Basic Model Theory of Algebraically Closed Fields

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The purpose of this mini-course given at the Moscow Higher School of Economics is to present the very first notions of model theory. It is introductory and focuses on extremely classical results which can be found in any book on the subject. There is no claim on originality, quite the opposite. Each of the four lectures is supposed to last about ninety minutes. The material was not only chosen in order to demonstrate the power and interest of model theory but also to explain the basic notions. Of course everything we shall do could be described algebraically without appealing to logic. What matters is not the theorems themselves but the way one deals with them. My hope is that this first introduction will be of sufficient interest to suggest some further reading.

References

[1] Elisabeth Bouscaren, ed. *Model theory and algebraic geometry.* Vol. 1696. Lecture Notes in Mathematics. An introduction to E. Hrushovski's proof of the geometric Mordell-Lang conjecture. Berlin: Springer-Verlag, 1998, pp. xvi+211. ISBN: 3-540-64863-1.

[2] C. C. Chang and H. J. Keisler. *Model theory.* Third. Vol. 73. Studies in Logic and the Foundations of Mathematics. Amsterdam: North-Holland Publishing Co., 1990, pp. xvi+650. ISBN: 0-444-88054-2.

[3] David Marker. *Model theory.* Vol. 217. Graduate Texts in Mathematics. An introduction. New York: Springer-Verlag, 2002, pp. viii+342. ISBN: 0-387-98760-6.

[4] Bruno Poizat. *A course in model theory.* Universitext. An introduction to contemporary mathematical logic, Translated from the French by Moses Klein and revised by the author. New York: Springer-Verlag, 2000, pp. xxxii+443. ISBN: 0-387-98655-3.

Each of [2, 3, 4] provides an introduction to model theory; [2] is classical, [3] has become the recent benchmark. Poizat's [4] is a reference to French people; it is available in Russian translation at http://math.univ-lyon1.fr/~poizat/ The far more advanced [1] is for further reading.

Prerequisites

In order to read these notes one must know what an algebraically closed field is and a little more. One must also know that for each prime number p and each power p^n there is a unique field $\mathbb{F}_{p^n}$ of finite cardinality p^n. Every $\mathbb{F}_{p^n}$ can be realized as a subfield of the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$, which is then the union of the $\mathbb{F}_{p^n}$'s. The null-characteristic closest-looking analogue of $\overline{\mathbb{F}}_p$ is the field of algebraic numbers $\overline{\mathbb{Q}}$. It resembles $\overline{\mathbb{F}}_p$ in so far as it has no transcendental element: every $x \in \overline{\mathbb{Q}}$ has a minimal polynomial over the prime subfield, which certainly does not hold of $\mathbb{C}$. Actually this minimal polynomial (or the absence thereof) prescribes the isomorphism type of $\mathbb{Q}(x)$. This is exactly the method one uses, over $\mathbb{F}_p$, in order to construct $\mathbb{F}_{p^n}$.

We certainly do not recommend attempting at reading these notes if any of this did not sound perfectly familiar. Of course being acquainted as well with transcendence bases and the transcendence degree of a field extension cannot harm, but the reader may temporarily survive ignoring these. In case he is not familiar yet with transcendental extensions he will admit the following crucial fact.

Fact (Steinitz)**.** *Let $\mathbb{K}, \mathbb{L}$ be two algebraically closed fields of same characteristic and same transcendence degree over their prime field. Then $\mathbb{K} \simeq \mathbb{L}$.*

Contents

1 First-Order Logic

Our starting point will be the following: sometimes two mathematical objects, although non-isomorphic, share exactly the same properties. An example is the famous Lefschetz "transfer principle" which will serve as a guideline until we prove it.

Transfer Principle (Lefschetz). *$\overline{\mathbb{Q}}$ and $\mathbb{C}$ are "the same" when viewed as fields.*

We hope to have surprised the reader since for instance $\overline{\mathbb{Q}}$ is countable whereas $\mathbb{C}$ is not. So the claim is worth an explanation. We do not consider all possible mathematical sentences here but merely the *relevant* ones. The statement on cardinalities clearly belongs to set theory, which is out of the scope of the algebraist studying fields. Of course we now need a precise notion of relevance, and this is classically explained in terms of first-order logic.

1.1 First-Order Formulas

Roughly speaking the idea of first-order logic is to quantify only elements of the mathematical object under study, never subsets.

Here is a geometric motivation. Let $\mathbb{K}$ be a field. In naive algebraic geometry only some subsets of $\mathbb{K}^n$ are of interest. The basic shapes are of course the sets of zeros of polynomials, that is those defined by an equation $P(\underline{X}) = 0$ (where $\underline{X} = (X_1, \ldots, X_n)$ is a tuple of indeterminates and P a polynomial over $\mathbb{K}$). Now if one wishes to take intersections, one has to extend to systems of polynomials $P_1(\underline{X}) = \cdots = P_m(\underline{X}) = 0$. In order to understand the affine space this is not enough. One may of course look at unions and complements as well. This gives rise to more complicated shapes. In logical terms, we wish to take negations, conjunctions, disjunctions. But one should also be interested in projections of such sets. Now observe that if $Y \subseteq \mathbb{K}^n \times \mathbb{K}$ is of interest and $Z = \pi(Y)$ is the projection of Y onto the n first coordinates, then Z is characterized by:

$$\underline{z} \in Z \quad \text{iff} \quad \text{there exists } y \text{ in } Y \text{ such that } (\underline{z}, y) \in Y$$

So in order to capture projections, one should allow quantification over $\mathbb{K}$. There are iterated quantifications but past this point we have captured all algebraically "relevant" shapes. Geometers call them the "constructible sets". Logicians prefer the phrase "definable sets".

This construction lives naturally at the level of first-order logic; it is actually slightly more general and the first definitions are consequently tedious.

Definition 1.1. A first-order language consists of:

- a collection of constant symbols;
- a collection of relation symbols of various arities;
- a collection of function symbols of various arities.

The arity of a relation/function symbol is the number of arguments it takes.

Among the relations there is always - even if implicitely - the symbol "=" for a binary relation which will always stand for equality.

We shall also use variables $x, y, \dots$

Example 1.2. The language of rings is $\mathcal{L}_{\text{rings}} = \{0, 1, +, -, \cdot\}$, where 0 and 1 are constants, and $+, -, \cdot$ are binary functions. Observe that $=$ (binary relation) remains implicit.

Not all expressions of a first-order language are potentially meaningful.

Definition 1.3. We define the collection of terms of a first-order language $\mathcal{L}$:

- if c is a constant symbol, then c is a term;
- if x is a variable symbol, then x is a term;
- if f is an n-ary function symbol and $t_1, \dots, t_n$ are terms, then $f(t_1, \dots, t_n)$ is a term.

Example 1.4. $x \cdot (1 + 1) - 0 \cdot x - x - x$ is a term of the language of rings; as it makes no sense so far, one may not simplify. Notice however that we are writing $1 + 1$ instead of $+(1, 1)$.

Definition 1.5. We define the collection of formulas of $\mathcal{L}$:

- if R is an n-ary relation symbol and $t_1, \dots, t_n$ are terms, then $R(t_1, \dots, t_n)$ is a formula ("atomic formula");
- if φ is a formula, then $\neg\varphi$ ("not-phi") is a formula;
- if φ and ψ are formulas, then $\varphi \wedge \psi$ ("phi-and-psi"), $\varphi \vee \psi$ ("phi-or-psi"), $\varphi \to \psi$, and $\varphi \leftrightarrow \psi$ are formulas;
- if φ is a formula and x is a variable, then $\forall x\ \varphi$ and $\exists x\ \varphi$ are formulas.

Example 1.6. Consider the language $\{0, 1, +, -, \cdot\}$ of rings. Then $\forall x \forall y \forall z\ (x+y)+z = x+(y+z)$ is a formula.

Remark 1.7. There is no way to say "there are infinitely many".

- A first attempt may be to write $\exists x_1 \exists x_2 \dots$ and leave the dots unended. But they must stop somewhere because formulas have finite length.
- The second attempt may be more clever. A set is infinite if it is in bijective correspondence with a proper subset. But saying this amounts to quantifying subsets, which is forbidden.

So there really is no general way to say "there are infinitely many".

1.2 Structures

We now start explaining what is the meaning of a language.

Definition 1.8. Let $\mathcal{L}$ be a first-order language. An $\mathcal{L}$-structure $\mathcal{M}$ is a non-empty set M with:

- for each constant symbol c, an element $c^{\mathcal{M}}$ of M;
- for each n-ary relation symbol R, a subset $R^{\mathcal{M}}$ of M^n
  (and $=^{\mathcal{M}}$ is always the diagonal of M^2);
- for each n-ary function symbol f, a function $f^{\mathcal{M}}$ from M^n to M.

Notice that we shall write $\mathcal{M}$ in order to distinguish the structure from its underlying set M; very quickly the risk of a confusion will disappear.

Example 1.9. The ring $\mathbb{Z}$ is the set of integers equipped with $0^{\mathbb{Z}} = 0$, $1^{\mathbb{Z}} = 1$, $+^{\mathbb{Z}}, -^{\mathbb{Z}}, \cdot^{\mathbb{Z}}$ the standard addition, subtraction, multiplication.

All this is fine; the formula $\forall x \forall y \forall z\ x + (y + z) = (x + y) + z$ is obviously meaningful (and true) in $\mathbb{C}$. But since we introduced the language before the structure, writing $e + \pi = \pi + e$ is meaningless: e and π are not in the language we started with. In order to remedy this one introduces parameters.

Definition 1.10. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $\underline{x}$ be a tuple of variables, $\underline{a}$ a tuple of the same length consisting of elements of M. We define the interpretation $t^{\mathcal{M}[\underline{x}:=\underline{a}]}$ of a term t in $\mathcal{M}$ with parameters $[\underline{x} := \underline{a}]$:

- if c is a constant symbol, its interpretation is $c^{\mathcal{M}}$ (defined above);
- if x_i is a variable of $\underline{x}$, its interpretation is the corresponding a_i from $\underline{a}$;
- if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, the interpretation of $f(t_1, \ldots, t_n)$ is $f^{\mathcal{M}}(t_1^{\mathcal{M}[\underline{x}:=\underline{a}]}, \ldots, t_n^{\mathcal{M}[\underline{x}:=\underline{a}]})$.

Example 1.11.

- The interpretation of $1 + 1$ in the ring $\mathbb{Z}$ is 2, regardless of the parameters.
  Conversely, we shall always write n as an abbreviation for $1 + \cdots + 1$ in $\mathcal{L}_{\text{rings}}$.
- The interpretation of $x \cdot x$ in $\mathbb{Z}$ with parameters $[x := 3]$ is 9.

Definition 1.12. Let $\mathcal{M}$ be an $\mathcal{L}$-structure; $\underline{x}$ a tuple of variables; $\underline{a}$ a tuple of M. We define satisfaction of a formula with parameters $[\underline{x} := \underline{a}]$, denoted $\mathcal{M}[\underline{x} := \underline{a}] \models \varphi$:

- if φ is an atomic formula $R(t_1, \ldots, t_n)$, then $\mathcal{M}[\underline{x} := \underline{a}] \models \varphi$ iff $(t_1^{\mathcal{M}[\underline{x}:=\underline{a}]}, \ldots, t_n^{\mathcal{M}[\underline{x}:=\underline{a}]}) \in R^{\mathcal{M}}$;
- if φ is $\neg\psi$, then $\mathcal{M}[\underline{x} := \underline{a}] \models \varphi$ iff $\mathcal{M}[\underline{x} := \underline{a}] \not\models \psi$;
- if φ is $\psi_1 \wedge \psi_2$, then $\mathcal{M}[\underline{x} := \underline{a}] \models \varphi$ iff $\mathcal{M}[\underline{x} := \underline{a}] \models \psi_1$ and $\mathcal{M}[\underline{x} := \underline{a}] \models \psi_2$;
- if φ is $\psi_1 \vee \psi_2$, then $\mathcal{M}[\underline{x} := \underline{a}] \models \varphi$ iff $\mathcal{M}[\underline{x} := \underline{a}] \models \psi_1$ or $\mathcal{M}[\underline{x} := \underline{a}] \models \psi_2$;
- if φ is $\psi_1 \to \psi_2$, then $\mathcal{M}[\underline{x} := \underline{a}] \models \varphi$ iff $\mathcal{M}[\underline{x} := \underline{a}] \models \neg\psi_1 \vee \psi_2$;
- if φ is $\psi_1 \leftrightarrow \psi_2$, then $\mathcal{M}[\underline{x} := \underline{a}] \models \varphi$ iff $\mathcal{M}[\underline{x} := \underline{a}] \models (\psi_1 \to \psi_2) \wedge (\psi_2 \to \psi_1)$;
- if φ is $\exists y\ \psi$, then $\mathcal{M}[\underline{x} := \underline{a}] \models \varphi$ iff there is some $n \in M$ such that $\mathcal{M}[\underline{x}, y := \underline{a}, n] \models \psi$;
- if φ is $\forall y\ \psi$, then $\mathcal{M}[\underline{x} := \underline{a}] \models \varphi$ iff for any $n \in M$, $\mathcal{M}[\underline{x}, y := \underline{a}, n] \models \psi$.

Observe that if φ has no free variables, then the satisfaction of φ does not depend on parameters.

Definition 1.13. A sentence is a formula with no free variable.

Convention 1.14.

- *When φ is a formula whose free variables are among $\underline{x}$ we write $\varphi(\underline{x})$.*
  From now on φ (without reference to $\underline{x}$) will denote a sentence.
- *If $\varphi(\underline{x})$ is a formula with free variables among $\underline{x}$ and $\underline{a}$ is a tuple of parameters, then we write $\mathcal{M} \models \varphi(\underline{a})$ instead of $\mathcal{M}[\underline{x} := \underline{a}] \models \varphi$.*

There is no risk of confusion.

Example 1.15.

- $\mathbb{R} \models \forall x\ (x \geq 0 \to \exists y\ x = y^2)$.
- $\mathbb{R} \not\models \exists x\ x^2 = -\pi$. (This of course means: $\mathbb{R} \not\models \exists x\ x^2 = y$ with parameters $[y := -\pi]$.)

1.3 Theories and Models

Definition 1.16. Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{L}$-structures. $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent, denoted $\mathcal{M} \equiv \mathcal{N}$, if $\mathcal{M}$ and $\mathcal{N}$ satisfy the same sentences.

Example 1.17.

- $(\mathbb{R}, 0, 1, +, -, \cdot) \not\equiv (\mathbb{C}, 0, 1, +, -, \cdot)$
- $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$
- $(\overline{\mathbb{Q}}, 0, 1, +, -, \cdot) \equiv (\mathbb{C}, 0, 1, +, -, \cdot)$: this is Lefschetz' principle
- $(\overline{\mathbb{Q}} \cap \mathbb{R}, 0, 1, +, -, \cdot) \equiv (\mathbb{R}, 0, 1, +, -, \cdot)$ (Tarski-Seidenberg; non-trivial)
- Let F_n be the free group on n generators. Then for any $m, n \geq 2$, $(F_n, 1, \cdot, ^{-1}) \equiv (F_m, 1, \cdot, ^{-1})$ (Sela).

This suggests that abstract sets of sentences have a role to play, especially those coming from some structure.

Definition 1.18. An $\mathcal{L}$-theory is a set of $\mathcal{L}$-sentences T. It is consistent if there is an $\mathcal{L}$-structure $\mathcal{M}$ with for all $\varphi \in T$, $\mathcal{M} \models \varphi$. One says that $\mathcal{M}$ is a model of T and writes $\mathcal{M} \models T$.

Remark 1.19. This is a digression. Technically speaking one should say "satisfiable" instead of "consistent", as the latter term belongs to proof theory. But Gödel's completeness theorem asserts that in the case of first-order logic, both notions are equivalent.

Example 1.20. The theory of commutative rings is given by:

- $\forall x \forall y \forall z\ (x + y) + z = x + (y + z)$
- $\forall x \forall y\ x + y = y + x \wedge x \cdot y = y \cdot x$
- $\forall x\ x + 0 = x \wedge x \cdot 1 = x$
- $\forall x\ x - x = 0$
- $\forall x \forall y \forall z\ x \cdot (y + z) = x \cdot y + x \cdot z$

A model of this theory is precisely a commutative ring. Exercise 1.29

Notation 1.21. For $q \in \mathcal{P} \cup \{0\}$, let ACF_q denote the theory of algebraically closed fields of characteristic q.

Definition 1.22. A (consistent) theory T is complete if all its models are elementarily equivalent. Exercise 1.30

Example 1.23. The theory of algebraically closed fields is not complete as the characteristic is not fixed. But for any $q \in \mathcal{P} \cup \{0\}$, ACF_q is complete as we shall prove (for $q = 0$ this is exactly Lefschetz's principle).

1.4 Isomorphisms and Categoricity

Definition 1.24. Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{L}$-structures. An isomorphism (of $\mathcal{L}$-structures) is a bijection $\sigma : M \to N$ such that: Exercise 1.31

- for every constant c, $\sigma(c^{\mathcal{M}}) = c^{\mathcal{N}}$;
- for every relation R and every tuple $\underline{a} \in M$, $\underline{a} \in R^{\mathcal{M}}$ iff $\sigma(\underline{a}) \in R^{\mathcal{N}}$;
- for every function f and every tuple $\underline{a} \in M$, $f^{\mathcal{N}}(\sigma(\underline{a})) = \sigma(f^{\mathcal{M}}(\underline{a}))$.

Lemma 1.25. *Two isomorphic structures are elementarily equivalent.*

Proof. Suppose $\mathcal{M} \simeq \mathcal{N}$. We wish to prove that whenever $\mathcal{M}$ satisfies a sentence φ, so does $\mathcal{N}$. It is tempting to proceed by induction on the complexity of φ. Unfortunately, doing so one will not have a sentence anymore. So we shall prove a little better.

> Suppose $\sigma : \mathcal{M} \simeq \mathcal{N}$ is an isomorphism. Then for any formula with parameters, $\mathcal{M} \models \varphi(\underline{a})$ iff $\mathcal{N} \models \varphi(\sigma(\underline{a}))$.

Verification. The basic case of a formula of the form $R(t_1, \dots, t_n)$ is by definition of an isomorphism. The case of the connectives $\neg, \wedge, \vee, \to$ is easy. Now suppose $\varphi(\underline{a})$ is of the form $\exists x\ \psi(x, \underline{a})$ (the case of $\forall$ is similar).

- If $\mathcal{M} \models \varphi(\underline{a})$, then there is $\alpha \in M$ with $\mathcal{M} \models \psi(\alpha, \underline{a})$. By induction, $\mathcal{N} \models \psi(\sigma(\alpha), \sigma(\underline{a}))$, so $\mathcal{N} \models \exists x\ \psi(x, \sigma(\underline{a}))$, that is $\mathcal{N} \models \varphi(\sigma(\underline{a}))$.
- Suppose $\mathcal{N} \models \varphi(\sigma(\underline{a}))$. Then there is $\beta \in N$ with $\mathcal{N} \models \psi(\beta, \sigma(\underline{a}))$. But σ is onto, so there is $\alpha \in M$ such that $\sigma(\alpha) = \beta$. Hence $\mathcal{N} \models \psi(\sigma(\alpha), \sigma(\underline{a}))$, whence by induction $\mathcal{M} \models \psi(\alpha, \underline{a})$. It follows that $\mathcal{M} \models \varphi(\underline{a})$. □

This of course implies the Lemma.

Definition 1.26. Let κ be an infinite cardinal. A theory is κ-categorical if it has exactly one model cardinal κ up to isomorphism. Exercise 1.32

We shall not prove the following useful criterion due to lack of time. It is actually straightforward once one knows the so-called Löwenheim-Skolem theorems.

Theorem 1.27 (Vaught)**.** *If T is categorical in some cardinal, then it is complete.* Exercise 1.33

Corollary 1.28 (Lefschetz's principle)**.** $\mathbb{C} \equiv \overline{\mathbb{Q}}$ *in the language of rings.*

Proof. The theory ACF_0 is categorical in any uncountable cardinal. By Vaught's principle, it is complete. Since $\overline{\mathbb{Q}}$ and $\mathbb{C}$ are two models, they are elementarily equivalent. □

Of course one may not be entirely convinced at this point: we used both categoricity and Vaught's Theorem. Later we shall give a full proof of Lefschetz's principle.

1.5 Exercises

Exercise 1.29. *Axiomatize the following theories in $\mathcal{L}_{\text{rings}}$:*

- *fields;*
- *fields of characteristic p;*
- *fields of characteristic 0;*
- *algebraically closed fields of given characteristic.*

Exercise 1.30. *Prove that T is complete iff for every sentence φ, either $T \models \varphi$ or $T \models \neg\varphi$.*

Exercise 1.31. *Let $\mathcal{L}$ be a first-order language and $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $A \subseteq M$ be a set which will provide parameters for our formulas.*

A subset $X \subseteq M^k$ is definable with parameters in A (or "over A") if there is a formula $\varphi(\underline{x}, \underline{a})$ with parameters $\underline{a}$ in A such that for all $\underline{m} \in M^k$, one has: $\underline{m} \in X$ iff $\mathcal{M} \models \varphi(\underline{m}, \underline{a})$.

1. *Prove that if $f : \mathcal{M} \to \mathcal{M}$ is an automorphism of $\mathcal{M}$ fixing A pointwise and X is definable over A, then $f(X) = X$ setwise.*

2. *Deduce that $\mathbb{R}$ is not definable in the field $\mathbb{C}$.*

Exercise 1.32. *For each infinite cardinal κ, determine the number of models of ACF_q of size κ.*

Exercise 1.33. *Consider the language $\mathcal{L} = \{<\}$ and the theory DLO ("dense linear orderings") given by the axioms:*

- $\forall x\ \neg(x < x)$;
- $\forall x \forall y \forall z\ (x < y) \wedge (y < z) \to (x < z)$;
- $\forall x \forall y\ x < y \vee x = y \vee y < x$;
- $\forall x \forall y \exists z\ (x < y) \to (x < z \wedge z < y)$;
- $\forall x \exists y \exists z\ y < x < z$.

Show that DLO is $\aleph_0$-categorical. Deduce that it is complete. Is it $\aleph_1$-categorical?

2 Ultraproducts

We shall now discuss a fascinating construction where one "smoothes" the behaviour of a class of structures. It has extremely important applications.

2.1 Filters and Ultrafilters

Definition 2.1 (filter)**.** Let $X \neq \emptyset$ be a set. $\mathcal{F} \subseteq P(X)$ is a filter on X if:

- $\emptyset \notin \mathcal{F}$;
- if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$;
- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Example 2.2.

- Recall that Y cofinite in X means that $X \setminus Y$ is finite.

  If X is infinite, then $Fr\acute{e} = \{Y \subseteq X : Y \text{ is cofinite in } X\}$ is a filter on X ("Fréchet filter").
- In a topological space, the collection of neighborhoods of a given point forms a filter.

Remark 2.3. Let $\mathcal{B}$ be a family of subsets of X having the finite intersection property. Then there is a filter $\mathcal{F}$ such that $\mathcal{B} \subseteq \mathcal{F}$.

Verification. Let $\mathcal{F} = \{Y \subseteq X : \text{ there exist } B_1, \ldots, B_n \text{ in } \mathcal{B} \text{ such that } B_1 \cap \cdots \cap B_n \subseteq Y\}$, i.e. $\mathcal{F}$ is the smallest family of sets containing every finite intersection of members of $\mathcal{B}$. This clearly meets the conditions defining a filter. Notice that $\emptyset \notin \mathcal{F}$ by the finite intersection property.

Definition 2.4 (ultrafilter)**.** A maximal filter on X is called an ultrafilter.

Lemma 2.5. *Let $\mathcal{U}$ be a filter on X. Then $\mathcal{U}$ is an ultrafilter iff for all A in $P(X)$, $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.*

Proof. Let $\mathcal{U}$ be a filter, and bear in mind that $\mathcal{U}$ is closed under finite intersections.

Suppose that $\mathcal{U}$ is an ultrafilter, and let $A \in P(X)$. There are three cases. If for all $Y \in \mathcal{U}$, $A \cap Y \neq \emptyset$, then $\mathcal{U} \cup \{A\}$ is a family as in Remark 2.3, so it is contained in a filter. By maximality of $\mathcal{U}$ as a filter, it follows $A \in \mathcal{U}$. If for all $Y \in \mathcal{U}$, $(X \setminus A) \cap Y \neq \emptyset$, then $X \setminus A \in \mathcal{U}$ similarly. So it remains the case where there are $Y_1, Y_2 \in \mathcal{U}$ such that $A \cap Y_1 = (X \setminus A) \cap Y_2 = \emptyset$. But then $Y_1 \cap Y_2 = \emptyset$, a contradiction.

Suppose that for all $A \in P(X)$, $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$. Let $\hat{\mathcal{U}}$ be a filter extending $\mathcal{U}$: we show $\mathcal{U} = \hat{\mathcal{U}}$. Let $Y \in \hat{\mathcal{U}}$. If $Y \in \mathcal{U}$ we are done. Otherwise $X \setminus Y \in \mathcal{U} \subseteq \hat{\mathcal{U}}$, so $\emptyset = Y \cap (X \setminus Y) \in \hat{\mathcal{U}}$, a contradiction. □

Ultrafilters trivially exist.

Definition 2.6 (principal ultrafilter)**.** Let $a \in X$. The principal ultrafilter on a is $\mathcal{P}_a = \{Y \subseteq X : a \in Y\}$. Exercise 2.15

In the interesting case (X infinite), do non-principal ultrafilters exist? Assuming AC (or "Zorn's Lemma"), they do; AC is actually slightly stronger than necessary to show this. In fact, there is an Ultrafilter Axiom which is independent of ZF (and weaker than AC) saying that every filter is included in an ultrafilter. Applying this to the Fréchet filter on X, we find non-principal ultrafilters. This is however highly non-constructive.

Lemma 2.7 (AC, or "Ultrafilter Axiom")**.** *Every filter is included in an ultrafilter.* Exercise 2.16

Lemma 2.8 (AC)**.** *If $\mathcal{B}$ has the finite intersection property, there is an ultrafilter $\mathcal{U}$ such that $\mathcal{B} \subseteq \mathcal{U}$.*

Proof. As $\mathcal{B}$ has the finite intersection property, it is contained in a filter $\mathcal{F}$ by Remark 2.3. Then Lemma 2.7 gives an ultrafilter $\mathcal{U}$ extending $\mathcal{F}$. □

Exercise 2.17

2.2 Ultraproducts

Definition 2.9 (ultraproduct)**.** Let $I \neq \emptyset$ be a set, $\mathcal{U}$ be an ultrafilter on I, and for each $i \in I$ let $\mathcal{M}_i$ be an $\mathcal{L}$-structure. We define the ultraproduct of the $\mathcal{M}_i$'s (with respect to $\mathcal{U}$), denoted $\prod_I \mathcal{M}_i/U$. For simplicity, we refer to it as $\mathcal{M}^*$.

- The base of $\mathcal{M}^*$ is the set $\prod_I M_i$ modulo the equivalence relation:

$$(m_i) \sim (n_i) \quad \text{if} \quad \{i \in I : m_i = n_i\} \in \mathcal{U}$$

- We interpret constants in $\mathcal{L}$ by:

$$c^{\mathcal{M}^*} = \left[(c^{\mathcal{M}_i})_{i \in I}\right]$$

- We interpret relations by:

$$R^{\mathcal{M}^*}\left(\left[(m_i^1)\right], \dots, \left[(m_i^k)\right]\right) \quad \text{if} \quad \left\{i \in I : R^{\mathcal{M}_i}(m_i^1, \dots, m_i^k)\right\} \in \mathcal{U}$$

- Similarly for functions:

$$f^{\mathcal{M}^*}\left(\left[(m_i^1)\right], \dots, \left[(m_i^k)\right]\right) = \left[(m_i^{k+1})\right] \quad \text{if} \quad \left\{i \in I : f^{\mathcal{M}_i}\left(m_i^1, \dots, m_i^k\right) = m_i^{k+1}\right\} \in \mathcal{U}$$

Checking that these are well-defined is very similar to showing the transitivity of the equivalence relation: if we have two equivalent sequences, then they agree on a set in $\mathcal{U}$, which means that the functions or relations agree on a set containing a set in $\mathcal{U}$, hence lying in $\mathcal{U}$.

Theorem 2.10 (Łoś)**.** *Let φ be a sentence. Then $\mathcal{M}^* \models \varphi$ iff $\{i \in I : \mathcal{M}_i \models \varphi\} \in \mathcal{U}$.*

Proof. As in the case of Lemma 1.25 one must actually prove something stronger:

Let $\varphi(\underline{a}^*)$ be a formula with parameters $\underline{a}^*$ in M^*. Take representatives so that $\underline{a}^* = [(\underline{a}_i)]$. Then $\mathcal{M}^* \models \varphi(\underline{a}^*)$ iff $\{i \in I : \mathcal{M}_i \models \varphi(\underline{a}_i)\} \in \mathcal{U}$.

This we prove by induction.

- If $\varphi(\underline{a}^*)$ is atomic then it is clear from our definition of $\mathcal{M}^*$.
- Now suppose that φ is $\neg\psi$. If $\mathcal{M}^* \models \neg\psi(\underline{a}^*)$, then $\mathcal{M}^* \not\models \psi(\underline{a}^*)$. By induction, this means that $\{i \in I : \mathcal{M}_i \models \psi(\underline{a}_i)\} \notin \mathcal{U}$, and since $\mathcal{U}$ is an ultrafilter (so it always contains a set or its complement), this means that $\{i \in I : \mathcal{M}_i \not\models \psi(\underline{a}_i)\} \in \mathcal{U}$, i.e. that $\{i \in I : \mathcal{M}_i \models \neg\psi(\underline{a}_i)\} \in \mathcal{U}$. Hence $\mathcal{M}^* \models \varphi(\underline{a}^*)$. All of these steps are reversible, so the iff holds.
- Next assume that φ is $\psi \wedge \chi$. Then using that $\mathcal{U}$ is an ultrafilter:

$$\begin{aligned} & \mathcal{M}^* \models (\psi \wedge \chi)(\underline{a}^*) \\ \text{iff} \quad & \mathcal{M}^* \models \psi(\underline{a}^*) \text{ and } \mathcal{M}^* \models \chi(\underline{a}^*) \\ \text{iff} \quad & \{i \in I : \mathcal{M}_i \models \psi(\underline{a}_i)\},\ \{i \in I : \mathcal{M}_i \models \chi(\underline{a}_i)\} \in \mathcal{U} \\ \text{iff} \quad & \{i \in I : \mathcal{M}_i \models \psi(\underline{a}_i)\} \cap \{i \in I : \mathcal{M}_i \models \chi(\underline{a}_i)\} \in \mathcal{U} \\ \text{iff} \quad & \{i \in I : \mathcal{M}_i \models (\psi \wedge \chi)(\underline{a}_i)\} \in \mathcal{U} \end{aligned}$$

  The cases of $\vee$ and $\to$ are similar.
- Finally, assume that φ is $\exists x\ \psi$.
  - If $\mathcal{M}^* \models \exists x\ \psi(x, \underline{a}^*)$, there is $\alpha^* \in M^*$ such that $\mathcal{M}^* \models \psi(\alpha^*, \underline{a}^*)$. Say $\alpha^* = [(\alpha_i)]$ for some representatives. Then by induction, $\{i \in I : \mathcal{M}_i \models \psi(\alpha_i, \underline{a}_i)\} \in \mathcal{U}$. In particular, $\{i \in I : \mathcal{M}_i \models \exists x\ \psi(x, \underline{a}_i)\} \in \mathcal{U}$.
  - Now suppose that $J = \{i \in I : \mathcal{M}_i \models \exists x\ \psi(x, \underline{a}_i)\}$ is in $\mathcal{U}$. For $i \in J$, let $\alpha_i \in M_i$ be such that $\mathcal{M}_i \models \psi(\alpha_i, \underline{a}_i)$. For $i \in I \setminus J$, take arbitrary α_i. Let $\alpha^* = [(\alpha_i)]$. As $J \subseteq \{i \in I : \mathcal{M}_i \models \psi(\alpha_i, \underline{a}_i)\}$ is in $\mathcal{U}$, we find by induction that $\mathcal{M}^* \models \psi(\alpha^*, \underline{a}^*)$. Hence $\mathcal{M}^* \models \varphi(\underline{a}^*)$.

  The case of $\forall$ is similar. □

2.3 The Compactness Theorem

Theorem 2.11 (compactness)**.** *A first-order theory is consistent iff it is finitely consistent, i.e. every finite subtheory is consistent.*

Proof. Fix a finitely consistent first-order theory Σ. Let I be the collection of finite subsets of T. For each $i \in I$, there is a structure $\mathcal{M}_i$ satisfying T_i by the definition of finite consistency.

For each $\varphi \in T$, let $A_\varphi = \{i \in I : \mathcal{M}_i \models \varphi\}$. Then the family $\{A_\varphi : \varphi \in T\}$ has the finite intersection property, since the intersection of a finite family corresponds to $A_{\varphi_1 \wedge \cdots \wedge \varphi_n}$, which must be nonempty since T is finitely consistent. By Lemma 2.8, there is an ultrafilter $\mathcal{U}$ extending $\{A_\varphi : \varphi \in T\}$. We let $\mathcal{M}^* = \prod \mathcal{M}_i / \mathcal{U}$ (i.e. $\mathcal{M}^*$ is the ultraproduct of the $\mathcal{M}_i$'s with respect to $\mathcal{U}$).

Let $\varphi \in T$. We have (using Łoś' Theorem and our definitions):

$$\mathcal{M}^* \models \varphi \quad \Leftrightarrow \quad \{i \in I : \mathcal{M}_i \models \varphi\} \in \mathcal{U} \quad \Leftrightarrow \quad A_\varphi \in \mathcal{U}$$

and the latter is true by construction, so $\mathcal{M}^* \models T$. □

2.4 An application: Cross-Characteristic Transfer

The following is an easy consequence of Steinitz's principle (see the introduction).

Fact 2.12 (uncountable categoricity of ACF_q)**.** *Let $q \in \mathcal{P} \cup \{0\}$ and κ be an uncountable cardinal. Then the algebraically closed field of characteristic q and cardinal κ is unique up to isomorphism.*

Theorem 2.13. *Let $\mathcal{U}$ be any non-principal ultrafilter on the set $\mathcal{P}$ of prime numbers. Then $\prod_{\mathcal{P}} \overline{\mathbb{F}}_p/\mathcal{U} \simeq \mathbb{C}$.*

Proof. Let $\mathbb{K} = \prod_{\mathcal{P}} \overline{\mathbb{F}}_p/\mathcal{U}$. Since every $\overline{\mathbb{F}}_p$ is a field, so is $\mathbb{K}$ by Łoś' theorem. An integer n being fixed, recall that there is a first-order formula stating that every polynomial of degree $\leq n$ has a root in $\overline{\mathbb{F}}_p$. By Łoś' theorem again, every polynomial of degree $\leq n$ has a root in $\mathbb{K}$. Since this is true of any n, $\mathbb{K}$ is algebraically closed. (Observe that we could not do it in one step since there is no single first-order formula expressing algebraic closedness). Exercise 2.18

Let q be a prime. Then for $p \neq q$ any other prime, one has $\overline{\mathbb{F}}_p \not\models 1 + \cdots + 1 = 0$ (q times). So cofinitely often $\overline{\mathbb{F}}_p \models q \neq 0$. Since $\mathcal{U}$ is non-principal, it contains $\mathcal{P} \setminus \{q\}$. By Łoś' theorem, $\mathbb{K} \models q \neq 0$. This holds of any prime q so $\mathbb{K}$ has characteristic 0.

In order to conclude one will use categoricity in uncountable power. It suffices to show that $\mathbb{K}$ and $\mathbb{C}$ have the same uncountable cardinality, that is to show that $\mathbb{K}$ has cardinality continuum. The rest of the proof is a little combinatorics and may be omitted.

Since we are only interested in cardinalities we entirely forget the algebraic structure: we replace the index set $\mathcal{P}$ and each $\overline{\mathbb{F}}_p$ by $\mathbb{N}$. It is then clear that $\operatorname{Card} \mathbb{K} \leq \operatorname{Card} \prod_{\mathbb{N}} \mathbb{N} = \operatorname{Card} \mathbb{N}^{\mathbb{N}} = \operatorname{Card} \mathbb{C}$. Only the reverse equality must be proved.

We now have various copies of $\mathbb{N}$ indexed by $\mathbb{N}$. In the n^{th} copy we keep only the set $\{0,1\}^n$ of sequences of 0 and 1 of length n.

Let us consider the set of coherent sequences:

$$C = \{(\sigma_n)_n \in \prod_{\mathbb{N}} \{0,1\}^n : \forall (i,j) \in \mathbb{N}^2,\ i < j \Rightarrow (\sigma_j)_{|i} = \sigma_i\}$$

C consists of the familiers of finite sequences extending each other, such as $(\emptyset; 0; 01; 010; \dots)$. It is well-known that C has cardinality $2^{\aleph_0}$.

We contend that C embeds into $\mathbb{K}$. This is because if (σ_n) and (τ_n) are two coherent sequences mapped to the same point, then they are related modulo $\sim_{\mathcal{U}}$, that is: $I = \{n \in \mathbb{N} : \sigma_n = \tau_n\} \in \mathcal{U}$. Since $\mathcal{U}$ is non-principal this set is therefore infinite and contains arbitrarily large integers. But by coherence I is closed downwards, hence $I = \mathbb{N}$ and $(\sigma_n) = (\tau_n)$ as families.

So C injects into $\mathbb{K}$, which proves the assertion on cardinalities. Categoricity finishes the proof.

□ Exercise 2.19

Corollary 2.14 (cross-characteristic transfer)**.** *Let φ be a first-order sentence in the language of rings. Then $\mathbb{C} \models \varphi$ iff $\overline{\mathbb{F}}_p \models \varphi$ for all but finitely many prime numbers.*

Proof. Let $I = \{p \in \mathcal{P} : \overline{\mathbb{F}}_p \models \varphi\}$. Suppose that I is cofinite. Let $\mathcal{U}$ be any non-principal ultrafilter on $\mathcal{P}$. Then $I \in \mathcal{U}$. Since $\mathbb{C} \simeq \prod_{\mathcal{P}} \overline{\mathbb{F}}_p/\mathcal{U}$, one has by Łoś: $\mathbb{C} \models \varphi$.

Suppose that I is not cofinite. Then $J = \mathcal{P} \setminus I$ is infinite. So there is an ultrafilter $\mathcal{V}$ containing J. Now $\mathbb{C} \simeq \prod_{\mathcal{P}} \overline{\mathbb{F}}_p/\mathcal{V}$ so $\mathbb{C} \not\models \varphi$. □

All this is very nice but we have been using categoricity again. We shall provide another proof avoiding it.

2.5 Exercises

Exercise 2.15. *Show that if X is finite, then every ultrafilter on X is principal.*

Exercise 2.16. *Recall that $P(X)$ is a ring for $(\emptyset, X, \triangle, \cap)$. To $\mathcal{A}$ a family of $P(X)$ associate $\mathcal{A}' = \{X \setminus Y : Y \in \mathcal{A}\}$. Show that the involutive bijection of $P(X)$ which sends $\mathcal{A}$ to A' exchanges filters on X with ideals of the ring $(P(X), \Delta, \cap)$. Use Krull's Theorem to prove the existence of ultrafilters.*

Exercise 2.17 (Stone-Čech compactification)**.** *Let $\beta\mathbb{N}$ be the set of ultrafilters on $\mathbb{N}$. Equip $\beta\mathbb{N} \subseteq P(P(\mathbb{N})) \simeq 2^{P(\mathbb{N})}$ with the induced topology; basic open sets are, for $A \subseteq \mathbb{N}$: $O_A = \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}$.*

1. *Show that* $\beta\mathbb{N}$ *is a compact space and that the map* $i : x \mapsto \delta_x = \{A \in \mathbb{N} : x \in A\}$ *is continuous from* $\mathbb{N}$ *to* $\beta\mathbb{N}$*.*

2. *A compactification of* $\mathbb{N}$ *is any compact space* K *such that* $\mathbb{N}$ *is homeomorphic to a dense subset of* K*. Show that if* K *is a compactification of* $\mathbb{N}$*, then there is a continuous surjection* $\beta\mathbb{N} \to K$*.*

3. *Prove the following universal property of* $\beta\mathbb{N}$*: for every compact space* K *and every continuous map* $f : \mathbb{N} \to K$*, there is a unique continuous* $g : \beta\mathbb{N} \to K$ *with* $f = g \circ i$*.*

$$\begin{tikzcd} \mathbb{N} \arrow[rr, "f"] \arrow[dr, "i"'] & & K \\ & \beta\mathbb{N} \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

4. *Show that there are* $2^{2^{\aleph_0}}$ *ultrafilters on* $\mathbb{N}$*.*

Exercise 2.18. *An axiomatization of a theory* T *is any set* S *with* $S \models T$ *and* $T \models S$*.* T *is finitely axiomatizable if it admits a finite axiomatization.*

1. *Using compactness, show that if* T *is finitely axiomatizable and* S *is an axiomatization, then* S *contains a finite axiomatization.*

2. *Deduce that* ACF_0 *is not finitely axiomatizable.*

3. *Prove that* ACF_p *is not finitely axiomatizable.*

Exercise 2.19. *Prove that any non-principal ultraproduct of sets has cardinality finite or* $\geq 2^{\aleph_0}$*.*

Remarks 2.20.

- Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures and $\mathcal{U}$ be an ultrafilter on some index set I. Denote by $\mathcal{M}^{\mathcal{U}}$ the ultrapower of copies $(\mathcal{M})_{i \in I}$ modulo $\mathcal{U}$. It is obvious that $\mathcal{M} \equiv \mathcal{M}^{\mathcal{U}}$. As a consequence, if $\mathcal{M}^{\mathcal{U}} \simeq \mathcal{N}^{\mathcal{U}}$, then $\mathcal{M} \equiv \mathcal{N}$.

  Remarkably, Shelah proved the converse [3]: if $\mathcal{M} \equiv \mathcal{N}$, then there are an index set I and an ultrafilter $\mathcal{U}$ on I with $\mathcal{M}^{\mathcal{U}} \simeq \mathcal{N}^{\mathcal{U}}$.

- The relationships between ultraproducts and Arrow's Impossibility Theorem (in economics) was first noted in [2].

- An interesting trend in model theory is the study of so-called *pseudo-finite* structures, i.e. structures elementarily equivalent to ultraproducts of finite structures such as groups, fields, etc. In the case of fields, Ax' work [1] was seminal.

References

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[2] Alan P. Kirman and Dieter Sondermann. "Arrow's theorem, many agents, and invisible dictators". In: *J. Econom. Theory* 5.2 (1972), pp. 267–277. ISSN: 0022-0531.

[3] Saharon Shelah. "Every two elementarily equivalent models have isomorphic ultrapowers". In: *Israel J. Math.* 10 (1971), pp. 224–233. ISSN: 0021-2172.

3 Elementary extensions

Today we go back to some basic notions in order to prove Lefschetz's principle by naive methods.

3.1 Elementary extension and saturation

Definition 3.1. Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures. Suppose that $M \subseteq N$ and that the interpretation on $\mathcal{M}$ is induced by that on $\mathcal{N}$. The extension is elementary (written $\mathcal{M} \preceq \mathcal{N}$) if for every formula $\varphi(\underline{a})$ with parameters in M, one has $\mathcal{M} \models \varphi(\underline{a})$ iff $\mathcal{N} \models \varphi(\underline{a})$. Exercise 3.22

It is clear that if $\mathcal{M} \preceq \mathcal{N}$ then restricting to sentences, $\mathcal{M} \equiv \mathcal{N}$. Exercise 3.23

Remarks 3.2.

- If $\mathcal{M} \preceq \mathcal{N}$ and $\mathcal{M} \preceq \mathcal{P}$ then there is $\mathcal{M}'$ with $\mathcal{N} \preceq \mathcal{M}'$ and $\mathcal{P} \preceq \mathcal{M}'$.
- If $(\mathcal{M}_i)_{i \in I}$ is an ordered family of structures with $\mathcal{M}_i \preceq \mathcal{M}_j$ whenever $i \leq j$, then $\mathcal{N} = \bigcup_{i \in I} \mathcal{M}_i$ is a common elementary extension: $\mathcal{M}_i \preceq \mathcal{N}$ for all $i \in I$. Exercise 3.24

Lemma 3.3. *If $\mathbb{K} \models$ ACF is an algebraically closed field, then there exists $\mathbb{L} \succeq \mathbb{K}$ with infinite transcendence degree over the prime field.*

Proof. If $\mathbb{K}$ is uncountable then $\mathbb{L} = \mathbb{K}$ suffices. So suppose that $\mathbb{K}$ is countable. Let $\mathcal{U}$ be any non-principal ultrafilter on $\mathbb{N}$ and let $\mathbb{L} = \prod_{\mathbb{N}} \mathbb{K}/\mathcal{U}$. We know that $\mathbb{L}$ is an algebraically closed field of the same characteristic as $\mathbb{K}$, and we have proved in 2.13 that $\operatorname{Card} \mathbb{L}$ is the continuum. It therefore suffices to show that $\mathbb{K} \preceq \mathbb{L}$.

To that extent we construct the diagonal embedding:

$$\begin{array}{rccc} \iota: & \mathbb{K} & \to & \mathbb{L} \\ & k & \mapsto & [(k)] \end{array}$$

which takes any $k \in \mathbb{K}$ to the class of the constant sequence k. It is clear that ι embeds $\mathbb{K}$ into $\mathbb{L}$.

Let $\varphi(\underline{a})$ be a formula with parameters in $\mathbb{K}$. If $\mathbb{K} \models \varphi(\underline{a})$, then $\{i \in \mathbb{N} : \mathbb{K} \models \varphi(\underline{a})\} = \mathbb{N} \in \mathcal{U}$, so by Łoś' theorem (more precisely, its version with parameters), $\mathbb{L} \models \varphi(\iota(\underline{a}))$. Conversely, if $\mathbb{K} \not\models \varphi(\underline{a})$, then $\{i \in \mathbb{N} : \mathbb{K} \models \varphi(\underline{a})\} = \emptyset \notin \mathcal{U}$, in which case $\mathbb{L} \not\models \varphi(\iota(\underline{a}))$. $\square$ Exercises 3.25, 3.27

It will later be proved that actually *any* extension of algebraically closed fields is elementary, but we do not know it so far. Lemma 3.3 is the opportunity to introduce one of the key concepts of model theory. As far as today's techniques are concerned, the rest of this subsection may be regarded as a digression. It is actually the very centre of model theory.

Definition 3.4. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$ be a set of parameters. Let $\underline{x}$ have length k.

- A k-type $p(\underline{x})$ over A is a set of formulas with parameters in A such that: for some elementary extension $\mathcal{M} \preceq \mathcal{N}$ and $\underline{n} \in N^k$, $\mathcal{N} \models p(\underline{n})$ (meaning of course that for any $\varphi(\underline{x}, \underline{a}) \in p(\underline{x})$, one has $\mathcal{N} \models \varphi(\underline{n}, \underline{a})$).
- If $\underline{n}$ can be taken in M, the type is said to be realized in $\mathcal{M}$.

One of the most important questions one may ask in model theory is: how many (maximal) types over A does the structure have? As one sees, this may somehow relate to the number of Exercise 3.26
non-isomorphic elementary extensions. We do not dwell on the subject but rather introduce the model-theoretic notion of "richness" of a structure.

Definition 3.5. $\mathcal{M}$ is κ-saturated if any n-type over A, with $A \subseteq M$ of cardinal $< \kappa$, is realized in $\mathcal{M}$.

Remark 3.6. Any $\mathcal{M}$ admits a κ-saturated elementary extension (but its cardinal may be big unless one has control of the number of types). Exercise 3.27

3.2 Back-and-forth and Completeness

The following is a purely algebraic fact.

Lemma 3.7. *Let $\mathbb{K}, \mathbb{L} \models \mathrm{ACF}_q$ be two algebraically closed fields of the same characteristic. Suppose that $\mathbb{K}$ and $\mathbb{L}$ have infinite transcendence degree over their prime field. Let $\underline{a} \in \mathbb{K}$ and $\underline{b} \in \mathbb{L}$ be tuples satisfying the same quantifier-free formulas. Let $\alpha \in \mathbb{K}$. Then there is $\beta \in \mathbb{L}$ such that $\underline{a}, \alpha$ and $\underline{b}, \beta$ satisfy the same quantifier-free formulas.*

Proof. Observe that the field $\mathbb{K}_0 = \langle \underline{a} \rangle$ is isomorphic to the field $\mathbb{L}_0 = \langle \underline{b} \rangle$; let f be an isomorphism.

If α is algebraic over $\mathbb{K}_0$, then it has a minimal polynomial P over $\mathbb{K}_0$. Consider the polynomial $f(P)$ (this amounts to taking the image of the coefficients of P under the isomorphism). Since $\mathbb{L}$ is algebraically closed, $f(P)$ has a solution β. Now $\langle \underline{a}, \alpha \rangle \simeq \langle \underline{b}, \beta \rangle$, and $\underline{a}, \alpha$ and $\underline{b}, \beta$ satisfy the same formulas.

If on the other hand α is transcendental over $\mathbb{K}_0$, then take β to be any element of $\mathbb{L}$ transcendental over $\mathbb{L}_0$; by assumption and since $\mathbb{L}_0$ is finitely generated, there is such an element. $\square$

Corollary 3.8. ACF_q *is complete.*

Proof. We shall prove that any two models $\mathbb{K}, \mathbb{L} \models \mathrm{ACF}_q$ satisfy the same sentences, that is $\mathbb{K} \equiv \mathbb{L}$. First remember that there are $\mathbb{K} \preceq \mathbb{K}^*$ and $\mathbb{L} \preceq \mathbb{L}^*$ which have infinite transcendence degree over their prime field. In particular $\mathbb{K} \equiv \mathbb{K}^*$ and $\mathbb{L} \equiv \mathbb{L}^*$, so we may suppose $\mathbb{K} = \mathbb{K}^*$ and $\mathbb{L} = \mathbb{L}^*$.

We can prove a little more: two tuples $\underline{a} \in \mathbb{K}$ and $\underline{b} \in \mathbb{L}$ satisfying the same atomic formulas actually satisfy the same formulas.

By induction on formulas. The case where φ is atomic or obtained from shorter formulas by connectives $\neg, \wedge, \vee, \rightarrow$ is obvious.

So we may assume that $\varphi(\underline{a})$ is $\exists x\ \psi(x, \underline{a})$. Suppose $\mathbb{K} \models \varphi(\underline{a})$. Then there is $\alpha \in \mathbb{K}$ with $\mathbb{K} \models \psi(\alpha, \underline{a})$. But by Lemma 3.7 there is $\beta \in \mathbb{L}$ such that $(\alpha, \underline{a})$ and $(\beta, \underline{b})$ satisfy the same quantifier-free formulas. By induction, $\mathbb{L} \models \psi(\beta, \underline{b})$. So $\mathbb{L} \models \varphi(\underline{b})$, as desired. $\square$

3.3 Cross-Characteristic Transfer and Ax's Theorem

Theorem 3.9. *Let φ be a sentence. Then $\mathrm{ACF}_0 \models \varphi$ iff $\mathrm{ACF}_p \models \varphi$ for all but finitely many prime numbers p.*

Proof. Suppose that $\mathrm{ACF}_p \models \varphi$ cofinitely often. We claim that the theory $T = \mathrm{ACF}_0 \cup \{\varphi\}$ is consistent. By compactness (Theorem 2.11) it suffices to show that it is finitely consistent. But a finite subtheory $T' \subseteq T$ will mention only finitely many inequalities $p \neq 0$, so taking q large enough, $\mathrm{ACF}_q \models T'$. Hence T' is consistent and by compactness so is T. This means that $\mathrm{ACF}_0 \not\models \neg\varphi$. Since ACF_0 is complete, one has $\mathrm{ACF}_0 \models \varphi$.

Now suppose that $\mathrm{ACF}_p \models \varphi$ co-infinitely often. A similar argument shows that $\mathrm{ACF}_0 \cup \{\neg\varphi\}$ is consistent and $\mathrm{ACF}_0 \models \neg\varphi$. $\square$

Corollary 3.10 (Ax)**.** *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If f is injective, then it is surjective.*

Proof. Suppose this fails. Being a polynomial map, the graph of f can be defined by a formula $\varphi(\underline{x}, \underline{y}, \underline{a})$ with parameters $\underline{a} \in \mathbb{C}$. Observe that injectivity and non-surjectivity are first-order. So there is a formula $\psi(\underline{a})$ saying that f is injective and non-surjective. Now $\mathbb{C} \models \exists \underline{z}\ \psi(\underline{z})$.

In particular there is a prime number p such that $\overline{\mathbb{F}}_p \models \exists \underline{z}\ \psi(\underline{z})$. Hence there is a polynomial map $g : \overline{\mathbb{F}}_p^n \to \overline{\mathbb{F}}_p^n$ which is injective but not surjective. Let $\mathbb{F}_q$ be a finite field big enough to contain all coefficients of the polynomials involved. Being polynomial, g restricts to $h : \mathbb{F}_q^n \to \mathbb{F}_q^n$. Now h remains injective, but $\mathbb{F}_q^n$ is finite, so it is surjective as well. This is a contradiction. $\square$

Read the argument carefully and make sure that you cannot prove the converse.

3.4 Quantifier elimination

Definition 3.11. A theory T eliminates quantifiers if for every formula $\varphi(\underline{x})$ there is a quantifier-free formula $\psi(\underline{x})$ such that $T \models \forall \underline{x} \ (\varphi(\underline{x}) \leftrightarrow \psi(\underline{x}))$.

Lemma 3.12. *T eliminates quantifiers iff: whenever $\mathcal{M}, \mathcal{N} \models T$ are models and $\underline{a} \in M$, $\underline{b} \in N$ satisfy the same quantifier-free formulas, then $\underline{a}$ and $\underline{b}$ satisfy the same formulas.*

Exercises 3.28, 3.29

Theorem 3.13. ACF *eliminates quantifiers.*

Proof. Suppose $\mathbb{K}, \mathbb{L} \models$ ACF are two algebraically closed fields and $\underline{a} \in \mathbb{K}$, $\underline{b} \in \mathbb{L}$ satisfy the same quantifier-free formulas. Observe that the hypothesis and conclusions are preserved under elementary extension. So with Lemma 3.3 we may assume that $\mathbb{K}$ and $\mathbb{L}$ have infinite transcendance degree over their prime fields. Observe that since $\underline{a}$ and $\underline{b}$ satisfy the same equations, $\mathbb{K}$ and $\mathbb{L}$ have the same characteristic. So the proof of Corollary 3.8 shows that $\underline{a}$ and $\underline{b}$ satisfy the same formulas. By the previous criterion, ACF eliminates quantifiers. □

Corollary 3.14 (Chevalley-Tarski)**.** *Let $\mathbb{K}$ be an algebraically closed field. Then the boolean algebra generated by zero sets of polynomials in the various powers $\mathbb{K}^n$ is closed under projection.*

Proof. The zero sets of polynomials are exactly those defined by atomic formulas (with parameters). The boolean algebra they generate is exactly the collection of sets which are quantifier-free definable with parameters. By quantifier elimination, those are exacly all definable sets. □

Corollary 3.15. *Every extension of algebraically closed fields is elementary.*

Proof. Suppose $\mathbb{K} \subseteq \mathbb{L}$ are two algebraically closed fields. Let $\varphi(\underline{a})$ be a formula with parameters in $\mathbb{K}$. By quantifier elimination, there is a quantifier-free formula $\psi(\underline{x})$ such that ACF $\models \forall \underline{x} \ \varphi(\underline{x}) \leftrightarrow \psi(\underline{x})$. In particular, $\mathbb{K} \models \varphi(\underline{a})$ iff $\mathbb{K} \models \psi(\underline{a})$ and likewise for $\mathbb{L}$. But the equivalence $\mathbb{K} \models \psi(\underline{a})$ iff $\mathbb{L} \models \psi(\underline{a})$ is obvious since there are no quantifiers. □

Theorem 3.16 (Hilbert's Nullstellensatz)**.** *Let $\mathbb{K}$ be an algebraically closed field. Let $\underline{X} = (X_1, \ldots, X_n)$ be indeterminates and $I \triangleleft \mathbb{K}[\underline{X}]$ be an ideal. Then there is $\underline{a} \in \mathbb{K}^n$ such that $P(\underline{a}) = 0$ for all P in I.*

Proof. Let $\mathfrak{m}$ be a maximal ideal containing I. Let $\mathbb{L} = \overline{\mathbb{K}[\underline{X}]/\mathfrak{m}}$. Since $\mathfrak{m}$ is a proper ideal, $\mathbb{K}$ embeds into $\mathbb{L}$; we may suppose $\mathbb{K} \subseteq \mathbb{L}$. In particular, $\mathbb{K} \preceq \mathbb{L}$.

By noetherianity of $\mathbb{K}[\underline{X}]$, there are finitely many polynomials $P_1, \ldots, P_m$ generating $\mathfrak{m}$. Let $\underline{a} \in \mathbb{K}$ be a tuple encoding the coefficients. Then the existence of a solution to $\mathfrak{m}$ is a formula $\varphi(\underline{a})$ with parameters $\underline{a}$. Since $\mathbb{L}$ does have a solution (the image of $\underline{X}$ modulo $\mathfrak{m}$), it is the case that $\mathbb{L} \models \varphi(\underline{a})$. The extension is elementary so $\mathbb{K} \models \varphi(\underline{a})$ as well. This means that $\mathbb{K}$ has a solution of $\mathfrak{m}$; it is also a solution of I. □

3.5 Real Closed Fields

And now for something completely different: real closed fields.

Definition 3.17. In the language $\mathcal{L}_{\text{ordrings}} = \{0, 1, +, -, \cdot, \leq\}$ of ordered rings, consider the theory RCF given by:

- the field axioms;
- $\leq$ is a linear ordering compatible with the field structure;
- every polynomial satisfies the intermediate value theorem.

Of course $\mathbb{R}$ is an example.

Remark 3.18. RCF is not $2^{\aleph_0}$-categorical. For instance there is a model with same cardinality as $\mathbb{R}$ but with infinitesimals. (As a matter of fact there are many models of size $2^{\aleph_0}$, although $\mathbb{R}$ is the only complete one - completeness is a second-order property).

In spite of this, RCF has nice properties which are left as exercises.

Proposition 3.19 (Tarski-Seidenberg)**.** RCF *is complete and eliminates quantifiers.*

Remark 3.20. RCF could be expressed in the language of pure rings (replacing $a \leq b$ by $\exists x\; b-a = x^2$ everywhere). In the latter language, RCF does not eliminate quantifiers. It is however the case that any inclusion of real closed fields is elementary.

Corollary 3.21. *Every definable subset of* $(\mathbb{R}, 0, 1, +, -, \cdot)$ *is a finite union of points and intervals.*

The latter property is called *o*-minimality and is the beginning of quite a story.

3.6 Exercises and beyond

Exercise 3.22 (Tarski's test for $\preceq$)**.** *Suppose* $\mathcal{M} \subseteq \mathcal{N}$*. Show that* $\mathcal{M} \preceq \mathcal{N}$ *iff for every formula* $\varphi(x, \underline{a})$ *with parameters in* M *such that* $\mathcal{N} \models \exists x\; \varphi(x, \underline{a})$*, there is* $\alpha \in M$ *with* $\mathcal{N} \models \varphi(\alpha, \underline{a})$*.*

Exercise 3.23. *If* $\mathcal{M} \preceq \mathcal{N}$ *then* $\mathcal{M} \equiv \mathcal{N}$*. Prove that the converse may fail.*

Exercise 3.24. *Prove Remark 3.2:*

1. *Let* $\mathcal{M} \preceq \mathcal{N}$ *and* $\mathcal{M} \preceq \mathcal{N}$*. Add new constant symbols* c_n *and* c_p *for each* $n \in N$ *and* $p \in P$*. Considering* $\{\varphi(\underline{c}_n) : \mathcal{N} \models \varphi(\underline{n})\} \cup \{\psi(\underline{c}_p) : \mathcal{P} \models \psi(\underline{p})\}$*, construct a common elementary extension.*

2. *Let* $(\mathcal{M}_i)_{i \in I}$ *be an ordered family of structures with* $\mathcal{M}_i \preceq \mathcal{M}_j$ *whenever* $i \leq j$*, and* $\mathcal{N} = \bigcup_{i \in I} \mathcal{M}_i$ *with the induced interpretation. Show* $\mathcal{M}_i \preceq \mathcal{N}$ *for all* $i \in I$ *by induction on the structure of a formula with parameters.*

Exercise 3.25. *The idea of this exercise is to give another proof of Lemma 3.3 by important model-theoretic techniques.*

Let $\mathcal{M}$ *be an infinite* $\mathcal{L}$*-structure. For each* $a \in M$ *let* c_a *be a new constant symbol; let* $\mathcal{L}' = \mathcal{L} \cup \{c_a : a \in M\}$*. Observe that* $\mathcal{M}$ *is naturally an* $\mathcal{L}$*'-structure (the interpretation of* c_a *is* a*).*

1. *Let* $\mathrm{Th}(\mathcal{M}, M) = \{\varphi(c_{\underline{a}}) \in \mathcal{L}' : \mathcal{M} \models \varphi(\underline{a})\}$*. Show that models of* $\mathrm{Th}(\mathcal{M}, M)$ *(in* $\mathcal{L}'$*) are exactly elementary extensions of* $\mathcal{M}$ *(in* $\mathcal{L}$*).*

2. *Let* $p(x)$ *be a set of formulas with free variable* x *and parameters in* M*. Suppose that* $p(x)$ *is consistent with* $\mathrm{Th}(\mathcal{M})$*. (This is called a* 1*-type of* $\mathcal{M}$*). Show that there exist an elementary extension* $\mathcal{M} \preceq \mathcal{N}$ *and an element* $b \in N$ *satisfying all formulas of* $p(x)$*.*

3. *Back to Lemma 3.3. Let* $p(x)$ *express that* x *is transcendental over the prime field. Iterating, construct an elementary extension of* $\mathbb{K}$ *of infinite transcendence degree.*

If you have understood this exercise you certainly know how to construct κ*-saturated elementary extensions.*

Exercise 3.26.

1. *Count* n*-types over* $\emptyset$ *in* $(\overline{\mathbb{Q}}, 0, 1, +, -, \cdot)$*.*

2. *Count* 1*-types over* $\emptyset$ *in* $(\mathbb{N}, 0, 1, +, \cdot)$*.*

3. *Same question in* $(\mathbb{R}, 0, 1, +, -, \cdot)$*.*

Exercise 3.27. *Show that if* $(\mathcal{M}_n)_{n \in \mathbb{N}}$ *are* $\mathcal{L}$*-structures and* $\mathcal{U}$ *is a non-principal ultrafilter on* $\mathbb{N}$*, the ultraproduct* $\mathcal{M}^* = (\prod_n \mathcal{M}_n)/\mathcal{U}$ *is* $\aleph_1$*-saturated.*

Exercise 3.28. *Suppose that* $\varphi(\underline{x})$ *is a formula such that:*

> *whenever* $\mathcal{M}, \mathcal{N} \models T$ *and* $\underline{a} \in M$*,* $\underline{b} \in N$ *satisfy the same quantifier-free formula, then* $\mathcal{M} \models \varphi(\underline{a})$ *iff* $\mathcal{N} \models \varphi(\underline{b})$*.*

We shall show that φ is equivalent modulo T to a quantifier-free formula, that is there is quantifier-free $\psi(\underline{x})$ with $T \models \forall \underline{x}\ \varphi(\underline{x}) \leftrightarrow \psi(\underline{x})$. Let $\underline{c}, \underline{d}$ be new constant symbols.

1. *Find quantifier-free formulas $\psi_1, \ldots, \psi_n$ such that $T \cup \{\varphi(\underline{c})\} \cup \{\bigwedge_{i=1}^n \psi_i(\underline{c}) \leftrightarrow \psi_i(\underline{d})\} \models \varphi(\underline{d})$.*
2. *For $\mathcal{N} \models T$ and $\underline{b} \in N$ with $\mathcal{N} \models \varphi(\underline{b})$, let*

$$\chi_{\mathcal{N},\underline{b}} = \left(\bigwedge_{i:\mathcal{N} \models \psi_i(\underline{b})} \psi_i(\underline{x}) \right) \bigwedge \left(\bigwedge_{i:\mathcal{N} \models \neg\psi_i(\underline{b})} \neg\psi_i(\underline{x}) \right)$$

   (Note that the collection of formulas of the form $\chi_{\mathcal{N},\underline{b}}$ is finite.) Show that $T \cup \{\varphi(\underline{c})\} \cup \{\neg \bigvee \chi_{\mathcal{N},\underline{b}}(\underline{c})\}$ is inconsistent and conclude.

Exercise 3.29.

1. *Show that the theory DLO (see Exercise 1.33) has quantifier elimination and is complete.*
2. *Now consider the theory DiLO ("discrete linear orderings") in the same language:*
   - $\forall x\ \neg(x < x)$;
   - $\forall x \forall y \forall z\ (x < y) \wedge (y < z) \to (x < z)$;
   - $\forall x \forall y\ x < y \vee x = y \vee y < x$;
   - $\forall x \exists y \forall z\ (x < y) \wedge ((x < z) \to \neg(z < y))$;
   - $\forall x \exists y \forall z\ (y < x) \wedge ((z < x) \to \neg(y < z))$;
   - $\forall x \exists y \exists z\ y < x < z$.

   Show that DiLO is complete but does not *eliminate quantifiers.*
3. *For each $n \geq 1$ let d_n be a binary relation symbol. Add axioms saying that $d_n(x, y)$ holds iff x and y are at distance n, obtaining a theory DiLO'. Show that in $\mathcal{L}' = \mathcal{L} \cup \{d_n : n \geq 1\}$, DiLO' eliminates quantifiers.*

Remarks 3.30.

- It is entirely non-trivial that any structure elementarily equivalent to another o-minimal structure is in turn o-minimal [1].
- The proof relies on the existence of a nice cell decomposition for definable subsets of M^k (where $\mathcal{M}$ is o-minimal).
- Alex Wilkie proved that $(\mathbb{R}, 0, 1, +, -, \cdot, \leq, \exp)$ is o-minimal [2].
- o-minimality has since found celebrated applications to number theory.

o-minimality is however another world. The final lecture will (in spirit) be about algebraically closed fields.

References

[1] Julia F. Knight, Anand Pillay, and Charles Steinhorn. "Definable sets in ordered structures. II". In: *Trans. Amer. Math. Soc.* 295.2 (1986), pp. 593–605. ISSN: 0002-9947. DOI: 10.2307/2000053. URL: http://dx.doi.org/10.2307/2000053.

[2] A. J. Wilkie. "Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function". In: *J. Amer. Math. Soc.* 9.4 (1996), pp. 1051–1094. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-96-00216-0. URL: http://dx.doi.org/10.1090/S0894-0347-96-00216-0.

4 What was it all about?

This lecture will be more advanced; I shall describe general phenomena but give no proofs. The real invitation to model theory starts here. For everything I say today, see Marker's book.

It is a commonplace in mathematics to say that every field ultimately returns to geometry. I wish to illustrate this commonplace. To the question "what is model theory?" a couple of answers may have been given along the years.

- If the phrase "model theory" made any sense in the thirties, one would certainly have considered it a part of logic.
- A mathematician of the fifties may have argued that model theory is universal algebra.
- But from the seventies on, model theory often was related to combinatorics.
- Nowadays, and this is at least what I wish to defend in this talk, model theory may be the language for geometries.

Categoricity has played an essential role in the well-behavedness of the theory ACF_q: our first proof of Lefschetz' principle, our first proof of cross-characteristic transfer relied on this algebraic fact. We could dispense using it by giving other proofs, but it was there. Similarly the transcendence degree was omnipresent. Transcendence degree is something fascinating: it relies on a naive notion of algebraic independence and then builds a dimension theory out of it, precisely like in vector spaces.

In today's lecture I wish to put the pieces together and convince you that all the apparently algebraic facts are actually model-theoretic.

Convention 4.1. *We fix a "big" model $\mathcal{M}$ (technically speaking, sufficiently saturated). For instance if one starts with $\mathcal{M}_0$, then an ultrapower $\mathcal{M} = (\prod_I \mathcal{M}_0)/\mathcal{U}$ modulo some non-principal ultrafilter may do.*

We consider $\mathcal{M}$ in a countable language.

4.1 Categoricity

Recall that given an infinite cardinal κ, a first-order theory is κ-categorical iff it has up to isomorphism a unique model of cardinal κ.

Example 4.2.

- The theory of an infinite set with pure equality is κ-categorical for any κ.
- Let $\mathbb{K}$ be any field. There is a natural way to axiomatize $\mathbb{K}$-vector spaces in the language $\mathcal{L}_{\mathbb{K}\text{-vs}} = \{0, +, -\} \cup \{\lambda_k : k \in \mathbb{K}\}$ (observe that $\mathbb{K}$ is fixed throughout). The resulting theory $T_{\mathbb{K}\text{-vs}}$ is κ-categorical for any $\kappa > \operatorname{Card} \mathbb{K}$.
- ACF_q is not $\aleph_0$-categorical but it is κ-categorical for any $\kappa \geq \aleph_1$.

The reader should keep in mind these three examples.

In a more abstract direction, Vaught had conjectured the following.

Theorem 4.3 (Morley, 1965)**.** *Let T be theory in a countable language. Then T is κ-categorical for* some *uncountable κ iff it is κ-categorical for* all *uncountable κ.*

Remark 4.4.

- In the language given above, the theory of $\mathbb{K}$-vector spaces is countable only if $\mathbb{K}$ is at most countable.
- Does not work with $\kappa = \aleph_0$. $\aleph_0$-categoricity is another world which we shall not enter.

- Saharon Shelah generalized this to the uncountable case.

  As a matter of fact, Shelah tackled the "spectrum problem", namely the following Promethean question: "given T, what are the possible behaviours of the function $\kappa \mapsto$ #models of size κ?", leading to what is called classification theory. (He almost provided the full answer, which was given in its final form by Hart, Hrushovski, Laskowski.)

Morley's theorem is the beginning of a fascinating story. In the case of ACF_q one knows that isomorphism of large models is entirely described by the cardinality of a transcendence basis (just like in large $\mathbb{K}$-vector spaces over fixed $\mathbb{K}$). In particular, categoricity and dimension seem to have something to do. As a matter of fact Morley introduced in his proof a highly valuable tool.

4.2 Towards Geometries: Dimension

Definition 4.5. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A subset $X \subseteq \mathcal{M}^k$ is definable (with parameters $\underline{a}$) if there is a formula $\varphi(\underline{x}, \underline{a})$ such that for any $\underline{m} \in M^k$, $\underline{m} \in X$ iff $\mathcal{M} \models \varphi(\underline{m}, \underline{a})$.

By convention, "definable" means with parameters. In particular, all singletons, all finite sets, are definable.

Definition 4.6. The Morley rank of a definable set $X \subseteq M^k$ is either an ordinal or ∞ and is defined as follows:

- $\mathrm{MR}\, X \geq 0$ iff X is non-empty;
- $\mathrm{MR}\, X \geq \alpha+1$ iff there are infinitely many disjoint, definable subsets $Y_i \subseteq X$ with $\mathrm{MR}\, Y \geq \alpha$;
- $\mathrm{MR}\, X \geq \alpha$ for limit α iff $\mathrm{MR}\, X \geq \beta$ for all $\beta < \alpha$.

Remark 4.7. One should be cautious. This is the correct definition only if $\mathcal{M}$ is big enough (see above); in general one should compute the Morley rank in elementary extensions. If $\mathcal{M}$ is already big enough, the Morley rank does not depend on $\mathcal{M} \preceq \mathcal{N}$.

In general this has no reason to be an ordinal.

Example 4.8.

- Let $(\mathbb{K}, 0, 1, +, -, \cdot)$ be an algebraically closed field. By quantifier elimination, any definable subset of $\mathbb{K}$ is either finite or cofinite. Hence $\mathrm{MR}\, \mathbb{K} \not\geq 2$: $\mathrm{MR}\, \mathbb{K} = 1$.
- Consider $(\mathbb{R}, 0, 1, +, -, \cdot)$. Then $<$ is definable, and so is $(0, 1)$, which can be put in definable bijection with $\mathbb{R}$. Hence $\mathrm{MR}\, \mathbb{R} = \infty$ (bigger than any ordinal).

  Note. This example is not perfectly fair as $\mathbb{R}$ is not saturated enough for a honest computation. But we wanted to give the general idea.
- Let $(\mathbb{K}, 0, 1, +, -, \cdot) \models \mathrm{ACF}_q$. Let $X \subseteq \mathbb{K}^n$ be a definable subset. Then $\mathrm{MR}\, X$ is exactly the Zariski dimension of X (as an affine algebraic variety).

Lemma 4.9. *If $\mathrm{MR}\, X = \alpha < \infty$, then there is an integer k such that X contains at most k disjoint, definable subsets $Y_i \subseteq X$. Optimal such k is called the Morley degree of X.*

It sometimes happen that all definable sets have ordinal $< \infty$ rank. This is equivalent to the theory having "very few" types, in the sense of Shelah's classification theory. But there is an even better world: the case where every definable set has rank a *finite* ordinal.

Theorem 4.10 (Baldwin, 1973)**.** *If $\mathcal{M}$ is $\aleph_1$-categorical, then every definable set has finite Morley rank.*

At this point it is clear that one should focus on rank 1, degree 1 pieces.

In case the notion of Morley rank is too elaborate, here is another approach. We wish to analyze the basic blocks of ($\aleph_1$-categorical) model-theoretic nature. An attempt at defining "basic bricks" would certainly be the following: X is minimal if any two infinite definable subsets must meet. Alas one can find minimal sets which are "small" but no "essentially small", meaning that it is not a property of their theory.

Example 4.11. $(\mathbb{N}, \leq)$ is minimal, but if $(\mathcal{M}, \leq) \equiv (\mathbb{N}, \leq)$ is not isomorphic to $(\mathbb{N}, \leq)$, then $\mathcal{M}$ is not minimal.

Definition 4.12. $\mathcal{M}$ is strongly minimal if any $\mathcal{N} \equiv \mathcal{M}$ is minimal: two infinite, definable subsets of $\mathcal{N}$ must meet.

Checking it for all elementarily equivalent structures can be avoided by assuming that $\mathcal{M}$ was sufficiently "rich" (saturated) in the first place.

Example 4.13.

- An infinite set with no structure is strongly minimal.
- An infinite-dimensional $\mathbb{K}$-vector space ($\mathbb{K}$ fixed) is strongly minimal.

  Exercise: prove quantifier elimination and deduce that every definable set is either finite or cofinite.
- Let $(\mathbb{K}, 0, 1, +, -, \cdot)$ be an algebraically closed field. Then $\mathbb{K}$ is strongly minimal.

4.3 Towards geometries: Independence

All above suggests that in strongly minimal sets one should retrieve some form of geometry. Model theory handles linear and field-theoretic independence in the same way.

Remark 4.14. Here again we deal only with a special case. Model theory (under the impulse of Shelah) can study an abstract form of independence in a variety of contexts: this is called "(non-)forking" and can be generalized to very elaborate settings.

Definition 4.15. Let $A \subseteq \mathcal{M}$. Some $m \in M$ is algebraic over A if there is a formula $\varphi(x, \underline{a})$ with parameters $\underline{a}$ in A such that:

- $\mathcal{M} \models \varphi(m, \underline{a})$;
- the set $\{n \in M : \mathcal{M} \models \varphi(n, \underline{a})\}$ is finite.

Example 4.16. Let $(\mathbb{K}, 0, 1, +, -, \cdot)$ be a field. Then the model-theoretic notion coincides with the field-theoretic notion (caution: only in this language).

Definition 4.17. The algebraic closure of A, $\mathrm{acl}(A)$, is the set of elements algebraic over A.

Proposition 4.18. *Suppose $\mathcal{M}$ is a strongly minimal structure. Let $A \subseteq M$ be a set of parameters and $m, n \in M$. If $m \in \mathrm{acl}(A \cup \{n\}) \setminus \mathrm{acl}(A)$, then $n \in \mathrm{acl}(A \cup \{m\})$.*

As a consequence of this Steinitz-like exchange principle, one can reconstruct abstract notions of independence, basis, dimension.

Definition 4.19.

- A subset B is independent over A if no $b \in B$ lies in $\mathrm{acl}(A \cup B \setminus \{b\})$.
- A basis (over A) is a maximal independent subset.
- By the exchange principle, any two bases have the same cardinality.

The following result is a key step in the proof of Morley's categoricity Theorem.

Theorem 4.20. *Suppose $\mathcal{M} \equiv \mathcal{N}$ are strongly minimal and have the same dimension. Then $\mathcal{M} \simeq \mathcal{N}$.*

Here is a final word on dimension. One can extend the notion of Morley rank to tuples: $\mathrm{MR}(\underline{m}/A)$ is the least Morley rank of an A-definable set containing $\underline{m}$ (in M^k).

Proposition 4.21. *($\mathcal{M}$ always strongly minimal.)* $\mathrm{MR}(\underline{m}/A) = \dim(\mathrm{acl}(\underline{m})/A)$.

Example 4.22. There may be some risk of confusion as we are handling two different computations. So let me give an example. Let $\mathcal{M}$ be an infinite-dimensional $\mathbb{K}$-vector space, as above. Let $a, b \in \mathcal{M}$.

- First suppose that a and b are collinear: $b = \lambda_k(a)$ for some $k \in \mathbb{K}$.

  $\mathrm{MR}(a, b/\emptyset)$ is the least Morley rank of an $\emptyset$-definable subset of $\mathcal{M}^2$ containing both a and b. The formula $y = \lambda_k(x)$ does; it has Morley rank 1.

  On the other hand, $\mathrm{acl}(ab/\emptyset)$ is exactly $\mathrm{acl}(a) = \mathrm{acl}(b) = \mathbb{K}a$, which has dimension 1.

- The case where a and b are ($\mathbb{K}$-linearly) independent is an exercise.

Hence, uncountable categoricity and (combinatorial) geometries seem to be tightly linked.

4.4 Towards Geometries: Zilber's Conjecture

Question 4.23 (Zilber's trichotomy conjecture, early 80's)**.** *Let $\mathcal{M}$ be strongly minimal. Is $\mathcal{M}$ "essentially" of one of the three kinds:*

- *a set with (almost) no structure*

  technically: $\mathrm{acl}(A) = \cup_{a \in A} \mathrm{acl}(a)$

- *a vector space*

  technically: $\mathcal{M}$ is essentially an abelian group in which every definable subset is a boolean combination of cosets of definable subgroups.

- *an algebraically closed field*

  technically: "defines" and "is defined by" some $(\mathbb{K}, 0, 1, +, -, \cdot) \models \mathrm{ACF}$.

Theorem 4.24 (Hrushovski, 1993)**.** *No.*

Hrushovski's proof involved combinatorial geometry and of course a good deal of model theory. The negative answer to Zilber's original conjecture is not the end of the story for two reasons:

- Zilber's trichotomy conjecture has been proved to hold in several special contexts, in particular in so-called Zariski geometries (Hrushovski-Zilber);
- Hrushovski's method has been used to construct interesting objects.

Question 4.25. *Can one construct a field structure* $(\mathbb{K}, 0, 1, +, -, \cdot, R)$ *with finite Morley rank and R is an infinite, coinfinite subset of $\mathbb{K}$? (Of course one does not require $\mathbb{K}$ to be strongly minimal.)*

- *Extra requirement: $R \leq \mathbb{K}_+$ is an additive subgroup. Answer: yes, only in char $p > 0$ (Poizat).*
- *Extra requirement: $R \leq \mathbb{K}^\times$ is a multiplicative subgroup. Answer: yes in char 0 (Baudisch, Hils, Martin-Pizarro, Wagner), open in char p.*
- *If there is such a structure in characteristic p with R a multiplicative subgroup, then there are only finitely many p-Mersenne prime numbers (Wagner).*