Groups and Representations of finite Morley rank

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Abstract

Notes for a tutorial given during the *Models and Groups 6* meeting held at the Istanbul Centre for Mathematical Sciences on 24–26 March 2016. There were three lectures of 75 minutes each.

Lecture 1 — The setting

1 Model-theoretic background

Let us first describe the setting for our lectures: universes of finite Morley rank. This requires a minimal model-theoretic framework which is better understood in view of the theory of linear algebraic groups; knowledge of the latter helps, but is not a prerequisite.

1.1 Definable sets

Definition 1.1 (structure)**.** A structure is a set M equipped with relations. Each relation R_i is a subset of M^{n_i} for some n_i which can depend on R_i. Using the graph trick we may also allow functions $f_i : M^{n_i} \to M$. We write $\mathcal{M} = (M; \{R_i\})$ to denote the structure.

A group structure is a structure $(G; \cdot, =, \dots)$, with *possibly more* relations than just the group law. Likewise a field structure is a structure $(\mathbb{K}; +, \cdot, =, \dots)$ with *possibly more* relations. The phrase *pure group* (or *pure field*) helps emphasize the other case.

Definition 1.2 (definable sets)**.** The definable class is the smallest collection:

- containing all singletons, all Cartesian powers M^n, and all relations R_i;
- stable under Boolean combinations (viz. finite intersections, finite unions, and taking complements; infinitary combinations *not* allowed);
- stable under projection (viz. if $A \subseteq M^{n+1}$ is definable, so is $\pi(A) = \{\overline{x} \in M^n : \exists y \in M, (\overline{x}, y) \in A\}$; and likewise for the other projections);
- stable by quotient (viz. if $A \subseteq M^n$ is definable and $E \subseteq A^2$ is an equivalence relation on A, then A/E is definable).

Example 1.3.

- For model-theorists. In proper model-theoretic terminology, definable sets ought to be called "interpretable with parameters in M", or equivalently "definable in (M^{eq}, M)".
- For algebraists. Consider an algebraically closed field $\mathbb{K}$ *with no extra structure*. Then by Chevalley's Theorem the definable class is exactly the constructible class. (If $\mathbb{K}$ has extra structure there are more basic relations, so not all definable sets need be constructible; we shall return to the topic regularly.)
- If G is a group structure and $x \in G$, then $x^G = \{y \in G : \exists g \in G, y = g^{-1}xg\}$ and $C_G(x) = \{y \in G : xy = yx\}$ are definable. So are $Z(G) = \{x \in G : \forall y \in G, xy = yx\}$ and $G/Z(G)$. Even the group law on $G/Z(G)$ is definable: so $G/Z(G)$ is a definable group.
- More generally, if $X \subseteq G$ is definable, then $C_G(X)$ and $N_G(X)$ are definable. It need not be so if X is not.
- Likewise, the *set* of commutators is always definable, but the commutator *subgroup* need not be.

Here is a more detailed example.

Lemma 1.4. *Let $\mathbb{K}$ be an algebraically closed field and $G = \mathrm{GL}_n(\mathbb{K})$ as a group. Then the maps J_s and J_u mapping $g \in \mathrm{GL}_n(\mathbb{K})$ to is semisimple and unipotent parts in the Jordan decomposition are definable.*

Proof. It so happens that the definable sets in the group (with no extra structure) are exactly the constructible sets (i.e., definable in $\mathbb{K}$, a field with no extra structure). As the constructible point of view brings nothing here, we adopt definable terminology.

A function is definable iff its graph is. Due to uniqueness of the Jordan decomposition, it suffices to show that the following set of triples is definable:

$$\{(g, s, u) \in G^3 : (g = s \cdot u) \ \wedge\ (su = us) \ \wedge\ (s \text{ is semisimple}) \wedge\ (u \text{ is unipotent})\}$$

And clearly it suffices to show that the sets of semisimple elements on the one hand, of unipotent elements on the other hand, are definable.

It is *not* a good idea to think in terms of eigenvalues, as G is an abstract group and we have lost the action on $\mathbb{K}^n$. We need something more intrinsincally group-theoretic.

Take a diagonal matrix d with distinct entries. Then $C(d)$ is exactly the diagonal subgroup, and $\bigcup_{g \in G} C(d^g)$ is the set of semisimple elements, which is therefore definable.

Being unipotent in characteristic $p > 0$ could easily be defined by: having order dividing p^n. But this does not work in characteristic 0. In any case, being unipotent means being conjugate to the strictly upper-triangular group:

$$U := \left\{ \begin{pmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{pmatrix} \right\}$$

To prove definability of $\bigcup_{g \in G} U^g$ it suffices to prove that of U.

The associative $\mathbb{K}$-algebra $M_n(\mathbb{K})$ is around; the group does not know this but we do. Observe that $C_{M_n(\mathbb{K})}(X)$ is always a $\mathbb{K}$-vector space. By the descending chain condition on vector subspaces (which relates to dimension theory and we shall return to this shortly), $C_{M_n(\mathbb{K})}(U)$ is a *finite* intersection $C_{M_n(\mathbb{K})}(u_1, \dots, u_k)$ for some tuple $(u_1, \dots, u_k) \in U^k$. In particular $C_G(U) = G \cap C_{M_n(\mathbb{K})}(U) = C_G(u_1, \dots, u_k)$ is definable. A computation shows that:

$$\hat{U}_{1,n} := C_G(U) = \{\lambda I + \mu E_{1,n} : (\lambda, \mu) \in \mathbb{K}^\times \times \mathbb{K}_+\}$$

where $E_{i,j}$ is the matrix with only one non-zero entry, in cell (i, j).

Now it is a fact that for any $i \neq j$, the group $\hat{U}_{i,j} := \{\lambda I + \mu E_{i,j} : (\lambda, \mu) \in \mathbb{K}^\times \times \mathbb{K}_+\}$ is G-conjugate to $\hat{U}_{1,n}$, hence definable as well. Consider the group they generate:

$$\hat{U} := \left\langle \hat{U}_{i,j} : i < j \right\rangle = \left\{ \begin{pmatrix} \lambda & * & * \\ & \ddots & * \\ & & \lambda \end{pmatrix} : \lambda \in \mathbb{K}^\times \right\}$$

One should be careful in general with generation but here again, by dimension theory, finite products will do: so $\hat{U}$ is definable. Therefore so is its normaliser:

$$B := N_G(\hat{U}) = \left\{ \begin{pmatrix} * & * & * \\ & \ddots & * \\ & & * \end{pmatrix} \right\}$$

It so happens that $B' = U$. One should be careful in general with commutator subgroups, but here again finite products of commutators do (actually U is even the *set* of commutators of B). Hence U is definable. $\square$

As Jordan decomposition will play a role in the last lecture, it is a good idea to do the following.

Exercise 1.5.

- Find your own argument.
- Find parameter-free definitions of semisimplicity and unipotence (we had to fix matrices d and u_i with good properties).

1.2 Morley rank

A certain rank function was introduced by logician Michael Morley for pure model theory. It is named after him, although very little model theory is needed now for the definition. The rough idea is that we want a dimension function enabling the arguments of Lemma 1.4.

Definition 1.6 (rank function)**.** Let M be a structure and $\mathcal{D}$ be its definable class. A rank function is a finite-valued map $\mathrm{rk} : \mathcal{D} \to \mathbb{N}$ with the following properties, in which A, B stand for definable sets and $f : A \twoheadrightarrow B$ is a definable surjection:

- $\mathrm{rk}\, A \geq n + 1$ iff there are infinitely many disjoint definable $B_i \subseteq A$ with $\mathrm{rk}\, B_i \geq n$ ("monotonocity");
- for every integer k, the set $F_k = \{b \in B : \mathrm{rk}\, f^{-1}(b) = k\}$ is definable ("definability");
- if $B = F_k$ for some k then $\mathrm{rk}\, A = k + \mathrm{rk}\, B$ ("additivity");
- there is an integer ℓ such that for each $b \in B$, either $f^{-1}(b)$ has at most ℓ elements or is infinite ("elimination of infinity").

The first axiom has the following consequence: $\mathrm{rk}\, A = 0$ iff A is finite.

Example 1.7.

- For algebraists. Consider an algebraically closed field with no extra structure, and a linear algebraic group $\mathbb{G}_\mathbb{K}$. Then the Zariski dimension is a rank function (on the constructible class).
- For model-theorists. Let G be a group structure. Then G has a rank function iff its theory has finite Morley rank (not true for other structures).

- Any divisible torsion-free group, seen as a pure group, has a $\{0,1\}$-valued rank function. As a matter of fact, every definable set there is either finite or cofinite (trivial if you are a model theorist; not so otherwise).

In the following subsection we shall see a few easy consequences of the rank axioms. We finish with a more theoretical remark.

Remark 1.8. Having finite Morley rank is inherited by definable sets (and definable algebraic structures). Hence a definable group, a definable field, in a structure of finite Morley rank, have finite Morley rank themselves.

2 Groups of finite Morley rank

Before turning to definable representations of algebraic groups, we must say a few words of groups with finite Morley rank in general.

2.1 Basic properties

Lemma 2.1 (and definition)**.** *Let G be a group of finite Morley rank. Then there is a smallest definable subgroup of finite index. It is definably characteristic in G and called the* connected component *of G, denoted G°.*

Remark 2.2. Be very careful that there is no general notion of connected components for definable sets: there is no topology here.

Proof. For standard reasons (see the notion of multiplicity in geometry), there is a bound n such that every definable subgroup of G of finite index has index at most n. Now let G° be a definable subgroup of maximal finite index. If H is another definable subgroup of finite index, then so is $G^\circ \cap H$: by maximality of the index, $G^\circ \leq H$. In particular, it is easily seen that G° is definably characteristic in G. □

Corollary 2.3 (descending chain condition)**.** *Let G be a group of finite Morley rank. Then any descending chain of definable subgroups is stationary.*

Proof. Let (H_i) be such a chain. At some stage n, the rank must become stationary, so for any i, one has $\operatorname{rk} H_{n+i} = \operatorname{rk} H_n$. Because ranks are equal, H_{n+i} must have finite index in H_n, showing that $H_{n+i} \geq H_n^\circ$. Now each H_{n+1} is a union of finitely many translates of H_n°: clearly the chain must become stationary. □

Remark 2.4. There is an ascending chain condition only for definable, *connected* subgroups.

Corollary 2.5 (and definition)**.** *Let $X \subseteq G$ be any subset. Then there is a smallest definable subgroup containing X, called the* hull *of X and denoted by $X \subseteq d(X) \leq G$.*

Be careful that there is no general analogue of the Zariski closure, i.e. no general definition of a "definable closure" in our sense (the phrase exists in model theory but means something else). This can only produce definable *groups* — because the DCC works only for definable *subgroups.*

2.2 Fields of finite Morley rank

Let us now start talking about the atoms of our nature. We start with the algebraic phenomenon.

Theorem 2.6 (not used — and actually not so trivial)**.** *The "basic" linear algebraic groups are $\mathbb{K}_+$ and $\mathbb{K}^\times$; more precisely:*

- *the only linear algebraic groups of dimension* 1 *and multiplicity* 1 *(= constructibly connected) are $\mathbb{K}_+$ and $\mathbb{K}^\times$;*
- *the only infinite linear algebraic groups with no infinite, algebraic, proper subgroups are $\mathbb{K}_+$ and $\mathbb{K}^\times$.*

We will now ask whether this generalises to groups of finite Morley rank.

Question 2.7 (and definition)**.**

- *Can one classify connected groups of rank* 1*?*

- *Say that a group of finite Morley rank is* minimal *if it is infinite and has no infinite,* definable*, proper subgroup. Can one classify minimal groups?*

Remark 2.8.

- A minimal group is connected.
- A connected group of Morley rank 1 is minimal, but the converse need not hold (there are counterexamples in model-theory).
- The hull $d(X)$ of X need not be minimal.

Example 2.9.

- The quasi-cyclic Prüfer p-group $\mathbb{Z}/p^\infty\mathbb{Z} := \bigcup_n \mathbb{Z}/p^n\mathbb{Z}$ is connected and has Morley rank 1.
- More generally, $(\mathbb{Z}/p^\infty\mathbb{Z})^2 \bigoplus \left(\mathbb{Z}_{/}q^\infty\mathbb{Z}\right)^3 \bigoplus (\oplus_I\mathbb{Q})$ for any set (possibly infinite!) I, is connected and has Morley rank 1.
- Try to generalise the latter example. Be careful, this is *not* as general as $\bigoplus_p \left(\oplus_{I_p}\mathbb{Z}/p^\infty\mathbb{Z}\right)$ for arbitrary sets I_p (p a prime).

In short, the situation is pretty bad.

In example 2.9 we gave minimal groups which do not look like $\mathbb{K}_+$ nor $\mathbb{K}^\times$. Can one at least prove that $\mathbb{K}_+$ and $\mathbb{K}^\times$ have Morley rank 1, or are minimal?

Example 2.10. If $\mathbb{K}$ is an algebraically closed field *with no extra structure*, then $\operatorname{rk}\mathbb{K} = 1$.

What can one say of fields of finite Morley rank in general?

Theorem 2.11 (Macintyre)**.** *If $\mathbb{K}$ is an infinite field of finite Morley rank, then $\mathbb{K}$ is algebraically closed.*

Proof. This involves a bit of Galois theory. The proof is extremely interesting in its own right but not in the spirit of the lectures. □

Remark 2.12. The result means in particular that our setting does *not* encompass "real algebraic geometry", which can look surprising as one might have expected $\mathbb{R}$ to be a nice field, perhaps even of finite Morley rank.

It is *not* as a simple argument shows: suppose $\mathbb{R}$ has finite Morley rank n. Notice that $\mathbb{R}_{\geq 0}$ is definable by "being a square". Therefore so is the interval $(0, 1)$. Now the latter is in definable bijection with $\mathbb{R}$ itself, so $\mathbb{R}$ and $(0, 1)$, as a matter of fact any $(k, k + 1)$, has Morley rank n. By the first axiom, $\mathbb{R}$ has rank $\geq n + 1$: a contradiction.

Remark 2.13 (Cherlin)**.** Any infinite (possibly non-commutative) domain of finite Morley rank is an algebraically closed field.

Fact 2.14.

- *If $\mathbb{K}$ is a field of finite Morley rank of characteristic zero, then $\mathbb{K}_+$ is minimal.*
- *There exists a field $\mathbb{K}$ of finite Morley rank of characteristic zero with $\mathbb{K}^\times$ non-minimal.*
- *There exists a field $\mathbb{K}$ of finite Morley rank of characteristic $p > 0$ with $\mathbb{K}_+$ non-minimal.*
- *One has strong evidence (but it is open) that there should be no field $\mathbb{K}$ of finite Morley rank of characteristic $p > 0$ with $\mathbb{K}^\times$ non-minimal.*
- *If $\mathbb{K}$ is a field of finite Morley rank of characteristic $p > 0$, then any definable subgroup of $\mathbb{K}^\times$ contains torsion.*

In particular, there is no reason why a field of finite Morley rank should have rank 1.

Proof. We shall prove only the first as the rest is highly non-trivial model-theory, proved by: Baudisch, Hils, Martin-Pizarro, Wagner, and Ziegler.

Let $A < \mathbb{K}_+$ be a infinite, definable, proper subgroup; we shall find a contradiction. Let $N = \{x \in \mathbb{K} : xA = A\}$. Clearly N is a definable subfield of $\mathbb{K}$. Since the characteristic is zero, N is infinite. By monotonocity (see Definition 1.6), the extension $\mathbb{K}/N$ is finite. But by Macintyre's Theorem, N is algebraically closed already: hence $N = \mathbb{K}$. So A is actually a proper ideal of the field $\mathbb{K}$, a contradiction. □

2.3 The Algebraic connection*

This section was not discussed and is included for culture.

We just discussed matter in groups of finite Morley rank and saw that minimal groups are more complicated than in the abelian category. However there are deep interactions between groups of finite Morley rank and algebraic groups. This section is a discussion of a classical topic not required for the rest of the tutorial.

Question 2.15 (and answer)**.** *Which linear algebraic groups are groups of finite Morley rank?*

If $G = \mathbb{G}_\mathbb{K}$, the group of $\mathbb{K}$-points of an algebraic group $\mathbb{G}$, has finite Morley rank as a group and $\mathbb{G}$ has sufficiently many $\mathbb{K}$-points, then it can be shown that $\mathbb{K}$ is definable in G. So by Macintyre's theorem $\mathbb{K}$ must be an algebraically closed field.

Conversely, such $G = \mathbb{G}_\mathbb{K}$ is a linear algebraic variety so it is definable in $\mathbb{K}$. As the latter has finite Morley rank (1, as a pure field), so does G.

We already saw in Example 2.9 that abelian groups of rank 1 are more complicated than in the linear algebraic case of Theorem 2.6. It is tempting to ask about non-abelian cases.

Question 2.16 (and answer)**.** *Is every non-abelian group of finite Morley rank of the form $\mathbb{G}_\mathbb{K}$?*

No. Baudisch has constructed a 2-nilpotent group of Morley rank 2 in which no infinite field is definable. In particular, it is not an object of algebraic geometry.

Question 2.17 (and answer)**.** *Is every non-nilpotent group of finite Morley rank of the form $\mathbb{G}_\mathbb{K}$?*

No again. There is as we know from Fact 2.14 a field $(\mathbb{K}, +, \cdot, T)$ of finite Morley rank where $T < \mathbb{K}^\times$ is an infinite, definable, proper *subgroup. Now $H = \mathbb{K}_+ \rtimes T$ has finite Morley rank, but H is not of the form $\mathbb{G}_\mathbb{K}$ — this can be seen since H has obviously more definable sets than there are "constructible" sets in algebraic geometry.*

Question 2.18 (and partial answer)**.** *Is every* simple *group of finite Morley rank of the form $\mathbb{G}_\mathbb{K}$?*

This is the Cherlin-Zilber conjecture. It is open. Although we shall not use this, it is important to mention the big (500 pages) theorem on the topic: an infinite simple group of finite Morley rank with an infinite elementary abelian 2-subgroup is *of the form $\mathbb{G}_\mathbb{K}$.*

Lecture 2 — Modules of finite Morley rank

Today we really start to work with group representations of finite Morley rank.

3 Definition and first properties

Definition 3.1. A *module of finite Morley rank* is a triple $(G, V, \cdot)$ where G is an infinite group, V is a *connected* abelian group, $\cdot$ is an action of G on V (say, a subset of $G \times V^2$), and all three are definable in some bigger structure of finite Morley rank.

Remark 3.2.

- The "bigger structure" can be taken to have supporting set $G \ltimes V$.
- Requiring V to be connected has consequences. The module is *irreducible* if there are no non-trivial definable, *connected*, proper, G-invariant subgroups. It also affects the notion of a composition series, and so on. Bear this in mind.

For instance, if $G = \mathbb{G}_\mathbb{K}$ is the group of $\mathbb{K}$-points of some algebraic group over some (algebraically closed) field, then any algebraic representation V of $\mathbb{G}$ taken over $\mathbb{K}$ will be a module of finite Morley rank for G. The main question is: are there other modules? Does model-theoretic representation theory strictly extend algebraic representation theory?

Question 3.3.

- *Can there be cross-characteristic phenomena?*
- *What does this even mean? (V not being assumed to be a vector space, its "characteristic" is not defined yet)*

3.1 Abelian groups

Answering the second question above requires a useful detour which is *not* a digression.

Theorem 3.4 (Macintyre)**.** *Let A be an abelian group of finite Morley rank. Then there are definable, characteristic subgroups $D, B \leq A$ such that:*

- *D is divisible and B has bounded exponent;*
- *$A = D + B$ and $D \cap B$ is finite.*

Proof. The proof uses the basic techniques, which is why we give it. We use additive notation. Consider the following chain of subgroups:

$$A \geq 2A \geq \cdots \geq n!A \geq \dots$$

By the descending chain condition, it must become stationary at say $n_0!A = D$, which is a definable, characteristic subgroup. We claim that D is divisible. This is by stationarity: $D = n_0!A = (n_0 + n)!A \leq n \cdot n_0!A = nD$, which is n-divisible.

Now let $B = \{a \in A : n_0!a = 0\}$ be the subgroup of elements of order dividing $n_0!$. Clearly B is a definable and characteristic subgroup of bounded exponent. We claim that $A = D + B$. For if $a \in A$, then $n_0!a \in D$ and since D is divisible there is $d \in D$ with $n_0!d = n_0!a$. So by construction $b = a - d \in B$, and $a = d + b$.

It remains to show that the intersection $I = D \cap B$ must be finite. Consider the definable homomorphism $f : D \to D$ with $f(d) = n_0!d$. The kernel of f is I. So all fibers of f have rank equal to $\operatorname{rk} I$. But D as we know is divisible, so f is onto. All this shows $\operatorname{rk} D = \operatorname{rk} D + \operatorname{rk} I$, and therefore by additivity $\operatorname{rk} I = 0$: so $I = D \cap B$ is finite. □

Exercise 3.5. Conjecture and prove something about the structure of D (the ingredients are in the proof).

Remark 3.6 (Nesin)**.** Let N be a nilpotent groups of finite Morley rank. Then there are definable, characteristic subgroups $D, B \leq A$ such that D is divisible and B has bounded exponent, $N = D \cdot B$ and $D \cap B$ is finite.

We shall not use this remark, which is a little too long to prove to be an exercise. But it is a very good take-home problem as you will require to develop a theory of nilpotent groups of finite Morley rank (general hint: try and adapt classical properties from the finite case).

3.2 Characteristic

Lemma 3.7 (irreducible modules have a characteristic)**.** *Let $(G, V, \cdot)$ be an irreducible module with G infinite and faithful. Then either V is divisible and torsion-free, or V has prime exponent.*

Proof. It will be useful for an abelian group A to let A_n denote the subgroup of elements of order dividing n and A_{n^∞} the subgroup of elements of order dividing a power of n. A_n is always definable; A_{n^∞} need not be.

We use Macintyre's Theorem 3.4 to write $V = D + B$ where both are definable and characteristic, hence G-invariant. Clearly there are two cases.

If $D < V$ then $D = 0$ and $V = B$ has bounded exponent. There is p such that B_{p^∞} is infinite; since the exponent is bounded, $B_{p^\infty} = B_{p^n}$ for some n is actually definable. As it is G-invariant, by irreducibility $B = B_{p^n}$. So far we know that B has bounded exponent a power of p, and we introduce $f : B \to B$ which maps b to pb. Notice that f is definable and G-covariant. So $\operatorname{im} f$ is G-invariant. If $f \neq 0$ then $\operatorname{im} f$ is infinite, so by irreducibility f is onto, against B having exponent p^n.

The other case is when $D = V$ is divisible; we also prove that it is torsion-free and this will use a little model theory. Suppose there is an element in D of prime order p. We claim that for any k, the p^k-torsion subgroup D_{p^k} is finite. This is by induction: if the claim is known for $k \geq 1$, then the map $f : D_{p^{k+1}} \to D_{p^k}$ mapping d to pd is onto, with kernel D_p.

Remember that D_{p^∞} need not be definable. But it is characteristic, and G acts on D_{p^∞}. Now D_{p^∞} is also countable, so it has at most continuum many definable automorphisms, which is a bounded cardinal. It is a principle in model theory that everything bounded is finite. So G, which is infinite, contains two elements $f \neq g$ which coincide on D_{p^∞}. Now $D_{p^\infty} \leq \ker(f - g)$.

Let $\Delta = d(D_{p^\infty})$ be the hull (see Corollary 2.5) of D_{p^∞}; remember that by definition it is the smallest definable subgroup of V containing D_{p^∞}. Since $\ker(f - g)$ is definable, one has

$\Delta \leq \ker(f-g)$. But T is characteristic in V, so Δ is G-invariant. Now by irreducibility one has $\Delta = V \leq \ker(f-g)$, against $f \neq g$. □

If V has exponent p we shall say it has *characteristic* p; if V is divisible and torsion-free, we shall say that it has *characteristic* 0.

4 Linearity

4.1 The main result

We now prove an important result about linear structures. Nothing in the definition forces modules to be vector spaces, even if $G = \mathbb{G}_{\mathbb{K}}$. The following remedies this in many situations.

Theorem 4.1. *Let $(G, V, \cdot)$ be a module of finite Morley rank with G infinite. Suppose the action is faithful and irreducible. Suppose in addition one of the two:*

- *V has characteristic zero;*
- *G has an infinite centre.*

Then there is an algebraically closed field $\mathbb{K}$ with $V \simeq \mathbb{K}_+^n$ and $G \hookrightarrow \mathrm{GL}(V)$, definably.

Proof. This is a definable version of Schur's Lemma. We would like to introduce the collection of covariant endomorphisms of V, and prove that it is a skew-field. But as we work in the definable category, it is better to restrict oneself to *definable* ones.

Notation 1. Let:

$$\mathrm{DefEnd}(V) = \{\lambda : V \to V \text{ a definable endomorphism}\} \quad \text{and} \quad C = C_{\mathrm{DefEnd}(V)}(G)$$

Be very careful that $\mathrm{DefEnd}(V)$ *need not be definable* (even in the end it will not be quite clear). We aim at showing that C is a definable skew-field, and then rely on the Macintyre-Cherlin analysis of such objects (Theorem 2.11 and Remark 2.13). There are three steps.

Step 2. There are an integer n and some $w_0 \in V$ with $V = G_n \cdot w_0$, where G_n denotes the set of at most n elements of G.

Proof: A little model theory. Notice that this is almost trivial if V has characteristic zero, since in that case $G \cdot w_0$ is infinite, and the sum has to stop by finiteness of the rank. In characteristic p this uses classical techniques for which we have no time. ♢

Notation 3. Let:

$$L = \bigcap_{\substack{h \in \langle G \rangle: \\ w_0 \in \ker h}} \ker h$$

For $w_1 \in L$, let:

$$\begin{array}{cccc} \lambda_{w_0 \to w_1} : & V & \to & V \\ & v = f(w_0) & \mapsto & f(w_1) \end{array}$$

where $f \in G_n$.

L is a form of double centraliser. The map $\lambda_{w_0 \to w_1}$ is conveniently thought of as a *replacement map* insofar as it replaces the argument in the function f.

Step 4. This is well-defined; moreover $C = \{\lambda_{w_0 \to w_1} : w_1 \in L\}$.

Proof: Notice that since there always is such an f *and since G_n is a definable set*, the map is definable.

Well-definition requires a word. But if $v = f(w_0) = g(w_0)$ for f and g in $\langle G \rangle$, then $f - g \in \langle G \rangle$ vanishes at w_0. So by definition, $f - g$ vanishes at w_1: hence $f(w_1) = g(w_1)$ and the map is well-defined. It is left as an exercise to check that $\lambda_{w_0 \to w_1}$ is even an endomorphism.

We claim that $\lambda_{w_0 \to w_1} \in C$. For simplicity, just write λ. Now if $g \in G$ and $v = f(w_0) \in V$ with $f \in G_n$, then $g(v) = f'(w_0)$ for some other sum of at most n operators f'. Notice that $gf - f' \in \langle G \rangle$ vanishes at w_0, so it must vanish at w_1 as well. Hence:

$$g(\lambda(v)) = g(f(w_0)) = g(f(w_1)) = f'(w_1) = \lambda(f'(w_0)) = \lambda(g(v))$$

as desired. This means $\lambda \in C$. The converse inclusion is left as an exercise. ♢

Step 5. C is a definable, algebraically closed field; V is a $C[G]$-module.

Proof: The previous step proves definability of C. It remains to show that C is an infinite skew-field. In characteristic zero, C contains $\mathbb{Z}\,\mathrm{Id}_V$. In characteristic p, it contains the centre of G. So it remains to prove that it is a skew-field. This is a good exercise in the spirit of Schur's Lemma and using some of the techniques of Lemma 3.7.

As a conclusion, C is an infinite definable skew-field, hence by Macintyre-Cherlin' result, an algebraically closed field. Now V is a vector space over C, hence finite-dimensional, and the action of G is linear (by definition of C). We are done. ◊

This concludes the proof of the theorem. □

Question 4.2 (open, and unlikely). *Can one extend Theorem 4.1 to the case where the characteristic is p without assuming that G has an infinite centre?*

As a corollary we retrieve the famous result which started the whole business of groups of finite Morley rank.

Corollary 4.3 (Zilber's Field theorem). *Let* $(G, V, \cdot)$ *be a module of finite Morley rank with* G *infinite* and abelian. *Suppose the action is faithful and irreducible. Then there is an algebraically closed field* $\mathbb{K}$ *with* $V \simeq \mathbb{K}_+$ *and* $G \hookrightarrow \mathbb{K}^\times$*, definably.*

Zilber's Field Theorem drew interest to the model-theory of $\mathbb{K}^\times$; we now know (Fact 2.14) that it need not be minimal, i.e. that the embedding $G \hookrightarrow \mathbb{K}^\times$ may be proper in finite Morley rank.

4.2 Matter, continued

We return to the question of cross-characteristic phenomena: can an algebraic group $\mathbb{G}_\mathbb{K}$ over a field of characteristic q have a non-trivial module of characteristic $q' \neq q$?

Corollary 4.4 (the structure of matter for faithful modules of characteristic p). *Let* (G, V) *be a module of finite Morley rank where* G *is faithful and* V *has characteristic* $p > 0$ *(we do not need irreducibility here). Then every definable, soluble subgroup of* G *has the form* $B = U \rtimes T$*, where* U *is a definable, nilpotent p-group of bounded exponent, and* T *is a divisible, abelian group with no p-element.*

Proof. This uses a classical structure theorem for soluble groups of finite Morley rank. We want to give an idea. Suppose B contains an infinite elementary abelian q-group for $q \neq p$, say $U \neq 1$. We shall prove a contradiction.

Take a U-composition series $V = V_n > \cdots > V_0 = 0$, so that the V_i's are definable, connected, U-invariant, and $W_i = V_i/V_{i-1}$ is U-irreducible. Consider the faithful action of the abelian, connected group $U/C_U(W_i)$ on the irreducible module W_i. If $U/C_U(W_i) \neq 1$ then by Theorem 4.1 we find an algebraically closed field with $W_i \simeq \mathbb{K}_+^n$ and $U/C_U(W_i) \hookrightarrow \mathrm{GL}_n(\mathbb{K})$. Because of V, $\mathbb{K}$ must have characteristic p; this prevents $\mathrm{GL}_n(\mathbb{K})$ from having an infinite elementary abelian q-group, hence $U/C_U(W_i) = 1$.

In other words, U centralises all quotients in the composition series. It easily follows that $U \ltimes V$ is nilpotent, and as in the finite case, it can be proved that the p-torsion subgroup commutes with the q-torsion subgroup. So U centralises V, and by faithfulness $U = 1$. □

Exercise 4.5. Finish the proof, and say something sensible if $p = 0$.

As a consequence, *no cross-characteristic phenomena occur* in our setting: if $G = \mathbb{G}_\mathbb{K}$ is the group of $\mathbb{K}$-points of an algebraic group in characteristic q (possibly 0), then non-trivial modules have characteristic q.

Remark 4.6. It can even be shown (but this requires some model-theory) that T is the hull of its (non-definable) torsion subgroup.

Lecture 3 — Algebraicity issues

The following inclusions sum up our setting and goals:

- {algebraic geometry } ⊆ {model theory} (a controversial statement, but why not?);
- {constructible sets } ⊆ {definable sets} (Fact 2.14: inclusion is proper);
- {linear algebraic groups } ⊆ {groups of finite Morley rank} (equality, in the simple case, is called the Cherlin-Zilber conjecture);
- for a group $G = \mathbb{G}_{\mathbb{K}}$, where $\mathbb{G}$ is an algebraic group and $\mathbb{K}$ a field of finite Morley rank, {algebraic representations of G} ⊆ {modules of finite Morley rank for G}.

We wish to ask whether some form of equality can be reached in the last inclusion, or at least what might be missing. Linearity has been studied in the last lecture (Theorem 4.1) and although it does not provide a full positive answer, today we shall handle the other end of the problem.

Question 4.7. *Suppose (G, V) is a module of finite Morley rank and we know that definably, V is a $\mathbb{K}$-vector space for which $G \leq \mathrm{GL}(V)$. Is this linear configuration related to algebraic geometry?*

5 Characteristic 0

5.1 Jordan Decomposition

Theorem 5.1 (Poizat)**.** *Let $\mathbb{K}$ be a field of finite Morley rank* of characteristic zero *and $G \leq \mathrm{GL}_n(\mathbb{K})$ be a definable subgroup. Then G is stable under Jordan decomposition: if $x = s \cdot u \in G$ with commuting semisimple s and unipotent u, then $s, u \in G$.*

Bear in mind that a field structure can be more complicated than just the two laws. So we must distinguish two levels: the *definable* subsets of $\mathrm{GL}_n(\mathbb{K})$, which use the full structure, and the *constructible* subsets, which only use the algebraic part (the pure field structure); hence "constructible" means "definable in the pure field".

Proof. The core idea is that the finite Morley rank setting, being close to classical algebraic geometry, should not encode field exponentials. This will make sense at some point, and certainly involves definable homomorphisms between $\mathbb{K}_+^a$ and $(\mathbb{K}^\times)^b$. They will appear in time.

Write $g = us$; we may suppose $u \neq 1$. Conjugating, we may suppose that u is strictly upper-triangular and that s is diagonal. Let $Y = d(u)$ be the hull of u and $\Theta = d(s)$ be the hull of s. As we know, the upper-triangular subgroup U and the diagonal subgroup T are constructible (we did that in the proof of Lemma 1.4), so they certainly are definable: hence $Y \leq U$ and $\Theta \leq T$. We also introduce $\Gamma = d(g) \leq G$ (since G is definable). Now of course $\Gamma \leq Y \times \Theta$. We shall show that Γ, as a graph, defines a map $\Theta \to Y$. Since Γ is a group, this map will be a group homomorphism.

We shall prove that $Y \simeq \mathbb{K}_+$. Consider the Lie algebra:

$$\mathfrak{u} = \left\{ \begin{pmatrix} 0 & * & * \\ & \ddots & * \\ & & 0 \end{pmatrix} \right\}$$

and the partial exponential function:

$$\begin{array}{rccc} \exp: & \mathfrak{u} & \to & U \\ & x & \mapsto & \sum_n \frac{x^n}{n!} \end{array}$$

The good thing with working with nilpotent elements is that the sum is actually finite: so $\exp : \mathfrak{u} \to U$ is definable (even constructible). It also is a bijection, so it exchanges definable subsets of $\mathfrak{u}$ with definable subsets of U. Now since Y is abelian, its preimage (intuitively, something like its Lie algebra, but Y is not yet known to be constructible) $\mathfrak{y} = \exp^{-1}(Y)$ is a definable subgroup of $\mathfrak{u}$. Applying the idea in the proof of Fact 2.14 to $N_{\mathbb{K}}(\mathfrak{y})$, and finally using that the characteristic is 0, we see that $\mathfrak{y}$ is a vector space over $\mathbb{K}$. (This is not unexpected from an ordinary Lie algebra, but Y was not known to be topologically closed: so some model theory was involved here.) There is more: let $\ell = \exp^{-1}(u)$. Since exp exchanges definable sets, $\mathfrak{y} = \exp^{-1}(Y) = \exp^{-1}(d(u)) = d(\ell)$ is actually a one-dimensional vector space. As a conclusion, $Y \simeq \mathfrak{y} \simeq \mathbb{K}_+$ definably, and in particular Y is minimal (see Question 2.7 and Fact 2.14) and torsion-free.

With this information we finally get a map. Consider $\{y \in Y : (y, 1) \in \Gamma\}$, a definable subgroup of Y. If it equals Y, then certainly $u \in \Gamma \leq G$ and we are done. Otherwise it is trivial: this means that Γ defines a map $\Theta \to Y$, which is a group homomorphism.

We derive a contradiction. Remember that $\Theta \leq T \simeq (\mathbb{K}^\times)^n$. It suffices to show that there are no definable homorphisms from definable subgroups of $(\mathbb{K}^\times)^n$ to $\mathbb{K}_+$. This is left as an exercise (hint: induction). □

5.2 Structure of definably linear groups in characteristic 0

Corollary 5.2 (Macpherson-Pillay ; Poizat). *Let $\mathbb{K}$ be a field of finite Morley rank* of characteristic zero *and $G \leq \mathrm{GL}_n(\mathbb{K})$ be a quasi-simple (= simple modulo a finite centre), definable subgroup. If G is not constructible, then G contains only semisimple elements.*

Proof. Suppose that G contains more than just semisimple elements; so there is a non-trivial Jordan decomposition $g = us$. We know from the previous theorem that $u, s \in G$; in particular, G contains a non-trivial unipotent element. Let $Y = d(u)$ and argue like in the previous theorem. When we proved that $\mathfrak{y}$ was a vector space over $\mathbb{K}$, we proved that it was *constructible.* Now since $\exp : \mathfrak{u} \to U$ is constructible too, so is Y.

We conclude with a classical result from algebraic geometry: a group generated by irreducible, constructible subgroups, is constructible as well. By quasi-simplicity, $G = \langle Y^g : g \in G\rangle$ is constructible, i.e. definable in the pure field. □

Remark 5.3. It is a significant open question whether there can indeed exists a simple, definable subgroup of $\mathrm{GL}_n(\mathbb{K})$ which is not constructible. A little more is known: it would have no involutions, and its connected soluble subgroups would be abelian. Quite pathological!

The answer in known to be negative positive characteristic (Theorem 6.2 below).

In particular, a configuration contradicting Question 4.7 must be so pathological that the acting group cannot come from algebraic geometry.

Corollary 5.4 (Cherlin-D). *Let (G, V) be a module of finite Morley rank where G has the form $\mathbb{G}_\mathbb{K}$ for a simple algebraic group $\mathbb{G}$ and an algebraically closed field $\mathbb{K}$* of characteristic zero.

Then V is definably isomorphic to an algebraic representation of G as an algebraic group; i.e., (G, V) already lives in the algebraic category.

Proof. Linearising by Theorem 4.1 we find a definable, algebraically closed field $\mathbb{L}$ of characteristic 0 such that $V \simeq \mathbb{L}_+^n$ and $G \hookrightarrow \mathrm{GL}_n(\mathbb{L})$ definably. Now G is quasi-simple and does not consist in semisimple elements, so it is definable in the pure field $\mathbb{L}$ by Corollary 5.2. By classical methods, $\mathbb{K}$ is constructible in $\mathbb{G}_\mathbb{K}$. Hence $\mathbb{K}$ is constructible in $\mathbb{L}$, and by model theory $\mathbb{K} \simeq \mathbb{L}$ constructibly. Hence we have a constructible subgroup of $\mathrm{GL}_n(\mathbb{K})$ acting naturally on $V \simeq \mathbb{K}_+^n$: we are done. □

This is the end of the story in characteristic 0.

6 Characteristic p — A Conjecture

We ask the same questions in positive characteristic.

As far as the Jordan decomposition goes, things are outrageously good.

Exercise 6.1. Let $\mathbb{K}$ be a field of finite Morley rank *of positive characteristic* and $G \leq \mathrm{GL}_n(\mathbb{K})$ be *any* (not necessarily definable) subgroup. Then G is stable under Jordan decomposition.

However the proof of Corollary 5.2 cannot be adapted. But it can be substituted with something much harder.

Theorem 6.2 (Poizat). *Let $\mathbb{K}$ be a field of finite Morley rank* of positive characteristic *and $G \leq \mathrm{GL}_n(\mathbb{K})$ be a quasi-simple, definable subgroup. Then:*

- *G is definably isomorphic to a constructible subgroup of* $\mathrm{GL}_n(\mathbb{K})$*;*
- *G is definable in the* pure *field $\mathbb{K}$ augmented by a finite number of definable field automorphisms.*

Proof. The proof is rather intricate and we have no time to explain it. It uses highly non-trivial model theory and the classification of the locally finite simple groups (which relies on the classification of the finite simple groups). □

Remark 6.3. Field automorphisms did not appear in the characteristic 0 case, for if $\mathbb{K}$ if a field of finite Morley rank of characteristic 0 and φ is a definable field automorphism, then $\mathbb{K}^\varphi$ is an infinite, definable subfield: we know that $\mathbb{K}^\varphi = \mathbb{K}$ so φ is the identity.

Question 6.4. *Is there an analogue of Corollary 5.4 in positive characteristic?*

There are two issues. First, field automorphisms might complicate matters a bit. Second, and perhaps even more importantly, one cannot linearise a priori in positive characteristic.

So the only strategy one can imagine so far is to try and reconstruct "weight spaces", that is understand the action of the algebraic torus.

Conjecture 6.5 (a Steinberg Tensor Theorem analogue; quite open)**.** *Let $\mathbb{G}$ be a reductive algebraic group, $\mathbb{K}$ be a field of finite Morley rank of characteristic p, and $G = \mathbb{G}_\mathbb{K}$ as a group of finite Morley rank. Let V be an irreducible G-module of finite Morley rank with $C_G(V)$ finite.*

Then there are irreducible representations W_i of G as an algebraic group *and* definable *automorphisms $\varphi \in \mathrm{Aut}(\mathbb{K})$ such that $V \simeq \otimes_i {}^{\varphi_i} W_i$ (twist-and-tensor).*