Tutorial on Model Theory and Groups

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1. Prove that an abelian free group cannot have finite Morley rank.
   Prove that a free group cannot have finite Morley rank.

2. Consider $G = \mathrm{PSL}_2(\mathbb{C})$. Let

$$T = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\}, \qquad F = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\}, \qquad B = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\}.$$

   Find (explicit !) definitions for these groups.

Cherlin-Zilber Conjecture
Any infinite simple group of finite Morley rank is an algebraic group over some algebraically closed field.

3. Find (non-simple) groups of finite Morley rank that cannot be algebraic.

4. Prove that if there is some infinite, finitely generated group of finite Morley rank, then the Cherlin-Zilber conjecture cannot hold.

Some general results

5. Let D be an abelian divisible group of finite Morley rank. Let $n \in \mathbb{N}$. Then D has finitely many elements of order $|n$.
   Actually, it is easy to prove that there are integers n_p ($p \in \mathcal{P}$) and some (possibly finite !) set I such that

$$D = \left(\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_{p^\infty}^{n_p} \right) \bigoplus \left(\bigoplus_{I} \mathbb{Q} \right).$$

6. **Macintyre's Theorem**
   Let A be an abelian group of finite Morley rank. Then $A = D \oplus B$ where D is a definable, characteristic and divisible subgroup, and B has bounded exponent. If one allows finite intersection, B can be chosen definable.

7. As a corollary, if x is some element of a group of finite Morley rank, then $d(x) = D \oplus C$ where D is definable and divisible, and C is a finite, cyclic group.

8. **An "Ax's principle"**
   Let G be a group of finite Morley rank and $\varphi : G \to G$ some definable, injective morphism. Then φ is surjective.

9. Let G be a group of finite Morley rank. Assume $H < G$ is a solvable subgroup. Prove that $d(H)$ is solvable. (Also true for nilpotency.)

10. Let G be a connected group of finite Morley rank. Assume $Z(G)$ is finite. Prove that $G/Z(G)$ is centerless.

11. Let N be a nilpotent group of finite Morley rank. Let $H \lhd N$ be an infinite, normal subgroup. Prove that $H \cap Z(N)$ is infinite.

The field Theorem

12. Let $\mathbb{K}$ be a field of finite Morley rank. Interpret $\mathbb{K}$ inside $G = \mathrm{GL}_2(\mathbb{K})$.

13. Let G be a simple group of finite Morley rank. Let α be some definable automorphism of order 2 of G.

    Assume $B < G$ is a definable, connected, and solvable subgroup. Let $A \leq B$ be some B-minimal subgroup of B and assume :

    No definable, connected and α-invariant subgroup H is s.t. $A \leq H < G$.

    Let $T = \{b \in B,\ b^\alpha = b^{-1}\}$ and assume $\mathrm{rk}\, T \geq \mathrm{rk}\, A$.

    Prove that there is an algebraically closed field $\mathbb{K}$ such that $\mathbb{K}_+ \simeq A$, and T is a group isomorphic to $\mathbb{K}^\times$.

    To prove that T is a group, show that $B \cap B^\alpha$ is abelian. You might also use the following fact :

    Let B be a solvable group of finite Morley rank. Then there is a

Around torsion

The two following noteworthy results are actually more inspired by finite group theory than by algebraic group theory.

14. Let G be a group of finite Morley rank with no element of order n. Prove that G is n-divisible. Prove that $x^n = y^n \Rightarrow x = y$. Let $H \leq G$ be a definable subgroup. Prove that $x^n \in H \Rightarrow x \in H$.

15. Prove the "lifting torsion lemma" :
    Lemma
    Let G be a group of finite Morley rank. Assume $N \lhd G$ is some definable, normal subgroup and let $x \in G$. Assume $\bar{x}$ is a p-element in G/N. Then the coset xN contains a p-element.

    (You might require Bézout's Theorem :"$\forall a, b \in \mathbb{Z},\ \gcd(a, b) \in (a, b)$".)

Something more recent

16. Let G be a group of finite Morley rank. Fix some involution i ; assume that $\forall g \in G, d(ii^g)$ has no 2-torsion. Prove that $\deg C(i) \leq \deg G$.

    Hint : define a surjective map $G \to C(i)$ with fibers all in bijection. This map will yield a fairly canonical "centralized part" of an element of the group.

    This is actually an analog to the *polar decomposition* in bilinear algebra : Let E be a finite dimensional $\mathbb{C}$-linear space with a scalar product ("hermitian space"). Take some definite positive endomorphism f of E (that is, f^*f has eigenvalues in $\mathbb{R}_{>0}$). Then f can be uniquely written as $f = us$, where u is an orthogonal automorphism, and s a symetric endomorphism.

    Of course no ordered structure such as the real line is interpretable ; the "no involution" hypothesis is here to provide square roots...