A First Encounter with Representations of Finite Groups

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Contents

List of Lectures

Introduction

These notes were written for a three-week introductory course taught at the Mathematics Village in Şirince. It contains very basic material on linear representations of finite groups and deals with it in a more than conventional way. The contents of the three weeks are independent altough the degree of abstraction increases.

- The treatment of Week 1 is in part borrowed from the beginning of Serre's [3]; starting from group actions we introduce representations and prove orthogonality relations, all in matrix form. This is inelegant but hopefully helpful to the beginner not at ease with higher algebra.
- During Week 2 essentially the same material is covered in a more adequate language; it is also the opportunity to learn about tensor products. Most arguments follow the exposition given by Fulton and Harris in the first part of [2].
- Week 3 covers two topics: Frobenius' celebrated complement theorem and the Frobenius-Schur indicator. The return of geometry provides the opportunity to discuss quaternions and a little tensor algebra.

The classical book by Curtis and Reiner [1] is strongly recommended to anyone willing to start learning representation theory.

Bibliography

[1] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. AMS Chelsea Publishing, Providence, RI, 2006. Reprint of the 1962 original.

[2] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[3] Jean-Pierre Serre. *Linear representations of finite groups.* Springer-Verlag, New York, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

Week 1: Elementary Representation Theory

Lecture 1 (Group actions; Burnside's formula)

1.1 Group Actions

We first start by simply looking at group actions. Some of the fundamental ideas of representation theory are already there.

1.1.1 Actions and Equivalence

Definition 1.1.1. Let G be a group. An action of G on a set X is an operation:

$$\begin{array}{ccc} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

satisfying the relations, for all $(g, h, x) \in G^2 \times X$:

$$e_G \cdot x = x, \qquad g \cdot (h \cdot x) = (gh) \cdot x$$

One sometimes says that X is a G-set, or that (G, X) is a permutation group. Abusing terminology, when X is clear from the context, one also says that G is a permutation group. As far as notations are concerned, when the action is clear from context, one way write gx instead of $g \cdot x$.

Example 1.1.2.

- The group G acts on itself by left translation: here $X = G$ and $g \cdot x = gx$ (group multiplication).
- The group G acts on itself by conjugation: here again $X = G$, and $g \cdot x = gxg^{-1}$.
- Let $H \leq G$ be a subgroup. Then G acts on the set of (left) cosets by left translation: $X = G/H = \{xH : x \in G\}$, and $g \cdot xH = (gx)H$.
- G also acts on the set of conjugates of H, $X = \{xHx^{-1} : g \in G\}$ by $g \cdot xHx^{-1} = (gx)H(gx)^{-1}$.

Definition 1.1.3. Let G be a group acting on a set X. Let $x \in X$.

- The stabilizer of x in G is $G_x = \mathrm{Stab}_G(x) = \{g \in G : g \cdot x = x\}$.

- The orbit of x under G is $G \cdot x = O_x = \{g \cdot x : g \in G\}$.
- The quotient set $G \backslash X$ is the set of orbits.

Example 1.1.4. We go back to the examples above.

- Consider the left translation action. Then for any $x \in G$, $\mathrm{Stab}_G(x) = \{e\}$ and $G \cdot x = G$.
- Consider the conjugacy action. Then for $x \in G$, $\mathrm{Stab}_G(x) = C_G(x) = \{g \in G : gx = xg\}$ is called the *centralizer* of x, and $G \cdot x = \{gxg^{-1} : g \in G\}$ is called the *conjugacy class* of x.
- Consider the action of G on G/H. Then for $xH \in G/H$, $\mathrm{Stab}_G(xH) = xHx^{-1}$, and $G \cdot xH = G/H$.
- Consider the action of G on $X = \{gHg^{-1} : g \in G\}$ and the element $H \in X$. Then $\mathrm{Stab}_G(H) = N_G(H) = \{g \in G : gHg^{-1} = H\}$ is called the *normalizer* of H, and $G \cdot H = X$.

Before going any further, the reader should remember the following facts:

Proposition 1.1.5.

- *The stabilizer of x is a subgroup of G.*
- *There is a canonical bijection between G/G_x and $G \cdot x$.*
- *The orbits form a partition of X.*
- *The class formula: if the x_i's represent the various orbits, then:*

$$\#X = \sum_i \frac{|G|}{|G_{x_i}|}$$

where $\#X$ is the cardinal of X and $|G|$ the order of G (one may write $\#G$, but one may not write $|X|$ since it is not a group).

Remark 1.1.6. This is the proof of Lagrange's theorem: let G be a finite group and $H \leq G$ a subgroup. Consider the action of G on G/H and the element $H \in G/H$. Then the orbit is $G \cdot H = G/H$, and the stabilizer is $\mathrm{Stab}_G(H) = H$. It follows that $\#(G/H) = \frac{|G|}{|H|}$, so the order of H divides the order of G.

Definition 1.1.7 (transitive, faithful)**.**

- The action is transitive if for all $x, y \in X$, there is $g \in G$ such that $g \cdot x = y$. This is equivalent to saying that there is only one orbit.
- The action is faithful if no element $g \neq 1$ fixes all elements in X.

Let X be a set. The symmetric group on X, denoted Sym_X or S_X, is the group of all bijections from X to itself. If X is finite, then $|\mathrm{Sym}_X| = (\#X)!$.

Proposition 1.1.8. *There is a correspondence between actions of G on X and group homomorphisms $G \to \mathrm{Sym}(X)$.*

Proof.

- We start with an action $*$ of G on X. For each $g \in G$, consider the map:

$$\begin{array}{rccc} \lambda_g : & X & \to & X \\ & x & \mapsto & g * x \end{array}$$

Then for any $(g,h) \in G^2$, one has $\lambda_{gh} = \lambda_g \circ \lambda_h$ as a function on X. Indeed, let any $x \in X$. Then:

$$\lambda_{gh}(x) = (gh) * x = g * (h * x) = \lambda_g(\lambda_h(x)) = \lambda_g \circ \lambda_h(x)$$

Moreover, it is easily seen that $\lambda_e = \mathrm{Id}$ by definition of a group action. In particular, each λ_g is a bijection of X, since $\lambda_g \lambda_{g^{-1}} = \lambda_{g^{-1}} \lambda_g = \lambda_e = \mathrm{Id}$. So we can consider the map:

$$\begin{array}{rccc} \Lambda_* : & G & \to & \mathrm{Sym}(X) \\ & g & \mapsto & \lambda_g \end{array}$$

This map is well-defined, and since $\lambda_{gh} = \lambda_g \circ \lambda_h$, Λ_* is a group homomorphism. This is what we had to prove.

- For the converse, suppose that we have a group homomorphism $\Lambda : G \to \mathrm{Sym}(X)$. Define the following:

$$g *_\Lambda x = (\Lambda(g))(x)$$

This defines an action of G on X. Indeed, since Λ is a group homomorphism, one has $\Lambda(e_G) = e_{\mathrm{Sym}_X} = \mathrm{Id}_X$, so $e *_\Lambda x = \mathrm{Id}_X(x) = x$ for all $x \in X$. Moreover, if $(g,h,x) \in G^2 \times X$, then using the fact that Λ is a morphism:

$$g *_\Lambda (h *_\Lambda x) = (\Lambda(g))(\Lambda(h)(x)) = (\Lambda(g) \circ \Lambda(h))(x) = \Lambda(gh)(x) = (gh) *_\Lambda x$$

So we have a group action $*_\Lambda$ associated to the morphism Λ.

- One needs to check that the two operations thus defined:

$$\{\text{group actions on } X\} \leftrightarrow \{\text{morphisms from } G \text{ to } \mathrm{Sym}(X)\}$$

are the converses of each other. It is an excellent exercise in order to check that you have understood the proof so far. □

Remark 1.1.9. The correspondence explains why when G acts faithfully on a set X, (G, X) is called a permutation group: we are now viewing G as a subgroup of Sym_X.

Remark 1.1.10. Using this correspondence, the action $\cdot$ of G on X is faithful iff the associated morphism $\Lambda_. : G \to \mathrm{Sym}(X)$ is injective.

As an application, we prove that every group naturally embeds into some symmetric group.

Theorem 1.1.11 (Cayley's Theorem)**.** *Let G be a group. Then G embeds into S_G, the symmetric group on (the underlying set of) G.*

Proof. To each $g \in G$, associate the map $\lambda_g : x \mapsto g \cdot x$ (left-multiplication by g). λ_g is a bijection of G. Then consider the map:

$$\begin{array}{rccc} \Lambda : & G & \to & S_G \\ & g & \mapsto & \lambda_g \end{array}$$

Λ is a group homomorphism as we know. It suffices to check that Λ is injective. Suppose indeed that $\Lambda(g) = \mathrm{Id}_G$. Then for any $h \in G$, one has $h = \mathrm{Id}_G(h) = (\Lambda(g))(h) = \lambda_g(h) = gh$, so $g = e_G$. So $\ker \Lambda = \{e_G\}$: the group homomorphism Λ is a group embedding. $\square$

One also needs to know when two actions are "the same".

Definition 1.1.12. Let G acts on X and Y. The actions are equivalent if there is a bijection $f : X \to Y$ which is covariant, in the sense that for all $(g, x) \in G \times X$:

$$f(\underbrace{g \cdot x}_{\text{action on } X}) = \underbrace{g \cdot f(x)}_{\text{action on } Y}$$

It should be clear that this notion is an equivalence relation on G-sets.

Lemma 1.1.13. *Let (G, X) be a transitive permutation group and $x \in X$. Then the action of G on X is equivalent to the action of G on the coset space G/G_x.*

Proof. This is exactly Proposition 1.1.5 1.1.5: to a coset aG_x (for $a \in G$), we assign the element $a \cdot x \in X$:

$$\begin{array}{rccc} f : & G/G_x & \to & X \\ & aG_x & \mapsto & a \cdot x \end{array}$$

This is well-defined, since if the coset $aG_x = bG_x$ is represented by two elements $a, b \in G$, then one has $b^{-1}a \in G_x$, so $b^{-1}a \cdot x = x$ and $b \cdot x = a \cdot x$.

We now prove that f is an equivalence of G-sets.

- f is injective: for if $f(aG_x) = f(bG_x$, then $a \cdot x = b \cdot x$, that is $b^{-1}a \in G_x$, and $aG_x = bG_x$.
- f is surjective: this is where we use transitivity. Let $y \in X$. By transitivity, there is $a \in G$ such that $a \cdot x = y$. Then $f(aG_x) = y$.
- f is covariant: if $(g, a) \in G^2$, then $f(g \cdot aG_x) = f(gaG_x) = ga \cdot x = g \cdot (a \cdot x) = g \cdot f(aG_x)$.

So f is an equivalence of G-sets from G/G_x to X. $\square$

As a good exercise, the reader may check the following fact:

Proposition 1.1.14. *Let G be a group and $H, K \leq G$ be two subgroups. Then G/H and G/K are equivalent iff H and K are conjugate inside G.*

1.1.2 A character-like formula

Definition 1.1.15. Let G be a finite group acting on a finite set X. For $g \in G$, let $\chi_X(g) = \#\{x \in X : g \cdot x = x\}$.

χ_X is sometimes called the character of the action, for reasons which will later become obvious.

Theorem 1.1.16. *The number of orbits of the action of the finite group G on the finite set X is:*

$$\frac{1}{|G|} \sum_{g \in G} \chi_X(g)$$

Proof. Let $S = \{(g, x) \in G \times X : g \cdot x = x\}$. We count S in two different ways:

$$\begin{aligned}
\#S &= \sum_{g \in G} \#\{x \in X : g \cdot x = x\} = \sum_{g \in G} \chi_X(g) \\
&= \sum_{x \in X} \#\{g \in G : g \cdot x = x\} = \sum_{x \in X} |\mathrm{Stab}_G(x)| = \sum_{x \in X} \frac{|G|}{\#(G \cdot x)} \\
&= \sum_{O \in G \backslash X} \#O \frac{|G|}{\#O} = \#(G \backslash X)|G|
\end{aligned}$$

which gives the desired formula. □

Corollary 1.1.17. *Let (G, X) be a finite transitive group with $\#X > 1$. Then there is $g \in G$ fixing no element in X.*

Proof. We want to show that there is $g \in G$ with $\chi_X(g) = 0$. Otherwise, $\chi_X(g) \geq 1$ for all g ; in addition, $\chi_X(1) = \#X$, so by Theorem 1.1.16:

$$1 = \frac{1}{|G|} \sum_{g \in G} \chi_X(g) = \frac{\#X}{|G|} + \frac{1}{|G|} \sum_{g \in G \backslash \{1\}} \chi_X(g) > 1$$

which is a contradiction. □

Corollary 1.1.18. *Let G be a finite group and $H < G$ a proper subgroup. Then the conjugates of H do not cover G.*

Proof. We let G act on the set G/H of left translates of H, that is $X = \{gH : g \in G\}$. The action is obviously transitive. Now suppose $G = \cup_{\gamma \in G} \gamma H \gamma^{-1}$. Then any element $g \in G$ will lie in some conjugate $\gamma H \gamma^{-1}$. In particular, $g \cdot (\gamma H) = \gamma H$, so every element fixes some element in X. This contradicts Corollary 1.1.17. So G cannot equal the union $\cup_{\gamma \in G} \gamma H \gamma^{-1}$. □

This is interesting: we started with an abstract group G, considered some group action of G, introduced a suitable counting function χ_X, and deduced something *about the abstract group G*. This is one of the aspects of the representation theory of finite groups: a new tool to study abstract groups.

During Week 3, we shall prove the following impressive result by Frobenius, Corollary 3.1.21:

> Let G be a finite group acting transitively on a set X. Suppose that every $g \neq e$ fixes at most one element of X. Then $N = \{\text{fixed-point free } g \in G\} \cup \{\{e\}$ is a normal subgroup of G.

This will require the full strength of representation theory and is out of reach for the moment.

End of Lecture 1.

Lecture 2 (n-transitivity; Permutation representations)

1.1.3 An application to higher transitivity

Notation 1.1.19. Let X be a set and $k > 0$ be an integer. Denote by $X^{[k]}$ the set of k-uples of distinct elements of X.

For example, $X^{[2]}$ is $X^2 \backslash \{(x, x) : x \in X\}$ ("the plane without the diagonal").

Remark 1.1.20. Suppose $\#X = n$. Then $\#X^{[k]} = n(n-1)\dots(n-k+1)$.

Observe that G acts naturally on X^k by $g \cdot (x_1, \dots, x_k) = (g \cdot x_1, \dots, g \cdot x_k)$. Now if the x_i's are pairwise distinct, then the $g \cdot x_i$'s are as well. This means that G naturally acts on $X^{[k]}$.

Definition 1.1.21. G acts k-transitively on X if G is transitive on $X^{[k]}$.

Unfolding the definition, this is equivalent to saying that if $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ are tuples of ditstinct elements of X, there is a common $g \in G$ such that $g \cdot x_i = y_i$ for all $i = 1 \dots k$.

Example 1.1.22.

- It is clear that S_n is n-transitive on $X = \{1, \dots, n\}$: if $\{k_1, \dots, k_n\}$ is another enumeration of X, then there is a (unique) bijection of X sending each i to k_i.
- On the other hand A_n is not n-transitive on X, since the bijection obtained in the case of S_n could have odd signature. Prove that A_n is nonetheless $(n-1)$-transitive on X.

Remark 1.1.23. If G is k-transitive, then $|G| = n(n-1)\dots(n-k+1)|G_{x_1,\dots,x_k}|$.

Proposition 1.1.24. *G is k-transitive iff*

$$\frac{1}{|G|} \sum_g \chi_X(g)(\chi_X(g) - 1)\dots(\chi_X(g) - k + 1) = 1$$

Proof. We shall simply compute the χ-function for the action of G on $X^{[k]}$. Now $(x_1, \dots, x_k) \in X^{[k]}$ is a fixed point of G iff every x_i is a fixed point iff $(x_1, \dots, x_k) \in (\text{Fix}g)^{[k]}$. Since $\text{Fix}g$ has $\chi_X(g)$ elements, we deduce that $\chi_{X^{[k]}}(g) = \#(\text{Fix}g)^{[k]} = \chi_X(g)(\chi_X(g) - 1)\dots(\chi_X(g) - k + 1)$.

It is then a straightforward application of Theorem 1.1.16. □

Corollary 1.1.25. *Suppose that G is transitive on X. Then G is 2-transitive iff* $\frac{1}{|G|} \sum_g \chi_X^2(g) = 2$.

Proof. We know that G is transitive, so that $\frac{1}{|G|}\sum_g \chi(g) = 1$. It follows that:

$$\frac{1}{|G|}\sum_g \chi_X(g)(\chi_X(g)-1) = \frac{1}{|G|}\sum_g \chi^2(g) - \frac{1}{|G|}\sum_g \chi(g) = \frac{1}{|G|}\sum_g \chi^2(g) - 1$$

is equal to 1 iff $\frac{1}{|G|}\sum_g \chi^2(g) = 2$. □

1.2 Representations and Subrepresentations

In order to motivate the definition of a representation, let us try and generalize what we have learnt so far.

- Group actions are useful.
- Given a group action, one can associate to it a magical function which seems to encode many properties.
- Group actions remain a bit difficult, because one easily falls into combinatorics.

The plan is simple: we replace group actions by something easier to study, and we find a magical function. What is easier than combinatorics? Well, linear algebra is a lot easier. Let us see how permutation groups can be encoded in matrix form.

1.2.1 Permutation representation

Definition 1.2.1. A permutation matrix is a matrix which has only one 1 in each row and column, and only zeros elsewhere.

Lemma 1.2.2. *The set of permutation matrices of size n is a group naturally isomorphic to S_n.*

Proof. To a permutation σ, associate the matrix :

$$M_\sigma = (\delta_{i,\sigma(j)})_{i,j}$$

Then M_σ is a permutation matrix. Moreover, any permutation matrix is of the form M_σ for some σ. And finally:

$$(M_\sigma M_\tau)_{i,j} = \sum_k \delta_{i,\sigma(k)}\delta_{k,\tau(j)} = \delta_{i,\sigma(\tau(j))} = (M_{\sigma\tau})_{i,j}$$

so we have a group homomorphism, which is clearly injective. □

Let e_i be the i^{th} column vector. Then for our embedding, we have:

$$M_\sigma \cdot e_i = e_{\sigma(i)}$$

What we have proved is simply that S_n embeds canonically into $\mathrm{GL}_n(\mathbb{Z})$ (and in particular into $\mathrm{GL}_n(\mathbb{C})$). To make sure that you have understood, write the permutation matrix M_γ associated to the permutation $\gamma = (123) \in S_3$, and see how it "acts" on an arbitrary matrix N: compute $M_\gamma N$, NM_γ, and $M_\gamma N M_\gamma^{-1}$.

We now extend group actions.

Definition 1.2.3 (permutation representation)**.** Suppose G acts on a finite set X. Consider the vector space $\mathbb{C}^X$ with basis $\{x_1, \ldots, x_n\} = X$. Then G acts on $\mathbb{C}^X$ by linearly extending the action defined on the basis. The action of G on $\mathbb{C}^X$ is called the permutation representation associated to the action of G on X.

Let us sum things up. If $G \to \mathrm{Sym}(X)$ is an action on X (see Proposition 1.1.8), then using permutation matrices and the embedding $\mathrm{Sym}(X) \hookrightarrow \mathrm{GL}_{|X|}(\mathbb{C})$, we find a group homomorphism: $\rho : G \to \mathrm{GL}_{|X|}(\mathbb{C})$. This means that G acts *linearly* on the $\mathbb{C}$-vector space $\mathbb{C}^X$:

$$\rho(g)(\lambda v_1 + v_2) = \lambda \rho(g)(v_1) + \rho(g)(v_2)$$

In matrix form: to each $g \in G$, we associate a (permutation) matrix M^g which "codes" the action of g, that is, for all $x \in X$:

$$M^g \cdot e_x = e_{g \cdot x}$$

Remark 1.2.4. We write M^g instead of M_g because we are going to work with coordinates $M = (m^g_{i,j})_{i,j}$. There is no risk of confusion with power or conjugacy.

Recall that the *trace* of an $n \times n$ matrix M is the sum:

$$\mathrm{Tr} M = \sum_{i=1}^{n} m_{i,i}$$

However easily defined it is, this number is extremely important. Remember (and prove again) that if M and N are two square matrices, then $\mathrm{Tr}(MN) = \mathrm{Tr}(NM)$. In particular, Tr is invariant under conjugation: if P is an invertible matrix, then $\mathrm{Tr}(PMP^{-1}) = \mathrm{Tr} M$. There is an explanation in terms of eigenvalues, to which we'll come later.

Lemma 1.2.5. *Let (G, X) be a permutation group. Consider the function χ_X from Definition 1.1.15, and the permutation matrices associated to the action. Then $\chi_X(g) = \mathrm{Tr}(M^g)$.*

Proof. Indeed, $x \in X$ is a fixed point iff $M^g \cdot e_x = e_x$ iff M^g has a 1 in cell number (x, x). So $\chi_X(g)$ =the number of fixed points of g, is exactly the number of 1's on the diagonal of M^g. Since M^g is a permutation matrix, if 1 is not in a cell, it is 0. So $\mathrm{Tr} M^g$ is the number of 1's on the diagonal, and $\mathrm{Tr} M^g = \chi_X(g)$. □

We shall now generalize the ideas here by considering abstract morphisms $\rho : G \to \mathrm{GL}_n(\mathbb{C})$, and computing their traces. This will prove surprisingly powerful.

1.2.2 Definitions

Given a (finite) group G, we are tempted to view G as a subgroup of $\mathrm{GL}(V)$. This is not always possible when V is fixed in advance: imagine $V \simeq \mathbb{C}$, and $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$. So we should relax our restriction of an embedding $G \hookrightarrow \mathrm{GL}(V)$, into a (group homo-)morphism. Embeddings will be a special case.

Definition 1.2.6. A representation of a group G in a vector space V is a morphism $\rho : G \to \mathrm{GL}(V)$.

Hence, if ρ is a representation of G in V, then G acts linearly on V, i.e. for all g we have a linear map $\rho(g) : V \to V$ and for every $g, h \in G$, one has:

$$\underbrace{\rho(g) \circ \rho(h)}_{\text{composition in } \mathrm{GL}(V)} = \rho(\underbrace{g \cdot h}_{\text{group law}})$$

If you wish to think matricially, fix a basis of $V \simeq \mathbb{C}^n$. Then to each $g \in G$ we associate a matrix $M_g \in \mathcal{M}_n(V)$ in such a way that:

$$\underbrace{M_g M_h}_{\text{matrix multiplication}} = M_{gh}$$

Beware of matrices; unnecessary computations tend to hide the ideas.

Definition 1.2.7. A (finite-dimensional, complex) representation (V, ρ) of a group G consists of a (finite-dimensional, complex) vector space V and a representation $\rho : G \to \mathrm{GL}(V)$.

Be careful that $\rho(g)$ does not denote a vector of V, but a linear bijection from V to itself! So we shall write things like $\rho(g)(v)$.

Notation 1.2.8. As said in Definition 1.2.7, a representation should be denoted by (V, ρ); in practice it is often denoted simply V, or simply ρ.

Let us insist that all representations will be complex and finite-dimensional.

Example 1.2.9. Any permutation representation in the sense of Definition 1.2.3 is a representation.

End of Lecture 2.

Lecture 3 (Examples; Maschke's Theorem)

Definition 1.2.10. Let G be a group. The trivial representation of G is:

$$\begin{array}{rccl} \mathrm{triv} : & G & \to & \mathrm{GL}_1(\mathbb{C}) \\ & g & \mapsto & 1 \end{array}$$

It has dimension 1.

Example 1.2.11. Let $G = S_3$, the symmetric group on $\{1, 2, 3\}$.

1. The trivial representation is:

   $$\begin{array}{rccl} \mathrm{triv} : & G & \to & \mathrm{GL}_1(\mathbb{C}) \simeq \mathbb{C}^\times \\ & g & \mapsto & 1 \end{array}$$

   hence if $g \in G$ and $z \in \mathbb{C}$, one has:

   $$[\mathrm{triv}(g)](z) = z$$

   or abusing notation, as suggested in Notation 2.1.5:

   $$g \cdot z = z$$

   I agree this is not very interesting.

2. The alternating representation is:

$$\begin{array}{rccl} \varepsilon: & G & \to & \mathrm{GL}_1(C) \simeq \mathbb{C}^\times \\ & g & \mapsto & \varepsilon(g) \end{array}$$

   where $\varepsilon(g)$ is the signature of g. Hence,

$$g \cdot z = \varepsilon(g) z$$

   Observe that in cases (1) and (2) the underlying vector space was the same: simply $\mathbb{C}$. But it's not the same representation!

   So far not much is happening; we have simply given morphisms $G \to (\{\pm 1\}, \times)$. You will now see a difference.

3. Let $\tau = (12)$ and $\gamma = (123)$. You know that τ and γ generate $G = S_3$; for instance, $(23) = \gamma\tau\gamma^{-1}$. One has:

$$S_3 = \{\mathrm{Id}, \tau, \underbrace{\gamma\tau\gamma^{-1}}_{(23)}, \underbrace{\gamma^{-1}\tau\gamma}_{(13)}, \gamma, \underbrace{\gamma^2}_{(321)}\}$$

   Let $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and in this basis, let $\rho(\tau)$ have matrix:

$$M_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   and $\rho(\gamma)$ have matrix:

$$M_\gamma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

   I say that this defines a representation. Indeed, one easily sees that $M_\tau^2 = M_\gamma^3 = \mathrm{Id}$ and $M_\tau M_\gamma M_\tau = M_\gamma^{-1}$, so ρ extends to all of S_3.

   Work out the details if you are not convinced. I hope you are now wondering how I built this one, which is *not* a permutation representation.

1.2.3 Covariance and equivalence of representations

Similarly to the case of group actions (Definition 1.1.12, there is a notion of equivalence relying on the notion of a covariant map. Be careful that we are doing linear algebra now, so all the maps we consider will be $\mathbb{C}$-linear maps!

Definition 1.2.12. Let (V_1, ρ_1) and (V_2, ρ_2) be two representations of a group G. A map $f : V_1 \to V_2$ is G-covariant if for every $g \in G, v_1 \in V_1$, one has $f([\rho_1(g)](v_1)) = [\rho_2(g)](f(v_1))$

Remark 1.2.13. f is G-covariant if and only if for all $g \in G$, $f \circ \rho_1(g) = \rho_2(g) \circ f$.

We may now define what it means for two representations to be the same.

Definition 1.2.14. Two representations (V_1, ρ_1) and (V_2, ρ_2) of G are equivalent if there is a G-covariant $\mathbb{C}$-linear isomorphism $f : V_1 \to V_2$.

Remark 1.2.15. You also require f to be linear! of course you want f to be an isomorphism of vector spaces in the first place!

1.3 Reducibility

1.3.1 Irreducible representations

The following should not be confused with "covariant". Covariance is for maps, invariance is for subspaces.

Definition 1.3.1. Let (V, ρ) be a representation of G. Let $W \leq V$ be a vector subspace. One says that W is G-invariant if for all $g \in G$, and all $w \in W$, $\rho(g)(w) \in W$.

This means that $\rho(g)$ maps W to W, or in other words, that $\rho(g)$ restricts to an automorphism of W. (In particular, $(W, \rho_{|W})$ may be viewed as a representation of G.)

Let us consider the permutation representation of $G = S_3$: $\rho : S_3 \to \mathrm{GL}_3(\mathbb{C})$, where $g \cdot e_k = e_{g\cdot k}$ ($k \in \{1, 2, 3\}$). Observe that the line generated by $(e_1 + e_2 + e_3)$ is fixed by G. In this sense, $\mathbb{C}^3$ contains a smaller representation: it is not minimal. In representation theory, the usual word (instead of minimal) is "irreducible".

Definition 1.3.2. (V, ρ) is an irreducible representation of G if the only two G-invariant subspaces are $\{0\}$ and V.

Example 1.3.3.

- The trivial representation, any representation of dimension 1, is irreducible.
- The dimension two representation given in Example 1.2.11 (3) is irreducible.

1.3.2 Complete reducibility

Theorem 1.3.4 (complete reducibility)**.** *Let G be a finite group and (V, ρ) be a complex, finite-dimensional representation. Then V is a direct sum of irreducible representations.*

Theorem 1.3.5 (Maschke's Theorem)**.** *Let G be a finite group and (V, ρ) be a complex, finite-dimensional representation. Let $W \leq V$ be a G-invariant subspace. Then W has a G-invariant complement.*

Proof. Let π be any projector with image W (this amounts to choosing a complement to W). Let:

$$\hat{\pi} = \frac{1}{|G|} \sum_{g\in G} \rho(g) \circ \pi \circ \rho(g^{-1})$$

$\hat{\pi}$ is clearly $\mathbb{C}$-linear. By construction, $\mathrm{im}\hat{\pi} \leq W$. Let $K = \ker \hat{\pi}$.

If $w \in W$, then by G-invariance of W, for any $g \in G$, $\rho(g^{-1})(w) \in W$ and $\pi(\rho(g^{-1})w) = \rho(g^{-1})(w)$, so that $(\rho(g) \circ \pi \circ \rho(g^{-1}))(w) = w$. In short, for any $w \in W$, $\hat{\pi}(w) = w$. In particular, $\mathrm{im}\hat{\pi} = W$ and $W \cap K = 0$. Moreover, if $v \in V$, then $v = \hat{\pi}(v) + (v - \hat{\pi}(v))$, and since $\mathrm{im}\hat{\pi} = W$, one finds:

$$\hat{\pi}(v - \hat{\pi}(v)) = \hat{\pi}(v) - \hat{\pi}\hat{\pi}(v) = \hat{\pi}(v) - \hat{\pi}(v) = 0$$

whence $v - \hat{\pi}(v) \in K$. This proves that $V = W \oplus K$. We shall show that K is G-invariant.

Fix $h \in G$. Then:

$$\begin{aligned}
\hat{\pi} \circ \rho(h) &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g^{-1}) \circ \rho(h) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho(h) \circ \rho(h^{-1}g) \circ \pi \circ \rho(g^{-1}h) \\
&= \rho(h) \frac{1}{|G|} \sum_{g \in G} \circ \rho(g) \circ \pi \circ \rho(g^{-1}) \\
&= \rho(h) \circ \hat{\pi}
\end{aligned}$$

so $\hat{\pi}$ is G-covariant. In particular, $K = \ker \hat{\pi}$ is G-invariant. □

Example 1.3.6. Consider the permutation representation of S_3 and consider the G-invariant line $L = \langle e_1 + e_2 + e_3 \rangle$. We apply the idea contained in the proof of Maschke's Theorem to explicitely determine a G-invariant complement. A (non G-invariant!) complement is $P = \langle e_1, e_2 \rangle$; the projector with image L and kernel P has matrix:

$$\pi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

So averaging, one finds:

$$\begin{aligned}
\hat{\pi} &= \frac{1}{6} \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right) \\
&= \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\end{aligned}$$

Now the kernel of $\hat{\pi}$ is spanned by $f_1 = 2e_1 - e_2 - e_3$ and $f_2 = -e_1 + 2e_2 - e_3$, and $\langle f_1, f_2 \rangle$ is S_3-invariant as one can check.

There is more. Let us compute the matrices of τ and γ in the basis (f_1, f_2). One finds:

$$\mathrm{Mat}_{(f_1,f_2)}\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{Mat}_{(f_1,f_2)}\gamma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

which is exactly Example 1.2.11 (3)!

(This is however *not* how I had constructed Example 1.2.11 (3); my construction relied on a "quotient vector space" from linear algebra.)

Proof of Theorem 1.3.4. We prove it by induction on $\dim V$. If V is irreducible, we are done. If V is not irreducible, then there is a subrepresentation $W \neq 0, V$. By Maschke's Theorem, Theorem 1.3.5, there is a G-invariant complement $K \leq V$ with $V = W \oplus K$. By induction, both W and K are direct sums of irreducible representations, so V is one as well. □

END OF LECTURE 3.

LECTURE 4 (SCHUR'S LEMMA; CHARACTERS)

Let us go back to complete reducibility, Theorem 1.3.4. Given a representation V, we may write V as a direct sum of irreducible representations.

Notation 1.3.7. Let $\{(V_i, \rho_i) : i \in I\}$ be the set of irreducible representations of G (up to equivalence of representations).

With this Notation, this means that there are integers $n_i \in \mathbb{N}$ such that:

$$V = \oplus_{i \in I} V_i^{n_i}$$

Two questions arise: determine the V_i's, and determine the n_i's! This will be possible thanks to *Character Theory*, §1.4.

1.3.3 Schur's Lemma

One last surprisingly powerful tool; talking about irreducible representations, is Schur's Lemma.

In order to prove it, one should remember what an eigenvalue is.

Definition 1.3.8. Let V be a $\mathbb{C}$-vector space and $f : V \to V$ a $\mathbb{C}$-linear map. An eigenvalue of f is a $\lambda \in \mathbb{C}$ such that there is $x \neq 0$ in V satisfying $f(x) = \lambda x$. Such an x is then called an eigenvector of f.

- In matrix terms, if $M = \mathrm{Mat}_{\mathcal{B}} f$ and X is the column of coordinates of x expressed in $\mathcal{B}$, one has $MX = \lambda X$.
- In more algebraic terms, λ is an eigenvalue of f iff $\ker(f - \lambda \mathrm{Id}) \neq 0$.

The reader should also remember the following important fact from linear algebra: if V is a finite dimensional, $\mathbb{C}$-vector space and $f : V \to V$ is a $\mathbb{C}$-linear map, then f has (at least) an eigenvalue.

Theorem 1.3.9 (Schur's Lemma)**.**

- *Let V_1 and V_2 be two irreducible representations. If V_1 and V_2 are not equivalent, then there is no non-zero G-covariant linear map $V_1 \to V_2$.*
- *Let V be irreducible and $u : V \to V$ be a G-covariant linear map. Then there is $\lambda \in \mathbb{C}$ such that $u = \lambda \mathrm{Id}$.*

Proof.

- Let $f : V_1 \to V_2$ be a G-covariant linear map. Then $\ker f$ is G-invariant, so by irreducibility of V_1, one has either $\ker = 0$ or $\ker V = V_1$. If $\ker f = V_1$ then $f = 0$, as desired. Otherwise f is injective. Now $\mathrm{im} f$ is G-invariant as well. If $\mathrm{im} f = 0$ then $f = 0$, as desired. Otherwise f is surjective, and V_1 and V_2 are equivalent.
- Let $f : V \to V$ be a G-covariant linear map. Since V is a finite-dimensional complex vector space, u has an eigenvalue λ. Hence $f - \lambda \mathrm{Id}$ is a non-injective, G-covariant linear map. So $\ker(f - \lambda \mathrm{Id})$ is a non-trivial, G-invariant subspace of V. By irreducibility of V, $f - \lambda \mathrm{Id} = 0$, as desired. □

Corollary 1.3.10. *Let V be a representation and $V = \oplus_{i\in I} V_i^{n_i}$ a decomposition into irreducible representations. Then the n_i's are unique.*

Proof. Let $V = \oplus_{i\in I} W_i^{m_i}$ be another decomposition. Consider the identity map $\mathrm{Id} : V \to V$. This is a $\mathbb{C}$-linear, G-covariant map. Restrict it to a term V_i: by Schur's Lemma, V_i is mapped to an equivalent representation.

Hence the image of each copy of V_i must be another copy of V_i. This proves that $n_i = m_i$. □

1.4 Characters

1.4.1 Definition

Remember how we obtained information on a group action (G, X) from the magical function χ_X, Theorem 1.1.16. Remember that we decided to view group actions as so-called "permutation representations", Definition 1.2.3, and that we had $\chi_X(g) = \mathrm{Tr} M^g$ (Lemma 1.2.5).

It is now time to generalize all this.

Definition 1.4.1. The character of a representation (V, ρ) of G is the function:

$$\begin{array}{rccc} \chi_\rho : & G & \to & \mathbb{C} \\ & g & \mapsto & \mathrm{Tr}\rho(g) \end{array}$$

This apparently useless definition will provide a most impressive theory. Let us first compute some examples.

Example 1.4.2. Go back to Example 1.2.11. Let ρ be the representation of 1.2.11 (3). All computations made, you should find:

χ	e	(12)	(13)	(23)	(123)	(132)
χ_ρ	2	0	0	0	-1	-1

Example 1.4.3. Let (G, X) be a permutation group, and consider the permutation representation ρ_{perm} with character χ_{perm}. Then $\chi_{\mathrm{perm}} = \chi_X$.

Lemma 1.4.4. *Two equivalent representations have the same character.*

Proof. Let (V_1, ρ_1) and (V_2, ρ_2) be equivalent; this means that there is a G-covariant linear bijection $f : V_1 \simeq V_2$

Let us work in matrix form: fix bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of V_1 and V_2, respectively. Let $M^g = \mathrm{Mat}_{\mathcal{B}_1}\rho_1(g)$ and $N^g = \mathrm{Mat}_{\mathcal{B}_2}\rho_2(g)$. Let $P = \mathrm{Mat}_{\mathcal{B}_1,\mathcal{B}_2} f$; since f is a bijection, P is an invertible matrix.

Now f is covariant, so this means that for all $g \in G$, one has:

$$f \circ \rho_1(g) = \rho_2(g) \circ f$$

In matrix form, $PM^g = N^g P$. In particular, $M^g = P^{-1} N^g P$, so taking traces:

$$\chi_1(g) = \mathrm{Tr}(M^g) = \mathrm{Tr}(P^{-1} N^g P) = \mathrm{Tr}(N^g) = \chi_2(g)$$

□

Question 1.4.5. *Will the converse hold?*

End of Lecture 4.

Lecture 5 (The orthogonality theorem)

1.4.2 The orthogonality theorem

Notation 1.4.6. Let $\{\chi_i : i \in I\}$ be the set of irreducible characters of G.

We shall equip the family of characters (and similar functions) with a complex scalar product. Before we give the definition, let us compute something involving complex conjugation.

Lemma 1.4.7. $\overline{\chi(g)} = \chi(g^{-1})$.

Proof. In terms of linear algebra, this is our hardest proof. Remember that working in matrix form, one has $\chi(g) = \mathrm{Tr}(M^g)$ (the matrix standing for $\rho(g)$). Bear in mind that the matrix M^g has inverse the matrix $M^{g^{-1}}$, because:

$$M^g M^{g^{-1}} = M^{g^{-1}} M^g = M^{e_G} = I_n$$

We now say that the matrix M^g is diagonalizable, meaning that there is an invertible matrix $P \in \mathrm{GL}_n(\mathbb{C})$ and complex numbers $\lambda_1, \dots, \lambda_n$ such that:

$$PM^gP^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

If you do not see at all why, then it is as easy to just admit it. If you already know about the minimal polynomial of a matrix, then observe how G being finite, there is an integer k such that $(M^g)^k = I_n$. In particular, the polynomial $X^k - 1$ annihilates M^g. Since $X^k - 1$ is a split polynomial with simple roots (over $\mathbb{C}$), M^g is diagonalizable.

So we start with the formula:

$$PM^gP^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Observe that in particular:

$$P(M^g)^{-1}P^{-1} = (PM^gP^{-1})^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^{-1} = \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix}$$

Since there is k such that $(M^g)^k = I_n$, one should also have:

$$\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^k = (PM^gP^{-1})^k = P(M^g)^kP^{-1} = I_n$$

so that $\lambda_i^k = 1$ for every $i = 1 \dots n$. In particular, the λ_i's are roots of unity in $\mathbb{C}$, and $\overline{\lambda_i} = \frac{1}{\lambda_i}$.

How does this relate to characters? Well, we have:

$$\chi(g) = \mathrm{Tr}(M^g) = \mathrm{Tr}(PM^gP^{-1}) = \mathrm{Tr}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \sum_{i=1}^{n} \lambda_i$$

and similarly:

$$\chi(g^{-1}) = \sum_{i=1}^{n} \lambda_i^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\sum_{i=1}^{n} \lambda_i} = \overline{\chi(g)}$$

□

Example 1.4.8. As an example, consider the group $\mathbb{Z}/4\mathbb{Z} = \langle 1, a, a^2, a^3 \rangle$. One of its representations is by letting a act on $\mathbb{C}$ as multiplication by i. So a^2 acts as -1 and a^3 as $-i$. Observe that since the representation is 1-dimensional (over $\mathbb{C}$), the character satisfies therefore $\chi(a) = i$, $\chi(a^{-1}) = \chi(a^3) = -i = \bar{i}$.

Definition 1.4.9. Let V be a finite-dimensional, complex vector space. A hermitian scalar product on V is a map $(.,.) : V \times V \to \mathbb{C}$ which is:

- sesquilinear: $(\lambda x + y, z) = \overline{\lambda}(x, z) + (y, z)$
- Hermite-symmetric: $(x, y) = \overline{(y, x)}$
- positive: $(x, x) \in \mathbb{R}_+$
- definite: $(x, x) = 0$ iff $x = 0$.

We need a vector space as well. Consider the set:

$$\mathcal{F} = \{\text{functions } G \to \mathbb{C} \text{ which are constant on conjugacy classes}\}$$

You may check that this is a complex vector space of finite dimension. Observe that all characters are in $\mathcal{F}$. Then on $\mathcal{F}$ we define:

$$\langle \alpha | \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$$

Lemma 1.4.10. *This defines a hermitian scalar product on $\mathcal{F}$.*

Proof. There are four things to prove.

- sesquilinearity:

$$\begin{aligned} \langle \lambda\alpha + \beta, \gamma \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{\lambda\alpha(g) + \beta(g)} \gamma(g) \\ &= \overline{\lambda} \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \gamma(g) + \frac{1}{|G|} \sum_{g \in G} \overline{\beta(g)} \gamma(g) \\ &= \overline{\lambda} \langle \alpha | \gamma \rangle + \langle \beta | \gamma \rangle \end{aligned}$$

- Hermite symmetry:

$$\begin{aligned} \langle \beta | \alpha \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{\beta(g)} \alpha(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\beta(g) \overline{\alpha(g)}} \\ &= \overline{\frac{1}{|G|} \sum_{g \in G} \alpha(g) \overline{\beta(g)}} \\ &= \overline{\langle \alpha | \beta \rangle} \end{aligned}$$

- Positivity:
$$\langle\alpha|\alpha\rangle = \frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)}\alpha(g) = \frac{1}{|G|}\sum_{g\in G}|\alpha(g)|^2 \in \mathbb{R}_+$$

- Definiteness:
$$\langle\alpha|\alpha\rangle = 0 \iff \frac{1}{|G|}\sum_{g\in G}|\alpha(g)|^2 = 0 \iff \forall g\in G, \alpha(g) = 0 \iff \alpha = 0$$

□

Theorem 1.4.11. *Irreducible characters form an orthonormal family, that is* $\langle\chi_i|\chi_j\rangle = \delta_{i,j}$ *for all irreducible characters* χ_i, χ_j.

Corollary 1.4.12. *Two representations are equivalent iff they have the same character.*

Proof. One implication is known. So suppose (W_1, ρ_1) and (W_2, ρ_2) are two representations with common character $\psi_1 = \psi_2$. We wish to prove that ρ_1 and ρ_2 are equivalent representations.

By complete reducibility, any representation is a direct sum of irreducible representations. So W_1 is of the form $\oplus_i V_i^{m_i}$ for some integers m_i; similarly, W_2 is of the form $\oplus V_i^{n_i}$ for other integers n_i. All we need to do is prove that $n_i = m_i$. But since $\psi_1 = \sum m_i\chi_i$, one has:
$$m_i = \langle\psi_1, \chi_i\rangle = \langle\psi_2, \chi_i\rangle = n_i$$
So the representations are equivalent. □

1.4.3 Proof of the orthogonality theorem

Proof. Let (V_1, ρ_1) and (V_2, ρ_2) be two representations; let χ_1 and χ_2 be their characters.

For any $\mathbb{C}$-linear map $f : V_1 \to V_2$, consider the map:
$$\tilde{f} = \frac{1}{|G|}\sum_{g\in G}\rho_2(g)\circ f\circ\rho_1(g^{-1})$$
which is again a $\mathbb{C}$-linear map from V_1 to V_2. We say that $\tilde{f}$ is G-covariant. Indeed, fix $h \in G$ and $v_1 \in V_1$. Then:
$$\begin{aligned}
\rho_2(h)\cdot\tilde{f}(v_1) &= \frac{1}{|G|}\sum_{g\in G}\rho_2(hg)\circ f\circ\rho_1(g^{-1})(v_1)\\
&= \frac{1}{|G|}\sum_{g\in G}\rho_2(hg)\circ f\circ\rho_1(g^{-1}h^{-1})\circ\rho_1(h)(v_1)\\
&= \frac{1}{|G|}\sum_{g\in G}\rho_2(g)\circ f\circ\rho_1(g^{-1})\circ\rho_1(h)(v_1)\\
&= \tilde{f}\circ\rho_1(h)(v_1)
\end{aligned}$$
which precisely means that $h\cdot\tilde{f}(v_1) = \tilde{f}(h\cdot v_1)$, that is $\tilde{f}$ is G-covariant.

We shally apply this construction together with Schur's Lemma to various maps, working in matrix form. So fix a basis $\mathcal{B}_1$ of V_1 and a basis $\mathcal{B}_2$ of V_2. Let $\rho_1(g)$ have matrix $M^g = (m^g_{i,j})$ and $\rho_2(g)$ have matrix $N^g = (n^g_{i,j})$.

- First suppose that ρ_1 and ρ_2 are not equivalent. Then by Schur's Lemma, the only $\mathbb{C}$-linear and G-covariant map from V_1 to V_2 is 0, that is $\tilde{f}$ is always 0. We take f to be the map whose matrix (in our bases) is the elementary matrix E_{k_0,ℓ_0}. The coefficient (i,j) in the matrix of $\tilde{f} = 0$ is then:

$$\begin{aligned} 0 &= \frac{1}{|G|}\sum_{g\in G}\sum_{k,\ell} n^g_{i,k}\delta_{k,k_0}\delta_{\ell,\ell_0} m^{g^{-1}}_{\ell,j} \\ &= \frac{1}{|G|}\sum_{g\in G} n^g_{i,k_0} m^{g^{-1}}_{\ell_0,j} \end{aligned}$$

  This is true for any (i,j), and in particular when $i = k_0$ and $j = \ell_0$, one finds:

$$0 = \frac{1}{|G|}\sum_{g\in G} n^g_{k_0,k_0} m^{g^{-1}}_{\ell_0,\ell_0}$$

  Now this is true for any (k_0, ℓ_0), so summing over k_0 and ℓ_0:

$$\begin{aligned} 0 &= \sum_{k_0,\ell_0}\frac{1}{|G|}\sum_{g\in G} n^g_{k_0,k_0} m^{g^{-1}}_{\ell_0,\ell_0} \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{k_0,\ell_0} n^g_{k_0,k_0} m^{g^{-1}}_{\ell_0,\ell_0} \\ &= \frac{1}{|G|}\sum_{g\in G}\left(\sum_{k_0} n^g_{k_0,k_0}\cdot\sum_{\ell_0} m^{g^{-1}}_{\ell_0,\ell_0}\right) \\ &= \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(N^g)\cdot\mathrm{Tr}(M^{g^{-1}}) \\ &= \frac{1}{|G|}\sum_{g\in G}\chi_2(g)\chi_1(g^{-1}) \\ &= \langle\chi_1|\chi_2\rangle \end{aligned}$$

- We move to the case where ρ_1 and ρ_2 are equivalent. We then know that they have the same character, that is $\chi_1 = \chi_2$. So to show that $\langle\chi_1|\chi_2\rangle = 1$, we may take $V_1 = V_2 = V$ and $\rho_1 = \rho_2 = \rho$: it won't change the result, but it simplifies the computation!

  Again we start with the $\mathbb{C}$-linear map f whose matrix is E_{k_0,ℓ_0}. Now the associated $\tilde{f}$ is by Schur's Lemma of the form $\lambda_{k_0,\ell_0}\mathrm{Id}$ (a priori λ_{k_0,ℓ_0} depends on k_0 and ℓ_0).

  The coefficient (i,j) in the matrix of $\tilde{f} = \lambda_{k_0,\ell_0}\mathrm{Id}$ is then:

$$\begin{aligned} \lambda_{k_0,\ell_0}\delta_{i,j} &= \frac{1}{|G|}\sum_{g\in G}\sum_{k,\ell} m^g_{i,k}\delta_{k,k_0}\delta_{\ell,\ell_0} m^{g^{-1}}_{\ell,j} \\ &= \frac{1}{|G|}\sum_{g\in G} m^g_{i,k_0} m^{g^{-1}}_{\ell_0,j} \end{aligned}$$

  This is true for any (i,j), and in particular when $i = k_0$ and $j = \ell_0$, one finds:

$$\lambda_{k_0,\ell_0}\delta_{k_0,\ell_0} = \frac{1}{|G|}\sum_{g\in G} m^g_{k_0,k_0} m^{g^{-1}}_{\ell_0,\ell_0}$$

Now we need to pay some extra attention to λ_{k_0,ℓ_0}. Since $\tilde{f}$ is $\lambda_{k_0,\ell_0}\mathrm{Id}$, we know $\mathrm{Tr}(\tilde{f}) = \lambda_{k_0,\ell_0}\dim(V)$. On the other hand, by definition,

$$\mathrm{Tr}(\tilde{f}) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}\left(\rho(g)\circ f\circ\rho(g^{-1})\right) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(f) = \mathrm{Tr}(f) = \delta_{k_0,\ell_0}$$

It follows that $\lambda_{k_0,\ell_0} = \frac{1}{\dim V}\delta_{k_0,\ell_0}$.

Combining with the previous equation, we find:

$$\frac{1}{\dim V}\delta_{k_0,\ell_0} = \frac{1}{|G|}\sum_{g\in G} m^g_{k_0,k_0} m^{g^{-1}}_{\ell_0,\ell_0}$$

Now this is true for any k_0 and ℓ_0, so summing over them:

$$\begin{aligned}
1 &= \sum_{k_0,\ell_0}\frac{1}{|G|}\sum_{g\in G} m^g_{k_0,k_0} m^{g^{-1}}_{\ell_0,\ell_0} \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{k_0,\ell_0} m^g_{k_0,k_0} m^{g^{-1}}_{\ell_0,\ell_0} \\
&= \frac{1}{|G|}\sum_{g\in G}\left(\sum_{k_0} m^g_{k_0,k_0}\cdot\sum_{\ell_0} m^{g^{-1}}_{\ell_0,\ell_0}\right) \\
&= \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(M^g)\cdot\mathrm{Tr}(M^{g^{-1}}) \\
&= \frac{1}{|G|}\sum_{g\in G}\chi(g)\chi(g^{-1}) \\
&= \langle\chi|\chi\rangle
\end{aligned}$$

which completes the proof. □

Remark 1.4.13. This proof is remarkably clumsy; as often with clumsy proofs, this simply means that some algebraic structure is hidden there. We shall come back to this during Week 2. We shall even prove more, namely that the irreducible characters actually form a basis of $\mathcal{F}$ - and in particular their number equals the number of conjugacy classes of G.

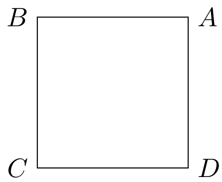
End of Lecture 5.

Lecture 6 ($D_{2\cdot 4}$ and $\mathbb{H}$)

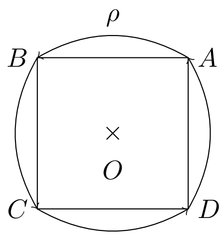
1.5 Two character tables

1.5.1 The square group

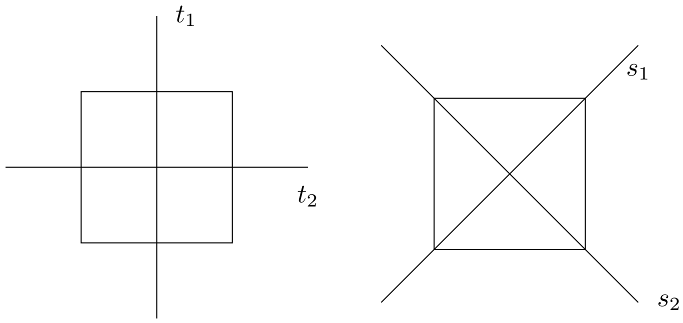
Let $D_{2\cdot 4}$ be the *square group*, that is the group of isometries of a square in the Euclidean plane:



Observe that any isometry must fix the center O of the square. There already are four rotations:



Here $\rho^2 = -\mathrm{Id}$, the central inversion of the plane. There are also the four symmetries having axes:



These are involutions, meaning that $s_1^2 = s_2^2 = t_1^2 = t_2^2 = \mathrm{Id}$.

One then easily shows that $D_{2\cdot 4} = \{\mathrm{Id}, -\mathrm{Id}, \rho, \rho^{-1}, s_1, s_2, t_1, t_2\}$. Here is a quick argument. Observe that the eight elements we have given so far form a subgroup of $D_{2\cdot 4}$. Observe that $D_{2\cdot 4} \hookrightarrow S_4$ since it permutes the vertices $\{A, B, C, D\}$. Finally observe that the embedding is proper, since one cannot fix A and send B to C: one would have $AB = AC$, which is not the case. So by Lagrange's Theorem, the order of $D_{2\cdot 4}$ is a multiple of 8 and divides 24 strictly; $D_{2\cdot 4}$ has order 8 and it therefore equals $\{\mathrm{Id}, -\mathrm{Id}, \rho, \rho^{-1}, s_1, s_2, t_1, t_2\}$.

There are five conjugacy classes: $\{\mathrm{Id}\}, \{-\mathrm{Id}\}, \{\rho, \rho^3\}, \{s_1, s_2\}, \{t_1, t_2\}$. We proceed to computing the character table.

There is of course the trivial character:

D	$1_{(\times 1)}$	$-1_{(\times 1)}$	$\rho_{(\times 2)}$	$s_{1(\times 2)}$	$t_{1(\times 2)}$
triv	1	1	1	1	1

Since $D \simeq \mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, there is also the "orientation representation":

D	$1_{(\times 1)}$	$-1_{(\times 1)}$	$\rho_{(\times 2)}$	$s_{1(\times 2)}$	$t_{1(\times 2)}$
ε	1	1	1	-1	-1

(This can as well be thought of by means of the embedding $D \hookrightarrow S_4$.)

We now turn our attention to the standard permutation representation (associated to the action on the vertices). It has character:

D	$1_{(\times 1)}$	$-1_{(\times 1)}$	$\rho_{(\times 2)}$	$s_{1(\times 2)}$	$t_{1(\times 2)}$
std	4	0	0	2	0

We then see that:

$$\langle \text{std}, \text{std} \rangle = \frac{1}{8}(4^2 + 2 \cdot 2^2) = \frac{24}{8} = 3$$

Writing std $= \sum n_i \chi$ (a sum of irreducible characters), one has $\langle \text{std}, \text{std} \rangle = \sum n_i^2 = 3$, so the only possibility here is that std is a sum of three distinct irreducible characters. To find of which, let us compute scalar products with irreducible characters we already know:

$$\langle \text{std}, \text{triv} \rangle = \frac{1}{8}(4 + 2 \cdot 2) = 1$$

and:

$$\langle \text{std}, \varepsilon \rangle = \frac{1}{8}(4 - 2 \cdot 2) = 0$$

This means that std $-$ triv is a sum of two irreducible characters (and none of them is ε).

We are stuck and need more geometrical insight. But D also acts on the set of diagonals, this gives a permutation action with character:

D	$1_{(\times 1)}$	$-1_{(\times 1)}$	$\rho_{(\times 2)}$	$s_{1(\times 2)}$	$t_{1(\times 2)}$
δ	2	2	0	0	2

Now $\langle \delta, \delta \rangle = \frac{1}{8}(2^2 + 2^2 + 2 \cdot 2^2) = 2$, so δ is a sum of two irreducible characters. Observe how:

$$\langle \delta, \text{triv} \rangle = \frac{1}{8}(2 + 2 + 2 \cdot 2) = 1 \quad \text{and} \quad \langle \delta, \varepsilon \rangle = \frac{1}{8}(2 + 2 - 2 \cdot 2) = 0$$

So $\delta -$ triv is an irreducible character γ:

D	$1_{(\times 1)}$	$-1_{(\times 1)}$	$\rho_{(\times 2)}$	$s_{1(\times 2)}$	$t_{1(\times 2)}$
γ	1	1	-1	-1	1

Now it is the case that $\chi_2 = \text{std} - \text{triv} - \gamma$ is an irreducible character:

D	$1_{(\times 1)}$	$-1_{(\times 1)}$	$\rho_{(\times 2)}$	$s_{1(\times 2)}$	$t_{1(\times 2)}$
χ_2	2	-2	0	0	0

It is now easy to find the missing irreducible character and form the complete character table.

$D_{2\cdot 4}$	$1_{(\times 1)}$	$-1_{(\times 1)}$	$\rho_{(\times 2)}$	$s_{1(\times 2)}$	$t_{1(\times 2)}$
triv	1	1	1	1	1
ε	1	1	1	-1	-1
γ	1	1	-1	-1	1
$\varepsilon\gamma$	1	1	-1	1	-1
χ_2	2	-2	0	0	0

1.5.2 The quaternion group

Let $\mathbb{H}$ be the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, subject to: -1 is a central involution, $i^2 = j^2 = k^2 = -1$, and $ij = k$, $jk = i$, $ki = j$. Then $\mathbb{H}$ has 8 elements in 5 conjugacy classes (i and $-i$ are conjugate). For the moment we know only one irreducible representation: the trivial one. We need four more, $\chi_2, \ldots, \chi_5$; one has $\sum d_i^2 = 8$, so all but one are one-dimensional; the last one has degree 2.

This suggests to "kill" the elements one after the other: in $V_i \simeq \mathbb{C}$, i will act trivially, and j and k invert V_i. Similarly in V_j and V_k. We can then determine the degree 2 representation by orthogonality, and find for the character table:

$\mathbb{H}$	$1_{(\times 1)}$	$-1_{(\times 1)}$	$i_{(\times 2)}$	$j_{(\times 2)}$	$k_{(\times 2)}$
triv	1	1	1	1	1
χ_i	1	1	1	-1	-1
χ_j	1	1	-1	1	-1
χ_k	1	1	1	-1	-1
ψ_2	2	-2	0	0	0

Caution. Observe that we have found two *non-isomorphic* groups which have the same character table.

End of Lecture 6.

Week 2: Basic Representation Theory

Lecture 7 (Introduction ; Basic notions)

You all know what a group is. This immediately raises, or should have long ago raised, two questions:

1. Why study groups?
2. How to study groups?

1. Why study groups? An excellent question. Simply because groups matter, because groups are everywhere in mathematics (and even out of mathematics: take for instance chemistry). This ubiquity should be rather surprising. Well, what *are* groups in the first place? Where do they come from? The answer is that groups naturally arise as sets of structure-preserving transformations:

- Consider a set X. The set of equality-preserving transformations (i.e., bijections) of X is a group, called the symmetric group of X.
- Consider a number field $\mathbb{K}$. The set of equation-preserving transformations (i.e., field automorphisms) of $\mathbb{K}$ is a group, called the Galois group of $\mathbb{K}/\mathbb{Q}$.
- Consider the space $\mathbb{R}^3$ equipped with an origin, and the usual metric. The set of linearity and distance-preserving transformations of $\mathbb{R}^3$ is a group, called the orthogonal group $O_3(\mathbb{R})$.

In short, whenever there is some form of "structure", the set of transformations preserving it is a group. This actually might be taken as a general definition for the somewhat vague notion of a "structure". Groups are everywhere in mathematics, simply because mathematics is the study of structures – precisely in this sense. Let us go back to the second question.

2. How to study groups? Abstract group theory is an option, although, historically speaking, a relatively recent one. Anyway it is quite difficult, and perhaps not that powerful. For instance, as soon as you wish to prove Sylow's Theorem, you "let the group act", which precisely means that you consider the group as a transformation group (of some permutation structure). A purely "inner" analysis of a group wouldn't yield that much information; if you want to understand a group you have to look at what it does "outside", when it acts on something else.

Hence a clearly better option is to recast groups in their natural context. *In order to efficiently study a group, view it as a structure-preserving transformation group.* The extra structure will give you a better grip on the group, because of the new constraints which arise. This is the line we take.

What are the structures we best understand, then?

- Not sets. First of all, infinite sets are complicated. We'll focus on finite groups, so this is not that much of an objection. Let us then take a group G and view it as a structure-preserving transformation group, where the only structure is equality, i.e. we view G as a group of bijections of some set. A classical way to do it is to let G act on its underlying set by say left translation; we obtain Cayley's Theorem, which says that any group embeds into a symmetric group.

  But you know that Cayley's Theorem is not that good. It reduces the study of groups to that of symmetric groups, but symmetric groups are tough, because they are close to combinatorics. Hence viewing a group as a bijection group is not very helpful.

- So, what is a structure one understands well? Vector spaces. We're extremely familiar with vector spaces, especially finite-dimensional, complex vector spaces. Perhaps you find linear algebra dull – precisely because you have perfect intuition of it. Let us then use this well-understood structure of a vector space.

  Representation theory is the art of recasting algebraic structures into linear algebra.

Or equivalently put, the art of understanding groups as transformation groups of vector spaces. We make three restrictions:

- We shall work only with finite groups.

  They are simpler.

- We shall work only with finite-dimensional vector spaces.

  Actually, when one works with a finite group, it is perfectly harmless (as one easily imagines and as we shall prove) to work in finite dimension.

- We shall work over $\mathbb{C}$.

  $\mathbb{C}$ has two interesting features. First, it is algebraically closed. One easily understands that linear algebra is easier over an algebraically closed field, because diagonalization is as simple as can be. On the other hand, some of you may know about $\overline{\mathbb{F}}$; but we want, and this will soon be clear, the characteristic to be 0 because we shall need to divide by integers.

Caution. Throughout these notes, G stands for a finite group and V for a finite-dimensional, complex vector space.

2.1 Representations and Complete Reducibility

2.1.1 Representations

Given a (finite) group G, we are tempted to view G as a subgroup of $\mathrm{GL}(V)$. This is not always possible when V is fixed in advance: imagine $V \simeq \mathbb{C}$, and $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$. So we should relax our restriction of an embedding $G \hookrightarrow \mathrm{GL}(V)$, into a (group homo-)morphism. Embeddings will be a special case.

Definition 2.1.1. A representation of a group G in a vector space V is a morphism $\rho : G \to \mathrm{GL}(V)$. The representation is faithful if ρ is injective.

Hence, if ρ is a representation of G in V, then G acts linearly on V, i.e. for all g we have a linear map $\rho(g) : V \to V$ and for every $g, h \in G$, one has:

$$\underbrace{\rho(g) \circ \rho(h)}_{\text{composition in } \mathrm{GL}(V)} = \rho(\underbrace{g \cdot h}_{\text{group law}})$$

Definition 2.1.2. A (finite-dimensional, complex) representation (V, ρ) of a group G consists of a (finite-dimensional, complex) vector space V and a representation $\rho : G \to \mathrm{GL}(V)$.

Definition 2.1.3. The degree of a representation (V, ρ) is $\dim V$.

Let us insist that all representations will be complex and finite-dimensional (hence, of finite degree).

Relaxing notations

Notation 2.1.4. As said in Definition 2.1.2, a representation should be denoted by (V, ρ); in practice it is often denoted simply V, or simply ρ.

Notation 2.1.5. If (V, ρ) is a representation of G, $g \in G$ and $v \in V$, then one ought to denote the action of g on v by $\rho(g)(v)$, or for extra clarity, $[\rho(g)](v)$. As this is rather clumsy, we shall simply write $g \cdot v$, at least when ρ is clear from the context. When it is not, let us agree on $\rho(g) \cdot v$.

Definition 2.1.6. Let G be a group. The trivial representation of G is:

$$\begin{array}{rccl} \mathrm{triv}: & G & \to & \mathrm{GL}_1(\mathbb{C}) \\ & g & \mapsto & 1 \end{array}$$

It has degree 1.

Definition 2.1.7. Let G be a finite group. Let $n = |G|$ and $\{e_g : g \in G\}$ be a basis of $\mathbb{C}^n$. The regular representation of G is:

$$\begin{array}{rccl} \mathrm{reg}: & G & \to & \mathrm{GL}_n(\mathbb{C}) \\ & g & \mapsto & \mathrm{reg}(g) \end{array}$$

where $\mathrm{reg}(g) \cdot e_h = e_{gh}$ (and extend linearly).

Remark 2.1.8. A decent alternative name would be the Cayley representation, as we have just been dressing the Cayley action in linear clothing.

Despite the naiveness of their definitions, the representations triv and reg are extremely useful. They are special cases of a more general construction.

Definition 2.1.9. Let G be a group acting on a set X. Let V be a vector space with basis $\{e_x : x \in X\}$. The permutation representation associated to (G, X) is given by $\mathrm{perm}(g) \cdot e_x = e_{g\cdot x}$.

On the other hand there is nothing new. We have simply added some "linear flesh" on the skeleton of the permutation action. This is pure make-up; mathematically speaking, nothing happened.

Example 2.1.10. Let $G = S_3$, the symmetric group on $\{1, 2, 3\}$.

1. The alternating representation is:

$$\begin{array}{rccl} \varepsilon: & G & \to & \mathrm{GL}_1(C) \simeq \mathbb{C}^\times \\ & g & \mapsto & \varepsilon(g) \end{array}$$

   where $\varepsilon(g)$ is the signature of g. Hence,

$$g \cdot z = \varepsilon(g)z$$

   Observe that the underlying vector space is the same as for the trivial representation: simply $\mathbb{C}$. But it's not the same representation!

   Time has now come for your first non-trivial example.

2. Let $\tau = (12)$ and $\gamma = (123)$. You know that τ and γ generate $G = S_3$; for instance, $(23) = \gamma\tau\gamma^{-1}$. One has:

$$S_3 = \{\mathrm{Id}, \tau, \underbrace{\gamma\tau\gamma^{-1}}_{(23)}, \underbrace{\gamma^{-1}\tau\gamma}_{(13)}, \gamma, \underbrace{\gamma^2}_{(321)}\}$$

   Let $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and in this basis, let $\rho(\tau)$ have matrix:

$$M_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   and $\rho(\gamma)$ have matrix:

$$M_\gamma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

   I say that this defines a representation. Indeed, one easily sees that $M_\tau^2 = M_\gamma^3 = \mathrm{Id}$ and $M_\tau M_\gamma M_\tau = M_\gamma^{-1}$, so ρ extends to all of S_3.

   Work out the details if you are not convinced. Here is how I constructed this one: I started with the usual action of S_3 on $\{1, 2, 3\}$, and moved to the permutation representation associated to it: a representation of S_3 in $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$. Then I identified that the line $L = \mathbb{C}(e_1 + e_2 + e_3)$ was centralized by all $g \in S_3$, and moved to the quotient space $\bar{V} = V/L$, with basis $\bar{e_1}, \bar{e_2}$. In the quotient, $\bar{e}_3 = -\bar{e}_1 - \bar{e}_2$.

   In $\mathbb{C}^3$, $\tau = (12)$ swaps e_1 and e_2; this still holds in the quotient. In $\mathbb{C}^3$, $\gamma = (123)$ sends e_1 to e_2 and e_2 to e_3; in the quotient, it sends $\bar{e}_1$ to $\bar{e}_2$ and $\bar{e}_2$ to $-\bar{e}_1 - \bar{e}_2$. This is exactly what I did to write the matrices!

This raises a series of questions:

- When is a vector subspace a subrepresentation?
- Can one go to quotients?
- Do quotients split, that is: if $W \leq V$ is a subrepresentation, is there a subrepresentation $K \leq V$ with $V = V \oplus K$?

End of Lecture 7.

Lecture 8 (Maschke and Schur)

2.1.2 Covariance and equivalence of representations

Let us go back to the regular representation, Definition 2.1.7. We have been acting left; of course one could easily imagine defining $\rho(g) \cdot e_h = e_{hg^{-1}}$ in order to have something one would call the "right-regular" representation. But we have a sense that this construction is equivalent to the regular one, in a way we can even make precise. First we need a notion of "morphism of representations".

Definition 2.1.11. Let (V_1, ρ_1) and (V_2, ρ_2) be two representations of a group G. A map $f : V_1 \to V_2$ is G-covariant if for every $g \in G, v_1 \in V_1$, one has $f([\rho_1(g)](v_1)) = [\rho_2(g)](f(v_1))$

This is a remarkably heavy notation; as there is no risk of confusion, let us not indicate the representations, since they should be clear from the context. The map f is G-covariant if and only if for all $g \in G, v_1 \in V_1$, one has:

$$f(\underbrace{g \cdot v_1}_{G\text{acting on } V_1}) = \underbrace{g \cdot f(v_1)}_{G\text{acting on } V_1}$$

which looks more civilized. Observe that we no longer need v_1.

Remark 2.1.12. f is G-covariant if and only if for all $g \in G$, $f \circ \rho_1(g) = \rho_2(g) \circ f$. It is convenient to represent this by a *commutative diagram*: on the diagram below, the two possible paths correspond to the same function.

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ \downarrow{\scriptstyle \rho_1(g)} & & \downarrow{\scriptstyle \rho_2(g)} \\ V_1 & \xrightarrow[f]{} & V_2 \end{array}$$

When you feel that there is no risk of confusion, you can start writing the covariance equation in the form $f \circ g = g \circ f$.

Caution. Observe that on the left-hand, g stands for $\rho_1(g)$, whereas on the right-hand it stands for $\rho_2(g)$. In practice, there is no risk of confusion.

Perhaps you are already familiar with the following notation.

Notation 2.1.13. Let V_1 and V_2 be two vector spaces. The set of linear maps from V_1 to V_2 is denoted $\mathrm{Hom}(V_1, V_2)$, or $\mathrm{Hom}_{\mathbb{K}}(V_1, V_2)$ if one wishes to put the stress on the base field (which will, for us, always be $\mathbb{C}$).

This suggests a convenient notation for the set of linear covariant maps.

Notation 2.1.14. Let V_1 and V_2 be two representations of G. The set of $\mathbb{C}$-linear, G-covariant maps from V_1 to V_2 is denoted $\mathrm{Hom}_G(V_1, V_2)$.

Example 2.1.15. In order to check that you are not confusing the two notions, take Example 1.2.11 (1) and (2). Then $\mathrm{Id} : \mathbb{C} \to \mathbb{C}$ is in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$, but not in $\mathrm{Hom}_G(\mathrm{triv}, \varepsilon)$, since $\mathrm{Id}(g \cdot 1) = 1$ but $g \cdot \mathrm{Id}(1) = \varepsilon(g)$.

We may now define what it means for two representations to be the same.

Definition 2.1.16. Two representations (V_1, ρ_1) and (V_2, ρ_2) of G are equivalent if there is a G-covariant $\mathbb{C}$-linear isomorphism $f : V_1 \to V_2$.

Remark 2.1.17. You also require f to be linear! of course you want f to be an isomorphism of vector spaces in the first place!

Example 2.1.18. Let (V, reg) be the regular representation, where V has basis $\{e_g : g \in G\}$. Let V' have basis $\{e'_g : g \in G\}$ and consider the right-regular representation (V', ρ'), with $\rho'(g) \cdot e'_h = e'_{hg^{-1}}$. Then (V, reg) and (V', ρ') are equivalent.

Proof. Consider the function:

$$\begin{array}{cccc} f: & V & \to & V' \\ & e_h & \mapsto & e'_{h^{-1}} \end{array}$$

extended linearly; f is a complex vector space isomorphism. It remains to show that f is G-covariant. Indeed, let $g, h \in G$; one has:

$$u(g \cdot e_h) = f([\mathrm{reg}(g)](e_h)) = f(e_{gh}) = e'_{(gh)^{-1}}$$

and on the other hand,

$$g \cdot f(e_h) = [\rho'(g)](e'_{h^{-1}}) = e'_{h^{-1}g^{-1}}$$

so f is G-covariant alright, hence an equivalence. □

Questions 2.1.19.

- *Can one classify representations up to equivalence?*
- *Is there a quick equivalence test?*

2.1.3 Complete reducibility

Let us consider again the permutation representation of $G = S_3$: $\rho : S_3 \to \mathrm{GL}_3(\mathbb{C})$, where $g \cdot e_k = e_{g \cdot k}$ ($k \in \{1, 2, 3\}$). Observe that the line generated by $(e_1 + e_2 + e_3)$ is fixed by G. In this sense, $\mathbb{C}^3$ contains a smaller representation.

Definition 2.1.20. (V, ρ) is an irreducible representation of G if the only two G-invariant subspaces are $\{0\}$ and V.

Example 2.1.21.

- The trivial representation, any representation of degree 1, is irreducible.
- The regular representation is *not* irreducible as soon as $G \neq \{1\}$. Indeed, the line generated by $\sum_{g\in G} e_g$ is G-invariant (and proper).
- The degree two representation given in Example 1.2.11 (3) is irreducible.

Question 2.1.22. *Is there a fast irreducibility test?*

Notation 2.1.23. Let $\{(V_i, \rho_i) : i \in I\}$ be the set of irreducible representations of G.

Question 2.1.24. *Is I finite? Can we say something about is cardinal?*

By the way, here is why we restrict ourselves to finite-degree representations.

Lemma 2.1.25. *Let G be a finite group, V a vector space, and $\rho : G \to \mathrm{GL}(V)$ be a morphism. If V is irreducible, then V is finite-dimensional.*

Proof. Let $v \neq 0$. Consider the vector space W generated by the finite family $\{g \cdot v : g \in G\}$. W is non-trivial, finite-dimensional, and G-invariant. By irreducibility, $W = V$. □

It is obvious from the dimension function that every finite-dimensional representation contains an irreducible representation. Here we are with one more question:

Question 2.1.26. *Is a finite-dimensional representation determined by the irreducible representations it contains?*

It is the first question we can answer.

Remark 2.1.27. You will now understand why we work in characteristic 0.

Theorem 2.1.28 (complete reducibility)**.** *Let G be a finite group and (V, ρ) be a complex, finite-dimensional representation. Then V is a direct sum of irreducible representations.*

It clearly suffices to show the following.

Theorem 2.1.29 (Maschke's Theorem)**.** *Let G be a finite group and (V, ρ) be a complex, finite-dimensional representation. Let $W \leq V$ be a G-invariant subspaces. Then W has a G-invariant complement.*

For the proof you need to remember what a hermitian scalar product is.

Definition 2.1.30. Let V be a finite-dimensional, complex vector space. A hermitian scalar product on V is a map $(.,.) : V \times V \to \mathbb{C}$ which is:

- sesquilinear: $(\lambda x + y, z) = \overline{\lambda}(x, z) + (y, z)$
- Hermite-symmetric: $(x, y) = \overline{(y, x)}$
- positive: $(x, x) \in \mathbb{R}_+$
- definite: $(x, x) = 0$ iff $x = 0$.

Proof. Let $(.,.)$ be any hermitian scalar product on V. Consider the new map:

$$\langle x, y \rangle = \frac{1}{|G|} \sum_{g \in G} (gx, gy)$$

I claim that $\langle .,. \rangle$ is yet another hermitian scalar product, which is readily checked. I also claim that $\langle .,. \rangle$ is G-covariant, in the sense that for any $h \in H$:

$$\langle hx, hy \rangle = \langle x, y \rangle$$

Now W is G-invariant, and so is $W^\perp$. But it is well-known that $W \oplus W^\perp = V$. □

Example 2.1.31. Consider Example 2.1.10 (2) again. We shall construct an invariant complement to the G-invariant line $\langle e_1 + e_2 + e_3 \rangle$ by this method.

We start with the standard complex scalar product; in the canonical basis, it has matrix I_3. Now if $\rho(g)$ has matrix M_g, then the scalar product $(x, y) \mapsto (\rho(g)(x), \rho(g)(y))$ has matrix $M_g^* I_3 M_g = M_g^* M_g$, where $M_g^* = \overline{M_g}^t$ is the complex adjoint matrix (conjugate-transpose). Observe how for any $g \in S_3$, one has $M_g^* = M_g^t = M_{g^{-1}}$, and in particular $M_g^* M_g = I_3$. So averaging, we find that the standard complex scalar product was G-invariant.

Then we compute $(e_1 + e_2 + e_3)^\perp = \langle e_1 - e_3, e_2 - e_3 \rangle$, which is therefore a G-invariant complement.

Maschke's Theorem can actually be extended to fields of characteristic not dividing $|G|$, but not beyond.

Counter-example 2.1.32. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{F}_2^2 = \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_2$ by:

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Then the line $\mathbb{F}_2 e_1$ is G-invariant, but there is no G-invariant complement: observe that neither e_2, nor $e_1 + e_2$, is fixed by G!

Let us go back to complete reducibility, Theorem 2.1.28. Given a representation V, we may write V as a direct sum of irreducible representations. With Notation 2.1.23, this means that there are integers $n_i \in \mathbb{N}$ such that:

$$V = \oplus_{i \in I} V_i^{n_i}$$

Question 2.1.33. *Is there some form of uniqueness in this decomposition?*

Uniqueness should not be taken in too strict a sense. Consider $\mathbb{C}^2$ equipped with a trivial action of G. Then any splitting of $\mathbb{C}^2$ as a sum of two distinct vector lines is actually a decomposition into irreducible representations! We shall however provide a decent answer to the question.

2.1.4 Schur's Lemma

One last surprisingly powerful tool; talking about irreducible representations, is the following observation.

Theorem 2.1.34 (Schur's Lemma)**.** *Let V_1 and V_2 be two irreducible representations. Then* $\mathrm{Hom}(\rho_1, \rho_2)$ *is either* 0 *or a skew-field. If* $\mathbb{K} = \mathbb{C}$*, then* $\mathrm{Hom}(\rho_1, \rho_2)$ *is eiher* 0 *or isomorphic to* $\mathbb{C}$*.*

Proof. Let $\varphi : V_1 \to V_2$ be a $\mathbb{K}$-linear, φ-covariant map. Then $\ker\varphi$ and $\mathrm{im}\varphi$ are φ-invariant $\mathbb{K}$-vector subspaces of V_1 and V_2, respectively. In particular, φ is either 0 or a $\mathbb{K}$-linear bijection.

Now suppose $\mathbb{K} = \mathbb{C}$ and $\mathrm{Hom}(\rho_1, \rho_2) \neq 0$. Since V_1 and V_2 are finite dimensional, $\mathrm{Hom}(\rho_1, \rho_2)$ is a finite-dimensional, complex algebra. But it also is a skew-field. Since $\mathbb{C}$ is algebraically closed, it follows $\mathrm{Hom}(\rho_1, \rho_2) = \mathbb{C}$. ☐

One easily imagines that even over $\mathbb{R}$, nothing of the kind will subsist. This is why we definitely prefer to work over $\mathbb{C}$.

Remark 2.1.35. In week 3, we shall prove a theorem implying that over $\mathbb{R}$, there are three possibilities: $\mathbb{R}, \mathbb{C}$, and the quaternion skew-field $\mathbb{H}$.

Corollary 2.1.36. *Let V be a representation and $V = \oplus_{i\in I} V_i^{n_i}$ a decomposition into irreducible representations. Then the n_i's are unique.*

Proof. Let $V = \oplus_{i\in I} V_i^{m_i}$ be another decomposition. Consider the identity map $\mathrm{Id} : V \to V$. The image of each copy of V_i must be another copy of V_i. This proves that $n_i = m_i$. ☐

Later we shall even say a bit more, proving that if the copies of V_i are individually not well-determined, however the block $V_i^{n_i}$ is.

Definition 2.1.37. Let V be a representation and V_i an irreducible representation. The multiplicity of V_i in V is the integer n_i defined in Corollary 2.1.36.

End of Lecture 8.

Lecture 9 (Algebraic constructions)

2.2 Some constructions

Given some representations of a group G, we know how to build "smaller ones" by restricting to G-invariant subspaces and quotienting out. Can we build "bigger" representations?

Definition 2.2.1. If (V_1, ρ_1) and (V_2, ρ_2) are two representations of G, one constructs a representation of G on $V_1 \oplus V_2$, denoted $\rho_1 \oplus \rho_2$, by letting:

$$[(\rho_1 \oplus \rho_2)(g)](v_1 + v_2) = [\rho_1(g)](v_1) + [\rho_2(g)](v_2)$$

Matricially speaking: one simply designs a block matrix. That was easy.

2.2.1 Dual and Hom representations

Let (V, ρ) be a representation of G. We wish to turn the dual space V^* into a representation (V^*, ρ^*), in a natural way. There is the natural pairing $(V^*, V) \to \mathbb{C}$ given by evaluation, i.e. for $(f, v) \in V^* \times V$,

$$\langle f, v \rangle = f(v)$$

We wish to preserve this pairing. This suggests to let ρ^* be such that:

$$\langle \rho^*(g) \cdot f, \rho(g) \cdot v \rangle = \langle f, v \rangle$$

or in other symbols,

$$(\rho^*(g) \cdot f)(\rho(g) \cdot v) = f(v)$$

Definition 2.2.2. Let (V, ρ) be a representation of G. The dual representation is (V^*, ρ^*) where $g \in G$ acts on a linear form $f \in V^*$ by:

$$[\rho^*(g)](f) = f \circ \rho(g^{-1})$$

In simpler notation, $g \cdot f = f \circ g^{-1}$. Matricially speaking, $\rho^*(g)$ has matrix $(M_g^t)^{-1} = (M_g^{-1})^t$.

Given two representations V_1 and V_2, we now wonder how G should act on the space of linear maps from V_1 to V_2, denoted $\mathrm{Hom}(V_1, V_2)$ (see Notation 2.1.13). Let $f : V_1 \to V_2$ be linear. Any $g \in G$ induces an endomorphism $\rho_1(g)$ of V_1, and an endomorphism $\rho_2(g)$ of V_2. Hence we have a diagram:

$$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(g)} & V_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g \cdot f} \\ V_2 & \xrightarrow[\rho_2(g)]{} & V_2 \end{array}$$

And a legitimate requirement is that this diagram commutes, so that for any $v_1 \in V_1$, one has:

$$(g \cdot f)(g \cdot v_1) = g \cdot f(v_1)$$

Definition 2.2.3. We let G act on $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ by:

$$g \cdot f = g \circ f \circ g^{-1}$$

Remark 2.2.4. f is G-covariant iff for all $g \in G$, $g \cdot f = f$.

Remark 2.2.5. The dual representation is a special case of the Definition 2.2.3, if $V_2 = \mathbb{C}$ is equipped with the trivial representation.

2.2.2 Universal Properties

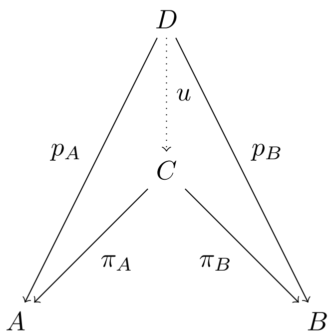
Sometimes in mathematics one works with "universal objects", defined by means of a "universal property".

Caution. A universal property is not a proof of existence.

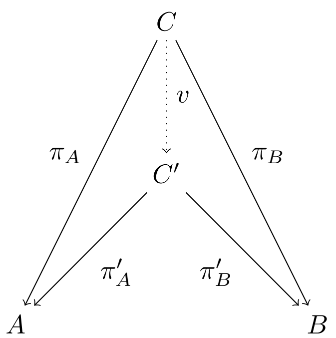
This is typically the following form of definition; here we give an example which should be well-known, just to show what it means to think in terms of universal objects.

Theorem 2.2.6. *Let A and B be two sets. Then there is a triple (C, π_A, π_B) where C is a set and $\pi_A : C \to A$, $\pi_B : C \to B$ with the following* universal property*:*

> *for any set D and maps $p_A : D \to A$, $p_B : D \to B$, there is a unique map $u : D \to C$ such that $p_A = \pi_A \circ u$, $p_B = \pi_B \circ u$.* (UP)



C is then unique up to a unique isomorphism, meaning that if (C', π'_A, π'_B) is another triple enjoying (UP), then there is a unique isomorphism $v : C \to C'$ such that the following diagram commutes:.



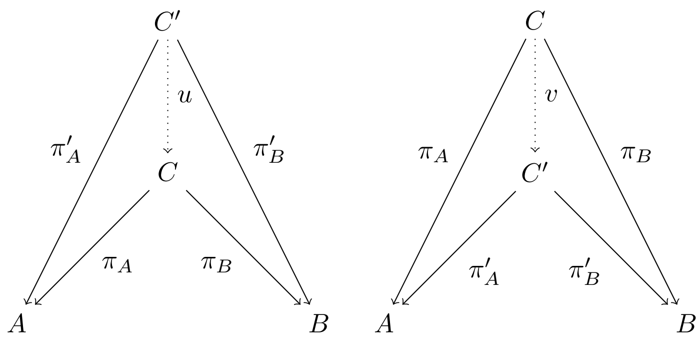
- As one knows, C is called the Cartesian product of A and B and denoted $A \times B$. We do not use this notation: our idea is to explain how people in category theory think about this object.
- An "isomorphism of sets" is simply a bijection. It is however convenient to think in terms of isomorphisms, because we will be considering more algebraic cases soon.

Proof. What is nice with universal properties is that they provide a very convenient way to deal with objects defined by their means. In particular, uniqueness is usually a quick consequence of the universal property. But existence is not, and one needs to provide a construction.

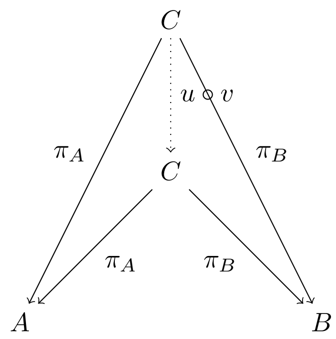
- Let $C = \{(a,b) : a \in A, b \in B\}$ and $\pi_A : C \to A$, $\pi_B : C \to B$ map (a,b) to a and b, respectively. We claim that (C, π_A, π_B) enjoys the universal property.

  Indeed, let (D, p_A, p_B) be another set D equipped with maps $p_A : D \to A$ and $p_B : D \to B$; we are looking for a map u as on the diagram. If there were one, it would satisfy $\pi_A \circ u(d) = p_A(d)$ and $\pi_B \circ u(d) = p_B(d)$ for all $d \in D$. So we let $u(d) = (\pi_A(d), \pi_B(d))$; notice that u works and that we didn't have another choice. In short, we have just proved existence and uniqueness of u. This means that (C, π_A, π_B) satisfies (UP).

- We now must prove that the "universal object" just constructed is unique up to a unique isomorphism. This part of the proof is pretty much invariable. Suppose (C', π'_A, π'_B) satisfies (UP) as well. Then consider the diagrams:



  The arrow u is obtained because C is universal; the arrow v because C' is universal as well. Now $u \circ v : C \to C$ satisfies the following:



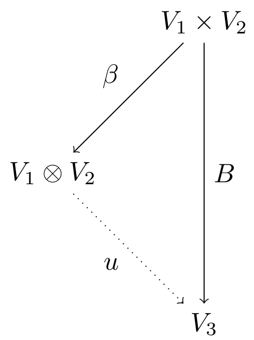
  But in the universal property, the map from D to C (here $D = C$) must be *unique*. Since Id_C works as well, this means that $u \circ v = \mathrm{Id}_C$. One proves $v \circ u = \mathrm{Id}_{C'}$ similarly. So C and C' are in bijection with each other, and since u and v are unique in the construction, one has uniqueness up to a unique isomorphism. □

2.2.3 The tensor product

In this subsection we do no representation theory but merely recall what the tensor product is. The basic motivation is to factorize bilinear maps.

Proposition 2.2.7. *Let V_1 and V_2 be $\mathbb{C}$-vector spaces. Then there is a $\mathbb{C}$-vector space, called the tensor product of V_1 and V_2 over $\mathbb{C}$, denoted $V_1 \otimes_{\mathbb{C}} V_2$, equipped with a bilinear map $\beta : V_1 \times V_2 \to V_1 \otimes V_2$, having the following universal property:*

> *for any bilinear map $B : V_1 \times V_2 \to V_3$ (V_3 a vector space), there is a unique linear map $u : V_1 \otimes V_2 \to V_3$ such that $B = u \circ \beta$.* (UP)



$(V_1 \otimes V_2, \beta)$ is unique up to a unique isomorphism: if (W, β) and (W', β') satisfy (UP), then there is a unique $\mathbb{C}$-isomorphism $f : W \simeq W'$ with $\beta' = f \circ \beta$.

Proof.

- We start with uniqueness. Suppose both (W, β) and (W', β') have the desired property. Since $\beta : V_1 \times V_2 \to W$ is a bilinear map, by universality of W', there is a unique $f' : W' \to W$ such that $\beta = f' \circ \beta'$. Similarly, there is a unique $f : W \to W'$ such that $\beta' = f \circ \beta$.

  Hence $\beta = f' \circ f \circ \beta$, which is a way to factor the bilinear map $\beta : V_1 \times V_2 \to W$. Another way is to write $\beta = \mathrm{Id}_W \circ \beta$. By uniqueness of factorizations in (UP), one has $f' \circ f = \mathrm{Id}_W$. And $f \circ f' = \mathrm{Id}_{W'}$ similarly. This shows that $W \simeq W'$, in a unique way compatible with β and β'. Draw diagrams for this argument to make sure you have understood it.

- We haven't done anything so far. It is time to construct the tensor product. Let $\{x_k : k \in K\}$ be a basis of V_1 and $\{y_\ell : \ell \in L\}$ be a basis of V_2. Let $V_1 \otimes V_2$ be a vector space with basis $\{x_k \otimes y_\ell : (k, \ell) \in K \times L\}$, where the $(x_k \otimes y_\ell)$'s are new vectors. Let β map (x_k, y_ℓ) to the tensor $(x_k \otimes y_\ell)$; one may extend β to a bilinear map $V_1 \times V_2 \to V_1 \otimes V_2$ by letting:

  $$\beta((\sum_{k \in K} \lambda_k x_k, \sum_{\ell \in L} \mu_\ell y_\ell)) = \sum_{(k,\ell) \in K \times L} \lambda_k \mu_\ell (x_k \otimes y_\ell)$$

  where all sums are finite.

  We claim that $(V_1 \otimes V_2, \beta)$ has the desired universal property. Let $B : V_1 \times V_2 \to V_3$ be a bilinear map. We shall factor B through β. Let $f(x_k \otimes y_\ell) = B(x_k, y_\ell)$ and extend linearly. Then clearly $B = f \times \beta$. It is not less clear that there was no other choice for f, proving unique factorization. Hence $(V_1 \otimes V_2, \beta)$ has the desired universal property. $\square$

In a nutshell, the tensor product is the space through which all bilinear maps factor. In practice one simply writes $V_1 \otimes V_2$, omitting β.

Caution. Not all elements of $V_1 \otimes V_2$ are of the form $v_1 \otimes v_2$! But it is true that such elements span $V_1 \otimes V_2$; actually, by construction, one can restrict v_1 to a basis of V_1 and v_2 to a basis of V_2.

Remark 2.2.8. The construction is much more general. When one tensors over a base ring which is not a field, difficulties can occur. Tensoring over $\mathbb{C}$ is perfectly trick-free.

Proposition 2.2.9. *Let V_1 and V_2 be two finite-dimensional, complex vector spaces. Then $V_1^* \otimes_{\mathbb{C}} V_2 \simeq \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$.*

Proof. We shall construct the isomorphism. To a tensor $(f \otimes v_2) \in V_1^* \otimes_{\mathbb{C}} V_2$, associate the linear map:

$$\begin{array}{rccc} \Phi(f \otimes v_2): & V_1 & \to & V_2 \\ & v_1 & \mapsto & f(v_1)v_2 \end{array}$$

The function:

$$\Phi : V_1^* \otimes_{\mathbb{C}} V_2 \to \mathrm{Hom}(V_1, V_2)$$

is linear. It is clearly injective. For dimension reasons, it is a bijection. □

2.2.4 Tensoring representations

If (V_1, ρ_1) and (V_2, ρ_2) are two representations of G, we wonder how G should naturally act on $V_1 \otimes V_2$. First of all, G should act on $V_1 \times V_2$ by $g \cdot (v_1, v_2) = (g \cdot v_1, g \cdot v_2)$; this is what we've done in Definition 2.2.1. Mapping to $V_1 \otimes_{\mathbb{C}} V_2$, this suggests the following definition.

Definition 2.2.10. Let (V_1, ρ_1) and (V_2, ρ_2) be two representations of G. Let $(V_1 \otimes V_2, \rho_1 \otimes \rho_2)$ be given by:

$$(\rho_1 \otimes \rho_2)(g) \cdot (v_1 \otimes v_2) = (\rho_1(g) \cdot v_1 \otimes \rho_2(g) \cdot v_2)$$

Proposition 2.2.11. *$V_1^* \otimes_{\mathbb{C}} V_2$ is equivalent to $\mathrm{Hom}(V_1, V_2)$.*

Proof. We show that the $\mathbb{C}$-isomorphism of Proposition 2.2.9 is actually an equivalence of representations. Indeed, let $(f \otimes v_2)$ be a tensor in $V_1^* \otimes_{\mathbb{C}} V_2$. Recall that Φ maps $(f \otimes v_2)$ to:

$$\begin{array}{rccc} \Phi(f \otimes v_2): & V_1 & \to & V_2 \\ & v_1 & \mapsto & f(v_1)v_2 \end{array}$$

that is, $\Phi(f \otimes v_2) = f(.)v_2$.

We know that:

$$g \cdot (f \otimes v_2) = (g \cdot f \otimes g \cdot v_2) = (f \circ g^{-1} \otimes g \cdot v_2)$$

which maps through Φ to:

$$v_1 \mapsto f(g^{-1} \cdot v_1) g \cdot v_2$$

On the other hand, g acts on $\Phi(f \otimes v_2) \in \mathrm{Hom}(V_1, V_2)$ by:

$$[g \cdot \Phi(f \otimes v_2)](v_1) = g \cdot (\Phi(f \otimes v_2)(g^{-1} \cdot v_1)) = f(g^{-1} \cdot v_1) g \cdot v_2$$

so Φ is also G-covariant. □

Too abstract? Things should be easier now.

End of Lecture 9.

Lecture 10 (Character Theory)

2.3 Characters

We now introduce the central tool of representation theory, a tool which will prove useful beyond imagination. Recall that a representation of G is a finite-dimensional, complex vector space V equipped with a linear action of G on V (equivalently, a morphism $\rho : G \to \mathrm{GL}(V)$). Is it possible to describe ρ?

- One may wish to describe it as a matrix. But this is cumbersome, and far from compact.
- every $\rho(g)$ is diagonalizable: since $g^n = 1$ for $n = |G|$, $\rho(g)^n = \mathrm{Id}$, meaning that the minimal polynomial of $\rho(g)$ over $\mathbb{C}$ divides $X^n - 1$, which is a split polynomial with simple roots. It is then well-known that $\rho(g)$ is diagonalizable.

  This suggests to try to describe $\rho(g)$ by its eigenvalues.
- And actually, however unrealistically optimistic this may sound, one can describe $\rho(g)$ simply by its trace.

2.3.1 Definition and examples

Definition 2.3.1. The character of a representation (V, ρ) of G is the function:

$$\begin{array}{rccc} \chi_\rho : & G & \to & \mathbb{C} \\ & g & \mapsto & \mathrm{Tr}\rho(g) \end{array}$$

This apparently useless definition will provide a most impressive theory. Let us first compute some examples.

Example 2.3.2. Go back to Example 1.2.11. Let perm be the permutation representation and ρ be the representation of 1.2.11 (3). All computations made, you should find:

χ	e	(12)	(13)	(23)	(123)	(132)
χ_{triv}	1	1	1	1	1	1
χ_ε	1	-1	-1	-1	1	1
χ_{perm}	3	1	1	1	0	0
χ_{reg}	6	0	0	0	0	0
χ_ρ	2	0	0	0	-1	-1

Remark 2.3.3. One always obviously has $\chi(e) = \deg \rho$.

Example 2.3.4.

- Let (G, X) be a permutation group and perm be the associated permutation representation. Then $\chi_{\mathrm{perm}}(g)$ is the number of fixed points of g in X.

- This can be viewed as a special case of the previous observation: consider the regular representation reg of G. Then $\chi_{\text{reg}}(e) = |G|$ and $\chi_{\text{reg}}(g) = 0$ for all $g \neq e$.

Lemma 2.3.5. *Two equivalent representations have the same character.*

Proof. Let (V_1, ρ_1) and (V_2, ρ_2) be equivalent; this means that there is a G-covariant linear isomorphism $f : V_1 \simeq V_2$, or equivalently, that for any $g \in G$, the diagram:

$$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(g)} & V_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ V_2 & \xrightarrow[\rho_2(g)]{} & V_2 \end{array}$$

is commutative. Since f is an isomorphism, $\text{Tr}\rho_1(g) = \text{Tr}\rho_2(g)$, as desired. □

The characters of irreducible representations will play a prominent role.

Definition 2.3.6. The irreducible characters of G are the characters of its irreducible representations.

Notation 2.3.7. Let $\{\chi_i : i \in I\}$ be the set of irreducible characters of G.

Here are some basic questions we shall all answer.

Questions 2.3.8.

- *How many irreducible characters are there?*
- *How do characters relate to irreducible characters?*
- *Do characters describe representations?*

2.3.2 Characters and constructions

How do characters behave with respect to the constructions of §2.2?

Proposition 2.3.9. *Let (V, ρ_1) and (V_2, ρ_2) be two representations of G, of characters χ_1 and χ_2. Then:*

1. *the character of $V_1 \oplus V_2$ is $\chi_1 + \chi_2$;*
2. *the character of V_1^* is $\overline{\chi_1}$;*
3. *the character of $V_1 \otimes V_2$ is $\chi_1 \chi_2$;*
4. *the character of $\text{Hom}(V_1, V_2)$ is $\overline{\chi_1}\chi_2$.*

Proof. Let $g \in G$. Let $\{x_k : 1 \leq k \leq \deg V_1\}$ be a basis of V_1 consisting of eigenvectors of g, and $\{y_\ell : 1 \leq \ell \leq \deg V_2\}$ similarly in V_2. By assumption, there are complex numbers λ_k and μ_ℓ such that $g \cdot x_k = \lambda_k x_k$ and $g \cdot y_\ell = \mu_\ell y_\ell$.

1. $\{x_k\} \cup \{y_\ell\}$ is a basis of $V_1 \oplus V_2$, actually an eigenbasis for g. So the trace of $(\rho_1 \oplus \rho_2)(g)$ is $\sum_k \lambda_k + \sum_\ell y_\ell = \chi_1(g) + \chi_2(g)$.

2. The dual basis $\{x_k^*\}$ is a basis of V_1. Recall that x_k^* is the linear form which maps x_ℓ to 1 if $\ell = k$, to 0 otherwise. Hence:

$$(g \cdot x_k^*)(x_\ell) = x_k^*(g^{-1} \cdot x_k) = \frac{1}{\lambda_k} x_k^*(x_\ell) = \frac{1}{\lambda_k}\delta_{k,\ell}$$

   which means that $g \cdot x_k^*$ is the linear form $\frac{1}{\lambda_k} x_k^*$; in other words, x_k^* is an eigenvector of g with eigenvalue λ_k^{-1}. But λ_k is a root of unity in $\mathbb{C}$, so $\lambda_k^{-1} = \overline{\lambda_k}$.

   It is then clear that the trace of g on V_1^* is $\sum_k \overline{\lambda_k} = \overline{\chi_1(g)}$.

3. We know that the tensors $x_k \otimes y_\ell$ form a basis of $V_1 \otimes V_2$; moreover each such tensor is an eigenvector for g with eigenvalue $\lambda_k \mu_\ell$. So clearly, the trace of g on $V_1 \otimes V_2$ is:

$$\sum_{k,\ell} \lambda_k \mu_\ell = \left(\sum_k \lambda_k\right)\left(\sum_\ell \mu_\ell\right) = \chi_1(g)\chi_2(g)$$

4. This follows from the last two steps. Indeed, we know from Proposition 2.2.11 that $\mathrm{Hom}(V_1, V_2)$ is equivalent to $V_1^* \otimes V_2$, the character of which is $\overline{\chi_1}\chi_2$. □

Corollary 2.3.10. *Let V be a representation with decomposition $V = \oplus_{i \in I} V_i^{n_i}$. Then $\chi_V = \sum_{i \in I} n_i \chi_i$.*

This is why one sometimes writes $V = \oplus_{i \in I} n_i V_i$, already with a tought towards characters.

2.3.3 Class Functions

It is clear that characters are constant on each conjugacy class. This is perhaps worth a general definition.

Definition 2.3.11. Let G be a group. A class function is a function $G \to \mathbb{C}$ which is constant on each conjugacy class.

These obviously form a complex vector space, of dimension the number of conjugacy classes in G.

Lemma 2.3.12. *Characters are class functions.*

Proof. $\chi(ghg^{-1}) = \mathrm{Tr}(ghg^{-1}) = \mathrm{Tr}(g^{-1}gh) = \mathrm{Tr}h = \chi(h)$. □

Notation 2.3.13. For two class functions α and β on G, let:

$$\langle \alpha, \beta \rangle_G = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$$

Then $\langle ., . \rangle_G$ is a complex scalar product on the space of class functions.

We shall often omit the subscript and simply write $\langle , \rangle$. But of course when two groups are present, subscripts are necessary.

2.3.4 The key theorem

The situation is as good as one could imagine, perhaps even better.

Theorem 2.3.14.

1. *The irreducible characters of G form an orthonormal basis of the space of class functions on G.*
2. *Every representation is determined by its character.*
3. *The multiplicty of an irreducible representation V_i in the regular representation* reg *is exactly* $\deg V_i$.

This subsection is entirely devoted to proving Theorem 2.3.14.

The space of fixed points and orthonormality

Notation 2.3.15. Let V be a representation of G. Let V^G denote the subspace of G-fixed points of V, i.e. $V^G = \{v \in V : \forall g \in G, g \cdot v = v\}$ (clearly a vector subspace of V).

Remarks 2.3.16.

- If V is irreducible, then $V^G = 0$ or $V^G = V$ is the trivial representation.
- $\mathrm{Hom}_G(V_1, V_2) = (\mathrm{Hom}(V_1, V_2))^G$.

Lemma 2.3.17. *Let V be a representation of character χ. Then $\dim V^G = \frac{1}{|G|}\sum_{g\in G}\chi(g)$.*

Proof. Consider the endomorphism of V:

$$\pi = \frac{1}{|G|}\sum_{g\in G} g$$

Notice that for $h \in G$, one has:

$$h \circ \pi = \frac{1}{|G|}\sum_{g\in G} h \circ g = \frac{1}{|G|}\sum_{g'\in G} g' = \pi$$

Hence for any $v \in V$, $\pi(v) \in V^G$. Now for $v \in V^G$, one has $\pi(v) = v$, which proves that π is a projector onto V^G. In particular, its trace is $\dim V^G$, as desired. □

Remark 2.3.18. This is a remark on Burnside's character-like formula obtained in Week 1, Theorem 1.1.16. Let (G, X) be a permutation group and (V_X, ρ_X) the associated permutation representation. Then the number of orbits is:

$$\frac{1}{|G|}\sum_{g\in G}\chi_X(g) = \frac{1}{|G|}\sum_{g\in G}\chi_{\rho_X}(g) = \langle 1, \chi_X\rangle = \dim(V_X)^G$$

which is the dimension of the space of fixed vectors. Indeed, for $O \subseteq X$ an orbit, $f_O = \sum_{x\in O} e_x$ is fixed by G, and every fixed vector is a linear combination of the f_O's (O an orbit).

Always keep Lemma 2.3.17 in mind, it is very useful.

Corollary 2.3.19. *Let V and W be two representations of the group G. Then* $\dim \mathrm{Hom}_G(V,W) = \langle \chi_V, \chi_W \rangle$.

Proof. The character of $\mathrm{Hom}(V,W)$ is $\overline{\chi_V}\chi_W$. So the previous Lemma yields:

$$\dim \mathrm{Hom}_G(V,W) = \dim(\mathrm{Hom}(V,W))^G = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_V(g)}\chi_W(g) = \langle \chi_V, \chi_W \rangle$$

□

Corollary 2.3.20. *The irreducible characters form an orthonormal family in the space of class functions on G.*

Proof. Let V_i and V_j be two irreducible representations (one may have $i = j$). Then by Schur's Lemma, Theorem 2.1.34, $\dim \mathrm{Hom}_G(V_i, V_j) = \delta_{i,j}$. But we just proved that $\langle V_i, V_j \rangle = \dim \mathrm{Hom}_G(V_i, V_j)$, so the irreducible characters form an orthonormal family. □

In particular, since the space of class functions is finite-dimensional, there are finitely many irreducible representations!

Corollary 2.3.21. *Any representation is determined by its character.*

Proof. Since the χ_i's are orthonormal, they are linearly independent. In particular, $\sum_{i\in I} n_i\chi_i$ uniquely defines the integers n_i. □

Corollary 2.3.22. *χ is irreducible iff $\langle \chi, \chi \rangle = 1$. If V is irreducible, then so is V^*.*

Proof. If $\chi = \sum_{i\in I} \chi_i$, then $\langle \chi,\chi \rangle = \sum_{i\in I} n_i^2$ which is 1 iff χ is a χ_i. If V is irreducible, then:

$$\langle \chi_{V^*}, \chi_{V^*} \rangle = \frac{1}{|G|}\sum_{g\in G} \overline{\overline{\chi_V(g)}\chi_V(g)} = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_V(g)}\chi_V(g) = \langle \chi_V, \chi_V \rangle = 1$$

□

Lemma 2.3.23. *Every irreducible character χ_i appears in the regular representation with multiplicity $\deg \chi_i$; in other words, $\mathrm{reg} = \sum_{i\in I} \deg \chi_i \chi_i$.*

Proof. Write $\mathrm{reg} = \sum_{i\in I} n_i\chi_i$. Then for fixed i:

$$n_i = \langle \mathrm{reg}, \chi_i \rangle = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_{\mathrm{reg}}(g)}\chi_i(g)$$

But $\chi_{\mathrm{reg}}(g) = |G|$ if $g = e$, and 0 otherwise. So it remains $n_i = \chi_i(e) = \deg V_i$, as desired. □

End of Lecture 10.

Lecture 11 (More orthogonality; Character tables)

Projections and class functions

Lemma 2.3.24. *Let V be a representation and χ_i be an irreducible character. Then the operator:*

$$\frac{\deg V_i}{|G|}\sum_{g\in G}\overline{\chi_i(g)}g$$

is the projector onto $V_i^{n_i}$ parallel to $\oplus_{j\neq i}V_j^{n_j}$. In particular, $V_i^{n_i}$ is well-defined as a submodule of V.

Proof. Let $\pi = \frac{\deg V_i}{|G|}\sum_{g\in G}\overline{\chi_i(g)}g$. This clearly is a $\mathbb{C}$-linear map. Moreover, it is G-covariant since for $h \in G$:

$$\begin{aligned}\pi \circ h &= \frac{\deg V_i}{|G|}\sum_{g\in G}\overline{\chi_i(g)}g\circ h = \frac{\deg V_i}{|G|}\sum_{g\in G}\overline{\chi_i(h^{-1}gh)}hh^{-1}gh \\ &= h\frac{\deg V_i}{|G|}\sum_{g'\in G}\overline{\chi_i(g')}g' = h\circ\pi\end{aligned}$$

Suppose $V = V_j$ is an irreducible representation (one may have $j = i$ so far). Then we know from Schur's Lemma that π is of the form $\lambda\mathrm{Id}$. If $j \neq i$ then $\lambda = 0$; otherwise $\mathrm{Tr}\pi = \deg V_i\lambda = \deg V_i\langle\chi_i,\chi_j\rangle$ whence $\lambda = 1$. So if $V = V_i$, then $\pi = \mathrm{Id}$.

Now to the general case. Restrict π to each $V_j^{n_j}$; when $j \neq i$ we get the zero map; when $j = i$ we get the identity. This shows that π is the projector onto $V_i^{n_i}$ with kernel $\oplus_{j\neq i}V_j^{n_j}$. In particular, $V_i^{n_i} = \mathrm{im}\pi$ is well-defined. □

This Lemma actually hides a more general idea. In order to show that π was G-covariant we used that χ_i is a class function; never did we use further properties.

Lemma 2.3.25. *Let $\lambda : G \to \mathbb{C}$ be any function. Then λ is a class function iff for any representation V, $\Lambda_V = \frac{1}{|G|}\sum_{g\in G}\lambda(g)g$ is G-covariant.*

Proof. We already know that if λ is a class function, then for any representation, Λ_V is G-covariant: we have just done that. So suppose that λ is not a class function. We look for a representation of G in which Λ_V is not G-covariant. Well, we already know of one representation which encodes all irreducible representations: let us try this one.

Consider the regular representation reg and Λ_{reg}. Is it G-covariant? By assumption on λ, there are $(x, y) \in G^2$ with $\lambda(x) \neq \lambda(yxy^{-1})$. We have a look at:

$$\Lambda_{\mathrm{reg}}\circ y\cdot e_1 = \frac{1}{|G|}\sum_{g\in G}\lambda(g)gy\cdot e_1 = \frac{1}{|G|}\sum_{g\in G}\lambda(g)e_{gy}$$

whose coefficient in e_{yx} is $\frac{1}{|G|}\lambda(yxy^{-1})$, whereas:

$$y\circ\Lambda_{\mathrm{reg}}\cdot e_1 = \frac{1}{|G|}\sum_{g\in G}\lambda(g)e_{yg}$$

whose coefficient in e_{yx} is $\frac{1}{|G|}\lambda(x)$, and both disagree by assumption. This shows that $\Lambda_{\mathrm{reg}}\circ y \neq y\circ\Lambda_{\mathrm{reg}}$: Λ_{reg} is not G-covariant. □

Corollary 2.3.26. *The irreducible characters span the space of class functions.*

Proof. Suppose $\lambda : G \to \mathbb{C}$ is a class function orthogonal to all irreducible characters. Let V be any representation and Λ_V as above. If $V = V_i$ is irreducible, then by Schur's Lemma Λ_V is of the form $\mu\mathrm{Id}$, and $\mathrm{Tr}\Lambda_V = \langle \overline{\lambda}, \chi_i \rangle = \langle \overline{\chi_i}, \mu \rangle = \langle \chi_{V_i^*}, \mu \rangle = 0$ since V_i^* is irreducible, so that Λ_V is trivial for each irreducible representation.

Now suppose that V is the regular representation. Since it is a direct sum of irreducible representations (as every representation), $\Lambda_{\mathrm{reg}} = 0$. But $\Lambda_{\mathrm{reg}} \cdot e_1$ has coefficient in e_g equal to $\frac{1}{|G|}\lambda(g) = 0$, so λ is the zero function. □

Theorem 2.3.14 is now proved.

2.4 Character Tables

We may encode all information on irreducible characters in a character table. The rows are the irreducible characters; the columns are the conjugacy classes. This gives us a square, orthogonal, matrix – provided one does not forget to indicate the number of elements in each conjugacy class!

As an example we give the character table of S_3.

Example 2.4.1. The character table of S_3 is:

	$e_{(\times 1)}$	$\tau_{(\times 3)}$	$\gamma_{(\times 2)}$
χ_{triv}	1	1	1
χ_ε	1	-1	1
χ_ρ	2	0	-1

2.4.1 S4 and A4

S4 has 24 elements divided into 5 conjugacy classes, with representatives: (1) (1 element), (12) (6 elements), (123) (8 elements), (1234) (6 elements), and (12)(34) (3 elements). We already know that the trivial and alternating representations are (distinct) irreducible representations.

	$e_{(\times 1)}$	$(12)_{(\times 6)}$	$(123)_{(\times 8)}$	$(1234)_{(\times 6)}$	$(12)(34)_{(\times 3)}$
triv	1	1	1	1	1
ε	1	-1	1	-1	1

What are the other irreducible characters? We can already start with the permutation representation and factor out the line spanned by $e_1 + e_2 + e_3 + e_4$. This amounts to subtracting triv from perm, and we get a character:

	$e_{(\times 1)}$	$(12)_{(\times 6)}$	$(123)_{(\times 8)}$	$(1234)_{(\times 6)}$	$(12)(34)_{(\times 3)}$
χ_3	3	1	0	-1	-1

This character satisfies $\langle \chi_3, \chi_3 \rangle = \frac{1}{24}(9+6+6+3) = 12$, so it is irreducible. We need two more. First, we know that $\sum_{i=1}^{5}(\deg \chi_i)^2 = 24$, so we need two representations with $d_4^2 + d_5^2 = 13$, which forces a representation of degree 2 and one of degree 3. As for the representation of degree 3, we already have one. The character χ_3 has real values, so there is little point in taking the dual. But we can try and tensor with ε. We get a new character

	$e_{(\times 1)}$	$(12)_{(\times 6)}$	$(123)_{(\times 8)}$	$(1234)_{(\times 6)}$	$(12)(34)_{(\times 3)}$
φ_3	3	-1	0	1	-1

which is again irreducible. This is a general fact.

Lemma 2.4.2. *If V is irreducible and L has degree 1, then $V \otimes L$ is irreducible.*

Proof. Let χ be the character of V and ψ be the character of L; then $V \otimes L$ has character $\chi\psi$, and:

$$\langle \chi\psi, \chi\psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)\psi(g)} \chi(g)\psi(g)$$

But L has degree 1, so $\rho_L(g)$ is (multiplication by) a complex root of unity. It follows that $\psi(g)$ is this complex root of unity, so that $\overline{\psi(g)} = \frac{1}{\psi(g)}$. It remains:

$$\langle \chi\psi, \chi\psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \chi(g) = 1$$

and $V \otimes L$ is irreducible as well. □

We still need one more character, of degree 2. We can use the orthogonality relations to determine it.

	$e_{(\times 1)}$	$(12)_{(\times 6)}$	$(123)_{(\times 8)}$	$(1234)_{(\times 6)}$	$(12)(34)_{(\times 3)}$
triv	1	1	1	1	1
ε	1	-1	1	-1	1
χ_3	3	1	0	-1	-1
φ_3	3	-1	0	1	-1
ψ_2	2	a	b	c	d

Since $\langle \chi_3, \psi_2 \rangle = 0$, one has $6 + 6a - 6c - 3d = 0$. Since $\langle \varphi_3, \psi_2 \rangle = 0$, one has $6 - 6a + 6c - 6d = 0$. Combining the two, $d = 2$ and $c = a$. Since $\langle \text{triv}, \psi_2 \rangle = 0$, one has $8 + 12a + 8b = 0$; since $\langle \varepsilon, \psi_2 \rangle = 0$, one has $8 - 12a + 8b = 0$. Combining the two, $a = 0$ and $b = -1$. The full character table is then:

S_4	$e_{(\times 1)}$	$(12)_{(\times 6)}$	$(123)_{(\times 8)}$	$(1234)_{(\times 6)}$	$(12)(34)_{(\times 3)}$
triv	1	1	1	1	1
ε	1	-1	1	-1	1
χ_3	3	1	0	-1	-1
φ_3	3	-1	0	1	-1
ψ_2	2	0	-1	0	2

Remark 2.4.3. Observe that we can now describe the irreducible representation behind ψ_2; it is 2-dimensional. Since $(12)(34)$ has character 2, it acts as the identity matrix. So the normal subgroup consisting of bitranspositions $K = \{\text{Id}, (12)(34), (13)(24), (14)(23)\}$ acts trivially. So we actually have an irreducible action of $S_4/K \simeq S_3$, which is its non-trivial 2-dimensional representation.

End of Lecture 11.

Lecture 12 (More character tables)

A4 has 12 elements, but the conjugacy class in S_4 of (123) becomes two distinct conjugacy classes, as follows:

$$e_{(\times 1)} \quad (123)_{(\times 4)} \quad (132)_{(\times 4)} \quad (12)(34)_{(\times 3)}$$

There is of course the trivial representation. And apart from that? Well, we could have a look at what A_4 does to the irreducible representations of S_4. But one must be careful as there is no reason for them to remain irreducible. Talking about that, observe that ε, a representation of S_4, has degree 1; so that the restriction of ε to A_4 remains irreducible. But it's the trivial representation of A_4!

Caution. Let G be a group and $H \leq G$ a subgroup.

- If V is an irreducible representation of G, then V need not be irreducible as a representation of H.
- If V and W are representations of G which are not G-equivalent, then they may be equivalent as representations of H.

So we proceed with care and compute in each case $\langle \chi, \chi \rangle$ with respect to A_4. χ_3 takes the values $3, 0, 0, -1$, so $\langle \chi_3, \chi_3 \rangle = 1$; χ_3 remains irreducible. φ_3 takes the values $3, 0, 0, -1$ as well - but it's the same. Hence, the restrictions to A_4 of φ_3 and χ_3 become A_4-equivalent. We still need two more representations.

There is ψ_2. Now in A_4, $\langle \psi_2, \psi_2 \rangle = 2$, so ψ_2 has become the sum of two irreducible representations, which must be of degree 1. We could determine them using the orthogonality relations. But it is simpler to view them as morphisms $\rho, \sigma : A_4 \to \mathbb{C}^\times$. Since $K = \{\mathrm{Id}, (12)(34), (13)(24), (14)(23)\}$ is normal in A_4 and $A_4/K \simeq \mathbb{Z}_3\mathbb{Z} \hookrightarrow \mathbb{C}^\times$, we simply let $\rho(123) = j$ (with $j^3 = 1$) and $\sigma(123) = j^2$. Here is the character table of A_4.

A_4	$e_{(\times 1)}$	$(123)_{(\times 4)}$	$(132)_{(\times 4)}$	$(12)(34)_{(\times 3)}$
triv	1	1	1	1
χ_3	3	0	0	-1
ρ	1	j	j^2	1
σ	1	j^2	j	1

Observe how $\sigma = \rho^*$, the dual representation.

2.4.2 S5

We shall treat S_5 by brute force, but different and better approaches are possible.

1. One should use what one knows about interesting actions of S_5 to create permutation representations (action on the Sylow 5-subgroups, action on a regular polyhedron). This approach requires more mathematical culture.
2. At some point, we shall go to a tensor square, which is too brutal; a symmetric or alternating power (see §3.2.3 below) will be better. We shall come to this later.

S_5 has 120 elements; as in every symmetric group, conjugacy classes only depend on the type of the permutation. We thus have seven conjugacy classes, and already two irreducible representations:

	e $(\times 1)$	(12) $(\times 10)$	(123) $(\times 20)$	(1234) $(\times 30)$	$(12)(34)$ $(\times 15)$	(12345) $(\times 24)$	$(12)(345)$ $(\times 20)$
triv	1	1	1	1	1	1	1
ε	1	-1	1	-1	1	1	-1

We need five more representations; it is unrealistic at this point to compute the dimensions. So we take the permutation representation perm; the character is the number of fixed points and is readily computed:

	e $(\times 1)$	(12) $(\times 10)$	(123) $(\times 20)$	(1234) $(\times 30)$	$(12)(34)$ $(\times 15)$	(12345) $(\times 24)$	$(12)(345)$ $(\times 20)$
χ_{perm}	5	3	2	1	1	0	0

In particular, $\langle \chi_{\text{perm}}, \chi_{\text{perm}} \rangle = \frac{1}{120}(25 + 90 + 80 + 30 + 15) = 2$. Moreover, $\langle \chi_{\text{perm}}, \chi_{\text{triv}} \rangle = \frac{1}{120}(5+30+40+30+15) = 1$, so $\chi_4 = \chi_{\text{perm}} - \chi_{\text{triv}}$ is orthogonal to χ_{triv}, and easily proved to be irreducible. So is $\chi_4 \chi_\varepsilon$. For the moment the table is as follows:

	e $(\times 1)$	(12) $(\times 10)$	(123) $(\times 20)$	(1234) $(\times 30)$	$(12)(34)$ $(\times 15)$	(12345) $(\times 24)$	$(12)(345)$ $(\times 20)$
triv	1	1	1	1	1	1	1
ε	1	-1	1	-1	1	1	-1
χ_4	4	2	1	0	0	-1	-1
$\chi_4\chi_\varepsilon$	4	-2	1	0	0	-1	1

We need three more irreducible characters. Let a, b, c be their degrees; one has $a^2 + b^2 + c^2 = 120 - 1 - 1 - 16 - 16 = 86$. Thus $\{a, b, c\} = \{5, 5, 6\}$ or $\{1, 6, 7\}$. The latter does not seem very likely. First, we shall prove in week 3 that the degrees must divide the order of G; for the moment this is no argument. Second, a degree 1 representation amounts to a normal subgroup with cyclic quotient, and you know perhaps that A_5 is the only non-trivial normal subgroup of S_5 (which corresponds to ε), so there can be no extra degree 1 representation. Again, the argument involved a bit of group theory, so let us pretend we do not know the degrees.

Where should we look? We need to find a bigger representation and decompose it. The regular representation is way to large for this purpose. So we try another approach (to which we shall come later with more accurate tools), and go to a tensor square. Let $\Phi = \chi_4^2$; the values are easily computed:

	e $(\times 1)$	(12) $(\times 10)$	(123) $(\times 20)$	(1234) $(\times 30)$	$(12)(34)$ $(\times 15)$	(12345) $(\times 24)$	$(12)(345)$ $(\times 20)$
Φ	16	4	1	0	0	1	1

One then sees that:

- $\langle \Phi, \Phi \rangle = \frac{1}{120}(256 + 160 + 20 + 24 + 20) = 4$,
- $\langle \Phi, \chi_{\text{triv}} \rangle = \frac{1}{120}(16 + 40 + 20 + 24 + 20) = 1$,
- $\langle \Phi, \chi_\varepsilon \rangle = \frac{1}{120}(16 - 40 + 20 + 24 - 20) = 0$,

- $\langle \Phi, \chi_4 \rangle = \frac{1}{120}(64 + 80 + 20 - 24 - 20) = 1$,
- $\langle \Phi, \chi_4\chi_\varepsilon \rangle = \frac{1}{120}(64 - 80 + 20 - 24 + 20) = 0$

Hence $\Phi - \chi_{\text{triv}} - \chi_4$ is the sum of two new irreducible representations; but it has degree $16 - 1 - 4 = 11$, and this is why the three missing irreducible representations have degree $5, 5, 6$.

Let χ_6 be the one of degree 6. Since it is the only such, one must have $\chi_6 = \chi_6\chi_\varepsilon$ (which is again irreducible), so that the shape of χ_6 is:

	e $(\times 1)$	(12) $(\times 10)$	(123) $(\times 20)$	(1234) $(\times 30)$	$(12)(34)$ $(\times 15)$	(12345) $(\times 24)$	$(12)(345)$ $(\times 20)$
χ_6	6	0	a	0	b	c	0

We use the orthogonality relations to determine a, b, c:

- $0 = 120\langle \chi_6, \chi_{\text{triv}} \rangle = 6 + 20a + 15b + 24c$;
- $0 = 120\langle \chi_6, \chi_4 \rangle = 24 + 20a - 24c$;
- the other two yield nothing new.

We get $c = 1 + \frac{5}{6}a$ and $b = -2 - \frac{8}{3}a$. Now

$$\begin{aligned} 120 &= 120\langle \chi_6, \chi_6 \rangle \\ &= 36 + 20a^2 + 15b^2 + 24c^2 \\ &= 120 + 200a + 430a^2 \end{aligned}$$

which means that either $a = 0$ or $a = -\frac{20}{43}$. Of course we prefer the first solution, and we are right.

Lemma 2.4.4. *A character cannot take non-integral, rational values.*

Proof. Suppose $x = \frac{p}{q}$ where p and q are coprime is the value of a character. As the value of a character, x is a sum of roots of unity, hence an algebraic integer, meaning that there is a polynomial $P = X^n + a_{n-1} + \cdots + a_0 \in \mathbb{Z}[X]$ with leading coefficient 1 such that $P(x) = 0$ (if this is not clear for the moment, in particular the fact that P has leading coefficient 1, be reassured: we could come back to this topic later, in Chapter 3.

Multiplying by q^n one has:

$$p^n + a_{n-1}qp^{n-1} + \cdots + a_0q^n = 0$$

Since p and q are coprime, one has $q = 1$, that is, x is an integer. $\square$

So $a = 0$. It remains $c = 1$ and $b = -2$, whence:

	e $(\times 1)$	(12) $(\times 10)$	(123) $(\times 20)$	(1234) $(\times 30)$	$(12)(34)$ $(\times 15)$	(12345) $(\times 24)$	$(12)(345)$ $(\times 20)$
χ_6	6	0	0	0	-2	1	0

We need to determine the last two irreducible characters. But we already know that $\Phi - \chi_{\text{triv}} - \chi_4 - \chi_6$ will yield one, say:

	e $(\times 1)$	(12) $(\times 10)$	(123) $(\times 20)$	(1234) $(\times 30)$	$(12)(34)$ $(\times 15)$	(12345) $(\times 24)$	$(12)(345)$ $(\times 20)$
χ_5	5	1	-1	-1	1	0	1

and $\chi_5\chi_\varepsilon \neq \chi_5$ is the last irreducible character.

The character table of S_5 is thus:

S_5	e $(\times 1)$	(12) $(\times 10)$	(123) $(\times 20)$	(1234) $(\times 30)$	$(12)(34)$ $(\times 15)$	(12345) $(\times 24)$	$(12)(345)$ $(\times 20)$
triv	1	1	1	1	1	1	1
ε	1	-1	1	-1	1	1	-1
χ_4	4	2	1	0	0	-1	-1
$\chi_4\chi_\varepsilon$	4	-2	1	0	0	-1	1
χ_5	5	1	-1	-1	1	0	1
$\chi_5\chi_\varepsilon$	5	-1	-1	1	1	0	-1
χ_6	6	0	0	0	-2	1	0

2.4.3 A5

This will be relatively easier. Keep the character table of S_5 in mind as we shall use it. In A_5, the conjugacy classes and the trivial character are as follows:

	$(1)_{(\times 1)}$	$(123)_{(\times 20)}$	$(12)(34)_{(\times 15)}$	$(12345)_{(\times 12)}$	$(15432)_{(\times 12)}$
χ_{triv}	1	1	1	1	1

We need four more characters. Let ψ_4 be the restriction of χ_4 from S_5 to A_5; it takes the values:

	$(1)_{(\times 1)}$	$(123)_{(\times 20)}$	$(12)(34)_{(\times 15)}$	$(12345)_{(\times 12)}$	$(15432)_{(\times 12)}$
ψ_4	4	1	0	-1	-1

so that $\langle \psi_4, \psi_4 \rangle = \frac{1}{60}(16 + 20 + 12 + 12) = 1$: ψ_4 is irreducible. Restricting $\chi_4\chi_\varepsilon$ obviously gives the same irreducible character. Let us restrict χ_5 into a character ψ_5:

	$(1)_{(\times 1)}$	$(123)_{(\times 20)}$	$(12)(34)_{(\times 15)}$	$(12345)_{(\times 12)}$	$(15432)_{(\times 12)}$
ψ_5	5	-1	1	0	0

so that $\langle \psi_5, \psi_5 \rangle = \frac{1}{60}(25+20+15) = 1$; ψ_5 is irreducible, restricting $\chi_5\chi_\varepsilon$ yields the same character. For the moment the table looks like:

	$(1)_{(\times 1)}$	$(123)_{(\times 20)}$	$(12)(34)_{(\times 15)}$	$(12345)_{(\times 12)}$	$(15432)_{(\times 12)}$
χ_{triv}	1	1	1	1	1
ψ_4	4	1	0	-1	-1
ψ_5	5	-1	1	0	0

We need two more; let a, b be their degrees. Then $a^2+b^2 = 60-1-16-25 = 18$, so we need two 3-dimensional irreducible representations, say ψ_3 and φ_3:

	$(1)_{(\times 1)}$	$(123)_{(\times 20)}$	$(12)(34)_{(\times 15)}$	$(12345)_{(\times 12)}$	$(15432)_{(\times 12)}$
ψ_3	3	a	b	c	d
φ_3	3	a'	b'	c'	d'

At this point we use orthogonality:

- $0 = 60\langle \psi_3, \psi_{\text{triv}} \rangle = 3 + 20a + 15b + 12c + 12d$,

- $0 = 60\langle \psi_3, \psi_4 \rangle = 12 + 20a - 12c - 12d$,
- $0 = 60\langle \psi_3, \psi_5 \rangle = 15 - 20a + 15b$

From this follows $15 + 40a + 15b = 0 = 15 - 20a + 15b$, whence $a = 0$. Then $b = -1$ and $c + d = 1$. Moreover,

$$60 = 60\langle \psi_3, \psi_3 \rangle = 9 + 15 + 12c^2 + 12d^2$$

so $c^2 + d^2 = 3$. It follows $cd = -1$ and there are two possibilities: $\{c, d\} = \{\frac{1 \pm \sqrt{5}}{2}\}$ which give rise to the missing two irreducible characters. The character table of A_5 is thus as follows:

A_5	$(1)_{(\times 1)}$	$(123)_{(\times 20)}$	$(12)(34)_{(\times 15)}$	$(12345)_{(\times 12)}$	$(15432)_{(\times 12)}$
χ_{triv}	1	1	1	1	1
ψ_3	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
φ_3	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
ψ_4	4	1	0	-1	-1
ψ_5	5	-1	1	0	0

Observe how closely related are ψ_3 and φ_3. They are conjugate under some Galois action.

End of Lecture 12.

Week 3: Topics in Representation Theory

Lecture 13 (Refreshments; Induced representation)

3.1 Induced representations

We now consider the following setting: G is a group and $H \leq G$ a subgroup. Do representations of G and H relate to each other? Of course from G to H, one can always restrict.

Notation 3.1.1. Let V be a representation of G. Then V, as a representation of H, is wisely denoted $\mathrm{Res}^G_H V$, or $\mathrm{Res} V$.

As we have observed in Chapter 2, §2.4.1, V may be irreducible but not $\mathrm{Res}^G_H V$; V_1 and V_2 may be non-isomorphic but $\mathrm{Res}^G_H V_1 \simeq_{\mathbb{C}[H]} \mathrm{Res}^G_H V_2$.

3.1.1 Induced representations

Suppose G has a subgroup $H \leq G$ such that H acts on X. Is it possible to naturally extend the action of H to an action of G?

Counter-example 3.1.2. Consider the identity mapping $\varphi : S_4 \to S_4$. There is no $\Phi : S_5 \to S_4$ for a simple reason. Φ cannot be injective, so $\ker \Phi$ is a non-trivial, normal subgroup of S_5: we know from pure group theory that it must be either A_5 or S_5. But both meet S_4 non-trivially, a contradiction to the fact that φ is injective.

Now consider the natural action of S_4 on $X = \{1, 2, 3, 4\}$: it corresponds to the morphism $\varphi : S_4 \to S_4$ just described. Since there is no extension to S_5, it means that one cannot extend the action to an action of S_5 on X.

One should therefore not try to brutally extend the action. We go back to a simple case in order to gain some understanding. Suppose that G has a subgroup H acting transitively on X. Let $K = H_x$ be a stabilizer. Then we know that the action of H on X is equivalent to that of H on H/K. The natural way to extend this action to an action of G, is to let G act on G/K, which is bigger than $H/K = X$ "by a factor G/H". This suggests that in order to construct an action of G, one should in general expand X by some coset construction.

All this suggests the following. First we should write $g \in G$ as a member of a left coset: $g = ah$ with $h \in H$. Observe that neither is well-defined so we need to keep track of the operation. This can be done by chosing a set of representatives for G/H: $\{a_1, \ldots, a_r\}$ with $r = [G : H]$. Then $g = a_k h$ should act on $x \in X$ in a sense as h does, at least provided $a_H = e$. But we also need to keep track of a_k. This suggests to take r copies of X, indexed by G/H.

We have given the flavour of the extension for group actions. Of course for representations it is exactly the same idea. Here is the formal construction.

Definition 3.1.3. Let W be a representation of H. The induced representation of W from H to G, denoted $\mathrm{Ind}_H^G W$, is defined as follows. Fix a set of representatives $\{a_c : c \in G/H\}$, with $a_H = e$. Let $V = \oplus_{c \in G/H} a_c W$ be a vector space obtained as a direct sum of $[G : H]$ copies of W (denoted $a_c W$). Now for $g \in G$ and $v = a_c w \in V$ with $w \in W$, write $g a_c = a_{c'} h$ and:

$$g \cdot v = a_{c'}(h \cdot w)$$

Lemma 3.1.4. *This defines a representation of G; H acts on $a_H W$ as on W.*

Proof. Let $g \in G$ and $v = a_c w \in a_c W$; we write $g a_c = a_{c'} h$. Let $g' \in G$; write $g' a_{c'} = a_{c''} h'$. We compute:

$$g' \cdot (g \cdot v) = g' \cdot (a_{c'}(hw)) = a_{c''}(h' h w)$$

On the other hand, $(g'g) a_c = g' a_{c'} h = a_{c''} h' h$, so that:

$$(g'g)v = a_{c''}(h' h w)$$

as well, so we do have an action of G on V, which clearly is linear. Moreover, since $a_H = 1$, one has $h a_H = a_H h$ and:

$$h \cdot (a_H w) = a_H (hw)$$

meaning that H acts on $a_H W$ as on W. □

So identifying W with $a_H W$, we may consider that $W \leq \mathrm{Ind}_H^G W$. Of course one first has to wonder whether this depends on the choice of representatives. The answer is no.

Lemma 3.1.5. *The construction does not depend on the choice of representatives (provided of course $a_H = 1$).*

Proof. Let $V_1 = \oplus_{c \in G/H} a_c W$ and $V_2 = \oplus_{c \in G/H} b_c W$ for two sets of representatives $\{a_c\}$ and $\{b_c\}$ with $a_H = b_H = 1$. We shall construct a G-isomorphism between the two modules. For every $c \in G/H$, let $h_c = b_c^{-1} a_c \in H$; one has $h_H = 1$.

Let $\varphi(a_c w) = b_c(h_c w)$; this clearly is a vector space isomorphism, and only G-covariance must be proved. So suppose $g a_c = a_{c'} h_1$ and $g b_c = b_{c''} h_2$. Then on the one hand,

$$\varphi(g \cdot a_c w) = \varphi(a_{c'} h_1 w) = b_{c'}(h_{c'} h_1 w)$$

and on the other hand,

$$g \cdot \varphi(a_c w) = g \cdot (b_c(h_c w)) = b_{c''}(h_2 h_c w)$$

But by construction, for any class, $a_c = b_c h_c$, so that:

$$\underbrace{b_{c''} h_2}_{g b_c} h_c = g b_c h_c = g a_c = a_{c'} h_1 = b_{c'} h_{c'} h_1$$

so that $b_{c''} = b_{c'}$ and $h_2 h_c = h_{c'} h_1$, proving that φ is G-covariant. □

This is why we shall simply write cW instead of a_cW.

Example 3.1.6.

- Let W be the trivial representation of H. Then $\mathrm{Ind}_H^G W$ is the permutation representation associated to $(G, G/H)$.

  Hence be very careful that $\mathrm{Res}_H^G \mathrm{triv}_G = \mathrm{triv}_H$, but $\mathrm{Ind}_H^G \mathrm{triv}_H \neq \mathrm{triv}_G$.

- Let W be the regular representation of H. Then $\mathrm{Ind}_H^G W$ is the regular representation of G.

End of Lecture 13.

Lecture 14 (Frobenius reciprocity)

3.1.2 Frobenius' reciprocity formula

We now turn to the character of an induced representation.

Lemma 3.1.7. *Let W be a representation of H and $V = \mathrm{Ind}_H^G W$. Then:*

$$\chi_V(g) = \sum_{gc=c} \chi_W(a_c^{-1} g a_c)$$

where the sum is over cosets c such that $gc = c$, and a_c is any element of c.

Proof. Since χ_W is a class function on H, the right-hand term does not depend on $a_c \in c$, so we may as well take the ones used to define the induced representation. If $ga_c = a_{c'}h$, then g sends the subspace a_cW to $a_{c'}W$; in particular, since we are computing a trace, only the terms when $c' = c$ matter, that is we are summing only over classes with $gc = c$. Fix one such; g acts on cW. Moreover there is $h \in H$ such that $ga_c = a_ch$. Then $g \cdot (a_cw) = a_c(hw)$, so $\mathrm{Tr}(g_{|cW}) = \mathrm{Tr}(h_{|W}) = \chi_W(h) = \chi_W(a_c^{-1} g a_c)$ and we are done. □

Equivalent formulation: let $\tilde{\chi}_W$ coincide with χ_W on H, and be 0 on $G \setminus H$. Then:

$$\chi_V(g) = \frac{1}{|H|} \sum_{a \in G} \tilde{\chi}_W(a^{-1} g a)$$

Recall that, thanks to the requirement that $a_H = 1$, we could always view W as a subspace of $\mathrm{Ind}_H^G W$, identifying W with the copy associated to the trivial coset $1H = H$.

Proposition 3.1.8. *Let W be a representation of H and V be a representation of G. Let $f : W \to V$ be H-covariant. Then there is a unique $F : \mathrm{Ind}_H^G W \to V$ which is G-covariant and extends f.*

Proof. What does covariance imply? Considering $a_c \in G$ and $w \in W$, one must have:

$$F(a_cw) = F(a_c \cdot a_Hw) = a_c \cdot F(a_Hw) = a_c \cdot f(w)$$

since F must extend f. So we don't have much choice. We let $F(a_cw) = a_c \cdot f(w)$. This extends to a linear map; since $a_H = 1$, this does extend f. It remains to show that this is G-covariant. So let $g \in G$ and write $ga_c = a_{c'}h$. Then:

$$F(g \cdot a_cw) = F(a_{c'}hw) = a_{c'}f(hw) = a_{c'}hf(w)$$

by H-covariance of f, and it remains $F(g \cdot a_c w) = g a_c \cdot f(w) = g \cdot F(a_c w)$, as desired. ☐

Corollary 3.1.9 (Frobenius' reciprocity)**.** *Let W be a representation of H and V be a representation of G. Then $\langle \chi_{\mathrm{Ind}W}, \chi_V \rangle_G = \langle \chi_W, \chi_{\mathrm{Res}V} \rangle_H$.*

Caution. One scalar product is computed over H, and the other one over G.

Proof. Proposition 3.1.8 tells us that $\mathrm{Hom}_G(\mathrm{Ind}W, V)$ is isomorphic (as a vector space) with $\mathrm{Hom}_H(W, \mathrm{Res}V)$, in particular:

$$\dim \mathrm{Hom}_G(\mathrm{Ind}W, V) = \dim \mathrm{Hom}_H(W, \mathrm{Res}V)$$

Now bearing Lemma 2.3.17 (or Corollary 2.3.19 following it) in mind,

$$\langle \chi_{\mathrm{Ind}W}, \chi_V \rangle_G = \dim \mathrm{Hom}_G(\mathrm{Ind}W, V)$$

and:

$$\langle \chi_W, \chi_{\mathrm{Res}V} \rangle_H = \dim \mathrm{Hom}_H(W, \mathrm{Res}V)$$

so there is actually nothing to prove! ☐

Since characters linearly generate the space of class functions, Corollary 3.1.9 extends to arbitrary class functions, provided one uses a compatible definition of $\mathrm{Ind}_H^G \alpha$ for a class function α:

$$\mathrm{Ind}_H^G \alpha(g) = \sum_{gc=c} \alpha(a_c^{-1} g a_c)$$

where the sum is over cosets $c \in G/H$ such that $gc = c$, and a_c is any element of c.

3.1.3 Character kernels

Recall that the degree of a character is the dimension of the underlying space. It is equal to $\chi(1)$.

Notation 3.1.10. For a representation $\rho : G \to \mathrm{GL}(V)$, let $\ker \chi = \{g \in G : \chi(g) = \dim V\}$.

Lemma 3.1.11. $\ker \chi = \ker \rho$. *In particular,* $\ker \chi$ *is a normal subgroup of G.*

Proof. It is immediate that $\ker \rho \leq \ker \chi$, as $\mathrm{Tr}\mathrm{Id} = \dim V$. We prove the converse inclusion. Let $d = \dim V = \chi(1)$.

Let $g \in G$; recall that since g has finite order, so does $\rho(g)$. In particular, $\rho(g)$ is diagonalizable and its eigenvalues λ_k ($k = 1 \dots d$) are roots of unity. Suppose $g \in \ker \chi$, so that $\chi(g) = d$. Then:

$$\sum_{k=1}^{d} \lambda_k = d$$

where the λ_k's are roots of unity. It follows that for each k, $\lambda_k = 1$. In particular $\rho(g) = \mathrm{Id}$, and $g \in \ker \rho$. ☐

The rest of this subsection is an interesting digression, explaining how simplicity can be deduced from inspection of the character table.

Proposition 3.1.12. *Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. Then:*

$$\cap_{i=1}^r \ker \chi_i = \{e\}$$

Proof. Let K be the intersection, clearly a normal subgroup of G. Each χ_i is the character of a representation (V_i, ρ_i) of degree $n_i = \dim V_i$. Now i being fixed, $K \leq \ker \rho_i$, so one may factor $\rho_i : G \to \mathrm{GL}(V_i)$ via $\bar{\rho}_i : G/K \to \mathrm{GL}(V_i)$.

We claim that $\bar{\rho}_i$ is irreducible; actually, since K acts trivially, G-invariant subspaces correspond with G/K-invariant subspaces. Let ψ_i be the character of $\bar{\rho}_i$ (writing $\bar{\chi}_i$ would yield confusion with complex conjugation).

Moreover, $\psi_i \neq \psi_j$ for $i \neq j$, since G and G/K have the same action on V_i and the same action on V_j.

Now, the χ_i's are the irreducible characters of G, so:

$$\sum_{i=1}^r n_i^2 = |G|$$

On the other hand, the ψ_i's are among the irreducible characters of G/K, so:

$$\sum_{i=1}^r n_i^2 \leq |G/K|$$

This proves $K = \{e\}$. □

Proposition 3.1.13. *Let $N \leq G$ be a subgroup. Then N is normal if and only if there is $J \subseteq I$ such that $N = \cap_{j \in J} \ker \chi_j$.*

Proof. Since each $\ker \chi_i$ is normal in G, one implication is obvious. So suppose that N is normal in G, and let π be the projection $G \to G/N$.

Let $(W_1, \sigma_1), \ldots, (W_s, \sigma_s)$ be the irreducible representations of G/N; let ψ_k be the associated characters of G/N. We lift σ_k to a representation of G by letting:

$$\tau_k = \sigma_k \circ p : G \to \mathrm{GL}(W_k)$$

Each τ_k is irreducible as a G-invariant subspace must be G/N-invariant, hence 0 or W_k. Moreover, the τ_k's are distinct. Hence we have obtained s irreducible characters of G; say $\{\tau_1, \ldots, \tau_s\} = \{\chi_j : j \in J\}$ for $J \subseteq I$ a subset with s elements.

By the previous proposition applied to G/N, $\cap_{k=1}^s \ker \sigma_s = \{\bar{e}\}$, so:

$$\cap_{k=1}^s \ker \tau_s = N$$ □

Remark 3.1.14.

- In particular, G is simple iff for every irreducible character $\chi \neq 1$ and $g \neq e$, $\chi(g) \neq \chi(e)$.
- Using the character table, one may find whether G is solvable or not: one should look for normal series with abelian quotients.

End of Lecture 14.

Lecture 15 (Frobenius' complement theorem)

3.1.4 Frobenius' complement theorem

We now prove a remarkable application of representation theory.

Definition 3.1.15. Let G be a finite group and $H \leq G$ a subgroup. H is malnormal if for any $g \in G \setminus H$, one has $gHg^{-1} \cap H = \{e\}$.

Theorem 3.1.16 (Frobenius)**.** *Let $H \leq G$ be a malnormal subgroup and $N = \{e\} \cup (G \setminus \cup_{g \in G} gHg^{-1})$. Then N is a normal subgroup of G and $G = N \rtimes H$.*

Example 3.1.17. $\mathbb{K}_+ \rtimes \mathbb{K}^*$.

For the moment N is just a union of conjugacy classes; before the real ideas come in, we start with a very basic observation.

Lemma 3.1.18. $|N| = \frac{|G|}{|H|}$. *If M is a normal subgroup with $M \cap H = 1$, then $M \subseteq N$.*

Proof. Distinct conjugates of H intersect on the identity element; obviously $N_G(H) = H$, so $\cup_{g \in G} H^g$ has exactly $|G/H|(|H| - 1) + 1 = |G| - |G/H| + 1$ elements. Hence:

$$|N| = |G/H| = \frac{|G|}{|H|}$$

Now suppose $M \trianglelefteq G$ satisfies $M \cap H = 1$. Then for all $g \in G$, $M \cap gHg^{-1} = g(M \cap H)g^{-1} = 1$, and $M \subseteq N$. □

Proving Theorem 3.1.16 will require a few steps. The main idea is to show that every irreducible character of H extends to an irreducible character of G. Our first hint is to try Indψ. Unfortunately, ResInd$\psi \neq \psi$, since for instance Ind$\psi(1) = [G : H]\psi(1)$. This suggests to make the value at e be 0 by using an intergral-valued linear combination. This will prove successful.

Lemma 3.1.19. *Let α be a class function on H with $\alpha(1) = 0$. Let $\mathrm{Ind}_H^G \alpha$ be as in Lemma 3.1.7, that is:*

$$\mathrm{Ind}_H^G \alpha(g) = \sum_{gc=c} \alpha(a_c^{-1} g a_c)$$

where the sum is over cosets $c \in G/H$ such that $gc = c$, and a_c is any element of c. Then $\mathrm{Ind}_H^G \alpha$ extends α.

Proof. Let $\check{\alpha} = \mathrm{Ind}_H^G \alpha$; we wish to show that $\check{\alpha}$ extends α. Let $h \in H$. If $h = 1$, then by assumption $\alpha(1) = 0$, and $\check{\alpha}(h) = \sum_{c \in G/H} \alpha(1) = 0$ as well. We may assume that $h \neq 1$. Then:

$$\check{\alpha}(h) = \sum_{hc=c} \alpha(a_c^{-1} h a_c)$$

The sum is over those cosets $c = a_c H$ such that $ha_c H = a_c H$, in other words, such that $h \in H \cap a_c H a_c^{-1}$. So there is actually only term: the one for the coset H, and:

$$\check{\alpha}(h) = \alpha(a_H^{-1} h a_H) = \alpha(h)$$

since α is a class function. □

In particular, if H is malnormal in G, then $\text{Res}_H^G\text{Ind}_H^G\alpha = \alpha$ for any class function on H with $\alpha(1) = 0$.

Lemma 3.1.20. *Every irreducible character of H extends to an irreducible character of G.*

Proof. We shall use ψ for characters of H; in particular the trivial representation of H is denoted ψ_{triv}. Let $\psi \neq \psi_{\text{triv}}$ be a non-trivial irreducible character of H.

We wish to extend ψ to an irreducible character of G; as observed, $\text{Ind}\psi$ will not work, in particular because $\text{Ind}\psi(1) \neq 0$. So we let $\alpha = \psi - \psi(1)\psi_{\text{triv}}$, a class function with $\alpha(1) = 0$, and we consider $\text{Ind}\alpha$. A couple of computations are required.

By Frobenius' reciprocity (which extends to any central function with our notion of Ind),

$$\begin{aligned}\langle \text{Ind}_H^G\alpha, \text{Ind}_H^G\alpha\rangle_G &= \langle \alpha, \text{Res}_H^G\text{Ind}_H^G\alpha\rangle_H \\ &= \langle \alpha, \alpha\rangle_H \\ &= \langle \psi - \psi(1)\psi_{\text{triv}}, \psi - \psi(1)\psi_{\text{triv}}\rangle_H \\ &= \langle \psi, \psi\rangle_H - 2\psi(1)\langle \psi, \psi_{\text{triv}}\rangle_H + \psi(1)^2\langle \psi_{\text{triv}}, \psi_{\text{triv}}\rangle_H \\ &= 1 + \psi(1)^2\end{aligned}$$

using orthogonality. Moreover, denoting χ_{triv} the trivial character of G, always by Frobenius' reciprocity,

$$\langle \text{Ind}_H^G\alpha, \chi_{\text{triv}}\rangle_G = \langle \alpha, \psi_{\text{triv}}\rangle_H = \langle \psi - \psi(1)\psi_{\text{triv}}, \psi_{\text{triv}}\rangle_H = -\psi(1)$$

Let $\chi = \text{Ind}_H^G\alpha + \psi(1)\chi_{\text{triv}}$, a class function on G. Observe that $\text{Res}\chi = \text{Res}\text{Ind}\alpha + \psi(1)\text{Res}\chi_{\text{triv}} = \alpha + \psi_1\psi_{\text{triv}} = \psi$, so χ does extend ψ. Moreover, by construction, $\langle \chi, \chi_{\text{triv}}\rangle = 0$ and:

$$\begin{aligned}\langle \chi, \chi\rangle_G &= \langle \chi, \text{Ind}_H^G\alpha\rangle_G \\ &= \langle \text{Ind}_H^G\alpha, \text{Ind}_H^G\alpha\rangle_G + \psi(1)\langle \chi_{\text{triv}}, \text{Ind}_H^G\alpha\rangle_G \\ &= 1 + \psi(1)^2 - \psi(1)^2\end{aligned}$$

so that $\langle \chi, \chi\rangle_G = 1$. One cannot say that χ is an irreducible character yet: it is not known to be a character!

Now since α is a difference of characters, $\text{Ind}\alpha$ is one as well, so $\chi = \text{Ind}\alpha + \psi(1)\chi_{\text{triv}} = \text{Ind}\psi - \psi(1)\text{Ind}\psi_{\text{triv}} + \psi(1)\chi_{\text{triv}}$ is a difference of characters. Hence χ is of the form:

$$\chi = \sum_{i\in I} n_i\chi_i$$

with the n_i's in $\mathbb{Z}$ (the χ_i's are the irreducible characters of G). Now $\langle \chi, \chi\rangle_G = 1$ does imply that $\pm\chi$ is an irreducible character. But one has:

$$\chi(1) = \text{Ind}_H^G\alpha(1) + \psi(1)\chi_{\text{triv}}(1) = \alpha(1) + \psi(1) = \psi(1) \in \mathbb{N}$$

Hence χ is an irreducible character of G. Moreover, for $h \in H$,

$$\chi(h) = \text{Ind}_H^G\alpha(h) + \psi(1)\chi_{\text{triv}}(h) = \alpha(h) + \psi(1) = \psi(h)$$

which means that χ is an irreducible character of G extending ψ. ∎

Proof of Theorem 3.1.16. Let $\{\psi_j : j \in J^\#\}$ be the set of non-trivial irreducible characters of H and for each $j \in J^\#$, χ_j the irreducible character of G extending ψ_j constructed in Lemma 3.1.20. Let:

$$K = \cap_{J^\#} \ker \chi_j \trianglelefteq G;$$

we know that K is a normal subgroup. We claim that $K = N$.

If $h \in K \cap H$, then for every $j \in J^\#$, $\chi_j(h) = \psi_j(h) = 1$, so that $h \in \cap_{J^\#} \ker \psi_j = \{e\}$. It follows $K \subseteq N$ by Lemma 3.1.18. Now if $g \in N \setminus \{1\}$, then for every $j \in J^\#$,

$$\chi_j(g) = \mathrm{Ind}_H^G \alpha(g) + \psi_j(1)\chi_{\mathrm{triv}}(g)$$

Observe how for $g \in N \setminus \{1\}$, one has:

$$\mathrm{Ind}_H^G \alpha(g) = \sum_{c \in G/H : gc = c} \alpha(c^{-1}gc)$$

but $gc = c$ only when $c^{-1}gc \in H$, that is when $g \in cHc^{-1}$. Since $g \in N$, this is never the case. So the sum is empty, and all that remains is:

$$\chi_j(g) = \mathrm{Ind}_H^G \alpha(g) + \psi_j(1)\chi_{\mathrm{triv}}(g) = \psi_j(1) = \chi_j(1)$$

so $g \in K$. It follows $K = N$. Since $|K| = [G : H]$ and $K \cap H = \{e\}$, we have $G = N \rtimes H$. □

Corollary 3.1.21. *Let G be a finite group acting transitively on a set X. Suppose that every $g \neq e$ fixes at most one element of X. Then $N = \{$fixed-point free $g \in G\} \cup \{e\}$ is a normal subgroup of G.*

Proof. Let $x \in X$ and $H = \mathrm{Stab}_G(x)$. By assumption, H is malnormal. The Frobenius complement is N. □

Remark 3.1.22. There is no known full proof of Theorem 3.1.16 which avoids representation theory. Some special cases do exist, though.

End of Lecture 15.

3.2 Real representations

Lecture 16 (Quaternions; Pairings and bilinear forms)

3.2.1 The quaternion algebra

Theorem 3.2.1 (Frobenius). *Let $\mathbb{K}$ be a skew-field containing $\mathbb{R}$ in its center and such that $\dim_{\mathbb{R}} \mathbb{K}$ is finite. Then either $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Proof. Let $x \in \mathbb{K} \setminus \mathbb{R}$, if there is such a thing. Then $\mathbb{R}[x]$ is a (commutative) field and $\dim_{\mathbb{R}} \mathbb{R}[x]$ is finite. Hence $\mathbb{R}[x]$ is isomorphic to $\mathbb{C}$; we write it $\mathbb{C}$ (but $\mathbb{C}$ need not be central in $\mathbb{K}$, since perhaps x isn't). Let $i \in \mathbb{C}$ square to -1, and forget about x.

Since $\mathbb{C} \leq \mathbb{K}$, $\mathbb{K}$ is a (left-)complex vector space. We then consider the endomorphism of $\mathbb{K}$:

$$\begin{array}{rccc} f: & \mathbb{K} & \to & \mathbb{K} \\ & x & \mapsto & xi \end{array}$$

By associativity of $\mathbb{K}$, f is an endomorphism of the $\mathbb{C}$-vector space $\mathbb{K}$. Observe that $f^2 = -\mathrm{Id}$, so that f is diagonalizable with eigenvalues $\pm i$. Let $\mathbb{K}_i$ and $\mathbb{K}_{-i}$ be the associated eigenspaces.

If $x \in \mathbb{K}_i$, then x commutes with i and $\mathbb{R}$, so it commutes with $\mathbb{C}$: $\mathbb{C}[x]$ is a commutative field and a finite extension of $\mathbb{C}$, whence equal to $\mathbb{C}$. So $x \in \mathbb{C}$ and $\mathbb{K}_i = \mathbb{C}$. We shall therefore assume that $\mathbb{K} > \mathbb{K}_i$, and prove $\mathbb{K} \simeq \mathbb{H}$.

By assumption there is $x \in \mathbb{K}_{-i} \setminus \{0\}$. As above, $\mathbb{R}[x]$ is isomorphic to $\mathbb{C}$, but it is not our copy called $\mathbb{C}$. So $\mathbb{R}[x] \cap \mathbb{C}$, as a real vector space, can have dimension 0 or 1; it contains $\mathbb{R}$, so it has dimension 1, and one has $\mathbb{R}[x] \cap \mathbb{C} = \mathbb{R}$. On the other hand,

$$x^2 i = xxi = -xix = ix^2$$

so $x^2 \in \mathbb{K}_i = \mathbb{C}$, and also $x \in \mathbb{R}[x]$. Hence $x^2 \in \mathbb{R}$. In $\mathbb{R}[x] \simeq \mathbb{C}$, x^2 already has two roots: $\pm x \notin \mathbb{R}$. So x^2 can't be in $\mathbb{R}$ as there would be a third root. Hence $x^2 < 0$. We may adjust by a real scalar to find $j^2 = -1$ with $j \in \mathbb{K}_{-i}$, and forget about x.

Consider the $\mathbb{R}$-linear map:

$$\begin{array}{rccc} g: & \mathbb{K}_i & \to & \mathbb{K}_{-i} \\ & x & \mapsto & jx \end{array}$$

This is well-defined since for $y \in \mathbb{K}_i$, $jyi = jiy = -ijy$ and $jy \in \mathbb{K}_{-i}$ as desired. Moreover, if $z \in \mathbb{K}_{-i}$, then $jzi = -jiz = ijz$ so $jz \in \mathbb{K}_i$, and $z = j \cdot -jz \in \mathrm{im} g$. So g is a real vector space isomorphism, meaning that $\dim_{\mathbb{R}} \mathbb{K}_{-i} = \dim_{\mathbb{R}} \mathbb{K}_i = \dim_{\mathbb{R}} \mathbb{C} = 2$. Since $\mathbb{K} = \mathbb{K}_i \oplus \mathbb{K}_{-i}$, one has $\dim_{\mathbb{R}} \mathbb{K} = 4$.

So far $i^2 = j^2 = -1$ with $ij = -ji$. Let $k = ij$; clearly $k^2 = -1$ and $ki = -ik = j$, $jk = -kj = i$. Notice that j and k are linearly independent, since $i \notin \mathbb{R}$. Hence using $\mathbb{K}_i \cap \mathbb{K}_{-i} = 0$, also $\{1, i, j, k\}$ are linearly independent, and form therefore a basis of $\mathbb{K}$ over $\mathbb{R}$, which concludes the proof. □

The curtain opens on a real vector space.

3.2.2 Real, complex, and quaternionic representations

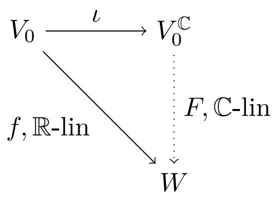
Definition 3.2.2. A representation (V, ρ) is real if there is a basis of V in which all matrices $\rho(g)$, $g \in G$, have real coefficients.

Remark 3.2.3. This means that there is a real vector space V_0 on which G acts "as on V", and that V is merely a complex version of V_0. This would be better explained in terms of tensor products over $\mathbb{R}$, but we can do it manually. Suppose V_0 has basis $\{x_k : 1 \leq k \leq n\}$ over $\mathbb{R}$, and add new vectors y_k, getting a $2n$-dimensional $\mathbb{R}$-vector space. Now impose $y_k = ix_k$, and let $z_k = x_k + y_k$. Then $\{z_k : 1 \leq k \leq n\}$ is the basis of an n-dimensional complex vector space containing V_0 as a real subspace. Make sure that the existence of a G-invariant, real subspace of V is equivalent to V being real as a representation!

It is tempting as well to introduce more terminology and abstraction. The following definition may be omitted.

Definition 3.2.4. Let V_0 be a real vector space. Then there is a pair $(V_0^{\mathbb{C}}, \iota)$ where $V_0^{\mathbb{C}}$ is a complex vector space and $\iota : V_0 \to V_0^{\mathbb{C}}$ is an injective, $\mathbb{R}$-linear map, with the following universal property:

for any complex vector space W and any $\mathbb{R}$-linear map $f : V_0 \to W$, there is a unique map $F : V_0^{\mathbb{C}} \to W$ such that $f = F \circ \iota$. (UP)



$V_0^{\mathbb{C}}$ is called the complexification of V_0.

Turning back to the question of real representations, for V to be real it is necessary that χ_V takes only real values, but the condition is not sufficient.

Counter-example 3.2.5. The character table of the quaternion group $\mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$ is:

$\mathbb{H}$	$1_{(\times 1)}$	$-1_{(\times 1)}$	$i_{(\times 2)}$	$j_{(\times 2)}$	$k_{(\times 2)}$
triv	1	1	1	1	1
χ_i	1	1	1	−1	−1
χ_j	1	1	−1	1	−1
χ_k	1	1	1	−1	−1
ψ_2	2	−2	0	0	0

We however claim that ψ_2 cannot be obtained on a real vector space. Suppose it is. Then we have a morphism $\mathbb{H} \to \mathrm{GL}_2(\mathbb{R})$; since the trace of -1 is -2, -1 does not map to Id, meaning that the morphism is injective. So $\mathbb{H}$ embeds into $\mathrm{GL}_2(\mathbb{R})$. Extend linearly: we get a morphism of algebras $\mathbb{H} \to M_2(\mathbb{R})$, which must be injective since $\mathbb{H}$ is a division algebra, and is a bijection for dimension reasons. So $\mathbb{H} \simeq M_2(\mathbb{R})$ as algebras; this is a contradiction since $M_2(\mathbb{R})$ is not a domain.

This shows that ψ_2 is not a real representation, although its character takes real values.

Proposition 3.2.6. *Let V be a finite-dimensional vector space. There is a bijection between the set of isomorphisms $V \simeq V^*$ and the set of non-degenerate bilinear forms on V.*

Proof. We exhibit two natural constructions; the reader will easily prove that they are converses of each other.

- Let $\varphi : V \simeq V^*$ be an isomorphism. Let:

$$\beta(v_1, v_2) = \underbrace{\varphi(v_1)}_{\in V^*}(v_2)$$

Clearly β is bilinear. We claim that it is non-degenerate.

  – Suppose v_1 is such that for any $v_2 \in V$, $\beta(v_1, v_2) = 0$. Then by definition, $\varphi(v_1)(v_2) = 0$ for all v_2, so $\varphi(v_1)$ is the zero linear form. Since φ is injective, it follows $v_1 = 0$ as desired.

– Now suppose v_2 is such that for any $v_1 \in V$, $\beta(v_1, v_2) = 0$. Then since φ is surjective, this means that for any linear form $\ell \in V^*$, $\ell(v_2) = 0$. So $v_2 = 0$.

- Conversely we start with a non-degenerate bilinear form β and attach to it an isomorphism $V \simeq V^*$. Indeed, for any $v_1 \in V$, consider the map:

$$\begin{array}{rccc} \ell_{v_1}: & V & \to & V \\ & v_2 & \mapsto & \beta(v_1, v_2) \end{array}$$

It is clear that ℓ_{v_1} is a form on V, and by non-degeneracy, $\ell_{v_1} = 0$ in V^* iff $v_1 = 0$ in V. Now consider the map:

$$\begin{array}{rccc} \varphi: & V & \to & V^* \\ & v_1 & \mapsto & \ell_{v_1} \end{array}$$

Here again, φ is clearly linear. It is injective by what we have just observed. Since $\dim V$ is finite, φ is a linear isomorphism. □

Remark 3.2.7. An isomorphism $\varphi : V \simeq V^*$ is G-covariant iff the attached bilinear form β is G-invariant. Indeed, supposing φ G-covariant, one has:

$$\begin{aligned} \beta(g \cdot v_1, g \cdot v_2) &= [\varphi(g \cdot v_1)](g \cdot v_2) \\ &= [g \cdot \underbrace{\varphi(v_1)}_{\in V^*}](g \cdot v_2) \\ &= [\varphi(v_1) \circ g^{-1}](g \cdot v_2) \\ &= \varphi(v_1)(v_2) \\ &= \beta(v_1, v_2) \end{aligned}$$

Prove the converse to make sure you remember about dual representations.

Lemma 3.2.8. *Suppose V is an irreducible representation with χ_V taking real values. Then there is a G-invariant, non-degenerate bilinear form on V; moreover, this form is either symmetric or skew-symmetric.*

Proof. Suppose χ_V takes only real values. This means that $\overline{\chi_V} = \chi_V$, so V^* and V have the same character. They are therefore isomorphic as representations of G, meaning that there is an equivalence of representations:

$$f : V \simeq V^*$$

This is called a pairing and induces a non-degenerate bilinear form on V; simply let:

$$B(v, w) = f(w)(v)$$

which is a bilinear map. Moreover, since f is G-covariant, one has: $f(g \cdot v) = g \cdot f(v) \in V^*$, so that $f(g \cdot v)(w) = f(v)(g^{-1} \cdot w)$. This exactly means that $B(g \cdot v, w) = B(v, g^{-1} \cdot w)$, or equivalently, that B is G-invariant.

It remains to show that B may be taken symmetric or skew-symmetric. But one way write $B = B_+ + B_-$, where:

$$B_+(v, w) = \frac{B(v, w) + B(w, v)}{2}, \quad B_-(v, w) = \frac{B(v, w) - B(w, v)}{2}$$

Both are G-invariant, bilinear forms; clearly B_+ is symmetric and B_- is skew-symmetric.

Now since V is irreducible, so is V^* (this is because χ and $\overline{\chi}$ have the same norm as class functions). So by Schur's Lemma, all G-covariant isomorphisms between V and V^* are collinear. In terms of bilinear forms, all non-degenerate G-invariant bilinear forms must be collinear. Hence B_+ and B_- are collinear. Since one is symmetric and the other one is skew-symmetric, one of them is zero, and B equals the other one. □

End of Lecture 16.

Lecture 17 (Real and quaternionic representations)

This suggests a criterion for "realness" of an irreducible representation.

Theorem 3.2.9. *Let V be an irreducible representation. Then the following are equivalent:*

1. *V is real;*
2. *there exists a G-invariant non-degenerate symmetric bilinear form on V;*
3. *there exists a G-covariant complex-conjugate $\mathbb{R}$-linear map f with $f^2 = \mathrm{Id}$.*

Proof. (1) ⇒ (2). If V is real, then we start with a real vector space V_0 contained in V, and on which G acts. On $V_0 \simeq \mathbb{R}^d$ we can pick a scalar product and average it as in the second proof of Maschke's Theorem, Theorem 2.1.29. In the end we get a G-invariant symmetric bilinear form β on V_0, which naturally extends to V by putting:

$$B(a_1 + ib_1, a_2 + ib_2) = (\beta(a_1, a_2) - \beta(b_1, b_2)) + i\,(\beta(a_1, b_2) + \beta(b_1, a_2))$$

where a_1, a_2, b_1, b_2 are vectors of the real vector space V_0. We claim that B is non-degenerate. For if $a + ib \in V$ is in the kernel of B, then for any $a' \in V_0$ one has $B(a + ib, a') = 0 = \beta(a, a') + i\beta(b, a')$, so in particular a and b are β-orthogonal to all $a' \in V_0$, proving $a = b = 0$ (in V_0) and $a + ib = 0$ (in V). So B is non-degenerate, as desired.

(2) ⇒ (3). We now suppose that there is a G-invariant non-degenerate symmetric bilinear form B on V. B induces a pairing:

$$\begin{array}{rccc} \Phi: & V & \simeq & V^* \\ & v & \mapsto & B(v, .) \end{array}$$

Now fix any G-invariant complex scalar product $\langle ., . \rangle$ on V. This induces a complex-conjugate isomorphism of $\mathbb{R}$-vector spaces:

$$\begin{array}{rccc} \Psi: & V & \simeq & V^* \\ & v & \mapsto & \langle v, . \rangle \end{array}$$

We consider the $\mathbb{R}$-linear map $f = \Psi^{-1} \circ \Phi : V \simeq V$ (isomorphism of real vector spaces). For $v \in V$, $f(v)$ is the unique vector such that $B(v, .) = \langle f(v), . \rangle$ as linear forms.

- f is $\mathbb{R}$-linear as observed;

- f is complex conjugate, meaning that $f(iv) = -if(v)$.

  Indeed, for any $w \in V$, one has:

$$\langle f(iv), w\rangle = B(iv, w) = B(v, iw) = \langle f(v), iw\rangle = \langle -if(v), w\rangle$$

- f is G-covariant, since for $g \in G$, $w \in V$,

$$\langle f(gv), w\rangle = B(gv, w) = B(v, g^{-1}w) = \langle f(v), g^{-1}w\rangle = \langle gf(v), w\rangle$$

It follows that f^2 is G-covariant as well, and $\mathbb{C}$-linear. Now V being irreducible, by Schur's Lemma, there is $\lambda \in \mathbb{C}$ with $f^2 = \lambda \mathrm{Id}$. Moreover,

$$\langle f(v), w\rangle = B(v, w) = B(w, v) = \langle f(w), v\rangle = \overline{\langle v, f(w)\rangle}$$

so:

$$\langle \lambda v, w\rangle = \langle f^2(v), w\rangle = \overline{\langle f(v), f(w)\rangle} = \langle v, f^2(w)\rangle = \langle v, \lambda w\rangle$$

proving that λ is a real number. Moreover, if $w = f(v)$, then:

$$\langle f(v), f(v)\rangle = \overline{\langle v, f^2(w)\rangle} = \lambda\langle v, v\rangle$$

proving that λ is positive. Up to rescaling $\langle ., .\rangle$, we may assume that $\lambda = 1$, so $f^2 = \mathrm{Id}$.

(3) $\Rightarrow$ (1). f, as an endomorphism of the underlying real vector space (of dimension $2n$) is diagonalizable, with eigenvalues ± 1. Let V_0 be the real eigenspace associated to eigenvalue 1. We claim that the real eigensepace associated to -1 is exactly iV_0. Indeed, since f is complex-conjugate,

$$f(v) = -v \Leftrightarrow f(iv) = -if(v) = v \Leftrightarrow iv \in V_0$$

But since f is G-invariant, so is V_0. The representation is real. □

Where did we use symmetry? only in the end. Actually, one can show in a similar way:

Theorem 3.2.10. *Let V be an irreducible representation. Then the following are equivalent:*

1. *there exists a G-invariant non-degenerate skew-symmetric bilinear form on V;*

2. *there exists a G-covariant complex conjugate $\mathbb{R}$-linear map f with $f^2 = -\mathrm{Id}$.*

Definition 3.2.11. A representation as in Theorem 3.2.10 is called quaternionic.

Of course the case remains where there is no G-invariant non-degenerate bilinear form. We know that in this case the character cannot be real-valued.

Definition 3.2.12. A representation with non-real character is called complex.

3.2.3 The Frobenius-Schur indicator

Theorem 3.2.13. *Let V be an irreducible representation. Then*

$$\frac{1}{|G|}\sum_{g\in G}\chi(g^2) = \begin{cases} 1 & \text{if } V \text{ is real} \\ 0 & \text{if } V \text{ is complex} \\ -1 & \text{if } V \text{ is quaternionic} \end{cases}$$

Example 3.2.14. Apply this to all characters of all groups computed so far. Observe that $D_{2\cdot 4}$ and $\mathbb{H}$ have the same character table, but in the case of $D_{2\cdot 4}$, the 2-dimensional character is real whereas in the case of $\mathbb{H}$, it is not. (This is simply because the square function does not behave in the same way in $D_{2\cdot 4}$ and $\mathbb{H}$.)

End of Lecture 17.

Lecture 18 (Proof of Frobenius-Schur; Sym and Alt)

Proving Theorem 3.2.13 will require a useful digression. One has to understand what $\sum \chi(g^2)$ means, preferably in terms of $V^* \otimes V^*$, which is the space of bilinear forms on V. Actually, we are interested in bilinear forms which are either symmetric or skew-symmetric. Does $V^* \otimes V^*$ split?

Definition 3.2.15. Let V be a vector space and k an integer. Let S_k be the symmetric group of $\{1, \ldots, k\}$.

- The k^{th} symmetric power of V is:

$$\mathrm{Sym}^k(V) = \left(\otimes^k V\right) \Big/ \left\langle \left\{ \begin{array}{r} v_1 \otimes \cdots \otimes v_k = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} : \\ (v_1, \ldots, v_k) \in V^k, \sigma \in S_k \end{array} \right\} \right\rangle$$

- The k^{th} exterior power of V is:

$$\mathrm{Alt}^k(V) = \left(\otimes^k V\right) \Big/ \left\langle \left\{ \begin{array}{r} v_1 \otimes \cdots \otimes v_k = \varepsilon(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} : \\ (v_1, \ldots, v_k) \in V^k, \sigma \in S_k \end{array} \right\} \right\rangle$$

Quite fortunately, for the moment we only need to understand Sym^2 and Alt^2. In Sym^2, one has $v_1 \otimes v_2 = v_2 = \otimes v_1$, whereas in Alt^2, one has $v_1 \otimes v_2 = -v_2 \otimes v_1$. This requires a more careful notation.

Notation 3.2.16.

- The image of $v_1 \otimes \cdots \otimes v_k$ in $\mathrm{Sym}^k V$ is denoted $v_1 \cdot \cdots \cdot v_k$.
- The image of $v_1 \otimes \cdots \otimes v_k$ in $\mathrm{Sym}^k V$ is denoted $v_1 \wedge \cdots \wedge v_k$.

Observe how $V \otimes V = \mathrm{Sym}^2 V \oplus \mathrm{Alt}^2 V$. Now each term is a representation of G, acting naturally.

Lemma 3.2.17. *Let V be a representation of G. Then:*

- $\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$
- $\chi_{\mathrm{Alt}^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$

Proof. Let $g \in G$ and let $\{v_k : 1 \leq k \leq n\}$ be a basis of V consisting of eigenvectors for g, with $g \cdot v_k = \lambda_k v_k$. Then $\{v_k \cdot v_\ell : k \leq \ell\}$ is a basis of eigenvectors of $\mathrm{Sym}^2 V$, and $g \cdot (v_k v_\ell) = \lambda_k \lambda_\ell (v_k v_\ell)$. Hence:

$$\chi_{\mathrm{Sym}^2(V)}(g) = \sum_{k \leq \ell} \lambda_k \lambda_\ell$$

Similarly, $\{v_k \wedge v_\ell : k < \ell\}$ is a basis of eigenvectors of $\mathrm{Alt}^2 V$, and:

$$\chi_{\mathrm{Alt}^2(V)}(g) = \sum_{k < \ell} \lambda_k \lambda_\ell$$

Now:

$$\chi_V(g)^2 = \left(\sum_k \lambda_k\right) = 2 \sum_{k<\ell} \lambda_k \lambda_\ell + \sum_k \lambda^2 = 2\chi_{\mathrm{Alt}^2(V)}(g) + \chi_V(g^2)$$

and we get the desired formulas. $\square$

Of course one finds $\chi_{\mathrm{Sym}^2 V} + \chi_{\mathrm{Alt}^2 V} = \chi_V^2$. Observe how $\chi_{\mathrm{Sym}^2 V}(e) = \frac{1}{2} \dim V (\dim V + 1)$ and $\chi_{\mathrm{Alt}^2 V}(e) = \frac{1}{2} \dim V (\dim V - 1)$, as a direct computation of a basis would have shown. We are now ready to prove Theorem 3.2.13.

Proof of Theorem 3.2.13. Recall from Lemma 2.3.17 that for any representation W, one has $\dim W^G = \frac{1}{|G|} \sum_{g \in G} \chi_W(g)$. Let V be an irreducible representation of G. Recall that V^* is irreducible as well.

We wonder whether there is a G-invariant non-degenerate bilinear form on V, that is, whether there is a G-covariant pairing $V^* \simeq V$. So we compute $\mathrm{Hom}(V^*, V) \simeq V^{**} \otimes V \simeq \otimes^2 V$ of character χ_V^2. Hence:

$$D = \dim \mathrm{Hom}_G(V^*, V) = \frac{1}{|G|} \sum_{g \in G} \chi_V^2(g)$$

which is 0 or 1 by Schur's Lemma, since V and V^* are irreducible.

In particular, the space of G-invariant non-degenerate symmetric bilinear forms on V has dimension 0 or 1 as well, meaning that:

$$d = \frac{1}{|G|} \sum_{g \in G} \frac{1}{2} \left(\chi_V^2(g) + \chi_V(g^2)\right)$$

is 0 or 1, and $d \leq D$.

- If $D = d = 0$, then there remains $\sum_{g \in G} \chi(g^2) = 0$, and V is complex.
- If $D = d = 1$, then there remains $\sum_{g \in G} \chi(g^2) = 1$, and V is real.
- If $D = 1$ and $d = 0$, then there remains $\sum_{g \in G} \chi(g^2) = 0$, and V is quaternionic. $\square$

3.2.4 More on Sym and Alt

In Chapter 2, §2.4.2, we built the character table of S_5, we introduced the tensor square of a representation, jumping from dimension 4 to dimension 16. Of course we had to go to a bigger space; on the other hand 16 was more than enough. We shall compute the character table of S_5 anew, using Sym^2 and Alt^2.

Take the irreducible character χ_4 (of the representation V_4), as follows:

	e $(\times 1)$	(12) $(\times 10)$	(123) $(\times 20)$	(1234) $(\times 30)$	$(12)(34)$ $(\times 15)$	(12345) $(\times 24)$	$(12)(345)$ $(\times 20)$
χ_4	4	2	1	0	0	-1	-1

Now consider the representation $\mathrm{Alt}^2 V_4$ and let Ψ be its character. We know that $\dim \mathrm{Alt}^2 V_4 = \frac{1}{2} \cdot 4 \cdot 3 = 6$ (which is much less than 16), and by Lemma 3.2.17 we can compute the values of Ψ:

	e $(\times 1)$	(12) $(\times 10)$	(123) $(\times 20)$	(1234) $(\times 30)$	$(12)(34)$ $(\times 15)$	(12345) $(\times 24)$	$(12)(345)$ $(\times 20)$
Ψ	6	0	0	0	-2	1	0

and computing $\langle \Psi, \Psi \rangle$, it appears that $\mathrm{Alt}^2 V_4$ is an irreducible representation of S_5 of degree 6.

We turn to $\mathrm{Sym}^2 V_4$ of dimension $\frac{1}{2} \cdot 4 \cdot 5 = 10$; let Φ be its character. Again, we can easily compute Φ:

	e $(\times 1)$	(12) $(\times 10)$	(123) $(\times 20)$	(1234) $(\times 30)$	$(12)(34)$ $(\times 15)$	(12345) $(\times 24)$	$(12)(345)$ $(\times 20)$
Φ	10	4	1	0	2	0	1

A few computations are required:

- $\langle \Phi, \Phi \rangle = \frac{1}{120}(100+40+20+60+20) = 3$ so Φ is the sum of thrre irreducible characters,
- $\langle \Phi, \chi_{\mathrm{triv}} \rangle = \frac{1}{120}(10 + 40 + 20 + 30 + 20) = 1$,
- $\langle \Phi, \chi_4 \rangle = \frac{1}{120}(40 + 80 + 20 - 20) = 1$

So $\chi_5 = \Phi - \chi_{\mathrm{triv}} - \chi_4$ is an irreducible character of degree $10 - 1 - 4 = 5$. We are done.

End of Lecture 18.