Tutorial on Groups of finite Morley rank

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Manchester, 14-18 July 2008

First tutorial

Groups of finite Morley rank first arose in a very model-theoretic context, the study of $\aleph_1$-categorical theories. Their algebraic properties were quickly recognized, so the main conjecture (Algebraicity Conjecture below) around them was soon formulated. It turned out a little later that this conjecture might be attacked by methods imported from the theory of finite groups. As a result, groups of finite Morley rank are at the connection of three domains: model theory, algebraic geometry, and finite group theory.

What is a good example of a group of finite Morley rank in the first place? an algebraic group over an algebraically closed field, equipped with the Zariski dimension. So groups of finite Morley rank generalize algebraic groups. It is very easy to construct a group of finite Morley rank which is not algebraic: consider $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\overline{\mathbb{F}}_p)$ - though the Morley rank remains finite, the notion of base field has vanished. And yet the previous example has something very artificial. This together with other reasons, such as Macintyre's theorem below, explains why in the 70's Cherlin and Zilber independently formulated the

Algebraicity Conjecture *Let G be an infinite* simple *group of finite Morley rank. Then G is an algebraic group over an algebraically closed field.*

Much has to be said about this conjecture.

- The field must be algebraically closed, by the following theorem of Macintyre which was the first hint that model theory might be a new language for algebraic geometry.

  Theorem (Macintyre) *Let $\mathbb{K}$ be an infinite field of finite Morley rank. Then $\mathbb{K}$ is algebraically closed.*

- Also notice that the Algebraicity Conjecture now looks very natural in view of the previous theorem.

- If one forgets the word "infinite", the Algebraicity Conjecture is not true, because of the huge

Classification of Finite Simple Groups *A finite, simple group is of one of the following forms:*

- $\mathbb{Z}/p\mathbb{Z}$, *p a prime.*
- Alt_n, $n \geq 5$.
- *a "group of Lie type" (say a finite analog of an algebraic group).*
- *one of the 26 exceptions known as the "sporadic groups".*

- On the other hand, with the word "infinite", the Algebraicity Conjecture looks very much like a "smooth" version of CFSG. Borovik has suggested to adapt the techniques from finite group theory to the finite Morley rank context.
- Gradually, groups of finite Morley rank have moved to algebra. Typical of this line of thought is the Borovik-Poizat axiomatization: the only reference to model-theory is the notion of a definable/interpretable set! And from there on, people in the domain have been working with generic *sets* instead of generic *types*.
  (Groups of finite Morley rank do require some big model-theoretic guns, but not in the study of their first properties.)
- As for the Conjecture itself, attempts at proving it have relied on a case division depending on the structure of the Sylow 2-subgroup. There are four cases two of which have been solved. One is open and the last one is dramatically open, in particular because of possible "bad groups" which, should they exist, would be really pathological objects. But the case-division takes us too far away from the lectures...

Recall that a group G is said *p-divisible* if $\forall x \in G\ \exists y \in G,\ y^p = x$. G is *divisible* if it is p-divisible for every prime number p.

Exercise 1 *Let A be a connected abelian group of finite Morley rank. Then A is p-divisible if and only if it has a finite number of elements of order p.*

Solution. Since A is abelian, we'll write the operation additively. Consider the definable function

$$\begin{array}{cccc} \varphi & : & A & \to & A \\ & & a & \mapsto & p.a \end{array}$$

As A is abelian, φ is a morphism. Now $\ker\varphi$ is the definable group of elements of A order $|p$. Since $\operatorname{im}\varphi = pA$, the group of p^{th} powers, we have a definable isomorphism $A/\ker\varphi \simeq pA$. It follows $\operatorname{rk} pA = \operatorname{rk} A - \operatorname{rk}\ker\varphi$.

In particular, if A is p-divisible, then $pA = A$ and $\operatorname{rk}\ker\varphi = 0$, i.e. $\ker\varphi$ is finite. Conversely, if $\ker\varphi$ is finite, then $\operatorname{rk} pA = \operatorname{rk} A$. As A is connected, it follows $pA = A$. $\square$

Comments.

- Connectedness is essential here, as the example of $\mathbb{Z}/p\mathbb{Z}$ shows.

One cannot generalize Exercise 1 to arbitrary groups of finite Morley rank:

- Recall that SL_2 is the group of 2×2 matrices with determinant 1. If the base field has characteristic not 2, then $Z(\mathrm{SL}_2) = \{\pm \mathrm{Id}\}$. It turns out that $-\mathrm{Id}$ is then the only involution of SL_2, and yet SL_2 is not 2-divisible, as the matrix $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ has no square root in SL_2!

- Recall that $\mathrm{PSL}_2 = \mathrm{SL}_2/Z(\mathrm{SL}_2)$, a very important simple group. You may check that it is 2-divisible, but has infinitely many involutions.

However, the following is true.

Exercise 2 *Let G be a group of finite Morely rank with no p-torsion. Then G is p-divisible, and p^{th} roots are unique.*

Solution. Let $x \in G$; we're looking for a root of x. If we were in the finite case, we would find a suitable power of x, as p does not divide $o(x)$. That is, we would search inside $\langle x \rangle$, a small abelian group containing x. In the finite Morley rank context x might have infinite order, in which case $\langle x \rangle$ would be isomorphic to $\mathbb{Z}$, and therefore non-definable (recall that $\mathbb{Z}$ does not have finite Morley rank). So playing the same trick might make us leave the definable category, and we have to approximate $\langle x \rangle$ in the following sense.

Consider $A = Z(C_G(x))$. This is an abelian group containing x; though not connected it contains no p-torsion, and a slight variation on Exercise 1 yields a p^{th} root y of x inside A. We prove that it is unique. If $z \in G$ is such that $z^p = x$, then $z \in C_G(x)$ and as $y \in Z(C_G(x))$, y and z commute. Now $(y^{-1}z)^p = 1$ implies $y = z$. $\square$

Comments.

- We have used the following fact:

  Lemma *Let φ be a definable endomorphism of a group of finite Morley rank G. If φ is injective then it is surjective.*

  This is rather obvious as φ must preserve rank and degree. More generally, Zilber has formulated the following

  Conjecture *Let f be a definable* function *of a group of finite Morley rank G. If φ is injective then it is surjective.*

  Of course this is reminiscent of Ax's principle: if V is an algebraic variety and $f : V \to V$ is an injective morphism of algebraic varieties, then f is surjective. So the above conjecture is inspired by the idea that groups of finite Morley rank are just a generalization of algebraic groups.

- We'll come back later (Exercise 5) to what a good approximation of $\langle x \rangle$ in the definable category is.

Exercise 3 *Let G be a connected group of finite Morley rank which has generically order 2, i.e. $X = \{x \in G,\ x^2 = 1\}$ is generic in G. Then G is abelian.*

Solution. If G has identically exponent 2, then it's a very classical trick: any x, y have order at most 2 and so does xy, so $[x, y] = x^{-1}y^{-1}xy = xyxy = (xy)^2 = 1$. The proof is a little more subtle if the equation only holds generically.

We show that involutions are central, i.e. $x \in X \Rightarrow x \in Z(G)$. Indeed, the two sets xX and X have maximal rank inside G of degree 1, so $Y = xX \cap X$ is generic too. Now if $y \in Y$, then $y = xz$ for some $z \in X$, and $[x, z] = (xz)^2 = y^2 = 1$. Hence $z \in C_G(x)$ and so does y. This means that $C_G(x)$ contains the generic set Y; by connectedness, $C_G(x) = G$ and this is true for any $x \in X$.

Hence $X \subseteq Z(G)$; as X is generic, it follows by connectedness $G = Z(G)$ and we are done. $\square$

Comments.

- On the other hand, not much is known about a group of finite Morley rank with generic exponent $p \neq 2$.
- Notice that the latter question is rather trivial in an algebraic group since one can use the Zariski closure.

Second tutorial

Exercise 4 (Macintyre) *Let A be an abelian group of finite Morley rank. Then there are definable subgroups $D, B \leq A$ which are characteristic in A, with D divisible, B of bounded exponent, and $A = D(+)B$ where $(+)$ means that the intersection is finite.*

Solution. Consider the decreasing chain of definable subgroups $A \geq 2A \geq 6A \geq \cdots \geq n!A \geq \ldots$. By DCC it must stabilize at some stage, say $D := n_0!A$. D is clearly definable and characteristic; we claim it is divisible. Letting indeed $n \in \mathbb{N}$ and $d \in D$, one has $d \in D = (n_0 + n)!A \leq n.n_0!A \leq nD$, so d is an n^{th} power in D.

By Baer's theorem on abelian divisible groups, which are the injective objects of the category of abelian groups, there is $K \leq A$ with $A = D \oplus K$. K has clearly exponent at most $n_0!$, but might not be definable as we'll show later. To avoid this problem, we define $B = \{a \in A,\ n_0!a = 0\}$. Then B is a characteristic, definable subgroup of A containing K, so $A = D + B$. By Exercise 1, the intersection $D \cap B$ must be finite. $\square$

Comments.

- Here is an example where K is not definable, and the intersection $D \cap B$ is non-trivial. Let Δ be a Pruefer 2-group (defined below) and i its involution. Let Γ be an infinite $\mathbb{F}_2$-vector space of Morley rank and degree 1. Pick $j \in \Gamma \setminus \{0\}$ and consider the group $A = (\Delta \oplus \Gamma)/\langle ij \rangle$ obtained by identifying i and j. Clearly our D is the image of Δ. But K is a hyperplane of Γ. As Γ is connected, no such thing is definable. In other words, the involution $i = j$ has to be in B.

- The structure of B is quite straightforward: there are finitely many prime numbers $p_1, \ldots, p_n$ such that $B = \oplus B_{p_i}$ where B_{p_i} is the subgroup of elements of B of order a power of p_i.

  The structure of D is known too. By general study of abelian divisible groups, it is of the form

  $$D = \oplus_{p \in \mathcal{P}} \mathbb{Z}_{p^\infty}^{n_p} \bigoplus \oplus_I \mathbb{Q}$$

  where $\mathbb{Z}_{p^\infty}$ is the *Pruefer quasi-cyclic p-group*, given as the inductive limit of the system $\mathbb{Z}/p\mathbb{Z} \hookrightarrow \ldots \hookrightarrow \mathbb{Z}/p^n\mathbb{Z} \hookrightarrow \ldots$, or more naively as $\{z \in \mathbb{C}^\times$, z has order a power of $p\}$.

  In the previous sum, n_p is an integer (finite by Exercise 1), but there is no reason for the sequence (n_p) to be bounded, as the group $\oplus_{p \in \mathcal{P}} \mathbb{Z}_{p^\infty}^p$ has Morley rank 1.

- There is a generalization of Macintyre's theorem to the case of nilpotent groups, emphasizing the fact that nilpotent groups of finite Morley rank are not essentially far from abelian groups.

  Theorem (Nesin) *Let N be an abelian group of finite Morley rank. Then there are definable subgroups $D, B \leq N$ which are characteristic in N, with D divisible, B of bounded exponent, and $N = D * B$ where the intersection is finite and $*$ means $[D, B] = 1$.*

- However this nice behaviour of nilpotent groups is relative. Baudisch has constructed a 2-nilpotent group of Morley rank 2 of exponent p which cannot come from an algebraic group. Notice that this does not refute the Algebraicity Conjecture, since the latter is only about simple groups.

Exercise 5 *Let G be a group of finite Morley rank. If $X \subseteq G$, then there is a smallest definable subgroup containing X. This is denoted $d(X)$ and called the* definable hull *of X. If $X = \{x\}$ is singleton, then $d(x) = D \oplus \langle s \rangle$, with D a definable, divisible abelian group and s an element of finite order.*

Solution. By DCC,

$$d(X) := \bigcap_{X \subseteq H, H \text{ definable in } G} H$$

is a definable group and will do.

In case X is singleton, we notice as in Exercise 2 that $x \in Z(C_G(x))$ which is abelian and definable. It follows $d(x) \leq Z(C_G(x))$ and in particular the definable hull of a singleton is abelian. Now by Exercise 4 we may write $d(x) = D \oplus K$ with D definable and divisible, and K of bounded exponent. So there is a decomposition $x = d + s$ with obvious notations. As s has finite order and $x \in D \oplus \langle s \rangle$ is definable, one has actually $d(x) = D \oplus \langle s \rangle$. □

Comments.

- Geometrically, the operation d has the effect of taking the smallest Zariski-closed group containing X.
- Try to prove the following: let $H \leq G$ be any subgroup of the group of finite Morley rank G. If H is solvable, so is $d(H)$. Prove the same with "nilpotent".

Exercise 6 (Torsion-lifting) *Let G be a group of finite Morley rank with a normal definable subgroup $H \lhd G$. Assume that there is $x \in G \setminus H$ such that $x^p \in H$. Then there is a p-element in the coset xH.*

Solution. In the finite case we would proceed as follows. Let n be the order of x; if p does not divide n then $x \in \langle x^p \rangle \leq H$, a contradiction. So $p|n$. Of course $x^{\frac{n}{p}}$ is a p-element, but we don't know in which coset it lives. So let us be more careful; we want to take a power $y = x^a$ of x such that $y \in xH$, that is $x^{a-1} \in H$. This suggests to take a congruent to 1 modulo p.

Write $n = p^k m$ with $p \not| m$. By Bézout's theorem, there are $u, v \in \mathbb{Z}$ such that $up + vm = 1$. Let $y = x^{vm}$. Then clearly $y^{p^k} = 1$, so y is a p-element. Moreover, $x^{-1}y = x^{vm-1} = x^{up} = (x^p)^u \in H$, so $y \in xH$ alright.

Let us adapt this to the finite Morley rank case. Let $x \in G$ as in the statement; by Exercise 5 $d(x) = D \oplus \langle s \rangle$ where D is divisible and s has finite order. As D is p-divisible, for any $d \in D$ there is $\delta \in D$ with $d = \delta^p$. Notice that $[\langle H, x \rangle : H] = p$. It follows $d \in H$; in other words, $D \leq H$. So $xH = sH$ and we can work with s, which does have finite order. □

Comments.

- As suggested by the proof, we do not really choose the order of the p-element: it might not have order exactly p. Here is an example. Consider $G = \mathrm{SL}_2(\mathbb{C})$ and $\zeta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G$. Then $\zeta^2 = -\mathrm{Id} \in Z(G)$ but $\zeta Z(G) = \{\zeta, \zeta^{-1}\}$ contains only elements of order 4, and none of order 2.
- Here is an important consequence of torsion-lifting.

  Lemma *Let G be a group of finite Morley rank with elements a, b such that $a^b = a^{-1}$. Then G contains an involution.*

  Proof. Consider the set $C_G^{\pm}(a) = \{x \in G,\ a^x = a^{\pm 1}\}$. This is a definable subgroup of G containing b. If a is an involution, we are done.

Otherwise $C_G(a) \lhd C_G^{\pm}(a)$ and $b^2 \in C_G(a)$, so by torsion-lifting, there is a 2-element in the coset $bC_G(a)$. Iterating this 2-element, we end up with an involution. $\square$

Third tutorial

Theorem (Reineke) *An infinite, definable, connected subgroup which is minimal as such is abelian. In particular, connected groups of rank* 1 *are abelian.*

Exercise 7 (Cherlin) *A group of Morley rank 2 is solvable-by-finite.*

Solution. A technical proof which gives the opportunity to use the many tools exposed during the lectures. Assume there is a non-solvable group G of rank 2. We shall eventually derive a contradiction, through careful study of the Borel subgroups. One should bear in mind several principles, like: in the case of definable groups, being connected is equivalent to having degree 1, a connected group acting on a finite set fixes it pointwise, the Morley rank is additive...

Step 1: getting a simple group.
As we are working up to finite extensions, we may assume G connected. If there is a definable, proper, infinite normal subgroup $N \lhd G$, then both N and G/N have rank 1, so both are abelian. Then G is abelian-by-abelian, hence solvable, a contradiction. So any definable proper normal subgroup of G must be finite. In particular $Z(G)$ is finite, and up to considering $G/Z(G)$ whose center is now trivial, we may assume that G is centerless.

If there is a proper, definable, normal subgroup $N \lhd G$ then it must be finite; now the connected group G acts by conjugation on the finite set N, it fixes it pointwise, i.e. $N \leq Z(G) = 1$. It follows that G has no non-trivial proper definable normal subgroup: it is *definably simple.*

One remark: a definably simple group of finite Morley rank is actually simple. We will not use this result which is easily derived from Zilber's indecomposability theorem.

Step 2: Borel subgroups have rank 1 *and are disjoint, generic and conjugate.*
If G were minimal as an infinite, definable group, it would be abelian by Reineke's Theorem. So there is an infinite definable subgroup of rank 1, say B_1; we may assume B_1 to be connected. Notice that B_1, as any connected group of rank 1, is abelian. B_1 is clearly a Borel subgroup as G is not solvable.

Let $B_2 \neq B_1$ be another Borel subgroup, we prove that it intersects B_1 trivially. Indeed, as both are abelian it follows $C := C_G^\circ(B_1 \cap B_2) \geq \langle B_1, B_2 \rangle$. C is of course is definable; if it has rank 1, then $B_1 = C = B_2$. So $\operatorname{rk} C = 2$, meaning $C = G$. As $Z(G) = 1$, one has $B_1 \cap B_2 = 1$. Hence two distinct Borel subgroups intersect trivially.

In particular B_1 has trivial intersection with its distinct conjugates. Those are indexed by $G/N_G(B_1)$; by simplicity, $N_G(B_1) < G$ so $[N_G(B_1) : B_1] < \infty$.

One remark: for any Borel subgroup B in any group of finite Morley rank G, one has $[N_G(B) : B] < \infty$. This is because the quotient $N_G(B)/B$, if infinite, must contain an infinite definable abelian group $\bar{A}$ by Reineke's Theorem, and the preimage of $\bar{A}$ in G is solvable-by-abelian, hence solvable, against maximality of B as such.

Letting $B_1^G = \{b^g,\ b \in B_1, g \in G\}$ one has:

$$\operatorname{rk} B_1^G = \operatorname{rk} \bigcup_{g \in G} B_1^g = \operatorname{rk} \bigcup_{g \in G/N_G(B_1)} B_1^g = \operatorname{rk} \coprod_{g \in G/N_G(B_1)} B_1^g$$

as the intersections are pairwise trivial. Now all B_1^g's have same rank 1, and by additivity of the rank we find:

$$\operatorname{rk} B_1^G = \operatorname{rk}(G/N_G(B_1)) + \operatorname{rk} B_1 = \operatorname{rk} G - \operatorname{rk} N_G(B_1) + \operatorname{rk} B_1 = 2.$$

hence B_1^G is generic in G.

Now let $B_2 \neq B_1$ be another Borel subgroup. The same applies to B_2, so B_2^G is generic too. As G has degree 1 by connectedness, $B_1^G \cap B_2^G$ is infinite: so there are $b_1, b_2, g_1, g_2 \in B_1 \times B_2 \times G^2$ with $b_1 \neq 1$ and $b_1^{g_1} = b_2^{g_2}$. This element lies both in $B_1^{g_1}$ and $B_2^{g_2}$ which are therefore not disjoint. It follows $B_1^{g_1} = B_2^{g_2}$, i.e. B_1 and B_2 are conjugate, and so are all Borel subgroups of G.

Step 3: $\forall g \in G \setminus \{1\}$, $B_g := C^\circ(g)$ is the only Borel subgroup containing g. We have to check that B_g is a Borel subgroup, i.e. has rank 1. As $Z(G) = 1$, the rank can't be 2. Assume it is 0, i.e. $C(g)$ is finite. Then letting g^G denote the conjugacy class of g, one has:

$$\operatorname{rk} g^G = \operatorname{rk} G - \operatorname{rk} C_G(g) = 2,$$

i.e. g^G is generic in G. So is B_1^G; by connectedness these two sets have to meet and it follows $\exists x \in G,\ g \in B^x$. But then as B^x is abelian, one has $B^x \leq C_G(g)$ which is infinite, a contradition.

So $B_g = C_G^\circ(g)$ is infinite alright. It remains to prove that B_g does contain g, which is not trivial: though $g \in C_G(g)$, one might have $C_G(g) > C_G^\circ(g)$. So we embark on the proof assuming that the set $X := \{x \in G,\ x \notin C^\circ(x)\}$ is non-empty. We shall prove a contradiction.

First, X is clearly definable and G-invariant. Being in X should be pathological; let us compute the rank. As $\forall b \in B_1 \setminus \{1\}$, $B_1 = C_G^\circ(b)$, one has $B_1 \cap X = \emptyset$. By G-invariance, $B_1^G \cap X = \emptyset$. As B_1^G is generic in G and G has degree 1, X cannot be generic. Our contradiction will be to prove that it is generic.

Let $x \in X$. Let $Y = xB_x$, a coset of B_x. Notice that if $n \in G$ normalizes Y, it must normalize all elements of the form $y_1^{-1}y_2$, which are all elements of B_x. In other words, $N_G(Y) \leq N_G(B_x)$.

We show that $Y \subseteq X$. Let $y \in Y$. As $B_x = C^\circ(x)$, one has $x \in C(B_x)$. Since B_x is abelian, one also has $B_x \leq C(B_x)$. So y centralizes B_x. Hence $B_y = B_x$ and therefore $y \notin B_y$, that is $y \in X$. So the whole coset $Y = xB_x$ is in X.

We show that distinct conjugates of the coset Y are disjoint. Indeed, if there is some $y \in Y \cap Y^g = xB_x \cap x^g B_{x^g}$, then similarly $B_x = B_y = B_{x^g}$. So Y and Y^g are two cosets of the same group with non-trivial intersection - hence equal.

From the discussion above results the following: X is G-invariant and contains Y; the latter has rank 1, is disjoint from its distinct conjugates, and $N_G(Y)$ has rank ≤ 1. It follows

$$\operatorname{rk} X \geq \operatorname{rk} Y^G = \operatorname{rk} \bigcup_{g \in G} Y^g = \operatorname{rk} \coprod_{g \in G/N_G(Y)} Y^g \geq 2 - 1 + 1 = 2,$$

so X is generic in G, against what we have said before.

Step 4: Borel subgroups are selfnormalizing.
We know that they have finite index in their normalizers, but one has to do better than that. Fix a Borel subgroup B and $g \in N_G(B)$. We shall prove $g \in B$. Consider the conjugacy class g^G: as $C_G(g)$ has rank 1, the class has rank $2 - 1 = 1$. As G has degree 1 and maps definably onto g^G, the class has degree 1. In other words g^G is strongly minimal.

Consider $X = \{x \in g^G,\ x \in N_G(B)\}$ and $Y = g^G \setminus X$. By strong minimality, exactly one of them is finite. But notice that B acts on both by conjugation.

If Y is finite, then the connected group B fixes it pointwise. So whenever $y \in Y$, one has $y \in C_G(B) \leq N_G(B)$, which is against the definition of Y. So in this case $Y = \emptyset$, meaning $X = g^G$. So all conjugates of g normalize G; since G is simple, g^G generates G, and it follows $G \leq N_G(B)$, a contradiction to the simplicity of G.

So Y is infinite, and X is finite. In particular the connected group B fixes it pointwise. Notice that $g \in X$ by our choice of g. Hence $B \leq C_G^\circ(g)$ and $B = B_g$. By step 3, $g \in B_g = B$.

Step 5: getting an involution.
Let B be a Borel subgroup and $g \notin B = N_G(B)$. Consider the definable map

$$\begin{array}{rccc} f & : & B \times B & \to & G \\ & & (x, y) & \mapsto & xgy \end{array}$$

We are looking for some genericity argument, and we would like f to be generically surjective. To do this we would have to show that the fibers are finite; it turns out that f is even injective.

Suppose $xgy = x_1 g y_1$ with $x, y, x_1, y_1 \in B$. Then $(x_1^{-1}x)^g = y_1 y^{-1} \in B \cap B^g = 1$ by disjunction of Borel subgroups. In particular $x_1 = x$ and $y_1 = y$, and f is injective. It follows that the double coset BgB is generic in G.

But the same applies to g^{-1}, so $Bg^{-1}B$ is generic too; by connectedness, they must meet: there are $x, y \in B$ with $g^{-1} = xgy$. Now $(gx)^2 = y^{-1}x \in B$, but $gx \notin B$. So the Lemma after Exercise 6 produces an involution in G.

Step 6: final contradiction.
Let $i \in G$ be an involution and $B_i = C^\circ(i)$ the only Borel subgroup containing it. Let $g \notin B_i = N_G(B_i)$: then $B_j = B_i^g \neq B_i$ is the only Borel subgroup containing $j := i^g$.

We form the strongly real element ij. It satisfies the magic formula:

$$(ij)^i = (ij)^{-1} = ji = (ij)^j.$$

In particular, both i and j normalize B_{ij}. As the latter is self-normalizing, it follows $i, j \in B_{ij}$. As B_i, resp. B_j, is the only Borel subgroup containing i, resp. j, one has $B_i = B_{ij} = B_j$, a final contradiction. $\square$

Comments.

- You may wonder what happens after rank 2. When trying to classify the *simple* groups of Morley rank 3 in the 70's, Cherlin tried to prove that the only such object was $\mathrm{PSL}_2(\mathbb{K})$, with $\mathbb{K}$ a field of Morley rank 1. But he could not kill the following monster: a simple group of rank 3 all of whose proper definable subgroups have rank ≤ 1 - this is the smallest bad group configuration. So the existence of bad groups has been open for thirty years.

- Of course a bad group cannot be algebraic, as all simple algebraic groups interpret PSL_2, a Borel subgroup of which is *not* nilpotent. Notice further that a bad group cannot interpret a field using Zilber's method, as all solvable subgroups are nilpotent. This makes them in a sense too smooth for local analysis.

- Only in the 80's could Nesin prove that a bad group has no involutions, a highly non-trivial result. In particular, a bad group would not only be a counter-example to the Algebraicity Conjecture, but also contradict a finite Morley rank version of the Feit-Thompson theorem.