Tutorial on Model Theory and Groups - Solutions

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June 2007

1. Let $A \simeq \mathbb{Z}^I$. Then $A > 2A > 6A > \dots$, all subgroups are definable, and this violates DCC.
   Now let $F = \mathcal{F}_X$. Pick some $x \in X$; then $C_F(x) = \langle x \rangle \simeq \mathbb{Z}$. Hence F interprets $(\mathbb{Z}, +)$, but the latter cannot have finite Morley rank thanks to the first part.

2. Let
$$i = \begin{pmatrix} i & \\ & -i \end{pmatrix}$$
   where i stands for a complex root of -1. (Any non-trivial element of T would suffice, but this one is so simple !)
   T looks fairly close to $C(i)$; actually $T = C^\circ(i)$, so we find a way to define the connected component. We notice that T is inverted by the involution
$$w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$
   and that $C(i) = T \rtimes \langle w \rangle$. In particular, one concludes that T is the set of squares in $C(i)$, that is T is defined by $\exists y\ [i, y] = 1 \wedge x = y^2$.

   One sees that $F \simeq (\mathbb{C}, +)$ is being acted upon by $T \simeq (\mathbb{C}^\times, \times)$, so it is reasonable to hope F will be the centralizer of any of its elements. It is actually the case, hence letting
$$a = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$
   one finds that F is defined by $[x, a] = 1$.

   Now it should be clear that $B = F \rtimes T$ ($\simeq \mathbb{C}_+ \rtimes \mathbb{C}^\times$), hence B is defined by $\exists f \in F\ \exists t \in T\ x = ft$.

The Cherlin-Zilber Conjecture

3.
   - $\mathbb{Z}_{p^\infty}$ (p is a prime): the Pruefer quasi-cyclic p-group.
   - $(\overline{\mathbb{F}}_2, +) \times (\overline{\mathbb{F}}_3, +)$.
   - $K_+ \rtimes T$, where $T < K$ is a bad field.

- The Baudisch group.
- A "bad group" (if it exists).

4. No algebraic group over an ACF can be finitely generated.
Suppose indeed that G is an algebraic group over $\mathbb{K}$. Taking generators $g_1, \ldots, g_n$ of G and representing them by matrices, one sees that all the elements of G would have coefficients in a field $\mathbb{K}_1$ with the following property: $\mathbb{K}_1$ is finitely generated as an algebra over its prime subfield. Hence $\mathbb{K}_1$ cannot be algebraically closed and $\mathbb{K}_1 < \mathbb{K}$. Now there is a result by Kramer, Roehrle and Tent saying that G must reinterpret $\mathbb{K}$, which is impossible.

So all we have to do is, starting with an infinite, finitely generated group of finite Morley Rank, get a simple one. If the Cherlin-Zilber Conjecture were true, it would be algebraic, which is impossible.

We use the "minimal (counter-)example" method. Though all (possible) groups having a property $\mathcal{P}$ might lie in different universes, i.e. carry "their own notion" of Morley rank, it is possible to choose a group satisfying $\mathcal{P}$ and having minimal rank and degree (minimize rank first). One can expect simplicity...

This is what we do. Let G be an infinite, finitely generated group of finite Morley rank; we assume that G has minimal rank as such; we assume that G has minimal degree as such. We want to prove that G is simple.

We use the following
Lemma
Let H be a *non-abelian* group of finite Morley rank. If H is definably-simple (that is, contains no proper non-trivial normal definable subgroup), then H is simple.

Proof

Let $N \triangleleft H$ be some proper normal subgroup; we prove $N = 1$. Because of definable-simplicity of H, H must be connected. Hence due to Zilber's indecomposable Theorem, the commutator $[N, H] \leq N < G$ is definable. As $[N, H]$ is normal in H, definable-simplicity implies $[N, H] = 1$, that is $N \leq Z(H)$. But the center being always definable and characteristic, non-abelianity of H implies $Z(H) = 1$; hence $N = 1$. $\square$

Obviously our group G won't be abelian (an infinite, finitely generated abelian group cannot have finite Morley rank). So we prove that G is definably-simple, and we are done.
Suppose that $1 \neq N \trianglelefteq G$ is some non-trivial definable normal subgroup. If N has infinite index in G, then the (interpretable) quotient G/N is an infinite, finitely generated group of finite Morley rank; this contradicts the minimality of G. Hence $[G : N] < \infty$ and now general group theory implies that N itself is finitely generated; this contradicts minimality again unless $N = G$. So G is definably simple.

General results

5. Consider the morphism $\varphi : x \mapsto nx$. Because D is divisible, φ must be surjective. Hence $\operatorname{rk}\ker\varphi = \operatorname{rk} D - \operatorname{rk}\operatorname{im}\varphi = 0$, and $\ker\varphi$ must be finite.

6. (We use additive notation.) Consider the chain $A > 2A > 6A > \dots$; by DCC it must terminate. Hence there is $n_0 \in \mathbb{N}$ and $D = n_0!A$ such that $nD = D$ for all n, i.e. D is divisible. Now as abelian divisible groups are the injective of the category of $\mathbb{Z}$-modules (Baer), there is $B \leq A$ such that $A = D \oplus B$; B needs not be definable. Nonetheless let $b \in B$, then $n_0!b \in D \cap B = \{0\}$, that is B has bounded exponent.

   Now let $B_1 = \{a \in A,\ n_0!a = 0\}$; this is obviously a definable subgroup containing B, hence $A = D + B_1$. The intersection is actually finite; this is because an abelian divisible group of finite Morley rank has finitely many elements of order n for each n.

7. $d(x)$ must be abelian, so we write $d(x) = D \oplus B$ like in Macintyre's Theorem. Now $x = (\delta, \beta)$ with obvious notations; it is clear that $d(x) = d(\delta) \oplus d(\beta) = D \oplus \langle\beta\rangle$.

8. Let G be a group and φ some definable, injective endomorpshim. Let $H_n = \operatorname{im}\varphi^n$; this is a definable subgroup. If $H_1 < G$ then we have a chain $G > H_1 > H_2 > \dots$, which violates DCC.

9. We can assume $G = d(H)$. We do induction on the solvability class of H.

   Assume H is abelian. Then by definability of centralizers, $d(H) \leq C(H)$, hence $H \leq C(d(H))$, hence $d(H) \leq C(d(H))$, that is $d(H)$ is abelian.

   Now assume H is solvable of class $c+1$, that is $H^{(c+1)} = 1$. Let $K = H' \neq 1$; K need not be definable, but it is solvable of class c. Since H normalizes $d(K)$, so does $d(H) = G$. Hence $d(K) \triangleleft G$.

   We remark the following: if φ is any definable morphism, then $d(\varphi(H)) = \varphi(d(H))$. Indeed, as $H \leq d(H)$ we find $\varphi(H) \leq \varphi(d(H))$; the latter is definable, hence $d(\varphi(H)) \leq \varphi(d(H))$. Now since $\varphi^{-1}(d(\varphi(H)))$ is definable and contains H, we get $d(H) \leq \varphi^{-1}(d(\varphi(H)))$. Turning to images, $\varphi(d(H)) \leq d(\varphi(H))$, whence equality follows.

   Now let φ be the projection modulo $d(K)$.

   We claim that $\varphi(H)$ is an abelian group. Indeed $\varphi(H') = (\varphi(H))'$; as H/K is abelian, $H' \leq K \leq d(K)$; hence $(\varphi(H))' = \varphi(H') \leq \varphi(d(K)) = 1$. So $\varphi(H)$ is abelian. By the class 1 case, $d(\varphi(H))$ is abelian too.

   Finally we get that $d(H)/d(K) = \varphi(d(H)) = d(\varphi(H))$ is an abelian group, hence $d(H)' \leq d(K)$. But by induction, $d(K)$ is solvable of class c, hence $d(H)$ is solvable of class $c+1$.

10. If you know group theory you understand: we have to prove $Z_2(G) = Z(G)$. By definition, denoting by π projection modulo $Z(G)$ and $\bar{G}$ the quotient group $G/Z(G)$, $Z_2(G) = \pi^{-1}(Z(\bar{G})$. As it is the group of elements

x such that $[x, G] \subseteq Z(G)$, the following map φ_x comes naturally into the picture.

Let $x \in Z_2(G)$. Let $\varphi_x(g) = [x, g]$. As $x \in Z_2(G)$, i.e. $\pi(x) \in Z(\bar{G})$, we have $\pi(\varphi_x(g)) = [\pi(x), \pi(g)] = 1$, that is $\varphi_x(g) \in Z(G)$. So φ_x is actually a map $G \to Z(G)$.

Because of the general formula $[a, bc] = [a, c][a, b]^c$ and the fact that $\varphi_x(G) \subseteq Z(G)$, one finds $\varphi_x(gh) = \varphi_x(g)\varphi_x(h)$. Hence φ_x is a morphism $G \to Z(G)$.

Now G is paritioned into a finite number of fibers (the sets $\varphi_x^{-1}(b)$, where $b \in \varphi_x(G)$) that are all in definable bijection because φ_x is a morphism. By connectedness of G, there can be only one such fiber, whence $\varphi_x(G)$ has only one element. This proves that $[x, G] = 1$, hence $x \in Z(G)$.

We have just proved that $Z_2(G) = Z(G)$, and going mod $Z(G)$ we get $Z(\bar{G}) = 1$.

11. Let (N, H) be a counterexample with $\operatorname{rk} N$ and $\deg N$ minimal. Let $\bar{N} = N/Z(N)$, which has rank smaller than N (because $Z(N)$ is infinite). By hypothesis, $\bar{H}$ is still infinite, and by minimality, $\bar{H}$ has infinite intersection with $Z(\bar{N}) = \overline{Z_2(N)}$. Hence H has infinite intersection with $Z_2(N)$. This proves that $(Z_2(N), K = N \cap Z_2(N))$ is a counterexample; hence $N = Z_2(N)$.

    Now for any $x \in N$, the map $\varphi_x : g \mapsto [x, g]$ is actually a group homomorphism $N \to Z(N)$. As $H \lhd N$, $\varphi(H) \leq H$ and it has to meet $Z(N)$ finitely. This means that $\varphi(H)$ is finite; from $\ker \varphi_x = C_N(x)$ we deduce $[H : C_H(x)] < \infty$.

    This is true for any $x \in N$. But by DCC on centralizers, $Z(N) = C_N(x_1, \ldots, x_m)$ for some elements $x_i \in N$. So $[H : H \cap Z(N)] < \infty$, whence H is finite, a contradiction.

The field Theorem

12. We look at the Borel subgroup:

$$B = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\}.$$

Of course we find $\mathbb{K}_+$ in the upper corner and $\mathbb{K}^\times$ on the diagonal. But there is a center, which of course cannot act, so we'll have to mod out $Z(B) = Z(G)$. Let

$$a = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \qquad A = C_G(a)/Z(G),$$

so that $A \simeq \mathbb{K}_+$. Now let

$$t = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix}, \qquad T = C_G(t)/Z(G).$$

$H = B/Z(G)$ is (definable and) solvable; $A \lhd H$ is H-minimal. No element of T centralizes A; both groups have the same rank. The Zilber field Theorem says that T embeds into $\mathbb{K}^\times$, but by rank equality we get $T \simeq \mathbb{K}^\times$ acting by multiplication on $A \simeq \mathbb{K}_+$.

13. Of course we will use the Zilber field Theorem. But we need to prove that T is a group !

    Suppose that $B \cap B^\alpha$ is abelian. As $T \subseteq B \cap B^\alpha$, letting $s, t \in T$ we get $(st^{-1})^\alpha = s^\alpha t^{-\alpha}) = s^{-1}t = (st^{-1})^{-1}$, hence T is a group. So it suffices to prove that $B \cap B^\alpha$ is abelian to prove that T is a group.

    Let $X = (B \cap B^\alpha)'$ and assume $X \neq 1$. Let $H = N^\circ(X)$. Of course H is definable, connected, and α-invariant; as G is simple, H is also proper. So we prove that $A \leq H$ and get a contradiction.

    We recall some properties of the connected Fitting subgroup:

    Proposition
    If B is a connected, solvable group of finite Morley rank then there exists a maximal definable, connected, nilpotent, normal subgroup denoted by $F^\circ(B)$. One has $B' \leq F^\circ(B)$ (alternatively, B' is nilpotent).

    We have $X \leq B' \leq F^\circ(B)$. Now $A \leq F^\circ(B)$ and must have infinite intersection with $Z(F^\circ(B))$. By B-minimality, this proves that $A \cap Z(F^\circ(B)) = A$ and $A \leq Z(F^\circ(B))$. So $A \leq C^\circ(X) \leq H$, and H contradicts the hypothesis on A.

    We have proved so far that T is an abelian group. Now let $t \in T^\#$ and assume that t centralizes A. Then letting $H = C^\circ(t)$ we have that H is proper, definable, connected, α-invariant, and contains A. As this contradicts the hypothesis on A we get $T \cap C_B(A) = 1$.

    One can finally apply the Zilber field Theorem in $A \rtimes B/C_B(A)$ and get some algebraically closed field $\mathbb{K}$ with $A \simeq \mathbb{K}_+$ and $B/C_B(A) \hookrightarrow \mathbb{K}^\times$. Now as $T \hookrightarrow B/C_B(A)$ and because we have assumed $\operatorname{rk} T \geq \operatorname{rk} A$, we actually get $T \simeq \mathbb{K}^\times$.

Around torsion

14. Let $b \in G$, we find some $a \in G$ such that $a^n = b$. As the definable closure $d(b)$ is abelian, the map $x \mapsto x^n$ restricts to an endomorphism of $d(b)$. Since there are no elements of order n in $d(b)$, this must be an injection, hence a surjection. So b is n-divisible.

    Now assume $x^n = y^n$. We can simplify only if x and y commute, so we look for some element t commuting both with x and y and such that $x^n = t^n = y^n$. There is no reason why x should belong to $C(y)$, so we look at $C(x^n)$. This contains both x and y; so we take the center and consider $H = Z(C(x^n))$. H is a definable abelian group containing x^n; it must be n-divisible due to the previous argument. Hence there is $t \in H$ such that

$t^n = x^n$. Now t commutes with x and y, hence $(tx^{-1})^n = t^n x^{-n} = 1$, and so $tx^{-1} = 1$. We conclude that $x = t = y$.

Eventually assume $x^n \in H$. By the first point, H is divisible; hence there is $y \in H$ such that $x^n = y^n$. With the second point, we get $x = y \in H$.

15. As often we look for some abelian subgroup, they're much easier to study. Let $H = d(x)$ and $K = H \cap N \triangleleft H$. It suffices to find a p-element in xK, and now we can work inside H which is abelian. So we'll use additive notation.

    Say $\bar{x}$ has order p^a in H/K, that is $p^a x \in K$. Now $d(p^a x) = D \oplus \langle \gamma \rangle$ for some definable, divisible group D and an element of finite order γ. Say $p^a x = d_1 + \gamma$ with $d_1 \in D$ and $|\gamma| = p^b r$ ($p \not| r$).

    Heuristics. What we look for is some element $k \in K$ such that $x - k$ has order p^c for some c; k will be of the form $k = d_2 + \ell\gamma$ for some integer ℓ. Assuming c large enough, we can write $p^c(x-k) = p^{c-a}d_1 + p^c d_2 + (p^{c-a} - p^c\ell)\gamma$. It is rather obvious that we will choose d_2 in order to kill the term involving d_1 and d_2; d_2 will be obtained by divisibility once c is known. We focus on annihilating $(p^{c-a} - p^c\ell)\gamma$. This is possible only if $p^b r | p^{c-a} - p^c\ell$. So taking $c = a + b$ looks very reasonable, and all we have to do is find ℓ such that $r | 1 - p^a\ell$.

    Resolution. As $\gcd(r, p^a) = 1$, there are integers ℓ and m such that $mr + \ell p^a = 1$. Let $c = a + b$, and let $z = \ell\gamma$. Now $p^c(x - z) = p^b x - p^c\ell\gamma = p^b d_1 + (p^b - p^c\ell)\gamma = p^b d_1 + p^b(1 - p^a\ell)\gamma = p^b d_1 \in D$. As D is divisible, there is $d_2 \in D$ such that $p^c d_2 = p^b d_1$. Now let $k = d_2 + z \in K$. We find $p^c(x - k) = 0$, the problem is solved.

    One must be careful with this result: a p-element need not be an element of order p ! For example, consider $\mathrm{SL}_2(\mathbb{K})$ and its center $Z(\mathrm{SL}_2(\mathbb{K})) \simeq \mathbb{Z}/2\mathbb{Z}$. In the quotient $\mathrm{PSL}_2(\mathbb{K}) = \mathrm{SL}_2(\mathbb{K})/Z(\mathrm{SL}_2(\mathbb{K}))$ there are involutions. Choose one and a preimage $x \in \mathrm{SL}_2(\mathbb{K})$ (hence $x^2 \in Z(\mathrm{SL}_2(\mathbb{K}))$. There are no involutions in $xZ(\mathrm{SL}_2(\mathbb{K}))$; the order of x is 4 !

Something more recent

16. Consider the expression $ii^g = ig^{-1}ig$ which we want to think of as f^*f in Hermitian algebra. Of course we would like * to be a contravariant involutive function on G. It seems quite natural to let * be $^{-i}$, that is let $x^* = ix^{-1}i$. Then $ii^g = g^*g$.

    To understand the analogy, notice that $S(*) = \{g \in G,\ g^* = g\} = \{g \in G,\ g^i = g^{-1}\}$ and $U(*) = \{g \in G,\ g^*g = 1\} = C_G(i)$. So we are looking for a polar decomposition indeed.

    In the bilinear case, one had to compute a square root for f^*f. Here as $d(g^*g) = d(i^g i)$ is by hypothesis 2-torsion free, there is in $d(g^*g)$ a unique square-root s_g of g^*g. Because $(s_g^*)^2 = s_g^2$ and $d(g^*g)$ has no 2-torsion, we have $s_g^* = s_g$.

The problem is that the association $\sigma : g \mapsto s_g$ need not be definable in general (the definable closures $d(g^*g)$ are not *uniformly* definable). There are some tricks to prove that σ is actually definable.

Now for any $g \in G$, consider $u_g = gs_g^{-1}$. We have $u_g^* u_g = (gs_g^{-1})^* g s_g^{-1} = s_g^{-*} g^* g s_g^{-1} = 1$, that is $u_g \in U(*)$. If you believe the association $g \mapsto s_g$ is definable, so will be the map $\tau : G \to C(i)$ which sends g to u_g.

Of course if $c \in C(i)$, one has $\tau(c) = c$, so τ is onto. But also $(cg)^*(cg) = g^* g$, hence $\sigma(cg) = \sigma(g)$ and $\tau(cg) = c\tau(g)$.

This proves that all fibers are in bijection, and hence $\tau : G \to C(i)$ is a definable surjection with constant fiber. One must have $\deg C(i) \leq \deg G$.