discussion of the status of the Cherlin-Zilber Conjecture let me refer to the first lecture (and final notes) of another class I gave—afterwards—in Los Andes [**Del18**].

Chain conditions and envelopes

Here again, see the final notes of lecture 2 of [Del18] for many generalisations.

Abelian (and nilpotent) groups of finite Morley rank

It is well-known amongst algebraists that a nilpotent group is nothing but an abelian group where commutativity fails. Macintyre's analysis of abelian groups of finite Morley rank was extended by Nesin.

Theorem (Nesin's structure theorem for nilpotent groups [**Nes91**]; see [Borovik-Nesin, Theorem 6.8])**.** *Let N be a nilpotent groups of finite Morley rank. Then there are definable, characteristic subgroups $D, B \leq A$ such that D is divisible and B has bounded exponent, $N = D \cdot B$ and $D \cap B$ is finite.*

And yet, the Baudisch group confirmed that any hope of classifying nilpotent groups of finite Morley rank would be hopeless, as we shall see in the next lecture.

Nesin's theorem is too long to be an exercise. But it is a very good take-home problem as you will require to develop a theory of nilpotent groups of finite Morley rank (general hint: try and adapt classical properties from the finite case).

Exercise. Find *parameter-free* definitions in $\mathrm{GL}_n(\mathbb{K})$ of the set of semisimple elements, of the set of unipotent elements (in our proof we had to fix matrices d and u_i with good properties).—Unipotence is difficult.

Exercise. Prove that $\mathrm{SL}_2(\mathbb{C})$ is definable in $\mathrm{GL}_2(\mathbb{C})$ (hint: Gauß' algorithm). Prove that a field isomorphic to $\mathbb{C}$ is definable in $\mathrm{SL}_2(\mathbb{C})$ (hint: write the upper-triangular subgroup as a semi-direct product).

The first is a consequence of the Chevalley-Zilber generation lemma; the second, of the Schur-Zilber field lemma—of course you have to do without those.

Exercise. The field $\mathbb{R}$ is *not* totally transcendental, i.e. $\mathrm{MR}(\mathbb{R})$ is greater than any ordinal.

Exercise. Let D be a divisible, abelian group of finite Morley rank. Prove that there are integers d_p indexed by the prime numbers and an arbitrary index set I such that:

$$D \simeq \bigoplus_p (\mathbb{Z}/p^\infty\mathbb{Z})^{d_p} \oplus \bigoplus_I \mathbb{Q}.$$

Exercise (the rigidity of tori)**.** Let G be a group of finite Morley rank and $T \leq G$ be a divisible abelian p-subgroup (we do *not* suppose definability). Then $[N_G(T) : C_G(T)]$ is finite and $N_G(T)$ is definable.

Hint: definability of $C_G(T)$ is by the DCC; now $T \simeq (\mathbb{Z}/p^\infty\mathbb{Z})^d$ for some d; then $N_G(T)/C_G(T)$ embeds into $N_G(T_{p^k})/C_G(T_{p^k})$ for k large enough, where $T_{p_k} = \{t \in T : t^{p^k} = 1\}$.

Exercise (torsion lifting)**.** Let p be a prime and $N \trianglelefteq G$ be two definable groups of finite Morley rank. If $g \in G$ satisfies $g^p \in N$, then there is a p-element $x \in gN$ (of order a power of p).

Hint: first prove it for finite G, using Bézout relations. Then prove that in the ranked case one has $\langle g \rangle_{\mathrm{def}} = D \oplus C$ where $D = \langle g \rangle_{\mathrm{def}}^\circ$ is divisible and C is a finite cyclic group.

[Poi10]: Bruno Poizat. 'Groups of small Cantor rank'. *J. Symbolic Logic* 75(1) (2010), pp. 346–354
[Del18]: Adrien Deloro. 'Groups of small Morley rank'. Lecture notes of a mini-course given at Univ. Los Andes. 2018
[Nes91]: Ali Nesin. 'Poly-separated and ω-stable nilpotent groups'. *J. Symbolic Logic* 56(2) (1991), pp. 694–699

Lecture 2 – Fields in Groups

In this lecture. We begin with the study of abstract ranked fields, which are algebraically closed (Macintyre's Theorem), but can be model-theoretically much more complicated than in geometry. Then we shall see how fields tend to spontaneously appear (definably) in ranked groups, or more precisely ranked modules: this is the Schur-Zilber Field Lemma.

References: Classical sources [Borovik-Nesin], [Poizat], [Marker] cover most of it. The Linearisation Theorem is however not in the canon.

Before we start, remember that any field interpretable in a ranked group must be ranked as well. Also bear in mind that by Poizat's Equivalence Theorem, fields of finite Morley rank and ranked fields are the same.

3 Abstract fields of finite Morley rank

Let us begin with properties of the base field in algebraic geometry. We shall address the following questions.

Questions.

1. Recall that a pure algebraically closed field has rank and degree 1 ("strong minimality": immediate from quantifier elimination).

   Conversely, *is an (infinite) field of finite Morley rank algebraically closed?* (luckily, YES).
2. *Must a field structure of finite Morley rank have rank* 1*?* (unfortunately, NO).
3. It is a non-trivial theorem by Poizat that an infinite field $\mathbb{L}$ definable in a pure algebraically closed field $\mathbb{K}$ must be isomorphic to $\mathbb{K}$ (and even definably so).

   Is the same true if $\mathbb{K}$ *is an arbitrary field of finite Morley rank?* (unfortunately, NO).
4. It is a consequence of dimension theory that if $\mathbb{K} \models \text{ACF}$ is a pure algebraically closed field, then $\mathbb{K}_+$ and $\mathbb{K}^\times$ are *minimal* (viz. contain no constructible, proper, infinite subgroups).

   Is the same true if $\mathbb{K}$ *is an arbitrary field of finite Morley rank?* (unfortunately, NO).

3.1 Good news

We first answer Question (Q1) above.

Theorem (Macintyre's Theorem on fields [**Mac71**b]; see [Borovik-Nesin, Theorem 8.1], [Marker, Theorem 7.2.10], or [Poizat, Theorem 3.1])**.** *If* $\mathbb{K}$ *is an infinite field of finite Morley rank, then* $\mathbb{K}$ *is algebraically closed.*

Proof. This involves a bit of Galois theory. The proof is extremely interesting in its own right but not in the spirit of the lectures. □

As a matter of fact, commutativity is not required.

Theorem (Zilber [Zil77], Shelah [**She75**], Cherlin [**Che78**])**.** *Any infinite (possibly non-commutative) domain of finite Morley rank is an algebraically closed field.*

Good news almost stop here, save for one little bit around Question (Q4).

[Mac71b]: Angus Macintyre. 'On ω_1-categorical theories of fields'. *Fund. Math.* 71(1) (1971), pp. 1–25
[She75]: Saharon Shelah. 'The lazy model-theoretician's guide to stability'. *Logique et Analyse (N.S.)* 18(71-72) (1975). Comptes Rendus de la Semaine d'Étude en Théorie des Modèles (Inst. Math., Univ. Catholique Louvain, Louvain-la-Neuve, 1975), pp. 241–308
[Che78]: Gregory Cherlin. 'Super stable division rings'. In: *Logic Colloquium '77 (Wroclaw, August 1–12, 1977).* Ed. by Angus Macintyre, Leszek Pacholski and Jeff Paris. Vol. 96. North-Holland, Amsterdam-New York, 1978, pp. 99–111

Definition (minimal group)**.** *Call a group* minimal *if it has no definable, infinite, proper subgroup.*

Examples.

- Any divisible, torsion-free, abelian group. As a matter of fact, any pure group of the form $\bigoplus_p (\mathbb{Z}/p^\infty\mathbb{Z})^{d_p} \oplus \bigoplus_I \mathbb{Q}$.
- As we know, in a pure algebraically closed field, $\mathbb{K}_+$ and $\mathbb{K}^\times$ are minimal; so are elliptic curves (which are *non-affine* algebraic groups).

Lemma (Zilber, but I could not source the original; see [Poizat, Corollary 3.3])**.** *If $\mathbb{K}$ is a field of finite Morley rank of characteristic zero, then $\mathbb{K}_+$ is minimal.*

Proof. Let $A < \mathbb{K}_+$ be a infinite, definable, proper subgroup; we shall prove $A = 0$. Let $N = \{x \in \mathbb{K} : xA = A\}$. Clearly N is a definable subfield of $\mathbb{K}$. Since the characteristic is zero, N is infinite. Since $\operatorname{rk} \mathbb{K}$ is finite, the extension $\mathbb{K}/N$ is finite. But by Macintyre's Theorem, N is algebraically closed already: hence $N = \mathbb{K}$. So A is a proper ideal of the field $\mathbb{K}$, forcing $A = 0$. □

3.2 Bad news

We turn to Question (Q3). Pure geometry, as mentioned, behaves nicely.

Fact (Poizat's Monosomy Lemma; first [Poi88, Lemme p. 129], see [Poizat, Theorem 4.15])**.** *Let $\mathbb{K} \models$ ACF be a* pure *algebraically closed field. Then any infinite field definable in $(\mathbb{K}; +, \cdot)$ is definably isomorphic to $\mathbb{K}$.*

Alas a ranked field structure can encode many fields—*even in Morley rank* 1, which makes it even worse.

Fact (Hrushovski [**Hru92**])**.** *There exists a strongly minimal structure (i.e. of Morley rank and degree* 1*) $(\mathbb{K}; \boxplus, \boxdot; \oplus, \odot)$ where $(\mathbb{K}; \boxplus, \boxdot)$ and $(\mathbb{K}; \oplus, \odot)$ are fields of different characteristic.*

This was among the first so-called "amalgam" constructions. Later when such constructions were better understood, more pathologies could be constructed, yielding bad answers to Questions (Q4) and (Q2).

Fact.

- *There exists a field $\mathbb{K}$ of finite Morley rank of characteristic zero with non-minimal $\mathbb{K}^\times$* [**Bau+09**]*.*
- *There exists a field $\mathbb{K}$ of finite Morley rank of characteristic $p > 0$ with non-minimal $\mathbb{K}_+$* [**BMZ07**]*.*
- *It is open whether there exists a field $\mathbb{K}$ of finite Morley rank of characteristic $p > 0$ with non-minimal $\mathbb{K}^\times$ (but regarded as unlikely by number-theorists* [**Wag03**]*).*

In particular, there are fields of finite Morley rank > 1.

Remember however that in characteristic 0, $\mathbb{K}_+$ must be minimal.

$\operatorname{char} \mathbb{K}$	0	p
$\mathbb{K}_+$	minimal	not necessarily minimal
$\mathbb{K}^\times$	not necessarily minimal	open (but with strong constraints)

4 Presence of fields

There is one more observation from algebraic geometry and one more question.

[Hru92]: Ehud Hrushovski. 'Strongly minimal expansions of algebraically closed fields'. *Israel J. Math.* 79(2-3) (1992), pp. 129–151
[Bau+09]: Andreas Baudisch et al. 'Die böse Farbe'. *J. Inst. Math. Jussieu* 8(3) (2009), pp. 415–443
[BMZ07]: Andreas Baudisch, Amador Martin-Pizarro and Martin Ziegler. 'Red fields'. *J. Symbolic Logic* 72(1) (2007), pp. 207–225
[Wag03]: Frank Wagner. 'Bad fields in positive characteristic'. *Bull. London Math. Soc.* 35(4) (2003), pp. 499–502

Question.

5. It is a non-trivial fact in geometry that for $\mathbb{K}$ a pure algebraically closed field, any non-trivial, connected, non-abelian algebraic group of $\mathbb{K}$-points interprets $\mathbb{K}$, and no other infinite field.

   Does every infinite, non-abelian group of finite Morley rank interpret a field? A unique *field?*

The answer is not as good as one may wish. First, because of the negative answer to Question (Q3), uniqueness should not be expected in Question (Q5). What about existence?

4.1 Intepretation theorem

Here is a partial positive answer to field interpretability.

Theorem (Linearisation Theorem; largely inspired by [**Nes89**a])**.** *In a ranked universe, let G be a definable, infinite group acting definably on a definable, abelian group V. Suppose that the action is faithful and V is* irreducible*, i.e. has no definable, proper, infinite, G-invariant subgroup.*

Then $\mathbb{K} = C_{\mathrm{DefEnd}(V)}(G)$ is a definable field. If infinite then it is algebraically closed; V is finite-dimensional and $G \hookrightarrow \mathrm{GL}(V)$, definably.

Of course one has let $\mathrm{DefEnd}(V)$ be the ring of definable endomorphisms of V. Although its elements are, $\mathrm{DefEnd}(V)$ itself need not be definable.

Proof.

Step 1. *Either V has prime exponent $p > 0$, or it is divisible and torsion-free.*

Proof. Return to Macintyre's theorem on abelian groups. V decomposes as $D + B$, where both D and B are definable and characteristic, and consequently G-invariant. By irreducibility, either $V = B$, in which case it is easily seen to have prime exponent, or $V = D$, in which case it is divisible.

In the latter case it remains to show that V is actually torsion-free. But it can be proved that for any prime p, $\mathrm{Tor}_p(V) = \{v \in V : \exists n : p^n v = 0\}$ (this need not be definable) is isomorphic to $(\mathbb{Z}/p^\infty\mathbb{Z})^{d_p}$ for some integer $d_p \geq 0$. Now if $\mathrm{Tor}(V) \neq 0$ then there is p with $d_p > 0$. Observe how $\mathrm{Tor}_p(V)$ is then infinite countable; as it has at most $2^{\aleph_0}$ endomorphisms and G is infinite, there must be $f \neq g \in G$ which coincide on $\mathrm{Tor}_p(V)$. Then $\mathrm{Tor}_p(V) \leq \ker(f - g)$ which is definable, so $\langle \mathrm{Tor}_p(V) \rangle_{\mathrm{def}} \leq \ker(f - g)$. However $\mathrm{Tor}_p(V)$ is G-invariant, so its envelope as well. By irreducibility one has $\langle \mathrm{Tor}_p(V) \rangle_{\mathrm{def}} = G$, so $f = g$ in $\mathrm{End}(V)$, against faithfulness of G.

We carried the proof in a sufficiently saturated model: this is allowed since by Poizat's Equivalence Theorem, a ranked group always has a saturated elementary extension with the same rank function. ♢

The theorem is a definable version of Schur's Lemma. We would like to introduce the collection of covariant endomorphisms of V, and prove that it is a skew-field. As we work in the definable category, it is natural to restrict oneself to *definable* ones. So we aim at showing that:

$$\mathbb{K} = C_{\mathrm{DefEnd}(V)}(G) = \{f \in \mathrm{DefEnd}(V) : \forall g \in G \ g \circ f = f \circ g\}$$

is a definable skew-field, and then rely on our knowledge of ranked skew-fields (Macintyre's and Cherlin-Shelah-Zilber's Theorems). (If $\mathbb{K}$ is finite there is not much to do: it is easily seen to be a finite domain, hence a finite skew-field, hence a finite field, and definability is no issue. So we keep the infinite case in mind.)

The strategy followed in the present proof is not the most general, but it has its merits.

Step 2. *There are an integer n and some $w_0 \in V$ with $V = G_n \cdot w_0$, where G_n denotes the set of at most n elements of G.*

Proof. A little model theory. Notice that this is almost trivial if V is divisible and torsion-free, since in that case $G \cdot w_0$ is infinite, and the sum has to stop by finiteness of the rank. If V has exponent p this uses classical techniques reminiscent of the Chevalley-Zilber

[**Nes89a**]: Ali Nesin. 'Nonassociative rings of finite Morley rank'. In: vol. 11. 1989, pp. 117–137 generation lemma ("Indecomposability Theorem"), for which we have no time. ◊

Notation. *Let:*

$$L = \bigcap_{\substack{h \in \langle G \rangle:\\ w_0 \in \ker h}} \ker h$$

For $w_1 \in L$*, let:*

$$\begin{array}{rccc} \lambda_{w_0 \to w_1}: & V & \to & V \\ & v = f(w_0) & \mapsto & f(w_1) \end{array}$$

where $f \in G_n$*.*

L is a form of double centraliser. The map $\lambda_{w_0 \to w_1}$ is conveniently thought of as a *replacement map* insofar as it replaces the argument in the function f.

Step 3. *This is well-defined; moreover* $\mathbb{K} = \{\lambda_{w_0 \to w_1} : w_1 \in L\}$.

Proof. Notice that since there always is such an f *and since* G_n *is a definable set*, the map is definable.

Well-definition requires a word. But if $v = f(w_0) = g(w_0)$ for f and g in $\langle G \rangle$, then $f - g \in \langle G \rangle$ vanishes at w_0. So by definition, $f - g$ vanishes at w_1: hence $f(w_1) = g(w_1)$ and the map is well-defined. It is left as an exercise to check that $\lambda_{w_0 \to w_1}$ is even an endomorphism.

We claim that $\lambda_{w_0 \to w_1} \in \mathbb{K}$. For simplicity, just write λ. Now if $g \in G$ and $v = f(w_0) \in V$ with $f \in G_n$, then $g(v) = f'(w_0)$ for some other sum of at most n operators f'. Notice that $gf - f' \in \langle G \rangle$ vanishes at w_0, so it must vanish at w_1 as well. Hence:

$$g(\lambda(v)) = g(f(w_0)) = g(f(w_1)) = f'(w_1) = \lambda(f'(w_0)) = \lambda(g(v))$$

as desired. This means $\lambda \in \mathbb{K}$. The converse inclusion is left as an exercise. ◊

Step 4. $\mathbb{K}$ *is a definable, algebraically closed field;* V *is a* $\mathbb{K}[G]$*-module.*

Proof. Of course we assume that $\mathbb{K}$ is infinite. The previous step entails definability of $\mathbb{K}$. So it remains to prove that it is a skew-field. This is a good exercise in the spirit of Schur's Lemma.

As a conclusion, $\mathbb{K}$ is an infinite definable skew-field, hence by the Macintyre-Cherlin-Shelah-Zilber theorem, an algebraically closed field. Now V is a vector space over $\mathbb{K}$, hence finite-dimensional, and the action of G is linear (by definition of $\mathbb{K}$). We are done. ◊

This concludes the proof of the theorem. □

Question (open, and unlikely)**.** Can one extend the Linearisation Theorem to the case where V has exponent p without assuming that G has an infinite centraliser?

(For instance would something like: "for all $v \in V \setminus \{0\}$, $C_V(C_G(v))$ is infinite" be enough?)

As a first corollary we derive a famous result.

Corollary ("Zilber's Field Theorem"; [Borovik-Nesin, Theorem 9.1], [Poizat, Theorem 3.7], [Marker, Theorem 7.3.9])**.** *In a ranked universe, let* G *be a definable, infinite,* abelian *group acting definably on a definable, abelian group* V*. Suppose that the action is faithful and* V *is irreducible.*

Then there is an algebraically closed field $\mathbb{K}$ *with* $V \simeq \mathbb{K}_+$ *and* $G \hookrightarrow \mathbb{K}^\times$*, definably.*

Remarks.

- This corollary is but a very special case and should *not* be regarded as the fundamental phenomenon.
- Zilber's Field Theorem drew interest to the model-theory of $\mathbb{K}^\times$; we now know (see the "bad news" above, §3.2, answering Question (Q4)) that it need not be minimal, i.e. that in finite Morley rank the embedding $G \hookrightarrow \mathbb{K}^\times$ may be proper.
- As a corollary to the Chevalley-Zilber generation lemma ("Indecomposability Theorem"), any soluble, connected, non-nilpotent ranked group defines an algebraically closed field:

this was apparently first used in [**Zil84**].

However—and despite the proof we gave—the Schur-Zilber Field Lemma itself does *not* require irreducible generation.

We also retrieve another theorem which was deemed independent.

Corollary ([**LW93**])**.** *In a ranked universe, let G be a definable, infinite group acting definablly on a definable, abelian group V. Suppose that the action is faithful and V is irreducible. Suppose that V does* not *have bounded exponent. Then there is a definable algebraically closed field $\mathbb{K}$ over which $\mathbb{K}$ is a finite-dimensional vector space and $G \hookrightarrow \mathrm{GL}(V)$, definably.*

Let us mention one further consequence, which will be in the exercises.

Corollary (Nesin [**Nes89**b], isolated by Poizat [Poizat, Theorem 3.8])**.** *In a ranked universe, let G be a definable, infinite group acting definably on a definable, abelian group V. Suppose that the action is faithful and V is irreducible. Suppose that G is connected, and contains an infinite, definable, abelian, normal subgroup $A \trianglelefteq G$.*

Then there is a definable, algebraically closed field $\mathbb{K}$ over which V is a finite-dimensional vector space; $G \hookrightarrow \mathrm{GL}(V)$ while $A \hookrightarrow \mathbb{K}^\times \mathrm{Id}_V$.

4.2 Pathologies, known and potential

The Linearisation Theorem (more precisely, Zilber's version) yields a partial answer to Question (Q5). One cannot expect a full positive answer though.

Fact ([**Bau96**])**.** *There exists an infinite, connected, nilpotent group of finite Morley rank which cannot define an infinite field.*

Baudisch's group (which was obtained through a Hrushovski-style construction) is a periodic group, a group in which all elements have finite order. It is *not* an object of algebraic geometry: whence the title of Baudisch's publication.

Our knowledge ends here: beyond is the realm of fantasies.

Definition (bad group)**.** *A* bad group *would be a non-nilpotent group of finite Morley rank all of whose definable, connected, soluble subgroups would be nilpotent.*

We do not know whether there is a such a thing. Ever since Cherlin's first paper [Che79] on groups of finite Morley rank, they drew attention as the worst possible pathology. However one should be careful with terminology.

- First: no relationship with what was once called *bad fields*. This is just an instance of unimaginative (not to say: bad) terminology.
- Then: the very notion of a bad group has changed over the years.
- Last but not least: the definition is concerned with the case where *soluble interpretation à la Zilber* fails to produce a field. So far this is the main tool, but obsession with nilpotence of soluble subgroups tends to hide the possibility to retrieve a field using other methods.

  There is no proof that a bad group in this sense would necessarily be a negative answer to Question (Q5).

I personally tend to avoid the terminology as highly unclarifying, and use either "asomic group" for a group not defining an infinite field, or "group with nilpotent Borel subgroups" for a group whose definable, connected, soluble subgroups (its *Borel subgroups*) are nilpotent.

[Zil84]: Boris Iossifovitch Zilber. 'Some model theory of simple algebraic groups over algebraically closed fields'. *Colloq. Math.* 48(2) (1984), pp. 173–180
[LW93]: James Loveys and Frank O. Wagner. 'Le Canada semi-dry'. *Proc. Amer. Math. Soc.* 118(1) (1993), pp. 217–221
[Nes89b]: Ali Nesin. 'On solvable groups of finite Morley rank'. *J. Algebra* 121(1) (1989), pp. 26–39
[Bau96]: Andreas Baudisch. 'A new uncountably categorical group'. *Trans. Amer. Math. Soc.* 348(10) (1996), pp. 3889–3940

Final notes and exercises

Macintyre's Field Theorem

Algebraic closedness naturally occurs in the ω-stable context, where it was first proved; it was extend by Cherlin and Shelah to the superstable setting [**CS80**]; however, it is still open whether all stable fields are separably closed (already known in characteristic 0 though).

As we said, commutativity is not a necessary assumption, viz. skew-fields tend to be fields in model-theoretic context; see the exercises. (This also extends to superstable [CS80]; in the stable case, the latest result towards commutativity seems to be [**Mil11**].)

Around minimality

In algebraic geometry, minimal groups are easily classified.

Fact ([Humphreys, Theorem 20.5]). *Any connected, algebraic group of dimension* 1 *is one of the following:*

- $\mathbb{G}_a$*, viz. the additive group* $\mathbb{K}_+$*;*
- $\mathbb{G}_m$*, viz. the multiplicative group* $\mathbb{K}^\times$*;*
- *an elliptic curve.*

Only the first two are affine: they are the two sole atoms of linear algebraic group theory (which explains in a sense the Jordan decomposition); an exercise of Lecture 3 will provide a proof.

In the abstract theory of groups of finite Morley rank there is nothing similar. The reader is certainly familiar with Reineke's theorem.

Theorem ([**Rei75**]; see [Del18, §3.4] for a discussion). *A minimal group of finite Morley rank is abelian.*

It is hard to say anything sensible past that point. This is the reason why describing "matter" in abstract groups of finite Morley rank is so difficult—except in the definably linear case, where the *extrinsic* Jordan decomposition provided by linear algebra returns and interacts with the group structure. This will be seen in the next lecture.

Linearisation, field interpretation

- The version we stated can hardly be considered optimal; [**DW19**] goes well beyond. As a matter of fact, and contrary to the proof given here, *the Chevalley-Zilber generation lemma plays no role* in defining a field. [DW19] takes place in the context of *finite-dimensional theories*, which also encompasses the *o*-minimal case.
- We should mention another interpretation result.

  Fact (special case of [**GH93**]). *Any* torsion-free*, nilpotent, non-abelian stable group defines an infinite field.*

  In the finite Morley rank case, this non-trivial result reduces to an exercise below. Torsion-freeness is essential, as exemplified by the Baudisch groups.
- Last but not least, there is Hilbert's theorem that an arguesian projective plane defines a skew-field—see Lecture 4 of [Del18] for its use in finite Morley rank.

The Baudisch groups

Baudisch actually constructed $2^{\aleph_0}$ pairwise non-isomorphic such objects. Interestingly enough (and despite the lack of simplicity, so that Zilber's categoricity theorem does not apply) they are $\aleph_1$-categorical; also see [**Tan88**]. It is therefore hopeless to try to classify nilpotent groups, or even groups of rank 2. Of course one can still ask whether Baudisch constructed *all* possible such objects.

"Bad" groups

Frécon [**Fré18**b] proved the Cherlin-Zilber conjecture in rank 3; in particular there are no "bad groups" of rank 3. This is still open in rank 4 as Frécon's method does not generalise beyond specific configurations in rank $2n+1$ (Poizat and Wagner [**PW16**], [**Poi18**], [**Wag17**]).

We definitely refer to the Los Andes course [Del18].

In any case these objects are ill-named and one should promote more modern terminology.

[CS80]: Gregory Cherlin and Saharon Shelah. 'Superstable fields and groups'. *Ann. Math. Logic* 18(3) (1980), pp. 227–270

[Mil11]: Cédric Milliet. 'Stable division rings'. *J. Symbolic Logic* 76(1) (2011), pp. 348–352

[Rei75]: Joachim Reineke. 'Minimale Gruppen'. *Z. Math. Logik Grundlagen Math.* 21(4) (1975), pp. 357–359

[DW19]: Aldrian Deloro and Frank Wagner. 'Linearisation in model theory'. In preparation. 2019

[GH93]: Claus Grünenwald and Frieder Haug. 'On stable torsion-free nilpotent groups'. *Arch. Math. Logic* 32(6) (1993), pp. 451–462

[Tan88]: Katsumi Tanaka. 'Nonabelian groups of Morley rank 2'. *Math. Japon.* 33(4) (1988), pp. 627–635

[Fré18b]: Olivier Frécon. 'Simple groups of Morley rank 3 are algebraic'. *J. Amer. Math. Soc.* 31(3) (2018), pp. 643–659

[PW16]: Bruno Poizat and Frank Wagner. 'Commentaires sur un résultat d'Olivier Frécon'. 2016

[Poi18]: Bruno Poizat. 'Milieu et symétrie, une étude de la convexité dans les groupes sans involutions'. *J. Algebra* 497 (2018), pp. 143–163

[Wag17]: Frank Wagner. 'Bad groups'. In: *Mathematical Logic and its Applications.* Ed. by Makoto Kikuchi. Vol. 2050. RIMS Kôkyûroku. Kyoto: Kyoto University, 2017, pp. 57–66

Exercise. Prove the Cherlin-Shelah Theorem: any infinite (possibly non-commutative) domain of finite Morley rank is an algebraically closed field.

Hint: division ring by the DCC. Then take a minimal counter-example to commutativity and use Reineke's Theorem on minimal groups (take a definable subgroup of $\mathbb{K}^\times$) to derive a contradiction.

Exercise. Let $\mathbb{K}$ be a field of finite Morley rank.

1. If $\mathbb{K}$ has characteristic 0, then the only definable field automorphism is Id.
2. If $\mathbb{K}$ has characteristic $p > 0$, then every definable set of definable field automorphisms is finite (hint: bounded implies finite; study the restriction to $\overline{\mathbb{F}_p}$). Every definable *group* of definable automorphisms is trivial. Any two definable field automorphisms commute.
3. Any definable field automorphism is actually $\emptyset$-definable (hint: vary the parameters; by finiteness of the resulting family, there is $q = p^k$ large enough to distinguish them).

Exercise. Complete the proof of the interpretation theorem as follows.

1. Finish Step 3: prove that any $\mu \in \mathbb{K}$ is some $\lambda_{w_0 \to w_1}$.
2. Finish Step 4: show that $\mathbb{K}$ does act by automorphisms.

   Hint: to kill a finite kernel $\ker \lambda > 0$ with $\lambda \in \mathbb{K} \setminus \{0\}$, form $K = \bigcup_n \ker \lambda^n$, an infinite countable G-invariant subgroup of V, and use saturation like in the first step of the proof.

Exercise. The goal of this exercise is to prove the Nesin-Poizat linearisation theorem [Poizat, Theorem 3.8]. So let $G \neq 1$ be a definable, *connected* group acting definably, faithfully, and irreducibly on a definable, abelian group V. Suppose that there is an infinite, definable, abelian, normal subgroup $A \trianglelefteq G$.

1. Let $S = C_{\mathrm{DefEnd}(V)}(A)$, on which G acts. Also let $\Lambda = \{L \leq V$: definable, connected, A-invariant, and A-irreducible$\}$.

   Prove that for any $L \in \Lambda$, one can linearise the action of $A/C_A(L)$ on L (hint: although we did *not* assume A to be connected, $C_A(L)$ has infinite index in A because V is a finite sum of memebrs of Λ). Deduce that $\mathbb{K}_L = C_{\mathrm{DefEnd}(L)}(A)$ is an algebraically closed field.
2. Now let $\mathrm{Sp}_A(V) = \{\mathrm{Ann}_S(L) : L \in \Lambda\}$. Prove that $\mathrm{Sp}_A(V)$ is finite (hint: lines with different annihilators are in direct sum, this is *easier* than prime avoidance), and actually a singleton (hint: G permutes $\mathrm{Sp}_A(V)$).
3. Conclude by proving and using this lemma of general interest: if $\mathbb{K}$ is a definable field and $S \subseteq \mathbb{K}$ is a (non-necessarily definable) subring S which contains an infinite, definable set X, then $S = \mathbb{K}$.

The next three exercises use a corollary to the Chevalley-Zilber generation lemma.

Corollary ([Borovik-Nesin, Corollary 5.29]). *if G is a connected group of finite Morley rank and $X \subseteq G$ is any subset, then $[G, X]$ is definable and connected.*

Exercise (Zilber [Zil84, Corollary p. 175]). Let G be a non-trivial, connected, non-nilpotent, soluble group of finite Morley rank. Prove that G defines an algebraically closed field.

Exercise (Lie-Kolchin-Malcev theorem; apparently first proved by Zilber, but published in [Nes89b]; see [Poizat, Corollary 3.19], also [Humphreys, Theorems 17.6 and 21.2]). Let G be an infinite, connected, soluble group of finite Morley rank. Prove that G' is nilpotent.

Hint: first use induction to reduce to proving that $Z(G')$ is infinite. Now take $V \leq G'$ be definable, infinite, G-invariant, and minimal with these properties. Let $\Gamma = G/C_G(V)$ so we have a faithful module; let k be minimal killing the k-th commutator subgroup, $\Gamma^{(k)} = 1$, and suppose $k > 1$. Using the Nesin-Poizat theorem, linearise the action of Γ on V and study $\det : \Gamma^{(k-2)} \to \mathbb{K}^\times$ to see that $\Gamma^{(k-1)}$ acts by roots of unity. Find a contradiction to $k > 1$, and get the desired conclusion.

Exercise. Prove that a nilpotent, non-abelian, torsion-free group of finite Morley rank defines a (characteristic 0) field.

Hint: use torsion-lifting and Chevalley-Zilber generation to reduce to the 2-nilpotent case, $G = Z_2(G)$. Then fix a, b with $c = [a, b] \neq 1$ and show that:

$$[\langle a \rangle_{\mathrm{def}}, b] = \{[x, b] : x \in \langle a \rangle_{\mathrm{def}}\} = \langle c \rangle_{\mathrm{def}} = [a, \langle b \rangle_{\mathrm{def}}] = \{[a, x] : x \in \langle b \rangle_{\mathrm{def}}\}.$$

Now on $\mathbb{K} := \langle c \rangle_{\mathrm{def}}$ use $\cdot$ as addition, and as multiplication the law $*$ given by:

> For $c_1, c_2 \in \mathbb{K}$ there are $a_1 \in \langle a \rangle_{\mathrm{def}}, b_2 \in \langle b \rangle_{\mathrm{def}}$ with $c_1 = [a_1, b]$ et $c_2 = [a, b_2]$. Let $c_1 * c_2 = [a_1, b_2]$.

Exercise (Poizat, [Poizat, Corollary 3.32]). The ultimate bad-like configuration (open) would be the following: a simple group G with a definable, proper, connected, self-normalising subgroup $B < G$ whose conjugates partition G.

1. Prove that every finite subgroup $H < G$ would be contained in a conjugate of B. (Hint: let $B_1, \ldots, B_d$ be the various non-trivial intersections $H \cap B^g$; count the cardinal of $\bigcup_{h \in H} B_1^h$ and see that $d > 1$ is a contradiction.)
2. Deduce that G would satisfy a sentence *false in every* locally finite group. (Hint: $G \models \varphi(\underline{b})$, where $\underline{b}$ are used to define B, so $G \models \exists \underline{x}\ \varphi(\underline{x})$. Use the first question to find a suitable φ.)

Such groups *without the restriction on finiteness of the Morley rank* do exist: a *Tarski monster* as constructed by Olshanski [**Ols79**] (independently, Rips in work unpublished but cited by Collins [**Col90**] and Shelah [**She87**]) has all non-trivial, proper subgroups conjugate and isomorphic to $\mathbb{Z}/p\mathbb{Z}$, which implies the desired structure. If Tarski monsters are an overkill, there is something simpler in [**HW14**, §5.2].

The existence of similar bad-like configurations *under model-theoretic constraints* remains open.

Lecture 3 – Groups in Fields

In this lecture. Last time we studied why and which fields appear definably in groups of finite Morley rank. Today we change point of view and ask which groups can be defined in fields of finite Morley rank.

The most general notion is that of an *interpretable group*; in a pure algebraically closed field, this is the same as an algebraic group (Weil-Hrushovski Theorem). In an expanded field of finite Morley rank, Wagner [Wag01, Corollary 6] proved that interpretable reduces to definable (viz. elimination of imaginaries holds; see final notes of Lecture 1). But the class of definable groups is likely to be wild.

So in the expanded case we focus on the more reasonable class of *definably linear* groups, where linear algebra provides additional methods. In characteristic 0, simple, definably linear groups are either constructible or highly pathological (Macpherson-Pillay, Poizat; existence is open). In characteristic p, simple, definably linear groups are always algebraic up to isomorphism, but not necessarily constructible (Poizat).

References:

- For the Weil-Hrushovski Theorem, [Poizat, §4.5] or Marker's book [Marker, §7.4].
- The characteristic 0 study first appeared in [**MP95**].
- The characteristic p phenomenon was proved by Poizat [**Poi01**].

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5 Definable groups

5.1 Groups definable in pure fields

Theorem (Weil-Hrushovski: constructible groups are algebraic)**.** *Let* $\mathbb{K} \models \mathrm{ACF}$ *be a pure algebraically closed field and* G *be an interpretable group. Then there is a unique algebraic variety structure on* G *making it an algebraic group.*

Notice that by elimination of quantifiers and imaginaries, this geometrically amounts to stating: every constructible group can be made algebraic. The proof proceeds by taking a generic "chunk" of the group, and using it to define an algebraic variety structure on G. This is explained in [Poizat, §4.5] or [Marker, §7.4].

5.2 Groups definable in expanded fields

There is little hope of saying anything sensible here.

- First return to the "exotic" field of finite Morley rank constructed by Hrushovski: a strongly minimal structure $(\mathbb{K}; \boxplus, \boxdot, \oplus, \odot)$ consisting of two different field structures (see Question (Q3) in Lecture 2). Notice that $\mathbb{K}_{\boxplus} \times \mathbb{K}^{\odot}$ is then definable in a field expansion, but one hardly sees whether the base field should be $(\mathbb{K}; \boxplus, \boxdot)$ or $(\mathbb{K}; \oplus, \odot)$.

[**Ols79**]: Alexander Olshanski. 'Infinite groups with cyclic subgroups'. *Dokl. Akad. Nauk SSSR* 245(4) (1979), pp. 785–787

[**Col90**]: Michael Collins. 'Some infinite Frobenius groups'. *J. Algebra* 131(1) (1990), pp. 161–165

[**She87**]: Saharon Shelah. 'Uncountable groups have many nonconjugate subgroups'. *Ann. Pure Appl. Logic* 36(2) (1987), pp. 153–206

[**HW14**]: Pierre de la Harpe and Claude Weber. 'Malnormal subgroups and Frobenius groups: basics and examples'. *Confluentes Math.* 6(1) (2014). With an appendix by Denis Osin, pp. 65–76

[**MP95**]: Dugald Macpherson and Anand Pillay. 'Primitive permutation groups of finite Morley rank'. *Proc. London Math. Soc. (3)* 70(3) (1995), pp. 481–504

[**Poi01**]: Bruno Poizat. 'Quelques modestes remarques à propos d'une conséquence inattendue d'un résultat surprenant de Monsieur Frank Olaf Wagner'. *J. Symbolic Logic* 66(4) (2001), pp. 1637–1646

- Now take a ranked field with non-minimal $\mathbb{K}^\times$ (these do exist in characteristic 0), say $1 < T < \mathbb{K}^\times$ is an infinite, proper, definable subgroup. Then $T \ltimes \mathbb{K}_+$ is a definable group which can be viewed as the following linear group:

$$T \ltimes \mathbb{K}_+ \simeq \left\{ \begin{pmatrix} t & a \\ & 1 \end{pmatrix} : t \in T, a \in \mathbb{K} \right\}.$$

  This is *not* an algebraic group over any field, since one can define two minimal groups of distinct rank; which is impossible in algebraic geometry, because the two minimal affine algebraic groups are $\mathbb{G}_a$ and $\mathbb{G}_m$, both with dimension 1, hence with the same Morley rank.

6 Definably linear groups in expanded fields

Since the definable class is too wild, let us be less ambitious.

6.1 Definable linearity

One special case of algebraic groups is the class of *linear algebraic groups*, for which there are three equivalent definitions:

1. a linear algebraic group is a Zariski-closed subgroup of some $\mathrm{GL}_n(\mathbb{K})$;
2. a linear algebraic group is a constructible subgroup of some $\mathrm{GL}_n(\mathbb{K})$;
3. a linear algebraic group is an algebraic group whose underlying set is a Zariski-closed set of some $\mathbb{K}^n$ (when using this definition, one traditionally refers to an *affine algebraic group*).

The equivalence (i)$\Leftrightarrow$(ii) is routine; the implication (ii)$\Rightarrow$(iii) is not hard (use the "determinant trick", i.e. view GL_n as $\{(M, \lambda) \in M_n(\mathbb{K}) \times \mathbb{K} : \det M \cdot \lambda = 1\}$ to create a Zariski-closed subset of $\mathbb{K}^{n^2+1}$). But (iii)$\Rightarrow$(i) is a little algebraic geometry [Humphreys, Theorem 8.6], [Poizat, Proposition 4.11].

Linear algebraic groups are easier to study than general algebraic groups because of the usual tools of linear algebra, such as the Jordan decomposition. This suggests which class of groups definable in field expansions can reasonably be approached.

Definition (definably linear group)**.** *Let $\mathbb{K}$ be a field structure. A group structure G is* definably linear *if it is a definable subgroup $G \leq \mathrm{GL}_n(\mathbb{K})$ (here definable is in the* full *$\mathbb{K}$-structure induced on* $\mathrm{GL}_n(\mathbb{K})$*,* not *in the pure group* $\mathrm{GL}_n(\mathbb{K})$ *nor in* $\mathrm{GL}_n(\mathbb{K})$ *with the pure $\mathbb{K}$-structure).*

Besides the ability to use linear algebra, another advantage of working with definably linear groups is to remove the Hrushovski-style pathology $\mathbb{K}_{\boxplus} \times \mathbb{K}^{\odot}$ (see §3.2 in Lecture 2): it forces the structure to choose which base field it is about.

Remark. Notice that one could even study groups of the form H/N with $N \trianglelefteq H \leq \mathrm{GL}_n(\mathbb{K})$ are definable, and call them *interpretably linear*; it is not clear a priori whether H/N is then (isomorphic to something) definably linear or not.

This holds if $\mathbb{K}$ is a pure algebraically closed field, i.e. in the linear algebraic case; this is however non-trivial [Humphreys, Theorem 11.5]. (No, it is *not* the mere elimination of imaginaries: the reparametrisation/elimination function has no reason to preserve the group structure.)

So it looks more reasonable to stick to definably linear groups.

Finally let me mention an important fact, and an open question.

Fact (Rosenlicht [**Ros56**, Corollary 3 p. 431]; [Poizat, Theorem 4.14])**.** *Let G be a connected algebraic group. Then $G/Z(G)$ is a linear algebraic group.*

In particular, any simple algebraic group is actually linear, i.e. affine. In view of the Weil-Hrushovski theorem, this rephrases as: if G is a connected group definable in a *pure* field of finite Morley rank, then $G/Z(G)$ is definably linear.

Question. Is there anything similar for *expanded* fields of finite Morley rank?

The question is highly non-trivial since Rosenlicht's theorem is proved by letting G act on function germs at 1; it thus makes essential use of the (co-)adjoint action—something completely out of grasp in the finite Morley rank setting.

[Ros56]: Maxwell Rosenlicht. 'Some basic theorems on algebraic groups'. *Amer. J. Math.* 78 (1956), pp. 401–443

6.2 Characteristic 0

We shall prove that in characteristic 0, simple, definably linear subgroups $G \leq \mathrm{GL}_n(\mathbb{K})$ are algebraic: and even more than that, that they are constructible (constructible is stronger because it says something about the inclusion map, not only up to isomorphism). We first return to the Jordan decomposition.

Theorem (definably linear groups in characteristic 0 are stable under Jordan decomposition [**Poi12**, Lemme 2])**.** *Let $\mathbb{K}$ be a field of finite Morley rank* of characteristic zero *and $G \leq \mathrm{GL}_n(\mathbb{K})$ be a definable subgroup. Then G is stable under the Jordan decomposition: if $x = s \cdot u \in G$ with commuting semisimple s and unipotent u, then $s, u \in G$.*

Proof. The core idea is that the finite Morley rank setting, being close to classical algebraic geometry, should not encode field exponentials. This will make sense at some point, and certainly involves definable homomorphisms between $\mathbb{K}_+^a$ and $(\mathbb{K}^\times)^b$. They will appear in time.

Write $g = us$; we may suppose $u \neq 1$. Conjugating, we may suppose that u is strictly upper-triangular and that s is diagonal. Let $Y = \langle u \rangle_{\mathrm{def}}$ be the envelope of u and $\Theta = \langle s \rangle_{\mathrm{def}}$ be the envelope of s. We proved during the first lecture that the upper-triangular subgroup U and the diagonal subgroup T are definable: hence $Y \leq U$ and $\Theta \leq T$. We also introduce $\Gamma = \langle g \rangle_{\mathrm{def}} \leq G$ (since G is definable). Now of course $\Gamma \leq Y \times \Theta$. We shall show that Γ, as a graph, defines a map $\Theta \to Y$. Since Γ is a group, this map will be a group homomorphism.

We shall prove that $Y \simeq \mathbb{K}_+$. Consider the Lie algebra:

$$\mathfrak{u} = \left\{ \begin{pmatrix} 0 & * & * \\ & \ddots & * \\ & & 0 \end{pmatrix} \right\}$$

and the partial exponential function:

$$\begin{array}{cccc} \exp: & \mathfrak{u} & \to & U \\ & x & \mapsto & \sum_n \frac{x^n}{n!} \end{array}$$

The good thing with working with nilpotent elements is that the sum is actually finite: so $\exp : \mathfrak{u} \to U$ is definable (even constructible). It also is a bijection, so it exchanges definable subsets of $\mathfrak{u}$ with definable subsets of U. Now since Y is abelian, its preimage (intuitively, something like its Lie algebra, but Y is not yet known to be constructible) $\mathfrak{u} = \exp^{-1}(Y)$ is a definable subgroup of $\mathfrak{u}$. Applying the idea in the proof minimality of $\mathbb{K}_+$ in characteristic 0,, we see that $\mathfrak{u}$ is a vector space over $\mathbb{K}$. (This is not unexpected from an ordinary Lie algebra, but Y was not known to be topologically closed: so some model theory was involved here.) There is more: let $\ell = \exp^{-1}(u)$. Since exp exchanges definable sets, $\mathfrak{u} = \exp^{-1}(Y) = \exp^{-1}(\langle u \rangle_{\mathrm{def}}) = \langle \ell \rangle_{\mathrm{def}}$ is actually a one-dimensional vector space. As a conclusion, $Y \simeq \mathfrak{u} \simeq \mathbb{K}_+$ definably, and in particular Y is minimal and torsion-free.

With this information we finally get a map. Consider $\{y \in Y : (y, 1) \in \Gamma\}$, a definable subgroup of Y. If it equals Y, then certainly $u \in \Gamma \leq G$ and we are done. Otherwise it is trivial: this means that Γ defines a map $\Theta \to Y$, which is a group homomorphism.

We derive a contradiction. Remember that $\Theta \leq T \simeq (\mathbb{K}^\times)^n$. It suffices to show that there are no definable homorphisms from definable subgroups of $(\mathbb{K}^\times)^n$ to $\mathbb{K}_+$. This is left as an exercise. □

Corollary (Macpherson-Pillay [MP95, Theorem 1.4.a], Poizat [Poi01, Théorème 3]: the structure of simple, definably linear groups in char. 0)**.** *Let $\mathbb{K}$ be a field of finite Morley rank* of characteristic zero *and $G \leq \mathrm{GL}_n(\mathbb{K})$ be a quasi-simple (= simple modulo a finite centre), definable subgroup. If G is not constructible, then G contains only semisimple elements.*

Proof. Suppose that G contains more than just semisimple elements; so there is a non-trivial Jordan decomposition $g = us$. We know from the previous theorem that $u, s \in G$; in particular, G contains a non-trivial unipotent element. Let $Y = \langle u \rangle_{\mathrm{def}}$ and argue like in the previous theorem. When we proved that $\mathfrak{u}$ was a vector space over $\mathbb{K}$, we proved that it was

[Poi12]: Bruno Poizat. 'Groupes linéaires de rang de Morley fini'. *Ann. Sci. Math. Québec* 36(2) (2012), pp. 591–602

constructible. Now since $\exp : \mathfrak{u} \to U$ is constructible too, so is Y. Being a group it is closed.

We conclude with a classical result from algebraic geometry: a group generated by irreducible, closed subgroups, is closed as well (this is the usual, algebraic version of the Chevalley-Zilber generation lemma [Humphreys, Proposition 7.5], one does not need the full strength of Zilber's version). By quasi-simplicity, $G = \langle Y^g : g \in G \rangle$ is closed, i.e. definable in the pure field. ☐

6.3 Characteristic p

The study in characteristic 0 cannot be adapted, but it can be substituted with something much harder.

Theorem (simple, definably linear groups in char. p are algebraic [Poi01, Théorèmes 1 et 2])**.** *Let* $\mathbb{K}$ *be a field of finite Morley rank* of positive characteristic *and* $G \leq \mathrm{GL}_n(\mathbb{K})$ *be a quasi-simple, definable subgroup. Then:*

- *G is definably isomorphic to a constructible subgroup of* $\mathrm{GL}_n(\mathbb{K})$*;*
- *G is definable in the* pure *field* $\mathbb{K}$ *augmented by a finite number of definable field automorphisms.*

The proof is rather intricate and we have no time to explain it in detail. It uses highly non-trivial model theory (in particular a beautiful theorem on fields by Wagner [Wag01, Corollary 9], an article already cited in the notes of Lecture 1) and the classification of the locally finite simple groups (which relies on the classification of the finite simple groups) obtained by Thomas [**Tho83**]—also independently by Belyaev and by Hartley-Shute.

Proof. The main ideas are as follows. Using saturation, we may assume that G contains an infinite, definable, connected subgroup T *consisting of diagonal matrices*, $T \leq (\mathbb{K}^\times)^n$. By the Chevalley-Zilber generation lemma, $G = \langle T^g : g \in G \rangle$ is definable in $(\mathbb{K}; +, \cdot, T)$.

Observe how the Frobenius automorphism of $\mathbb{K}$ is then actually an automorphism of T. Let $\mathbb{F} = \mathbb{K} \cap \overline{\mathbb{F}_p}$, the subfield of $\mathbb{K}$ isomorphic to $\overline{\mathbb{F}_p}$ (bear in mind $\mathbb{K}$ is algebraically closed and has characteristic p).

By Wagner's theorem [Wag01] and since there is a non-trivial automorphism, we find $(\mathbb{F}; +, \cdot, T) \preceq (\mathbb{K}; +, \cdot T)$. Notice that linear groups definable in $(\mathbb{F}; +, \cdot, T)$ are locally finite; transferring, G turns out to be pseudo-locally-finite, viz. a model of the theory of locally finite groups.

Then by [Tho83], it is a Chevalley group over some field $\mathbb{L}$ which can be reconstructed from G; a Chevalley twist would be a definable field automorphism of $\mathbb{L}$, and this cannot happen in finite Morley rank, so our Chevalley group is untwisted: it is an algebraic group over $\mathbb{L}$.

Poizat then works a bit more to conclude that $\mathbb{L} \simeq \mathbb{K}$ definably, and that actually a finite number of definable automorphisms of $\mathbb{K}$ are needed to retrieve the identification. ☐

Remark. Be careful however that G need not be constructible (i.e. definable in the pure field $\mathbb{K}$). Let σ be a definable, non-constructible field automorphism (if there is such a thing: the question is open), and consider, as 4×4 matrices:

$$\left\{ \begin{pmatrix} A & \\ & A^\sigma \end{pmatrix} : A \in \mathrm{SL}_2(\mathbb{K}) \right\},$$

a group isomorphic to $\mathrm{SL}_2(\mathbb{K})$ but definitely not constructible in $\mathrm{GL}_4(\mathbb{K})$.

To our current knowledge there is insufficient control on definable automorphisms of ranked fields in characteristic p. This is what makes the fourth lecture of interest.

Final notes and exercises

Weil-Hrushovski Theorem

[Tho83]: Simon Thomas. 'The classification of the simple periodic linear groups'. *Arch. Math. (Basel)* 41(2) (1983), pp. 103–116

Weil's theorem [**Wei55**, Theorem (i), p. 375] is more general than the version we give: it is enough to have a *group chunk*. But Hrushovski proved something even stronger and much more model-theoretic, discussed in [Poizat, Theorem 5.23]; then Van den Dries observed that this would in particular yield a model-theoretic proof of Weil's classical result; see the very interesting historical notes in [Poizat, §4.8].

Reconstruction of a compatible topology works quite similarly in the *o*-minimal case [**Pil88**]; one can go pseudofinite [**HP94**]; the latest avatar of the method seems to be [**MOS18**].

Non-algebraic groups in expanded fields

Return to the example of a non-algebraic, non-nilpotent group of finite Morley rank: $T \ltimes \mathbb{K}_+$ where $1 < T < \mathbb{K}_+$ witnesses non-minimality of $\mathbb{K}^\times$. This group is non-algebraic, non-abelian but *not* simple (it is 2-soluble), so it does not refute the Cherlin-Zilber conjecture. There is a reason to that.

Fact ([**BMW16**])**.** *Let* $\mathbb{K}$ *be* one of the fields we know so far *with* $\mathbb{K}^\times$ *non-minimal. Let* G *be an infinite* simple *group definable in* $\mathbb{K}$*. Then* G *is an algebraic group.*

Be very careful however. *For the moment* no field expansion we know can define a non-algebraic simple group of finite Morley rank. What about the future?

- Perhaps some day, a new field expansion will be constructed which defines a non-algebraic simple group—hence refuting the Cherlin-Zilber conjecture in a rather strong sense.
- Another possibility is that there will be non-algebraic simple groups of finite Morley rank, but none of them will be definable in a field expansion: Cherlin-Zilber would be false, but not because of fields (this might be the case should "bad/asomic groups" exist).
- And of course there is the possibility that the Cherlin-Zilber conjecture is simply true.

Rosenlicht's Theorem

There is a dual statement (Barsotti-Chevalley-Rosenlicht, with an interesting story): if G is a connected algebraic group, then G' is a linear algebraic group [Ros56, Theorem 16]. It would be interesting to have an elementary proof, along the lines of the one Poizat gave for Rosenlicht's $G/Z(G)$ theorem [Poizat, Theorem 4.14]. And of course, there is nothing similar for abstract groups of finite Morley rank.

It is unclear what the contents of [**Fré18**a] are; at times it seems to try to take this direction.

Definably linear groups in characteristic 0

It is a significant open question whether there exists a simple, definable subgroup of $\mathrm{GL}_n(\mathbb{K})$ which is not constructible. A little more is known: its connected soluble subgroups would be abelian (already in [MP95, Theorem 1.4]), and it would have no involutions [**BB08**]—the latter was generalised in [**DW18**].

Some (in particular Poizat) have discussed the possibility to use a ranked field with non-minimal $\mathbb{K}^\times$ and infinite $1 < T < \mathbb{K}^\times$ to construct such a monster, where the maximal definable, connected subgroups would be conjugates of T. Apparently this was never realised.

Incidence geometries left aside (recently revived in [**BP19**]), Hrushovski amalgams mostly produced *fields*. The only groups of finite Morley rank obtained by such means are (so far) the Baudisch groups, which are nilpotent; so perhaps being able to amalgamate simple groups will require new developments in pure model theory.

Non-simple, definably linear groups

Mustafin [**Mus04**] provided the following generalisations; $R(G)$ denotes the *soluble radical* [Borovik-Nesin, §7.2].

Theorem ([Mus04, Théorème 2.6])**.** *Let* $\mathbb{K}$ *be a ranked field of positive characteristic and* $G \leq \mathrm{GL}_n(\mathbb{K})$ *be a definable, connected subgroup. Then* $G/R(G)$ *is* definably isomorphic *to a finite product of simple algebraic groups over* $\mathbb{K}$.

[Wei55]: André Weil. 'On algebraic groups of transformations'. *Amer. J. Math.* 77 (1955), pp. 355–391
[Pil88]: Anand Pillay. 'On groups and fields definable in *o*-minimal structures'. *J. Pure Appl. Algebra* 53(3) (1988), pp. 239–255
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[DW18]: Adrien Deloro and Joshua Wiscons. 'The Geometric Theorem (Paris Album No.1)'. In preparation. 2018
[BP19]: John Baldwin and Gianluca Paolini. 'Strongly minimal Steiner systems I: Existence'. Preprint. arXiv:1903.03541. 2019
[Mus04]: Yerulan Mustafin. 'Structure des groupes linéaires définissables dans un corps de rang de Morley fini'. *J. Algebra* 281(2) (2004), pp. 753–773

Theorem ([Mus04, Théorème 2.9]). *Let $\mathbb{K}$ be a ranked field of characteristic* 0 *and* $G \leq \mathrm{GL}_n(\mathbb{K})$ *be a definable, connected subgroup. Then $G/R(G)$ is* equal *to $C \times (A_1 \times \cdots \times A_k)$, the direct product of a definable group with no unipotent elements and finitely may simple algebraic groups over $\mathbb{K}$.*

Exercise. Let $\mathbb{K}$ be field of finite Morley rank.

1. Let U be a definable $\mathbb{K}$-vector space.

   Let $A \leq U$ be a definable, additive subgroup. Suppose that $N_{\mathbb{K}}(A) = \{k \in \mathbb{K} : kA \leq A\}$ is infinite. Prove that A is a vector subspace.
2. If $\mathbb{K}$ has characteristic 0, then any definable group homomorphism $\mathbb{K}_+^m \to \mathbb{K}_+^n$ is $\mathbb{K}$-linear.
3. There are no definable group homomorphisms $\mathbb{K}_+^m \to (\mathbb{K}^\times)^n$ nor $(\mathbb{K}^\times)^n \to \mathbb{K}_+^m$.

Exercise. Let $\mathbb{K}$ be any field *of positive characteristic* and $G \leq \mathrm{GL}_n(\mathbb{K})$ be *any* (not necessarily definable) subgroup. Suppose that G, as a group, has finite Morley rank. Then G is stable under the Jordan decomposition. Hint: study $\langle g \rangle_{\mathrm{def}}$.

Exercise. The goal of this exercise is to prove that the minimal linear algebraic groups are $\mathbb{G}_a$ and $\mathbb{G}_m$ [Humphreys, Theorem 20.5] using the theory of groups of finite Morley rank.

Let $\mathbb{K}$ be an algebraically closed field and $A \leq \mathrm{GL}_n(\mathbb{K})$ be a minimal linear algebraic group.

1. Show that we may assume that A consists of triangular matrices.
2. If A intersects the diagonal subgroup, then $A \simeq \mathbb{K}^\times$. (Hint: take a non-trivial coordinate map $\pi_{i,i} : A \to \mathbb{K}^\times$. To kill the finite kernel F show $A \simeq A/F$ using a power map.)

   From now on suppose that A consists of strictly upper-triangular matrices.
3. If $\mathbb{K}$ has characteristic 0, then $A \simeq \mathbb{K}_+$.

   From now on suppose that $\mathbb{K}$ has characteristic p.
4. Using the truncated exponential and logarithm, show that $A/F \simeq \mathbb{K}_+$ for some finite $F \leq A$.
5. Conclude by proving and using this lemma of general interest: if $\mathbb{K} \models \mathrm{ACF}_p$, $V = \mathbb{K}_+^n$, and $\alpha \in V$, then $V/\langle \alpha \rangle \simeq V$ definably (hint: Artin-Schreier map).

Exercise. The goal of this exercise is to prove Poizat's Monosomy Lemma (see §3.2, Lecture 2). Let $\mathbb{K}$ be a pure algebraically closed field and $\mathbb{L}$ be an infinite field definable in $\mathbb{K}$. We want to show that there is a definable field isomorphism $\varphi : \mathbb{K} \simeq \mathbb{L}$.

1. Prove that $\mathbb{L}_+$ is a strictly upper-triangular, linear algebraic group over $\mathbb{K}$. Hint: Weil-Hrushovski and Rosenlicht Theorems.
2. If $\mathbb{K}$ has characteristic 0 then $\mathbb{L}_+ \simeq \mathbb{K}_+$ (hint: this is yet another use of the Lie algebra).
3. If $\mathbb{K}$ has characteristic $p > 0$ then $\mathbb{L}_+ \simeq \mathbb{K}_+$ (hint: admit from algebraic geometry [Humphreys, Theorem 19.3] that $\mathbb{L}^\times \simeq (\mathbb{K}^\times)^n$ for some n. Studying torsion, conclude that $\operatorname{rk} \mathbb{L} = \operatorname{rk} \mathbb{K}$, so $\mathbb{L}_+/F \simeq \mathbb{K}_+$ for finite $F \leq \mathbb{L}_+$. Kill F à la Artin-Schreier).
4. Conclude by *admitting* and using this lemma of general interest: if $\mathbb{K}$ is a pure algebraically closed field, then any definable group of automorphisms of $\mathbb{K}_+$ is contained in $\mathbb{K}^\times$.

Be very careful that for two definable fields, having definable group isomorphisms $\mathbb{K}_+ \simeq \mathbb{L}_+$ and $\mathbb{K}^\times \simeq \mathbb{L}^\times$ does *not* guarantee $\mathbb{K} \simeq \mathbb{L}$ as fields.

Lecture 4 – Rank and Representations

In this lecture. Today we talk about representations of finite Morley rank.

References:

- [**BC08**] was the first article seriously considering groups of finite Morley rank as permutation groups.
- The most general text on representations of finite Morley rank is [**Del16**c].

[BC08]: Alexandre Borovik and Gregory Cherlin. 'Permutation groups of finite Morley rank'. In: *Model theory with applications to algebra and analysis. Vol. 2.* Ed. by Zoé Chatzidakis et al. Vol. 350. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, Cambridge, 2008, pp. 59–124

[Del16c]: Adrien Deloro. *Un regard élémentaire sur les groupes algébriques.* Mémoire d'habilitation à diriger des recherches. Paris, 2016

7 Generalities

7.1 Setting

One mostly represents algebraic structures into vector spaces. However in our setting it is more natural to consider action on abelian groups:

- such configurations arise naturally in the study of permutation groups ([MP95] ports the O'Nan-Scott Theorem from finite group theory to the finite Morley rank case);
- it is known, and compatible with the general ideology, that group actions often linearise spontaneously (see the Linearisation Theorem in §4.1, Lecture 2).

This we hope motivates the following as a decent investigation topic.

Definition (module of finite Morley rank)**.** *A* module of finite Morley rank *is a triple* $(G, V, \cdot)$ *where* G *is an infinite group,* V *is a* connected *abelian group,* $\cdot$ *is an action of* G *on* V *(say, a subset of* $G \times V^2$*), and all three are definable in some large structure of finite Morley rank.*

Remark.

- The "large structure" can be taken to have underlying set $G \ltimes V$.
- Requiring V to be connected has an effect. The module is *irreducible*, also known as G-*minimal*, if there are no non-trivial definable, *connected*, proper, G-invariant subgroups.

There are two main forms of reasonable questions, described as follows.

7.2 Simultaneous identification

Here one tries to identify both G and the action on V as in the following results.

Theorem ([**Del09**])**.** *Let* (G, V) *be an irreducible, faithful module of finite Morley rank with* G *non-soluble and* $\operatorname{rk} V = 2$*. Then:*

- *or* $G \simeq \mathrm{SL}_2(\mathbb{K})$ *in its natural action;*
- *or* $G \simeq \mathrm{GL}_2(\mathbb{K})$ *in its natural action.*

Theorem ([**BD16**])**.** *Let* (G, V) *be an irreducible, faithful module of finite Morley rank with* G *non-soluble and* $\operatorname{rk} V = 3$*. Then:*

- *either* $G \simeq \mathrm{PSL}_2(\mathbb{K})$ *in its adjoint action (the action on trace zero,* 2×2 *matrices);*
- *or* $G \simeq \mathrm{SL}_3(\mathbb{K})$ *in its natural action;*
- *or* $G \simeq \mathrm{GL}_3(\mathbb{K})$ *in its natural action.*

Interestingly enough, proving the latter required just every single piece of work on groups of finite Morley rank, featuring:

- the "even type" positive solution to Cherlin-Zilber by Altınel, Borovik and Cherlin [**ABC08**];
- Poizat-style study of definably linear groups ([Poi01], see previous lecture);
- all the involution-based technology;
- the thorough study of "small configurations of odd type" [**DJ16**];
- Frécon's result that bad groups of rank 3 do not exist [Fré18b].

As a result, I consider reading [BD16] an excellent way to learn about groups of finite Morley rank, both concrete and abstract.

[Del09]: Adrien Deloro. 'Actions of groups of finite Morley rank on small abelian groups'. *Bull. Symb. Log.* 15(1) (2009), pp. 70–90
[BD16]: Alexandre Borovik and Adrien Deloro. 'Rank 3 Bingo'. *J. Symb. Log.* 81(4) (2016), pp. 1451–1480
[ABC08]: Tuna Altınel, Alexandre Borovik and Gregory Cherlin. *Simple groups of finite Morley rank.* Vol. 145. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 2008. xx+556
[DJ16]: Adrien Deloro and Éric Jaligot. 'Involutive Automorphisms of $N_\circ^\circ$-groups of finite Morley rank'. *Pacific J. Math.* 285(1) (2016), pp. 111–184

7.3 Modules for algebraic groups

For instance, if $G = \mathbb{G}_{\mathbb{K}}$ is the group of $\mathbb{K}$-points of some algebraic group over some (algebraically closed) field, then any algebraic representation V of $\mathbb{G}$ taken over $\mathbb{K}$ will be a module of finite Morley rank for G. The main question is: are there other modules? Does model-theoretic representation theory extend algebraic representation theory beyond predictable twists?

This is what we discuss in the final section.

8 Definable representations of algebraic groups

8.1 Matter

Lemma (the characteristic of a module)**.** *Let (G, V) be a non-trivial, irreducible module of finite Morley rank. Then:*

- *either V has prime exponent;*
- *or V is divisible and torsion-free.*

We say that V has characteristic p *or* 0*, accordingly—and extend the definition to the non-irreducible case.*

Proof. This is exactly what we proved in the first step of the Linearisation Theorem (see §4.1, Lecture 2). □

Corollary (the structure of matter for faithful modules of characteristic p)**.** *Let (G, V) be a module of finite Morley rank where G is faithful and V has characteristic $p > 0$ (we do not need irreducibility here). Then every definable, soluble subgroup of G has the form $B = T \ltimes U$, where U is a definable, nilpotent p-group of bounded exponent, and T is a divisible, abelian group with no p-element.*

Proof. This mostly uses Nesin's classical structure theorem for nilpotent groups of finite Morley rank (see the final notes of Lecture 1). We want to give an idea. Suppose that B contains an infinite elementary abelian q-group for $q \neq p$, say $U \neq 1$. We shall prove a contradiction.

Take a U-composition series $V = V_n > \cdots > V_0 = 0$, so that the V_i's are definable, connected, U-invariant, and $W_i = V_i/V_{i-1}$ is U-irreducible. Consider the faithful action of the abelian, connected group $U/C_U(W_i)$ on the irreducible module W_i. If $U/C_U(W_i) \neq 1$ then by the Linearisation Theorem we find an algebraically closed field with $W_i \simeq \mathbb{K}_+^n$ and $U/C_U(W_i) \hookrightarrow \mathrm{GL}_n(\mathbb{K})$. Because of V, $\mathbb{K}$ must have characteristic p; this prevents $\mathrm{GL}_n(\mathbb{K})$ from having an infinite elementary abelian q-group, hence $U/C_U(W_i) = 1$.

In other words, U centralises all quotients in the composition series. It easily follows that $U \ltimes V$ is nilpotent, and as in the finite case, it can be proved that the p-torsion subgroup commutes with the q-torsion subgroup. So U centralises V, and by faithfulness $U = 1$. This rules out "q-unipotence" for $q \neq p$.

The same argument prevents B from containing a copy of $\mathbb{Z}/p^\infty\mathbb{Z}$. As a matter of fact, using also a Corollary to Wagner theorem on fields (see §6.3 in Lecture 3) discussed in the exercises, one can also prove that there is no non-trivial "0-unipotence" in B. Then thanks to the theory of soluble groups of finite Morley rank one may reach the conclusion. □

As a consequence, *no cross-characteristic phenomena occur* in our setting: if $G = \mathbb{G}_{\mathbb{K}}$ is the group of $\mathbb{K}$-points of a simple algebraic group in characteristic q (possibly 0), then non-trivial irreducible modules have characteristic q.

Remark. It can even be shown using Wagner's Theorem that T is the definable envelope of its (non-definable) torsion subgroup; T is called a *good torus*; see the final notes.

8.2 Characteristic 0

Corollary ([**CD12**, Lemma 1.4]; modules for alg. groups in char. 0 are algebraic)**.** *Let (G, V) be an irreducible, faithful module of finite Morley rank where G has the form $\mathbb{G}_\mathbb{K}$ for a simple algebraic group $\mathbb{G}$ and an algebraically closed field $\mathbb{K}$* of characteristic zero.

Then V is isomorphic to an algebraic representation of G as an algebraic group; i.e. up to isomorphism, (G, V) already lives in the algebraic category.

Proof. Using Zilber-style interpretation, the soluble, non-nilpotent subgroups of G (which exist by geometry [Humphreys, Proposition 21.4.B]) interpret a field $\mathbb{K}_1$; one does not really need Poizat monosomy to see that $\mathbb{K}_1$ has characteristic 0. In particular the structure of matter forces V to be divisible and torsion-free.

Thanks to the Linearisation Theorem we then find a definable, algebraically closed field $\mathbb{L}$ of characteristic 0 such that $V \simeq \mathbb{L}_+^n$ and $G \hookrightarrow \mathrm{GL}_n(\mathbb{L})$ definably. Now G is quasi-simple; since it has non-nilpotent, soluble subgroups, these do not consist of (geometrically) semisimple elements [Humphreys, Proposition 19.2]; finally since the characteristic of $\mathbb{L}$ is 0, we know from the structure of definably linear groups that G is constructible, i.e. definable in the pure field $\mathbb{L}$.

Then seeing G as an algebraic group *over* $\mathbb{L}$, we have a constructible subgroup of $\mathrm{GL}_n(\mathbb{L})$ acting naturally on $V \simeq \mathbb{L}_+^n$: we are done. □

Remark. Using Poizat's Monosomy Lemma (see §3.2, Lecture 2) *twice*, one finds $\mathbb{K} \simeq \mathbb{K}_1 \simeq \mathbb{L}$.

The resulting isomorphism $\mathbb{K} \simeq \mathbb{L}$ is however not definable in any structure smaller than the universe containing both $\mathbb{K}$ and $\mathbb{L}$. Since $\mathbb{K}_1$ is definable in G, which is definable in $\mathbb{K}$, the isomorphism $\mathbb{K} \simeq \mathbb{K}_1$ is $\mathbb{K}$-definable. Since $\mathbb{K}_1$ is definable in G, which is definable in $\mathbb{L}$, the isomorphism $\mathbb{K}_1 \simeq \mathbb{L}$ is $\mathbb{L}$-definable. But neither $\mathbb{K}$ nor $\mathbb{L}$ lives in the pure universe of the other: $\mathbb{L}$ is not definable in G but in (G, V); while in $\mathbb{L}$ only the copy $\mathbb{K}_1$ is definable.

This is the end of the story in characteristic 0.

8.3 Positive characteristic

Question. Can one classify representations of finite Morley rank in positive characteristic?

Conjecture ([Del16c])**.** *Let $\mathbb{G}$ be a reductive algebraic group, $\mathbb{K}$ be a field of finite Morley rank of characteristic p, and $G = \mathbb{G}_\mathbb{K}$ as a group of finite Morley rank. Let V be an irreducible G-module of finite Morley rank with $C_G(V)$ finite.*

Then there are irreducible representations W_i of G as an algebraic group *and* definable *automorphisms $\varphi \in \mathrm{Aut}(\mathbb{K})$ such that $V \simeq \otimes_i {}^{\varphi_i} W_i$ (twist-and-tensor).*

The conjecture is modelled after Steinberg's celebrated Tensor Product Theorem [**Ste63**].

Remark. As we know, definable field automorphisms of ranked fields do not exist *in characteristic* 0. So the conjecture is much simpler in that case, and positively settled by §8.2.

Here is what we know so far.

Fact (unpublished [**Del16**a])**.** *Definable in a ranked universe, let $\mathbb{K}$ be a field, $G \simeq \mathrm{SL}_2(\mathbb{K})$, and V be an irreducible G-module of Morley rank $\leq 4 \cdot \mathrm{rk}(\mathbb{K})$. Then one of the following holds:*

- $\mathrm{rk}\, V = 2\,\mathrm{rk}\,\mathbb{K}$*; $V \simeq \mathbb{K}^2$ in the natural action of G;*
- $\mathrm{rk}\, V = 3\,\mathrm{rk}\,\mathbb{K}$*; $V \simeq \mathbb{K}^3$ in the adjoint action of* $\mathrm{PSL}_2(\mathbb{K})$ *(i.e. action on homogeneous polynomials $\mathbb{K}[X^2, XY, Y^2]$);*
- $\mathrm{rk}\, V = 4\,\mathrm{rk}\,\mathbb{K}$*; $V \simeq \mathbb{K}^4$ in the rational representation of dimension* 4 *(i.e. action on homogeneous polynomials $\mathbb{K}[X^3, X^2Y, XY^2, Y^3]$);*
- $\mathrm{rk}\, V = 4\,\mathrm{rk}\,\mathbb{K}$*; there is a definable field automorphism $\varphi \in \mathrm{DefAut}(\mathbb{K})$ such that $V \simeq (\mathbb{K}^2) \otimes^\varphi (\mathbb{K}^2)$ (twist-and-tensor).*

The first two cases were already in [CD12], though obtained more clumsily. Notice that only the fourth case involves a tensor product. So the above conjecture is known to hold in the first non-trivial case (the Nat-by-twist-Nat representation). Beyond is a mystery.

[CD12]: Gregory Cherlin and Adrien Deloro. 'Small representations of SL_2 in the finite Morley rank category'. *J. Symbolic Logic* 77(3) (2012), pp. 919–933
[Ste63]: Robert Steinberg. 'Representations of algebraic groups'. *Nagoya Math. J.* 22 (1963), pp. 33–56
[Del16a]: Adrien Deloro. 'For a study of definable representations of algebraic groups'. Preprint. 2016

8.4 Tools

There are two issues to discuss.

1. First, field automorphisms make the conjecture a little beyond algebraic (bear in mind that the classification of definable field automorphisms is an open question).

   Let us return to Poizat's Theorem on definably linear groups in positive characteristic (§6.3, in Lecture 3). Suppose (G, V) is an irreducible module with $G = \mathbb{G}(\mathbb{K})$ a simple algebraic group in characteristic $p > 0$. *Suppose one could linearise*, i.e. find $V \simeq \mathbb{L}_+^n$ and $G \hookrightarrow \mathrm{GL}_n(\mathbb{L})$. By Poizat's Theorem, G is definably isomorphic to an algebraic group over $\mathbb{L}$; $\mathbb{K}$ and $\mathbb{L}$ are isomorphic, but finding a field isomorphism requires a finite number of automorphisms of $\mathbb{L}$. This is consistent with the spirit of the conjecture, but one does not see how to analyse V into a tensor product.

2. More importantly, one cannot linearise a priori in positive characteristic (this is a serious limit of the Linearisation Theorem).

   The only strategy one can imagine so far is to try to reconstruct "weight spaces", viz. to understand the action of the algebraic torus; this is how [Del16a] proceeds.

   Of course the question itself is non-trivial since we fall short of a linear structure a priori. Here is our best approximation of such spaces.

Theorem (Tindzogho Ntsiri [**Tin17**, Corollary 2.14]: complete reducibility of fixed-point free toral actions)**.** *In a universe of finite Morley rank, let T be a definable, connected, soluble, $p^\perp$ group and V be a T-module of characteristic p. Suppose $C_V(T) = 0$. Then every T-submodule admits a direct complement T-module.*

As a corollary, if T is the algebraic torus of an algebraic group G and V is a G-module, then $V = C_V(T) \oplus [T, V]$ and $[T, V]$ is *completely reducible*, i.e. a direct sum of T-minimal submodules W_i.

It is then quite tempting to call these the *weight spaces*. As a matter of fact, by the Linearisation Theorem, there is a definable field $\mathbb{L}_i$ with $W_i \simeq (\mathbb{L}_i)_+$ and T maps to an infinite subgroup of $\mathbb{L}_i^\times$. However:

- it is not clear whether the various $\mathbb{L}_i$ are isomorphic to $\mathbb{K}$;
- it is not even clear whether $\mathbb{L}_i \simeq \mathbb{L}_j$ must hold.

To achieve this, further tools (still in development) will be needed, and hopefully new monosomy properties.

Lemma. *If V is an irreducible $\mathrm{SL}_2(\mathbb{K})$-module and $T \simeq \mathbb{K}^\times$ denotes the algebraic torus, then $[T, V]$ is a direct sum of an even number of T-minimal modules* all of same rank as $\mathbb{K}$.

Then in rank $\leq 4 \operatorname{rk} \mathbb{K}$, some ugly group-theoretic computations work—but this is a small rank miracle. So to be honest, as of today, I can think of no general strategy to attack the next non-trivial question.

Question. Classify irreducible $\mathrm{SL}_2(\mathbb{K})$-modules of finite Morley rank (towards the conjecture).

This is something I will enjoy spending time on.

Final notes and exercises

Modules of finite Morley rank

It is not clear whether every simple infinite group of finite Morley rank has a non-trivial module; and although it would follow from the Cherlin-Zilber conjecture, in its absence it is not clear what the overlap between both problems is. So far only the remotely related question of *non-definable linearity* (viz. one requires V to be a vector space but one drops the definability requirement) has been tackled, in the nilpotent and soluble cases [**AW09**], [**AW11**].

Linearising in an algebraic group relies on its local coordinates, and involves the cotangent action [Poizat, Theorem 4.14], [Humphreys, Theorem 8.6].

How to define the tangent Lie ring of an *abstract* group of finite Morley rank, is a purely speculative question.

[**Tin17**]: Jules Tindzogho Ntsiri. 'The structure of an SL_2-module of finite Morley rank'. *Math. Log. Q.* 63(5) (2017), pp. 364–375

[**AW09**]: Tuna Altınel and John Wilson. 'On the linearity of torsion-free nilpotent groups of finite Morley rank'. *Proc. Amer. Math. Soc.* 137(5) (2009), pp. 1813–1821

[**AW11**]: Tuna Altınel and John Wilson. 'Linear representations of soluble groups of finite Morley rank'. *Proc. Amer. Math. Soc.* 139(8) (2011), pp. 2957–2972

Simultaneous identification

[BD16] may be seen as a step towards a general conjecture, which requires a definition.

Definition (generic transitivity)**.** *The action of a group G on a set X is* generically n-transitive *if G has a full-rank orbit in X^n.*

Conjecture (Borovik-Cherlin, [BC08, Problem 13])**.** *Let (G, V) be a faithful, irreducible module of finite Morley rank with* $\operatorname{rk} V = n$*. Suppose G acts generically n-transitively. Then there is a definable field $\mathbb{K}$ with $V \simeq \mathbb{K}_+^n$ and $G \simeq \mathrm{GL}_n(\mathbb{K})$.*

Notice that solving this conjecture would not require solving all of Cherlin-Zilber: for instance, a high generic transitivity degree immediately yields involutions (which disposes of some annoying cases in CZ). [BD16] gives a positive answer for $n \leq 3$; Borovik expects an inductive approach from there on. It has been solved in the *sharply* generically n-transitive case [**BB18**].

Incidentally, the conjecture is itself a step towards a larger one.

Conjecture (Borovik-Cherlin, [BC08, Problem 9])**.** *Let (G, X) be a faithful permutation group of finite Morley rank with* $\operatorname{rk} X = n$*. Suppose G acts generically $n+2$-transitively. Then there is a definable field $\mathbb{K}$ with $V \simeq \mathbb{P}^n(\mathbb{K})$ and $G \simeq \mathrm{GL}_{n+1}(\mathbb{K})$.*

Case $n = 1$ is a classical theorem by Hrushovski [**Hru89**]. Altınel and Wiscons [**AW18**] have recently solved case $n = 2$ and are pushing the matter further [**AW19**].

Matter in abstract groups

Matter is a remarkably delicate topic in the context of *abstract* groups of finite Morley rank.

- The notion of semisimplicity can be regarded as extremely satisfactory if there is divisible torsion, and quite disarming if there isn't.
  In the torsion case, one will definitely adopt Cherlin's good, or decent, tori [**Che05**], which have all expectable properties, including conjugacy.
  In the non-torsion case, one can use the theory of Carter subgroups instead; it is not clear to me how much Carter theory fits into the orthodox Borovik programme.
- The notion of unipotence is a nightmare. Torsion unipotence (i.e. if one imitates the behaviour of a linear algebraic group in positive characteristic) can be understood quickly—at a superficial level. Torsion-free unipotence (imitating characteristic 0) has proved a remakably subtle topic with Burdges' massive theory [**Bur04**], but with key applications to the classification of small groups [DJ16].

As a result, it is hard (though not hopeless [**ABF15**]) to define an abstract Jordan decomposition in general. It is remarkable how the issue vanishes in the presence of a module; which also suggests that the above question of the existence of modules for abstract groups is untractable.

The tensor conjecture

The unpublished [Del16a] tackles $\operatorname{Nat}\mathrm{SL}_2(\mathbb{K}) \otimes \operatorname{Nat}\mathrm{SL}_2(\mathbb{K})^\varphi$ at a considerable computational cost. As a matter of fact one could hope to push the method to $\operatorname{rk} V \leq 5 \operatorname{rk}\mathbb{K}$, but for serious geometric obstructions explained in [**Del16**b], it will not extend any furhter. So there is at present no general strategy even for $G = (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$ acting on a module V with $\operatorname{rk} V = 6 \operatorname{rk}\mathbb{K}$.

- One could try using the representation theory of the locally finite model, viz. consider the representation theory of $\mathrm{SL}_2(\mathbb{F}_q)$ for increasing q, hoping that the resulting module structures will match up well.
- One could also be more model-theoretic, and try to use more systematically the torus to produce weight spaces, and connect them using unipotent groups and commutator maps.

Exercise. Let (B, V) be a faithful module of finite Morley rank where B is connected and soluble, and V has characteristic 0.

Prove that B cannot contain an infinite group of bounded exponent; that for each prime $p > 0$, if B contains a copy of $(\mathbb{Z}/p^\infty\mathbb{Z})^d$, then $d \leq \operatorname{rk} V$ (one says that the *Prüfer p-rank of B* is bounded by $\operatorname{rk} V$; more specifically by $\ell_B(V)$, its length as a B-module).

[BB18]: Ayşe Berkman and Alexandre Borovik. 'Groups of finite Morley rank with a generically sharply multiply transitive action'. *J. Algebra* 513 (2018), pp. 113–132

[Hru89]: Ehud Hrushovski. 'Almost orthogonal regular types'. *Ann. Pure Appl. Logic* 45(2) (1989). Stability in model theory, II (Trento, 1987), pp. 139–155

[AW18]: Tuna Altınel and Joshua Wiscons. 'Recognizing PGL_3 via generic 4-transitivity'. *J. Eur. Math. Soc.* 20(6) (2018), pp. 1525–1559

[AW19]: Tuna Altınel and Joshua Wiscons. 'Toward the recognition of PGL_n via a high degree of generic transitivity'. *Comm. Algebra* 47(1) (2019), pp. 206–215

[Che05]: Gregory Cherlin. 'Good tori in groups of finite Morley rank'. *J. Group Theory* 8(5) (2005), pp. 613–621

[Bur04]: Jeffrey Burdges. 'Simple groups of finite Morley rank of odd and degenerate type'. PhD thesis. New Brunswick, New Jersey: Rutgers University, 2004. 225 pp.

[ABF15]: Tuna Altınel, Jeffrey Burdges and Olivier Frécon. 'Structure of Borel subgroups in simple groups of finite Morley rank'. *Israel J. Math.* 208(1) (2015), pp. 101–162

[Del16b]: Adrien Deloro. 'Symmetric powers of Nat $\mathrm{SL}_2(\mathbb{K})$'. *J. Group Theory* 19(4) (2016), pp. 661–692

The next two exercises rely on the following corollary to Wagner's theorem on fields.

Corollary (consequence of [Wag01, Corollary 9])**.** *If $\mathbb{K}$ is a field of finite Morley rank of positive characteristic, then every non-trivial definable subquotient of $\mathbb{K}^\times$ contains torsion.*

(In characteristic 0 this fails: one can produce $T < \mathbb{K}^\times$ which contains no torsion.)

Exercise. Let (A, V) be a faithful, irreducible module of finite Morley rank with A abelian. Suppose that V has characteristic $p > 0$ and that for all primes $q \neq p$, A contains a copy of $\mathbb{Z}/q^\infty\mathbb{Z}$. Prove that $\operatorname{rk} A = \operatorname{rk} V$.

Exercise. Let (B, V) be a faithful module of finite Morley rank where B is connected and soluble, and V has characteristic $p > 0$. Prove that all subquotients of B contain torsion.

Exercise. This exercise is key to understanding the "three field configurations" in [CD12, §1.3], [Tin17, §3.1], [Del16a, §2.2]. All objects are definable in a *common* ranked universe.

1. Let $\mathbb{K}, \mathbb{L}$ be definable fields. Suppose that $\mathbb{K} \times \mathbb{L}$ contains an infinite definable subring $\mathbb{F}$.
   Prove that the first projection map induces a field isomorphism $\mathbb{K} \simeq \mathbb{F}$. Deduce $\mathbb{K} \simeq \mathbb{L}$ definably.
2. Now we have a definable $\mathbb{K}$-vector space U and a definable $\mathbb{L}$-vector space V. We also have a definable group T and definable morphisms $\kappa : T \to \mathbb{K}^\times$, $\lambda : T \to \mathbb{L}^\times$; so U and V are T-modules.
   We even have a non-trivial, definable, additive morphism $\alpha : U \to V$ which is T*-covariant* (viz. $\alpha(t \cdot u) = t \cdot \alpha(u)$).
   Prove that $\ker \kappa = \ker \lambda$.
3. Prove that if $\operatorname{im} \kappa$ or $\operatorname{im} \lambda$ is infinite, then $\mathbb{K} \simeq \mathbb{L}$ definably.
4. Now suppose that α is contravariant instead, viz. $\alpha(t \cdot u) = t^{-1} \cdot \alpha(u)$. Prove that if $\operatorname{im} \kappa$ or $\operatorname{im} \lambda$ is infinite, then $\mathbb{K} \simeq \mathbb{L}$ definably.
   Hint: introduce $\mathbb{L}' = \mathbb{L} \setminus \{0\} \cup \{\infty\}$ with obvious multiplication and addition given by:
   $$a * b = \frac{ab}{a+b}.$$

Exercise. The goal of this exercise is to prove Tindzogho Ntiri's complete reducibility theorem. Let p be a prime. Let (T, V) be a module of finite Morley rank with T definable, connected, soluble, and $p^\perp$, and V has characteristic p. Suppose $C_V(T) = 0$. *We admit that the last assumption carries to subquotient modules.*

Let $0 < W < V$ be a T-submodule: we seek a direct complement in the category of T-modules.

1. Reduce to p-divisible, abelian T with $C_T(V) = 1$.
2. Using induction, reduce to the case where both W and V/W are T-irreducible, and W is the only non-trivial, proper T-submodule of V.
3. Let R be the ring generated by T inside $\operatorname{DefEnd}(V)$. Prove that $\mathfrak{m} := \operatorname{Ann}_R(W) = \operatorname{Ann}_R(V/W)$ is a maximal ideal. (Hint: for $f \in R$, play with $\ker^\circ f$ and $\operatorname{im} f$, which are T-invariant.)
4. Prove that R is p-divisible. Deduce that $\mathfrak{m} = 0$, and that R is a definable field. Conclude.

Remark. There should be a more direct proof, perhaps even one avoiding the $p > 0$ restriction.

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