

On the equivalence of two stability conditions of FB-modules

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Abstract. We give a proof of the equivalence of two stability conditions in the work of Thomas Church and Benson Farb on FB-modules ([1]): *representation stability* (RS) and *character polynomiality* (PC). While the implication (RS) \Rightarrow (PC) is a simple consequence of the Frobenius character formula, the converse seems not to have been documented, which motivated this work.

An FB-module is a countable family $\mathcal{W} := \{W_m\}_{m \in \mathbb{N}}$ of linear representations W_m of the symmetric groups \mathcal{S}_m , assumed, in these notes, to be finite dimensional over the field of rational numbers \mathbb{Q} , hereafter called *an \mathcal{S}_m -module*.

• The FB-module \mathcal{W} is said to be *representation stable* (RS in short), if there exists some $N \in \mathbb{N}$, such that, for all $m \geq N$, the decomposition in simple factors of W_m has the *stable* form :

$$W_m \sim \bigoplus_{\lambda} V_{\lambda[m]}^{\mathbf{n}_{\lambda}},$$

where

- ▷ $\lambda := (\lambda_1 \geq \dots \geq \lambda_{\ell})$ is a partition verifying $|\lambda| + \lambda_1 \leq N$;
- ▷ $V_{\lambda[m]}$ is the simple \mathcal{S}_m -module associated with $\lambda[m] := (m - |\lambda| \geq \lambda_1 \geq \dots \geq \lambda_{\ell})$;
- ▷ for $m \geq N$, the family of natural numbers $\{\mathbf{n}_{\lambda}\}_{\lambda}$ is *constant*.

The smallest such N is *the rank of (RS) of \mathcal{W}* , which we will denote by ‘ $\text{rank}_{\text{RS}}(\mathcal{W})$ ’.

• The FB-module \mathcal{W} is said to *have a polynomial character* (PC in short), if there exists $N \in \mathbb{N}$ and a polynomial $P_{\mathcal{W}} \in \mathbb{Q}[X_1, \dots, X_N]$, such that, for all $m \geq N$ and all $g \in \mathcal{S}_m$, one has :

$$\chi_{W_m}(g) = P_{\mathcal{W}}(X_1(g), \dots, X_N(g)),$$

where χ_{W_m} is the character of W_m , and $X_i(g)$ is the number of cycles of length i in the decomposition of g in \mathcal{S}_m as a product of disjoint cycles. The polynomial $P_{\mathcal{W}}$, which is unique, is *the polynomial character of \mathcal{W}* . The smallest such N is *the rank of character polynomiality of \mathcal{W}* , it will be denoted by ‘ $\text{rank}_{\text{PC}}(\mathcal{W})$ ’.

Regarding the equivalence of these conditions, we prove the following theorem.

Theorem (4.1.1). *Let \mathcal{W} be an FB-module.*

- a) $\text{rank}_{\text{PC}}(\mathcal{W}) \leq \text{rank}_{\text{RS}}(\mathcal{W})$.
- b) *If \mathcal{W} is (PC) and has polynomial character $P_{\mathcal{W}}$, then*

$$\text{rank}_{\text{RS}}(\mathcal{W}) \leq \max\{\text{rank}_{\text{PC}}(\mathcal{W}), 2 \deg_{\mathbf{w}}(P_{\mathcal{W}})\},$$

where $\deg_{\mathbf{w}}(P_{\mathcal{W}})$ denotes the polynomial degree of $P_{\mathcal{W}}$, when stipulating that $\deg_{\mathbf{w}}(X_i) := i$.

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Comments. (i) Theorem (2.3.4) exhibit FB-modules \mathcal{W} with $\text{rank}_{\text{PC}}(\mathcal{W}) = 0$ and arbitrarily big $\text{rank}_{\text{RS}}(\mathcal{W})$. It also shows that the bounds in (4.1.1) are optimal. (ii) Beyond FB-modules, we also look at FI-modules \mathcal{W} , which are FB-modules with additional structure (1.2). There is a corresponding representation stability condition for FI-modules, its rank is denoted by $\text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{W})$ (1.7.1). One has $\text{rank}_{\text{RS}}(\mathcal{W}) \leq \text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{W})$, but there is no equivalence with (PC) stability, in particular thm. 4.1.1-(b) for FI-modules and $\text{rank}_{\text{RS}}^{\text{FI}}$ is false.

An interesting by-product in this work is the fact that the *weight* $\mathbf{w}(W_m)$ of an \mathcal{S}_m -module W_m (see 1.5) coincides with the lowest possible degree $\deg_{\mathbf{w}}(P)$ of a polynomial P expressing the character χ_{W_m} (3.1.1). Moreover, if $\mathcal{W} := \{W_m\}$ is a (PC) FB-module, then $\deg_{\mathbf{w}}(P_{\mathcal{W}}) = \mathbf{w}(W_m)$, for all $m \geq \text{rank}_{\text{RS}}(\mathcal{W})$. The next theorem follows from these observations.

Proposition 5.2.1-(b). *If \mathcal{W}_1 and \mathcal{W}_2 are FB-modules which are (PC) $_{m \geq N}$, then*

$$\mathbf{w}(W_{1,m} \otimes W_{2,m}) = \mathbf{w}(W_{1,m}) + \mathbf{w}(W_{2,m}),$$

for all $m \geq 2(\mathbf{w}(W_{1,m}) + \mathbf{w}(W_{2,m}))$. In particular,

$$\mathbf{w}((\mathcal{W}_1 \otimes \mathcal{W}_2)_{\geq N}) = \mathbf{w}((\mathcal{W}_1)_{\geq N}) + \mathbf{w}((\mathcal{W}_2)_{\geq N}),$$

where $(\mathcal{W})_{\geq N}$ denotes the truncation of \mathcal{W} that replaces W_n by 0 for all $n < N$.

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The first two sections are elementary. They introduce notations and well-known properties of FI and FB-modules, which advanced readers can skip.

1. Preliminaries

1.1. General notations

- Given a group G , we denote by $G\text{-mod}$ the category of G -modules, *i.e.* of finite dimensional linear representations of G over the field of rational numbers \mathbb{Q} .
- We denote by \mathcal{S}_m the *symmetric group* defined over $\llbracket 1, m \rrbracket$. Its elements are the bijections of the interval of natural numbers $\llbracket 1, m \rrbracket$. For $n \leq m$, we write $\mathcal{S}_n \subseteq \mathcal{S}_m$ for the inclusion which identifies a permutation g of $\llbracket 1, n \rrbracket$ with its obvious extension to $\llbracket 1, m \rrbracket$ that fixes all $i > n$.
- \mathbb{Q}_m and $\epsilon\mathbb{Q}_m$ denote respectively the *trivial* and the *alternating or signature* \mathbb{Q} -linear representation of the group \mathcal{S}_m .
- $\mathcal{S}_a \boxtimes \mathcal{S}_b$ denotes the subgroup of \mathcal{S}_{a+b} stabilizing the subset $\llbracket 1, a \rrbracket$, or, equivalently, the subset $\llbracket a+1, a+b \rrbracket$. The subgroup $\mathbf{1}_a \boxtimes \mathcal{S}_b \subseteq \mathcal{S}_a \boxtimes \mathcal{S}_b$ is the *fixator* of $\llbracket 1, a \rrbracket$, *i.e.* the set of permutations fixing every $i \leq a$.
- If W_a is an \mathcal{S}_a -module and W_b is an \mathcal{S}_b -module, we denote by $W_a \boxtimes W_b$ the tensor product $W_a \otimes_{\mathbb{Q}} W_b$ endowed with the $\mathcal{S}_a \boxtimes \mathcal{S}_b$ -module structure defined by the componentwise action $(g_a, g_b)(w_a \otimes w_b) := g_a(w_a) \otimes g_b(w_b)$.
- A *partition* of $m \in \mathbb{N} \setminus \{0\}$ is any decreasing sequence of natural numbers $\lambda := (\lambda_1 \geq \dots \geq \lambda_\ell > 0)$ such that $m = \sum_i \lambda_i$. It is also denoted as the m -tuple $(1^{\mathbf{n}_1}, 2^{\mathbf{n}_2}, \dots, m^{\mathbf{n}_m})$ where $\mathbf{n}_k := \#\{i \mid \lambda_i = k\}$, so that $m = \sum_i i \mathbf{n}_i$. The notation $\lambda \vdash m$ says that λ is a partition of m , and $|\lambda|$ is used for the number partitioned by λ . The partition λ is *empty* if $|\lambda| = 0$.
- $\mathbb{Q}_{\text{cl}}(\mathcal{S}_m)$ denotes the \mathbb{Q} -algebra of *class functions* of \mathcal{S}_m . These are the functions $f : \mathcal{S}_m \rightarrow \mathbb{Q}$ constant along the conjugacy classes of \mathcal{S}_m , *i.e.* such that $f(gxg^{-1}) = f(x)$, $\forall g, x \in \mathcal{S}_m$. The scalar product

$$\langle _ | _ \rangle_{\mathcal{S}_m} : \mathbb{Q}_{\text{cl}}(\mathcal{S}_m) \times \mathbb{Q}_{\text{cl}}(\mathcal{S}_m) \rightarrow \mathbb{Q}$$

is defined by

$$\langle f_1 | f_2 \rangle_{\mathcal{S}_m} := \frac{1}{|\mathcal{S}_m|} \sum_{g \in \mathcal{S}_m} f_1(g) f_2(g^{-1}), \quad \forall f_1, f_2 \in \mathbb{Q}_{\text{cl}}(\mathcal{S}_m).$$

- If W_m is an \mathcal{S}_m -module, $\chi_{W_m} : \mathcal{S}_m \rightarrow \mathbb{Q}$ denotes its character. The *Schur's orthogonality relations* state that if V_1 and V_2 are *simple* \mathcal{S}_m -modules, then $\langle \chi_{V_1} | \chi_{V_2} \rangle_{\mathcal{S}_m}$ is equal to 1 if V_1 is isomorphic to V_2 , and to 0 otherwise.

Between here and the end of this preliminary section we recall concepts and terminology from the works of Church and Farb (*cf.* [1]).

1.2. The categories of FB and FI modules

- **FB** denotes the category of Finite sets and Bijections. An FB-*module* is, by definition, a *covariant* functor from the category FB to the category $\mathbf{Vec}_f(\mathbb{Q})$ of finite dimensional \mathbb{Q} -vector spaces and \mathbb{Q} -linear maps :

$$\mathcal{W} : \mathbf{FB} \rightsquigarrow \mathbf{Vec}_f(\mathbb{Q}).$$

To give an FB-module \mathcal{W} is equivalent to giving the countable collection $\{W_m := \mathcal{W}(\llbracket 1, m \rrbracket)\}_{m \in \mathbb{N}}$, where W_m is an \mathcal{S}_m -module. A morphism of FB-modules $f : \mathcal{W} \rightarrow \mathcal{Z}$ corresponds to a family $\{f_m : W_m \rightarrow Z_m\}_{m \in \mathbb{N}}$ of morphisms of \mathcal{S}_m -modules. We have a canonical identification :

$$\mathrm{Mor}_{\mathbf{FB}}(\mathcal{W}, \mathcal{Z}) = \prod_{m \in \mathbb{N}} \mathrm{Hom}_{\mathcal{S}_m}(W_m, Z_m)$$

Notation. The category of FB-modules will be denoted by FB-mod. It is a semi-simple abelian category.

- **FI** denotes the category of Finite sets and Injections. An FI-*module* is a *covariant functor* from FI to $\mathbf{Vec}_f(\mathbb{Q})$:

$$\mathcal{W} : \mathbf{FI} \rightsquigarrow \mathbf{Vec}_f(\mathbb{Q}).$$

To give an FI-module is equivalent to giving

FI-1) an FB-module $\mathcal{W} := \{W_m\}_{m \in \mathbb{N}}$;

FI-2) for all $m \in \mathbb{N}$, an *interior* map $\phi(\mathcal{W})_m : W_m \rightarrow W_{m+1}$ (in short ϕ_m), which is \mathbb{Q} -linear and such that, for all $g \in \mathcal{S}_m \subseteq \mathcal{S}_{m+1}$, one has

$$\phi_m(g \cdot w) = g \cdot \phi_m(w).$$

FI-3) for all $n \geq m$, the image of $\phi_{n,m} := \phi_{n-1} \circ \cdots \circ \phi_m$ satisfies :

$$\phi_{n,m}(W_m) \subseteq (W_n)^{\mathbf{1}_m \boxtimes \mathcal{S}_{n-m}}.$$

Under this equivalence, a morphism of FI-modules $f : \mathcal{W} \rightarrow \mathcal{Z}$ is simply a morphism of FB-modules which is compatible with the interior maps ϕ_m , *i.e.* such that the diagrams

$$\begin{array}{ccc} W_m & \xrightarrow{\phi(\mathcal{W})_m} & W_{m+1} \\ f_m \downarrow & & \downarrow f_{m+1} \\ Z_m & \xrightarrow{\phi(\mathcal{Z})_m} & Z_{m+1} \end{array}$$

are commutative.

Notation. The category of FI-modules will be denoted by FI-mod. It is an abelian category, which is *not* semi-simple.

1.2.1. Comments

- a) The category FB-mod is equivalent to the full subcategory of FI-mod of FI-modules whose interior maps are null. The *forgetful functor* associates to an FI-module $\mathcal{W} := \{\phi_m : W_m \rightarrow W_{m+1}\}$ the FB-module $\mathcal{W} := \{W_m\}$. It is an additive and exact functor which will be tacitly used.
- b) More interesting, the subcategory $\text{FI-mod}'$ of FI-modules whose interior maps are *injective* and (*eventually*) *exhaustive*, i.e. such that the image $\phi_m(W_m)$ generates W_{m+1} as \mathcal{S}_{m+1} -module for large m . Among these, the FI-modules \mathcal{V}_λ 's which bind together the simple modules $V_{\lambda[m]}$ (1.6.1). We will see that FB-modules \mathcal{W} which are (PC) or (RS) are asymptotically isomorphic to direct sums of \mathcal{V}_λ 's (4.1.2).

1.2.2. The truncations

(¹) The functor $(_)_{\geq \ell} : \text{FI-mod} \rightsquigarrow \text{FI-mod}$ that “truncates” an FI-module $\{W_m\}_{m \in \mathbb{N}}$ by replacing by $\mathbf{0}$ its terms W_m for $m < \ell$, is an additive exact functor. The *cokernel* of the natural inclusion $(_)_{\geq \ell} \hookrightarrow \text{id}_{\text{FI}}$ is the truncation $(_)_{< \ell}$, it replaces by $\mathbf{0}$ the terms W_m for $m \geq \ell$. We thus have short exact sequences

$$\mathbf{0} \rightarrow (\mathcal{W})_{\geq \ell} \twoheadrightarrow \mathcal{W} \twoheadrightarrow (\mathcal{W})_{< \ell} \rightarrow \mathbf{0},$$

which are natural with respect to \mathcal{W} . The full subcategory $\text{FI-mod}_{\geq \ell}$ of FI-modules \mathcal{W} such that the inclusion $\mathcal{W}_{\geq \ell} \hookrightarrow \mathcal{W}$ is an isomorphism is an abelian subcategory, and similarly for the full subcategory $\text{FI-mod}_{< \ell}$ of FI-modules \mathcal{W} such that the quotient $\mathcal{W} \twoheadrightarrow \mathcal{W}_{< \ell}$ is an isomorphism. One has,

$$\text{Ext}_{\text{FI}}^i(\text{FI-mod}_{\geq \ell}, \text{FI-mod}_{\leq \ell}) = 0, \quad \forall i > 0.$$

The intersection $\text{FI-mod}_{\geq \ell} \cap \text{FI-mod}_{\leq \ell}$ is the (semi-simple) category $\mathcal{S}_\ell\text{-mod}$.

1.3. Projective FI-modules

An obvious way to construct FI-modules of the type described in 1.2.1-(b) is to start with a representation W_n of some \mathcal{S}_n , and then set, for all $m \in \mathbb{N}$:

$$\mathcal{P}(W_n)_m := \begin{cases} \mathbf{0}, & \text{if } m < n, \\ \text{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m-n}}^{\mathcal{S}_m} (W_n \boxtimes \mathbb{Q}_{m-n}), & \text{otherwise.} \end{cases}$$

For each $m \geq n$, the composition of the following natural maps ι_m and κ_{m+1} :

$$\begin{array}{ccc} W_n \boxtimes \mathbb{Q}_{m-n} & \xhookrightarrow{\iota_m} & W_n \boxtimes \mathbb{Q}_{m+1-n} \\ & \searrow \psi_m & \downarrow \kappa_{m+1} \\ & & \text{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m+1-n}}^{\mathcal{S}_{m+1}} (W_n \boxtimes \mathbb{Q}_{m+1-n}) \end{array}$$

¹This truncations are frequently called *stupid* or *brutal*.

gives the map ψ_m whose image is invariant under $(\mathbf{1}_n \boxtimes \mathcal{S}_{m+1-n})$. In particular, ψ_m is $(\mathcal{S}_n \boxtimes \mathcal{S}_{m-n})$ -linear, inducing thus the map

$$\phi_m := \text{ind}(\psi_m) : \text{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m-n}}^{\mathcal{S}_m} (W_n \boxtimes \mathbb{Q}_{m-n}) \rightarrow \text{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m+1-n}}^{\mathcal{S}_{m+1}} (W_n \boxtimes \mathbb{Q}_{m+1-n}),$$

which satisfies the requirements making the family

$$\mathcal{P}(W_m) := \{\phi_m : \mathcal{P}(W_n)_m \rightarrow \mathcal{P}(W_n)_{m+1}\}_{m \in \mathbb{N}},$$

an FI-module.

All the previous constructions are natural on the category of \mathcal{S}_n -modules. As a consequence, we have defined a functor

$$\mathcal{P} : \mathcal{S}_n\text{-mod} \rightsquigarrow \text{FI-mod},$$

which is clearly additive and exact.

1.3.1. Proposition. *Let W_n be a representation of \mathcal{S}_n .*

- a) *The interior maps of $\mathcal{P}(W_n)$ are injective. They are exhaustive $\forall m \geq n$.*
- b) *There is a natural identification of functors*

$$\text{Mor}_{\text{FI}}(\mathcal{P}(W_n), _) = \text{Hom}_{\mathcal{S}_n}(W_n, (_)_n).$$

- c) *The FI-module $\mathcal{P}(W_n)$ is a projective FI-module, which moreover is simple projective if and only if W_n is a simple \mathcal{S}_n -module. All simple projective FI-modules are of this form.*
- d) *The category FI-mod has enough projective objects.*

Proof. Left to the reader. □

1.4. Young diagrams and Pieri's rule

We recall the re-parametrization of irreducible representations of the symmetric groups due to Church and Farb.

1.4.1. The socle and the weight of a partition

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0)$ be a non-empty partition of $m := \sum_i \lambda_i$.

- The *socle* of λ is the sub-partition $\underline{\lambda} := (\lambda_2 \geq \dots \geq \lambda_\ell)$.
- The *weight* of λ is the number $\mathbf{w}(\lambda) := |\underline{\lambda}| = \lambda_2 + \dots + \lambda_\ell$.

Notice that the map $\lambda \mapsto \underline{\lambda}$ is injective from the set of partitions of $m \in \mathbb{N}$, so that it amounts the same, giving $(\lambda \vdash m)$ or giving $(m, \underline{\lambda})$ with $m \geq |\underline{\lambda}| + \underline{\lambda}_1$.

In terms of Young diagrams, in order to get the socle $\underline{\lambda}$ of λ , one simply erases the first row of λ . The picture is thus :

$$\lambda := \begin{array}{cccccc} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array} \mapsto \underline{\lambda} := \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array}.$$

Conversely, given $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell)$ and any $m \geq |\lambda| + \lambda_1$, Church and Farb introduce the notation

$$\lambda[m] := (m - |\lambda| \geq \lambda_1 \geq \dots \geq \lambda_\ell)$$

which, in terms of Young diagrams, corresponds to adding a first row with as many boxes as is necessary to raise the total number of boxes from $|\lambda|$ to m .

$$\lambda := \begin{array}{cccccc} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array} \mapsto \lambda[m] := \begin{array}{cccccc} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array}.$$

Notice the following obvious notational facts :

$$\lambda[m] \vdash m, \quad \underline{\lambda[m]} = \lambda, \quad \lambda = \underline{\lambda[|\lambda|]}.$$

1.4.2. Irreducible representations of the symmetric groups

The irreducible representations of \mathcal{S}_m are parametrized by the partitions $\nu \vdash n$. The simple \mathcal{S}_m -module associated with ν is classically denoted by V_ν . The following proposition recalls a fundamental fact about the representations of the symmetric groups (see [2], chap. 4, thm. 4.3, pp. 46–).

1.4.3. Proposition. *The irreducible representations of the symmetric groups over a field k of characteristic zero are defined over the field of rational numbers. If W_m is a $k[\mathcal{S}_n]$ -module of finite dimension, the values of the character*

$$\chi_{W_m} : \mathcal{S}_m \rightarrow k, \quad g \mapsto \text{tr}(g : W_m \rightarrow W_m),$$

belong to the ring of integers $\mathbb{Z} \subseteq k$.

Hint. Because W_m is defined over \mathbb{Q} , the trace of $g \in \mathcal{S}_m$ belongs a priori to \mathbb{Q} , and because the trace is the sum of the eigenvalues, which are roots of unity, it is also an algebraic integer. It is therefore an integer. \square

1.4.4. Pieri's rule

Pieri's rule ⁽²⁾ gives the irreducible factors of the terms $\mathcal{P}(V_\nu)_m$, for any given partition $\nu \vdash n \leq m$. The rule says that in the decomposition

$$\mathcal{P}(V_\nu)_m := \text{ind}_{\mathfrak{S}_n \boxtimes \mathfrak{S}_{m-n}}^{\mathfrak{S}_m} (V_\nu \boxtimes \mathbb{Q}_{m-n}) = \bigoplus_{\mu \vdash m} V_\mu^{\mathbf{n}_\nu(\mu)}, \quad (1)$$

the nonzero multiplicities $\mathbf{n}_\nu(\mu)$ are all equal to 1, and the corresponding partitions μ are those obtained from ν by adding $(m-n)$ boxes in *different* columns. For example, if $\nu := (3, 2, 2)$ and $m \in \{8, 9, 10, 11\}$, we have

$$\begin{aligned} \text{ind}_{\mathfrak{S}_7 \boxtimes \mathfrak{S}_1}^{\mathfrak{S}_8} V_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} &= V_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} \\ \text{ind}_{\mathfrak{S}_7 \boxtimes \mathfrak{S}_2}^{\mathfrak{S}_9} V_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} &= V_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}} \\ \text{ind}_{\mathfrak{S}_7 \boxtimes \mathfrak{S}_3}^{\mathfrak{S}_{10}} V_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} &= V_{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array}} \\ \text{ind}_{\mathfrak{S}_7 \boxtimes \mathfrak{S}_4}^{\mathfrak{S}_{11}} V_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} &= V_{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}} \oplus V_{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}} \end{aligned}$$

where, after a key observation of Church and Farb, for $m \geq |\nu| + \nu_1 (= 10$ in this example), the set of socles of the Young diagrams appearing in the decomposition (1) is constant. The following lemma obviously follows.

1.4.5. Lemma. Let $\nu \vdash n > 0$.

a) For all $\mu \vdash m \geq n$, such that $\mathbf{n}_\nu(\mu) \neq 0$, the weight of μ verifies

$$|\underline{\nu}| = \mathbf{w}(\nu) \leq \mathbf{w}(\mu) \leq |\nu|,$$

$$\text{and} \quad \begin{cases} \mathbf{w}(\mu) = \mathbf{w}(\nu) & \iff \mu = \underline{\nu}[m] \\ \mathbf{w}(\mu) = |\nu| & \iff (m \geq |\nu| + \nu_1) \ \& \ (\mu = \nu[m]). \end{cases}$$

b) Let $m_0 = |\nu| + \nu_1$ and let \mathcal{P}_ν be a set of partitions μ such that

$$\text{ind}_{\mathfrak{S}_n \boxtimes \mathfrak{S}_{m_0-n}}^{\mathfrak{S}_{m_0}} (V_\nu \boxtimes \mathbb{Q}_{m_0-n}) = \bigoplus_{\mu \in \mathcal{P}_\nu} V_{\mu[m_0]}.$$

Then, $\nu[m_0] \in \mathcal{P}_\nu$ and $\nu[m_0]_1 = \nu[m_0]_2$.

For all $m > m_0$, we have

$$\text{ind}_{\mathfrak{S}_n \boxtimes \mathfrak{S}_{m-n}}^{\mathfrak{S}_m} (V_\nu \boxtimes \mathbb{Q}_{m-n}) = \bigoplus_{\mu \in \mathcal{P}_\nu} V_{\mu[m]}.$$

and, for all $\mu \in \mathcal{P}_\nu$, $\mu[m_0]_1 > \mu[m_0]_2$.

²For a thorough introduction to these rules, read §4.3, p. 54–62, and also Appendix A on the *Littlewood-Richardson rules*, p. 451, both in Fulton-Harris' book [2].

1.5. The *weight* of a representation and of an FB-module

The definition of *weight*, is extended from partitions (1.4.1) to representations and, more generally, to FB-modules.

- The *weight* of an \mathcal{S}_m -module W_m is the upper bound of the weights of the partitions associated with its irreducible factors, *i.e.* if

$$W_m \sim \bigoplus_{\mu \vdash m} V_\tau^{\mathbf{n}_\mu},$$

then

$$\mathbf{w}(W_m) := \sup \{ \mathbf{w}(\mu) \mid \mathbf{n}_\mu \neq 0 \}.$$

For example, as a consequence of lemma 1.4.5-(a), we have

$$\begin{cases} |\nu| = \mathbf{w}(\nu) \leq \mathbf{w}(\mathcal{P}(V_\nu))_m \leq |\nu|, & \forall m \geq |\nu|, \text{ and} \\ \mathbf{w}(\mathcal{P}(V_\nu))_m = |\nu|, & \forall m \geq |\nu| + \nu_1. \end{cases} \quad (*)$$

- The *weight* of an FB-module $\mathcal{W} := (W_m)$ is the upper-bound of the weights of its terms, *i.e.*

$$\mathbf{w}(\mathcal{W}) := \sup \{ \mathbf{w}(W_m) \}_{m \in \mathbb{N}}.$$

- The *weight at infinity* of an FB-module $\mathcal{W} := (W_m)$ is

$$\mathbf{w}_\infty(\mathcal{W}) := \lim_{N \rightarrow +\infty} \mathbf{w}(\mathcal{W}_{\geq N}).$$

–It is easy to see that $\mathbf{w}_\infty(\mathcal{W}) = \mathbf{w}(\mathcal{W}_{\geq N})$, for some $N \gg 0$.

–We have $\mathbf{w}(\mathcal{P}(V_\lambda)) = \mathbf{w}_\infty(\mathcal{P}(V_\lambda)) = |\lambda|$.

1.5.1. The *weight* truncations

Let $p \in \mathbb{N}$. Given an \mathcal{S}_m -module W_m , denote by $(W_m)_{\mathbf{w} > p}$ the sum of the irreducible factors of W_m of weight $> p$.

- By Pieri's rule 1.4.5-(a), if $\mathcal{W} := \{ \phi_m : W_m \rightarrow W_{m+1} \}_{m \in \mathbb{N}}$ is an FI-module, one has $\phi_m((W_m)_{\mathbf{w} > p}) \subseteq ((W_{m+1})_{\mathbf{w} > p})$, in which case, the family

$$\mathcal{W}_{\mathbf{w} > p} := \{ \phi_m(W_m)_{\mathbf{w} > p} \rightarrow (W_{m+1})_{\mathbf{w} > p} \}_{m \in \mathbb{N}}$$

is a sub-FI-module of \mathcal{W} .

- Let $\mathcal{W}_{\mathbf{w} \leq p} := \mathcal{W} / \mathcal{W}_{\mathbf{w} > p}$. The short exact sequence

$$\mathbf{0} \rightarrow \mathcal{W}_{\mathbf{w} > p} \rightarrow \mathcal{W} \rightarrow \mathcal{W}_{\mathbf{w} \leq p} \rightarrow \mathbf{0}$$

is natural with respect to \mathcal{W} .

1.5.2. Remark. The following are easy consequences of the definitions.

- a) The full subcategory $\text{FI-mod}_{\mathbf{w}>p}$ of FI-modules \mathcal{W} such that the inclusion $\mathcal{W}_{\mathbf{w}>p} \hookrightarrow \mathcal{W}$ is an isomorphism is an abelian subcategory.
- b) The full subcategory $\text{FI-mod}_{\mathbf{w}\leq p}$ of FI-modules \mathcal{W} such that the quotient $\mathcal{W} \twoheadrightarrow \mathcal{W}_{\mathbf{w}\leq p}$ is an isomorphism is an abelian subcategory.
- c) $\text{Ext}_{\text{FI}}^i(\text{FI-mod}_{\mathbf{w}>p}, \text{FI-mod}_{\mathbf{w}\leq p}) = 0$, for all $i \in \mathbb{N}$.

1.6. The FI-module \mathcal{V}_λ

Given a partition λ , consider the projective FI-module $\mathcal{P}(V_\lambda)$ of 1.3. Lemma 1.4.5-(a) says that for all $m \geq |\lambda|$ the smallest weight of the irreducible components of $\mathcal{P}(V_\lambda)_m$ is exactly $|\lambda|$ ($= \mathbf{w}(\lambda)$), which is the weight of a unique factor: the simple \mathcal{S}_m -module $V_{\underline{\lambda}[m]}$ with multiplicity 1. The following proposition results from this simple observation and the weight filtration 1.5.1.

1.6.1. Proposition. *Let λ be a nonempty partition.*

- a) *The terms of the quotient FI-module*

$$\mathcal{V}_\lambda := \mathcal{P}(V_\lambda)_{\mathbf{w}\leq \mathbf{w}(\lambda)} = \{\phi_m : \mathcal{V}_{\lambda,m} \rightarrow \mathcal{V}_{\lambda,m+1}\}_{m \in \mathbb{N}}$$

are

$$\mathcal{V}_{\lambda,m} = \begin{cases} \mathbf{0} & , \text{ for all } m < |\lambda|, \\ V_{\underline{\lambda}[m]} & , \text{ otherwise.} \end{cases}$$

The interior maps ϕ_m are injective, and are exhaustive for $m \geq |\lambda|$.

- b) *If $\mathcal{V}'_\lambda := \{\phi'_m : \mathcal{V}_{\lambda,m} \rightarrow \mathcal{V}_{\lambda,m+1}\}_{m \in \mathbb{N}}$ is an FI-module such that ϕ'_m is injective and is exhaustive for all $m \geq N$, then $(\mathcal{V}'_\lambda)_{\geq N}$ and $(\mathcal{V}_\lambda)_{\geq N}$ (cf. 1.2.2) are isomorphic FI-modules for $m \geq N$.*

1.7. Representation stability of FI and of FB modules

1.7.1. Representation stable FI-modules

An FI-module $\mathcal{W} = \{\phi_m : W_m \rightarrow W_{m+1}\}_{m \in \mathbb{N}}$ is said to be *representation stable* for $m \geq N$, in short $(\text{RS})_{m \geq N}$, if the following conditions are satisfied.

- RS-1) The interior maps ϕ_m are injective, and are exhaustive for $m \geq N$.
- RS-2) For all $m \geq N$, we have a stable decomposition in simple modules

$$W_m \sim \bigoplus_{|\lambda| \leq N} V_{\underline{\lambda}[m]}^{\mathbf{n}_\lambda},$$

where the \mathbf{n}_λ are independent of $m \geq N$.

We denote by $\text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{W})$, the smallest such N , and we call it the *rank of representation stability* of \mathcal{W} .

1.7.2. Examples

- a) $\mathcal{P}(V_\lambda)$ is (RS) $_{m \geq |\lambda| + \lambda_1}$ (lemma 1.4.5-(b)).
- b) $\mathcal{P}(W_n)$ (RS) $_{m \geq 2n}$, (consequence of lemma 1.4.5-(b)).
- c) \mathcal{V}_λ is (RS) $_{m \geq |\lambda|}$, (by definition).

1.7.3. Representation stable FB-modules

An FB-module $\mathcal{W} = \{W_m\}_{m \in \mathbb{N}}$ is said to be *representation stable for $m \geq N$* , in short (RS) $_{m \geq N}$, if the previous condition (RS-2) is satisfied.

In that case, we say that \mathcal{W} and $\bigoplus_{\lambda \in P} \mathcal{V}_\lambda^{n_\lambda}$ are *asymptotically isomorphic as FB-modules*, and we may write

$$\mathcal{W}_{\geq N} \sim \bigoplus_{|\lambda| \leq N} (\mathcal{V}_\lambda)^{n_\lambda} \quad (2)$$

We denote by $\text{rank}_{\text{RS}}(\mathcal{W})$, the smallest such N , and we call it the *rank of representation stability of \mathcal{W}* .

Warning. For an FI-module the stability rank *as FB-module* is, a priori, smaller than the stability rank as FI-module. It is very easy to construct examples where it is strictly smaller, for example take any (RS) FI-module and replace all its interior maps by the zero map.

1.7.4. Proposition

- a) $\text{rank}_{\text{RS}}(\mathcal{P}(V_\lambda)) = \text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{P}(V_\lambda)) = |\lambda| + \lambda_1.$
- b) *If the trivial representation \mathbb{Q}_n is contained in W_n , then*

$$\text{rank}_{\text{RS}}(\mathcal{P}(W_n)) = \text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{P}(W_n)) = 2n.$$

- c) $\text{rank}_{\text{RS}}(\mathcal{V}_\lambda) = \text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{V}_\lambda) = |\lambda|.$
- d) *If \mathcal{W} is an FB-module such that $\mathcal{Z} \subseteq \mathcal{W}$, where \mathcal{Z} is any of the FI-modules in the previous claims, then*

$$\text{rank}_{\text{RS}}(\mathcal{W}) \geq \text{rank}_{\text{RS}}(\mathcal{Z}).$$

Proof. In (c) there is nothing to be proved. Let us denote by \mathcal{Z} any of the other two FI-modules in the claims. We already gave the values of $\text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{Z})$ in 1.7.2. Next, because $\text{rank}_{\text{RS}}(\mathcal{Z}) \leq \text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{Z})$, the equality will follow if we show that for $m = \text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{Z})$ the \mathcal{S}_m -module Z_m contains an irreducible

factor corresponding to a Young diagram having its two first lines of equal lengths :

$$V_{\begin{array}{cccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array}} \dots \blacksquare \subseteq Z_m \quad (\diamond)$$

Indeed, if $\text{rank}_{\text{RS}}(\mathcal{Z}) < m$, we would have

$$Z_m \sim \bigoplus_{|\lambda| < m} V_{\underline{\lambda}[m]}^{n\lambda}, \quad (*)$$

and the length of the first line of the Young diagram $\underline{\lambda}[m]$ in $(*)$, would be strictly greater than that of the others, *i.e.* $\underline{\lambda}[m]_1 > \underline{\lambda}[m]_2$, which is contrary to our assumption (*cf.* 1.4.5-(b)).

Let us check (\diamond) in our cases. In (a) this results by Pieri's rule because $m = |\lambda| + \lambda_1$ is the smallest m such that λ is the socle of a diagram $\mu \vdash m$, which means that $\mu_1 = \mu_2$. In (b) the Young diagram corresponding to the trivial module \mathbb{Q}_n has only one line with n boxes, and $\mathcal{P}(\mathbb{Q}_n)_{2n}$ verifies (\diamond) for $m = 2n$.

(c) W_m contains some irreducible \mathcal{S}_m -module of the form (\diamond) . □

2. Character polynomiality of FI-modules

2.1. Character polynomiality

Let $\mathbb{Q}_{\text{cl}}(\mathcal{S}_m)$ denote the algebra of *rational class functions* on \mathcal{S}_m , *i.e.* functions $f : \mathcal{S}_m \rightarrow \mathbb{Q}$ which are constant on each conjugacy class of \mathcal{S}_m . Denote by $\mathbb{Q}[\bar{X}]$ the ring of polynomials with rational coefficients and countably many variables X_1, X_2, \dots , endowed with the grading 'deg_w' that stipulates that :

$$\text{deg}_w(X_i) := i.$$

2.1.1. The *weight* of a polynomial. To avoid confusion in the sequel with the usual degree $\text{deg}(P)$ of a polynomial $P \in \mathbb{Q}[\bar{X}]$ which stipulates that $\text{deg}(X_i) = 1$, we will call $\text{deg}_w(P)$ the *weight* of P .

2.1.2. Proposition. *Denote by $X_{m,i} : \mathcal{S}_m \rightarrow \mathbb{N}$ the class function which assigns to $g \in \mathcal{S}_m$, the number $X_{m,i}(g)$ of i -cycles in the decomposition of g as product of disjoint cycles in \mathcal{S}_m .*

a) *The map*

$$\rho_m : \mathbb{Q}[\bar{X}] \longrightarrow \mathbb{Q}_{\text{cl}}(\mathcal{S}_m) \quad (3)$$

$$X_i \longmapsto (g \mapsto X_{m,i}(g))$$

is an homomorphism of algebras whose kernel contains the polynomials

$$(X_1 + 2X_2 + \dots + mX_m - m) \quad \text{and} \quad (X_i(X_i - 1) \cdots (X_i - \lfloor m/i \rfloor)). \quad (4)$$

The restriction

$$\rho_m : \mathbb{Q}[X_1, \dots, X_{m-1}] \rightarrow \mathbb{Q}_{\text{cl}}(\mathcal{S}_m)$$

is surjective. In particular, the characters of \mathcal{S}_m are represented by polynomials with rational coefficients and variables X_1, \dots, X_{m-1} .

b) For $n \leq m$ and $g \in \mathcal{S}_n$, we have

$$\begin{aligned} \text{(i)} \quad & \rho_m(X_1)(\iota g) = \rho_n(X_1)(g) + (m - n), \\ \text{(ii)} \quad & \rho_m(X_i)(\iota g) = \rho_n(X_i)(g), \quad \forall i > 1. \end{aligned} \quad \begin{array}{c} \mathcal{S}_n \xrightarrow{\rho_n} \\ \downarrow \iota \\ \mathcal{S}_m \xrightarrow{\rho_m} \end{array} A$$

Hint. (a) That the polynomials (4) belong to $\ker(\rho_m)$ is clear. Next, to see that ρ_m is surjective, we need only show that the characteristic function of a conjugacy class of \mathcal{S}_m can be realized as a polynomial in X_1, \dots, X_m .

For $k \in \llbracket 1, m \rrbracket$, let $R_k(Z) := Z(Z-1) \cdots \widehat{(Z-k)} \cdots (Z-m)$ and consider

$$D_k(Z) := R_k(Z)/R_k(k) \in \mathbb{Q}[Z].$$

This polynomial has the property that

$$\rho_m(D_k(X_i)(g)) = \begin{cases} 1, & \text{if } X_i(g) = k, \\ 0, & \text{otherwise.} \end{cases}$$

So that, if $\sum_i i \mathbf{n}_i = m$, we get

$$\rho_m(D_{\mathbf{n}_1}(X_1)D_{\mathbf{n}_2}(X_2) \cdots D_{\mathbf{n}_m}(X_m))(g) = \begin{cases} 1, & \text{if } g \text{ is of type } (1^{\mathbf{n}_1}, \dots, m^{\mathbf{n}_m}) \\ 0, & \text{otherwise.} \end{cases}$$

(b) is clear. □

2.1.3. Convention. In order to alleviate notations, we will write $X_i(g)$ for $X_{m,i}(g)$ despite the possible ambiguity in ' $X_1(g)$ ' (cf. 2.1.2-(b-i)).

2.1.4. Definition. An FB-module $\mathcal{W} := \{W_m\}_{m \in \mathbb{N}}$ is said to have a *polynomial character* for $m \geq N$, in short $(\text{PC})_{m \geq N}$, if there exists a polynomial $P_{\mathcal{W}} \in \mathbb{Q}[\bar{X}]$ such that

$$\chi_{W_m} = \rho_m(P_{\mathcal{W}}), \quad \forall m \geq N.$$

We denote by $\text{rank}_{\text{PC}}(\mathcal{W})$, the smallest such N , and we call it the *rank of character polynomiality* of \mathcal{W} .

2.1.5. Proposition and definition. The polynomial $P_{\mathcal{W}}$ that asymptotically represents the characters of the terms W_m of an FB-module $\mathcal{W} := \{W_m\}_{m \in \mathbb{N}}$ is unique, it is called the *polynomial character* of the FB-module \mathcal{W} .

Proof. If $P'_{\mathcal{W}}$ were another polynomial representing χ_{W_m} for $m \gg 0$, the difference $Q := P_{\mathcal{W}} - P'_{\mathcal{W}}$, that we may assume to belong to $\mathbb{Q}[X_1, \dots, X_N]$, would be a polynomial representing the zero class function for all $m \gg N$.

If Q is not the null polynomial, we can write it as a polynomial in X_N with coefficients Q_i in $\mathbb{Q}[X_1, \dots, X_{N-1}]$:

$$Q = Q_0 + Q_1 X_N + Q_2 X_N^2 + \dots + Q_r X_N^r, \text{ and } Q_r \neq 0. \quad (*)$$

Now, for any family $\bar{a} := \{a_1, \dots, a_{N-1}\} \subseteq \mathbb{N}$, and any $i \in \mathbb{N}$, it is easy to find $m_i \gg N$ and $g_i \in \mathcal{S}_{m_i}$ such that $X_1(g_i) = a_1, \dots, X_{N-1}(g_i) = a_{N-1}$ and $X_N(g_i) > i$. In that case, the polynomial in one variable $Q(\bar{a}, X_N)$ has infinitely many roots and is, therefore, the null polynomial. In particular, in the decomposition (*), we would have $Q_r(\bar{a}) = 0$ for all choices of \bar{a} , but this implies that Q_r is the null polynomial in $\mathbb{Q}[X_1, \dots, X_{N-1}]$, contrary to its definition. The polynomial Q must therefore be the null polynomial. \square

2.2. Frobenius character formula

Given a partition $\lambda := (\lambda_1 \geq \dots \geq \lambda_\ell) \vdash m$, the celebrated Frobenius formula computes the character χ_{V_λ} of the simple \mathcal{S}_m -module V_λ . The important point for us about this formula is that it gives an expression of χ_{V_λ} as a polynomial *depending only on the socle* $\underline{\lambda}$ of λ . Consequently, the same polynomial expresses the characters of all the terms of the FI-module $\mathcal{V}_\lambda = \{V_{\lambda, m}\}_m$, for $m \geq |\lambda|$, which self-explains the property of character polynomiality of \mathcal{V}_λ .

2.2.1. Frobenius polynomial for χ_{V_λ}

Following Macdonald in his book [3] (ex. I.7.14, p. 122), let $y := \{y_1, \dots, y_\ell\}$ be a set of ℓ variables, where $\ell := \ell(\lambda)$. The *discriminant of y* is the antisymmetric homogeneous polynomial

$$\Delta(y) := \prod_{i < j} (y_i - y_j).$$

For $d \in \mathbb{N}$, the *d -power sum of y* is the symmetric homogeneous polynomial

$$P_d(y) := y_1^d + \dots + y_\ell^d.$$

The value $\chi_{V_\lambda}(g)$ for $g \in \mathcal{S}_m$, is the coefficient of the monomial

$$y_1^{\lambda_1 + (\ell-1)} y_2^{\lambda_2 + (\ell-2)} y_3^{\lambda_3 + (\ell-3)} \dots y_\ell^{\lambda_\ell},$$

in the development of the product

$$\Delta(y) \left(\prod_{d \geq 1} P_d(y)^{X_d(g)} \right). \quad (5)$$

This coefficient, denoted by \mathbf{X}_λ , is a polynomial in $\mathbb{Q}[\bar{X}]$, we call it *the Frobenius polynomial for χ_{V_λ}* .

2.2.2. Proposition. *The Frobenius polynomial \mathbf{X}_λ for χ_{V_λ} depends only on the socle $\underline{\lambda}$ of λ , and belongs to the ring $\mathbb{Q}[X_1, \dots, X_{\lambda_2+\ell-2}]$. Its weight is:*

$$\deg_{\mathbf{w}}(\mathbf{X}_\lambda) = \mathbf{w}(\lambda).$$

The characters of \mathcal{S}_m can be represented by polynomials in $\mathbb{Q}[X_1, \dots, X_{m-1}]$ of weights $\leq m - 1$.

Proof. Because the polynomial (5) is homogeneous, we can make $y_1 = 1$ without losing information. In that case, $\chi_{V_\lambda}(g)$ is the coefficient in

$$y_2^{\lambda_2+(\ell-2)} y_3^{\lambda_3+(\ell-3)} \dots y_\ell^{\lambda_\ell},$$

after the development of the product

$$\Delta(\tilde{y}) \left(\prod_{j>1} (1 - y_j) \right) \left(\prod_{d \geq 1} (1 + P_d(\tilde{y}))^{X_d(g)} \right) \quad (\ddagger\ddagger)$$

where $\tilde{y} := \{y_2, \dots, y_\ell\}$. But, in this product the first factor $\Delta(\tilde{y})$ is already homogeneous of total degree $(\ell - 2) + (\ell - 3) + \dots$, so that we must seek, in the development of the remaining factors, terms whose total degree is bounded by $|\underline{\lambda}| = \lambda_2 + \dots + \lambda_\ell$. But then, since we have

$$(1 + P_d)^{X_d} = 1 + \binom{X_d}{1} P_d + \binom{X_d}{2} P_d^2 + \binom{X_d}{3} P_d^3 + \dots$$

and because $\deg_{\text{tot}}(P_d^a) = ad$, we conclude that

- If $d > \lambda_2 + \ell - 2$, the factor $(1 + P_d)^{X_d}$ contributes only to χ_{V_λ} with its term 1^{X_d} , so can be ignored. The product symbol $\prod_{d \geq 1}$ in $(\ddagger\ddagger)$ can therefore be replaced by $\prod_{d=1}^{\lambda_2+\ell-2}$.
- The coefficient $\binom{X_d}{j}$ is a polynomial of degree j in X_d and appears attached to monomials in \tilde{y} of total degree jd . We can thus infer that, after development, the expression of χ_{V_λ} is a polynomial in $X_1, \dots, X_{\lambda_2+\ell-2}$ of weight $|\underline{\lambda}| = \mathbf{w}(\lambda)$. \square

The following corollary of proposition 2.2.2 is now immediate from the definition of representation stable FB-modules 1.7.3.

2.2.3. Corollary *If \mathcal{W} is an FB-module, then*

$$\text{rank}_{\text{PC}}(\mathcal{W}) \leq \text{rank}_{\text{RS}}(\mathcal{W}).$$

Proof. Left to the reader. \square

2.3. Basic examples of character polynomiality of FI-modules

2.3.1. The ℓ -cycles of $\llbracket 1, m \rrbracket$

Given $m, \ell \in \mathbb{N}$, we denote by Γ_ℓ^m the set of ℓ -tuples (i_1, \dots, i_ℓ) of pairwise distinct elements of $\llbracket 1, m \rrbracket$ modulo cyclic permutation, *i.e.* such that

$$(i_1, \dots, i_\ell) = (i_2, \dots, i_\ell, i_1) = (i_3, \dots, i_\ell, i_1, i_2) = \dots$$

The symmetric group \mathcal{S}_m acts on Γ_ℓ^m by

$$g \cdot (i_1, \dots, i_\ell) = (g(i_1), g(i_2), \dots, g(i_\ell)). \quad (6)$$

The elements of Γ_ℓ^m are called *the ℓ -cycles of $\llbracket 1, m \rrbracket$* .

2.3.2. Comments

a) Given $g \in \mathcal{S}_m$, the set $\llbracket 1, m \rrbracket$ is decomposed in $\langle g \rangle$ -orbits, each of which can be endowed with a cyclic order defined by g . For example, if $x \in \llbracket 1, m \rrbracket$, we may consider $(x \rightarrow g(x) \rightarrow g^2(x) \rightarrow \dots \rightarrow x)$, which gives the well-known *decomposition of $g \in \mathcal{S}_m$ as product of disjoint cycles*.

b) For $m \geq \ell$, there is a difference between the cases $\ell = 1$ and $\ell > 1$.

For all $\ell \geq 1$, define the *support* of an ℓ -cycle $\gamma := (i_1, \dots, i_\ell)$ the set $\{\{\gamma\}\}$ of its coordinates :

$$\{\{\gamma\}\} := \{i_1, \dots, i_\ell\} \subseteq \llbracket 1, m \rrbracket.$$

Then, define $\tilde{\gamma} \in \mathcal{S}_m$ by :

$$\tilde{\gamma} := \begin{cases} \tilde{\gamma}(i_j) := i_{j+1 \pmod{\ell}}, & \text{for } i_j \in \{\{\gamma\}\}, \\ \tilde{\gamma}(x) := x, & \text{for } x \notin \{\{\gamma\}\}. \end{cases}$$

i) For $\ell = 1$, we have $\Gamma_1^m = \llbracket 1, m \rrbracket$ and the action (6) of \mathcal{S}_m on Γ_1^m is the standard action of \mathcal{S}_m on $\llbracket 1, m \rrbracket$. The map $(\widetilde{\quad}) : \Gamma_1^m \rightarrow \mathcal{S}_m$ is uninteresting, as it is the constant map $\gamma \mapsto \text{id}_{\llbracket 1, m \rrbracket}$.

ii) For $\ell > 1$, the map $(\widetilde{\quad}) : \Gamma_\ell^m \subseteq \mathcal{S}_m$ is *injective*, and the action (6) of \mathcal{S}_m on Γ_ℓ^m is the conjugation action of \mathcal{S}_m on itself, *i.e.* :

$$\widetilde{g \cdot \gamma} = g \tilde{\gamma} g^{-1}.$$

We will identify γ and $\tilde{\gamma}$ if no confusion is likely to arise. In this sense, for $g \in \mathcal{S}_m$, the set of fixed points $(\Gamma_\ell^m)^g := \{\gamma \in \Gamma_\ell^m \mid g \cdot \gamma = \gamma\}$ is simply :

$$(\Gamma_\ell^m)^g = \{\ell\text{-cycles } \gamma \in \mathcal{S}_m \mid g\gamma = \gamma g\}. \quad (7)$$

iii) For all $m, \ell > 0$, we have :

$$|\Gamma_\ell^m| = \frac{m(m-1) \cdots (m-(\ell-1))}{\ell}.$$

2.3.3. The FI-modules \mathbb{E}_ν

For $m, \ell > 0$, let \mathbb{E}_ℓ^m be the vector space spanned by the set Γ_ℓ^m , i.e.

$$\mathbb{E}_\ell^m := \bigoplus_{\gamma \in \Gamma_\ell^m} \mathbb{Q} \cdot \gamma.$$

Endow it with the linear action of \mathcal{S}_m induced by its action on the basis Γ_ℓ^m .

Notice that :

- $\mathbb{E}_\ell^m = \mathbf{0}$, for all $m < \ell$.
- For all $m \geq n$, the set Γ_ℓ^m is a subset of Γ_ℓ^n , which is invariant under the action of $\mathbf{1}_n \boxtimes \mathcal{S}_{m-n}$, so that the natural inclusions $\Gamma_\ell^m \subseteq \Gamma_\ell^{m+1}$ induce the interior maps (clearly injective) of an FI-module

$$\boxed{\mathbb{E}_\ell := \{ \phi(\mathbb{E}_\ell)_m : \mathbb{E}_\ell^m \hookrightarrow \mathbb{E}_\ell^{m+1} \}}$$

where $\phi(\mathbb{E}_\ell)_m$ is exhaustive for $m \geq \ell$, since the natural map between orbit spaces: $\mathcal{S}_m \backslash \Gamma_\ell^m \rightarrow \mathcal{S}_{m+1} \backslash \Gamma_\ell^{m+1}$ is bijective for $m \geq \ell$.

More generally, if $\nu := (1^{n_1}, 2^{n_2}, \dots, N^{n_N}) = (\nu_1 \geq \nu_2 \geq \dots \geq \nu_\ell)$ is a nonempty partition, let

$$\begin{aligned} \mathbb{E}_\nu^m &:= (\mathbb{E}_1^m)^{\otimes n_1} \otimes (\mathbb{E}_2^m)^{\otimes n_2} \otimes \dots \otimes (\mathbb{E}_N^m)^{\otimes n_N}, \\ &:= \mathbb{E}_{\nu_1} \otimes \mathbb{E}_{\nu_2} \otimes \dots \otimes \mathbb{E}_{\nu_\ell}, \end{aligned}$$

and define $\phi(\mathbb{E}_\nu)_m : \mathbb{E}_\nu^m \rightarrow \mathbb{E}_\nu^{m+1}$ as the tensor product of interior maps

$$\begin{aligned} \phi(\mathbb{E}_\nu)_m &:= \phi(\mathbb{E}_1)^{\otimes n_1} \otimes \dots \otimes \phi(\mathbb{E}_N)^{\otimes n_N}, \\ &:= \phi(\mathbb{E}_{\nu_1}) \otimes \dots \otimes \phi(\mathbb{E}_{\nu_\ell}). \end{aligned}$$

The family

$$\boxed{\mathbb{E}_\nu := \{ \phi(\mathbb{E}_\nu)_m : \mathbb{E}_\nu^m \hookrightarrow \mathbb{E}_\nu^{m+1} \}_m}$$

is an FI-module with all interior maps injective, and exhaustive for $m \geq \nu_1$.

2.3.4. Proposition

a) For $m, \ell \in \mathbb{N}$, the character $\chi_{\mathbb{E}_\ell^m}$ is expressed by the following polynomial of $\mathbb{Q}[X_1, \dots, X_\ell]$ of weight ℓ and independent of $m \in \mathbb{N}$:

$$\mathbf{E}_\ell = \phi(\ell) X_\ell + \sum_{ed=\ell, e \neq 1} \phi(d) \frac{d^e}{\ell} X_d (X_d - 1) \cdots (X_d - (e-1)), \quad (*_\ell)$$

where ϕ is the Euler's totient function.

The FI-module $\mathbb{E}_\ell := \{ \mathbb{E}_\ell^m \hookrightarrow \mathbb{E}_\ell^{m+1} \}_m$ has the following ranks

$$\text{rank}_{\text{PC}}(\mathbb{E}_\ell) = 0 \quad \text{and} \quad \text{rank}_{\text{RS}}(\mathbb{E}_\ell) = \text{rank}_{\text{RS}}^{\text{FI}}(\mathbb{E}_\ell) = 2\ell.$$

- b) Given an ordered sequence $\bar{Z} := (Z_1, \dots, Z_N)$ of polynomials of $\mathbb{Q}[\bar{X}]$, and $d \in \mathbb{N}$, denote by $\mathbb{Q}^{\leq d}[Z_1, \dots, Z_N]$ the subspace of polynomials of weight $\leq d$ relative to \bar{Z} , i.e. the subspace spanned by the elements $Z_1^{a_1} \cdots Z_N^{a_N}$ where $\sum_i i a_i \leq d$. Then, for all $N \in \mathbb{N}$, the inclusion:

$$\mathbb{Q}^{\leq d}[\mathbf{E}_1, \dots, \mathbf{E}_N] \subseteq \mathbb{Q}^{\leq d}[X_1, \dots, X_N]$$

is an equality.

- c) For every nonempty partition $\nu := (1^{n_1}, 2^{n_2}, \dots, N^{n_N})$, we have

i) $\mathbf{w}(\mathcal{IE}_\nu^m) \leq \min\{m, |\nu|\}$.

- ii) The FI-module $\mathcal{IE}_\nu := \{\mathcal{IE}_\nu^m \xrightarrow{\gamma} \mathcal{IE}_\nu^{m+1}\}_m$ has the following ranks

$$\text{rank}_{\text{PC}}(\mathcal{IE}_\nu) = 0 \quad \text{and} \quad \text{rank}_{\text{RS}}(\mathcal{IE}_\nu) = \text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{IE}_\nu) = 2|\nu|.$$

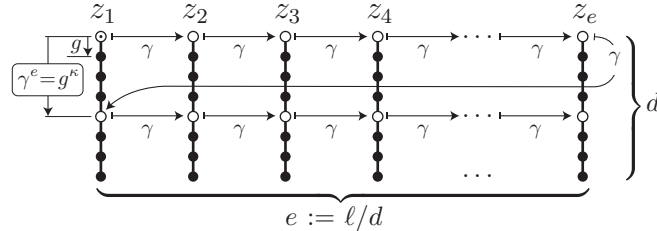
Proof. (a) Since the linear action of $g \in \mathcal{S}_m$ is induced by its action on the basis $\Gamma_\ell^m \subseteq \mathcal{IE}_\ell^m$, the trace $\chi_{\mathcal{IE}_\ell^m}(g)$ is the cardinality of $(\Gamma_\ell^m)^g$ (2.3.2-(b-ii)).

When $m < \ell$, the set Γ_ℓ^m is empty and $\mathcal{IE}_\ell^m = 0$. And, since in \mathcal{S}_m there is no permutation g such that $\rho_m(X_d)(g) \geq \ell/d$, we have $\rho_m(\mathbf{E}_\ell) = 0$, which states (a) when $m < \ell$.

When $m \geq \ell$, following 2.3.2-(b), we consider two cases:

– $\ell = 1$. Then $\Gamma_1^m = \llbracket 1, m \rrbracket$, $\chi_{\mathcal{IE}_1^m}(g) = |\llbracket 1, m \rrbracket^g| = X_1(g)$, and (a) is obvious.

– $\ell > 1$. The set $(\Gamma_\ell^m)^g$ identifies, after 2.3.2-(b-ii)-(7), with the set of ℓ -cycles $\gamma \in \mathcal{S}_m$ such that $g\gamma = \gamma g$. As a consequence, $g\{\{\gamma\}\} = \{\{\gamma\}\}$ and the set $\{\{\gamma\}\}$ appears endowed with two actions commuting to each other. But then, since the action of γ on $\{\{\gamma\}\}$ is transitive, the $\langle g \rangle$ -orbits share the same cardinality. Denote it by d and set $e := \ell/d$. Then, $g^e = \gamma^e$, for some $\kappa \in \llbracket 1, d \rrbracket$, relatively prime to d . These observations are gathered in the following image:



If we choose the point ‘ \circ ’ to stand for the smallest coordinate in γ , this graphic representation becomes one-to-one, and thus helpful for the computation of the cardinality of the set $(\Gamma_\ell^m)^g$. Indeed, it can be used to reconstruct $(\Gamma_\ell^m)^g$ by following the next steps.

1) Take an ordered sequence (z_1, \dots, z_e) of d -cycles in g , among the

$$X_d(g)^e := X_d(g)(X_d(g) - 1) \cdots (X_d(g) - (e-1)).$$

possible such sequences.

2) Shift to the first position the d -cycle containing the smallest coordinate in $\llbracket 1, m \rrbracket$. This means that $X_d(g)^e$ has to be divided by e .

3) Fix the vertical orders of z_2, \dots, z_e . There are d^{e-1} such orders.

4) Fix the number κ . There are $\phi(d)$ possibilities.

Therefore,

$$|(\Gamma_\ell^m)^g| = \sum_{de=\ell} \phi(d) \frac{d^e}{\ell} X_d(g)^e.$$

In both cases, $\ell = 1$ and $\ell > 1$, the previous expression is independent of $m \in \mathbb{N}$, which means that $\text{rank}_{\text{PC}}(\mathbb{E}_\ell) = 0$.

To estimate $\text{rank}_{\text{RS}}(\mathbb{E}_\ell)$, observe that, since the action of \mathcal{S}_m on Γ_ℓ^m is transitive, $\Gamma_\ell^m = \mathcal{S}_m \cdot (1, \dots, \ell)$, so that, denoting $\gamma := (1, \dots, \ell)$, we have an isomorphism of \mathcal{S}_m -spaces:

$$\Gamma_\ell^m = \mathcal{S}_m \cdot \gamma \simeq \mathcal{S}_m \times_{\langle \gamma \rangle \boxtimes \mathcal{S}_{m-\ell}} \{\text{pt}\},$$

because $\text{Stab}_{\mathcal{S}_m}(\gamma) = \langle \gamma \rangle \boxtimes \mathcal{S}_{m-\ell}$. Therefore,

$$\mathbb{E}_\ell^m = \text{ind}_{\mathcal{S}_\ell \boxtimes \mathcal{S}_{m-\ell}}^{\mathcal{S}_m} \left((\text{ind}_{\langle \gamma \rangle}^{\mathcal{S}_\ell} \mathbb{Q}) \boxtimes \mathbb{Q}_{m-\ell} \right).$$

and \mathbb{E}_ℓ is the projective FI-module $\mathcal{P}(\text{ind}_{\langle \gamma \rangle}^{\mathcal{S}_\ell} \mathbb{Q})$ for which we have already shown that it is $(\text{RS})_{m \geq 2\ell}$ (cf. 1.7.2-(b)).

We can be a more precise observing that the simple \mathcal{S}_ℓ -module \mathbb{Q}_ℓ appears with multiplicity 1 in $\text{ind}_{\langle \gamma \rangle}^{\mathcal{S}_\ell} \mathbb{Q}$, which is true since:

$$\text{Hom}_{\mathcal{S}_\ell}(\text{ind}_{\langle \gamma \rangle}^{\mathcal{S}_\ell} \mathbb{Q}, \mathbb{Q}_\ell) = \text{Hom}_{\langle \gamma \rangle}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}.$$

Thus, \mathbb{E}_ℓ contains $\mathcal{P}(\mathbb{Q}_\ell) = \mathcal{P}(V_{0[\ell]})$ as the sub-FI-module, and, thanks to 1.7.4-(d),

$$\text{rank}_{\text{RS}}(\mathbb{E}_\ell) = \text{rank}_{\text{RS}}^{\text{FI}}(\mathbb{E}_\ell) = 2\ell.$$

(b) In (a), the equation $(*_\ell)$ shows that $\mathbf{E}_\ell = \phi(\ell) X_\ell$ modulo a polynomial of weight ℓ in the variables X_d for $d|\ell$ and $d \neq \ell$. (In particular, $\mathbf{E}_1 = X_1$.) Now, given $N \geq 1$, the system of equations $(*_\ell)$, for $\ell \leq N$, can be inverted by recursively introducing some polynomials $Q_\ell^m(Z_1, \dots, Z_\ell) \in \mathbb{Q}[\bar{Z}]$ of weight ℓ , such that

$$X_\ell = Q_\ell^m(\mathbf{E}_1, \dots, \mathbf{E}_\ell), \quad \forall \ell \leq N.$$

The equality,

$$\mathbb{Q}[X_1, \dots, X_N] = \mathbb{Q}[\mathbf{E}_1, \dots, \mathbf{E}_N],$$

but also,

$$\mathbb{Q}^{\leq d}[X_1, \dots, X_N] = \mathbb{Q}^{\leq d}[\mathbf{E}_1, \dots, \mathbf{E}_N], \quad \forall d \in \mathbb{N},$$

then follows easily.

(c) Since the action of \mathcal{S}_m on $\mathbb{I}\mathbf{E}_\nu^m := \mathbb{I}\mathbf{E}_{\nu_1}^m \otimes \cdots \otimes \mathbb{I}\mathbf{E}_{\nu_\ell}^m$ is induced by its component-wise action on the basis $\Gamma_\nu^m := \Gamma_{\nu_1}^m \times \Gamma_{\nu_2}^m \times \cdots \times \Gamma_{\nu_\ell}^m$, the structure of $\mathbb{Q}[\mathcal{S}_m]$ -module of $\mathbb{I}\mathbf{E}_\nu^m$ follows from the structure of \mathcal{S}_m -space of Γ_ν^m . Given $\bar{\gamma} := (\gamma_1, \dots, \gamma_\ell) \in \Gamma_\nu^m$, we have

$$\mathcal{S}_m \cdot \bar{\gamma} = \mathcal{S}_m / \text{Stab}_{\mathcal{S}_m}(\bar{\gamma}) = \mathcal{S}_m \times_{\mathcal{S}_{\{\{\bar{\gamma}\}\}}} (\mathcal{S}_{\{\{\bar{\gamma}\}\}} / \text{Stab}_{\mathcal{S}_{\{\{\bar{\gamma}\}\}}}(\bar{\gamma})) \quad (\ddagger)$$

where we set $\{\{\bar{\gamma}\}\} := \{\{\gamma_1\}\} \cup \cdots \cup \{\{\gamma_\ell\}\}$.

As a consequence the $\mathbb{Q}[\mathcal{S}_m]$ -module $\mathbb{I}\mathbf{E}_\nu^m$ is a direct sum of induced modules of the form:

$$\text{ind}_{\mathcal{S}_{\{\{\bar{\gamma}\}\}} \boxtimes \mathcal{S}_{\mathbb{C}^{\{\{\bar{\gamma}\}\}}}^{\mathcal{S}_m}} W_n \boxtimes \mathbb{Q}_{m-n},$$

where we can choose to have $\{\{\bar{\gamma}\}\} := \llbracket 1, n \rrbracket$, for $n := |\{\{\bar{\gamma}\}\}| \leq \min\{m, |\nu|\}$, in which case

$$W_n := \mathbb{Q}[\mathcal{S}_n / \text{Stab}_{\mathcal{S}_n}(\bar{\gamma})] = \text{ind}_{\text{Stab}_{\mathcal{S}_n}(\bar{\gamma})}^{\mathcal{S}_n} \mathbb{Q}_n. \quad (\ddagger\ddagger)$$

When $m \geq |\nu|$, the corollary 1.4.5-(a) to Pieri's rule, gives the inequality

$$\mathbf{w}(\nu) \leq \mathbf{w}(\mathbb{I}\mathbf{E}_\nu^m) \leq |\nu|,$$

and (c-i) follows.

(c-ii) Since $\chi_{\mathbb{I}\mathbf{E}_\nu^m} = (\chi_{\mathbb{I}\mathbf{E}_1^m})^{n_1} \cdots (\chi_{\mathbb{I}\mathbf{E}_N^m})^{n_N}$ and since $\text{rank}_{\text{PC}}(\mathbb{I}\mathbf{E}_\ell) = 0$, after (a), it follows immediately that $\text{rank}_{\text{PC}}(\mathbb{I}\mathbf{E}_\nu) = 0$.

To finish, we notice that, on the one hand, the number of terms of the form (\ddagger) is exactly de number of \mathcal{S}_m -orbits in Γ_ν^m , and, on the other hand, the sequence of inclusions $\mathcal{S}_m \setminus \Gamma_\nu^m \subseteq \mathcal{S}_{m+1} \setminus \Gamma_\nu^{m+1}$ is *strict increasing* until it stabilizes for $m \geq |\nu|$. As a consequence, the truncated FI-module $(\mathbb{I}\mathbf{E}_\nu)_{\geq |\nu|}$ is isomorphic to the sum of truncated projective FI-modules $\mathcal{P}(W_n)_{\geq |\nu|}$, all of which being $(\text{RS})_{m \geq 2|\nu|}$, after 1.7.2-(b). We therefore have proved that

$$\text{rank}_{\text{RS}}(\mathbb{I}\mathbf{E}_\nu) \leq \text{rank}_{\text{RS}}^{\text{FI}}(\mathbb{I}\mathbf{E}_\nu) \leq 2|\nu|.$$

To see that these are equalities, it suffices, by 1.7.4, to show that, for $m = |\nu|$, one has :

$$\mathcal{P}(\mathbb{Q}_m) \subseteq \mathbb{I}\mathbf{E}_\nu. \quad (\ddagger)$$

But, this is clear if we choose $\bar{\gamma} \in \Gamma_\nu^m$ such that $|\{\{\bar{\gamma}\}\}| = m$ (as in $(\ddagger\ddagger)$), in which case (\ddagger) follows, proceeding as in (a), from the inclusion:

$$\mathbb{Q}_m \subseteq \text{ind}_{\text{Stab}_{\mathcal{S}_m}(\{\{\bar{\gamma}\}\})}^{\mathcal{S}_m}(\mathbb{Q}_m). \quad \square$$

3. Polynomial weight vs. Representation weight

3.1. On the minimal polynomial weight of a character

The Frobenius polynomial $\mathbf{X}_\lambda \in \mathbb{Q}[\overline{X}]$ is of weight $\mathbf{w}(\lambda)$, and there are infinitely many polynomials $P_\lambda \in \mathbb{Q}[X_1, \dots, X_{m-1}]$ such that $\rho_m(P_\lambda) = \rho_m(\mathbf{X}_\lambda) = \chi_{V_\lambda}$. In this section, we advance the study of the relationship between weights of polynomials and weights of representations. We will see that $\mathbf{w}(W_m)$ is the lowest possible weight for a polynomial $P_{W_m} \in \mathbb{Q}[X_1, \dots, X_{m-1}]$ verifying $\rho_m(P_{W_m}) = \chi_{W_m}$. For example, the polynomial $\mathbf{X}_{W_m} := \sum_\lambda \mathbf{n}_\lambda \mathbf{X}_\lambda$, where \mathbf{X}_λ is the Frobenius polynomial of an irreducible factor V_λ of multiplicity \mathbf{n}_λ in W_m . Moreover, we will see that this polynomial \mathbf{X}_{W_m} is the only one satisfying these weights conditions, if and only if $\mathbf{w}(W_m) \leq m/2$.

3.1.1. Theorem

a) For $d, m \in \mathbb{N}$, denote by $\rho_{m|d}$ the restriction of $\rho_m : \mathbb{Q}[\overline{X}] \rightarrow \mathbb{Q}_{\text{cl}}(\mathcal{S}_m)$ to the subspace $\mathbb{Q}^{\leq d}[\overline{X}]$ of polynomials of weight $\leq d$. The image of the map

$$\rho_{m|d} : (\mathbb{Q}^{\leq d}[\overline{X}]) \rightarrow \mathbb{Q}_{\text{cl}}(\mathcal{S}_m)$$

is the subspace

$$\mathbb{Q}_{\text{cl}}^{\leq d}(\mathcal{S}_m) = \langle \chi_{V_\lambda} \mid (|\lambda| = m) \ \& \ (\mathbf{w}(\lambda) \leq d) \rangle_{\mathbb{Q}}, \quad (8)$$

and $\ker(\rho_{m|d}) = 0$ if and only if $d \leq m/2$.

b) Let $P \in \mathbb{Q}[\overline{X}]$.

- i) The scalar product $\langle\langle 1 \mid P \rangle\rangle_{\mathcal{S}_m}$ is constant for $m \geq \deg_{\mathbf{w}}(P)$.
- ii) The scalar product $\langle\langle \chi_{V_\lambda} \mid P \rangle\rangle_{\mathcal{S}_m}$ vanishes for $\lambda \vdash m$ s.t. $\mathbf{w}(\lambda) > \deg_{\mathbf{w}}(P)$.
- iii) Let W_m be a representation of \mathcal{S}_m . If $\rho_m(P) = \chi_{W_m}$, then:

$$\deg_{\mathbf{w}}(P) \geq \mathbf{w}(W_m).$$

c) Let $W_m = \bigoplus_{\lambda \vdash m} V_\lambda^{\mathbf{n}_\lambda}$ be a representation of \mathcal{S}_m . The polynomial

$$\mathbf{X}_{W_m} := \sum_{\lambda \vdash m} \mathbf{n}_\lambda \mathbf{X}_\lambda \in \mathbb{Q}[X_1, \dots, X_{m-1}]$$

where \mathbf{X}_λ is the Frobenius polynomial for χ_{V_λ} (2.2.2), verifies

$$\rho_m(\mathbf{X}_{W_m}) = \chi_{W_m} \quad \text{and} \quad \deg_{\mathbf{w}}(\mathbf{X}_{W_m}) = \mathbf{w}(W_m),$$

and it is the only polynomial verifying simultaneously these two equalities, if and only if $\mathbf{w}(W_m) \leq m/2$.

Proof. (a) In (8), the inclusion ‘ \supseteq ’ is 2.2.2. For the converse, it suffices to restrict oneself to the case of a monomial $\rho_m(X_1^{n_1} \cdots X_d^{n_d})$ for $\sum_i i \mathbf{n}_i \leq d$,

or, thanks to 2.3.4-(b), what amounts to the same thing, to show that

$$(\chi_{\mathbb{E}_1^m})^{\mathbf{n}_1} \cdots (\chi_{\mathbb{E}_d^m})^{\mathbf{n}_d} \in \langle \chi_{V_\lambda} \mid (|\lambda| = m) \ \& \ (\mathbf{w}(\lambda) \leq d) \rangle_{\mathbb{Q}}.$$

Here, one recognizes, on the left, the character of \mathbb{E}_ν^m , for $\nu := (1^{\mathbf{n}_1}, \dots, d^{\mathbf{n}_d})$. Now, since we already know after 2.3.4-(c), that $\mathbf{w}(\mathbb{E}_\nu^m) \leq |\nu|$, we conclude that $\mathbf{w}(\mathbb{E}_\nu^m) \leq d$. The irreducible factors of \mathbb{E}_ν^m are therefore of the form V_λ with $\mathbf{w}(\lambda) \leq d$, which settles the inclusion ‘ \subseteq ’.

For the last claim about $\ker(\rho_{m|d})$, notice that the space $\mathbb{Q}^{\leq d}[X_1, \dots, X_d]$ admits as a basis the set of monomials

$$M(d) := \{ X_1^{\mathbf{n}_1} X_2^{\mathbf{n}_2} \cdots X_d^{\mathbf{n}_d} \mid \sum_{i=1}^d i \mathbf{n}_i \leq d \},$$

while the space $\mathbb{Q}_{\text{cl}}^{\leq d}(\mathcal{S}_m)$ has a basis indexed by the following set of partitions:

$$L(d) := \{ \underline{\lambda} := (1^{\mathbf{n}_1}, 2^{\mathbf{n}_2}, \dots, \lambda_2^{\mathbf{n}_{\lambda_2}}) \mid (\sum_{i=1}^{\lambda_2} i \mathbf{n}_i \leq d) \ \& \ (\sum_{i=1}^{\lambda_2} i \mathbf{n}_i + \lambda_2 \leq m) \},$$

because $\{\chi_{V_\lambda} \mid \lambda \vdash m\}$ is linearly independent after Schur orthogonality.

Since $\lambda_2 \leq d$, the map $\xi : L(d) \rightarrow M(d)$, which associates $(1^{\mathbf{n}_1}, 2^{\mathbf{n}_2}, \dots, \lambda_2^{\mathbf{n}_{\lambda_2}})$ with the monomial $X_1^{\mathbf{n}_1} X_2^{\mathbf{n}_2} \cdots X_{\lambda_2}^{\mathbf{n}_{\lambda_2}}$ is well defined and injective. The condition for ξ to be bijective is that λ_2 be able to take the value d , in which case $\sum_{i=1, \dots, \lambda_2} i \mathbf{n}_i = d$, and that condition appears to be simply that $2d \leq m$.

(b-i) Again, thanks to 2.3.4-(b), we need only prove that the number:

$$\langle\langle 1 \mid \chi_{\mathbb{E}_1^m}(g)^{\mathbf{n}_1} \chi_{\mathbb{E}_2^m}(g)^{\mathbf{n}_2} \cdots \chi_{\mathbb{E}_r^m}(g)^{\mathbf{n}_r} \rangle\rangle_{\mathcal{S}_m}, \quad (\diamond)$$

is constant for all $m \geq \sum_{i=1}^r i \mathbf{n}_i$. But (\diamond) is simply the dimension of the subspace of invariant tensors in the \mathcal{S}_m -module

$$\mathbb{E}_\nu^m := (\mathbb{E}_1^m)^{\otimes \mathbf{n}_1} \otimes (\mathbb{E}_2^m)^{\otimes \mathbf{n}_2} \otimes \cdots \otimes (\mathbb{E}_r^m)^{\otimes \mathbf{n}_r},$$

where $\nu := (1^{\mathbf{n}_1}, 2^{\mathbf{n}_2}, \dots, r^{\mathbf{n}_r})$. Let $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_\ell)$. The canonical basis \mathcal{B}_ν^m of \mathbb{E}_ν^m is the set of tensors

$$\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_\ell,$$

where γ_i is a cycle of length ν_i of \mathcal{S}_m . The basis \mathcal{B}_ν^m is clearly in bijection with the set \mathcal{T}_ν^m of Young *tableaux* τ of shape ν , with the i 'th row filled with elements of $\llbracket 1, m \rrbracket$ so that they represent a cycle of length ν_i of \mathcal{S}_m . The action of \mathcal{S}_m on \mathcal{B}_ν^m induces in \mathcal{T}_ν^m the natural action of \mathcal{S}_m on Young tableaux.

Because of these identifications, the dimension of $(\mathbb{E}_\nu^m)^{\mathcal{S}_m}$ is the cardinality of the orbit space $\mathcal{T}_\nu^m / \mathcal{S}_m$, and the stability we are seeking to prove is

equivalent to the natural map

$$\mathcal{T}_\nu^m / \mathcal{S}_m \rightarrow \mathcal{T}_\nu^{m+1} / \mathcal{S}_{m+1}, [\tau \pmod{\mathcal{S}_m}] \mapsto [\tau \pmod{\mathcal{S}_{m+1}}],$$

being a bijection. But the necessary and sufficient condition for this is precisely that the total number of boxes $|\nu|$ be smaller or equal to m , since, in that case, a single permutation $g \in \mathcal{S}_m$ will allow renumbering of all boxes simultaneously with numbers in the interval $\llbracket 1, |\nu| \rrbracket$. Hence (\diamond) is constant for $m \geq |\nu| = \sum_i i \mathbf{n}_i$.

(b-ii) After (a), $\rho_m(P)$ belongs to the linear span of the characters χ_{V_ν} of \mathcal{S}_m , with $\mathbf{w}(\nu) \leq \deg_{\mathbf{w}}(P)$, and, thanks to Schur orthogonality, these characters are orthogonal to any χ_{V_λ} with $\mathbf{w}(\lambda) > \deg_{\mathbf{w}}(P)$, hence the claim.

(b-iii) Indeed, if we had $\deg_{\mathbf{w}}(P) < \mathbf{w}(W_m)$ then, for any irreducible factor V_λ of W_m of weight $\mathbf{w}(V_\lambda) = \mathbf{w}(W_m)$, we would get, after (b-ii):

$$0 = \langle\langle \chi_{V_\lambda} | P \rangle\rangle_{\mathcal{S}_m} = \langle\langle \chi_{V_\lambda} | W_m \rangle\rangle_{\mathcal{S}_m} \neq 0,$$

which is a contradiction. Hence $\deg_{\mathbf{w}}(P) \geq \mathbf{w}(W_m)$.

(c) Immediate consequence of (b-iii) and the study of $\ker(\rho_{m|d})$ in (a). \square

3.1.2. Comments

- The proposition 2.3.4-(a) showed that, for fixed $m \geq N$, the character of the tensor product $\mathbf{E}_\nu^m := (\mathbf{E}_1^m)^{\otimes n_1} \otimes (\mathbf{E}_2^m)^{\otimes n_2} \otimes \dots \otimes (\mathbf{E}_N^m)^{\otimes n_N}$, as an \mathcal{S}_m -module, is the polynomial: $\mathbf{E}_\nu := \mathbf{E}_1^{n_1} \mathbf{E}_2^{n_2} \dots \mathbf{E}_N^{n_N}$ of $\mathbb{Q}[X_1, \dots, X_N]$, whose weight $\deg_{\mathbf{w}}(\mathbf{E}_\nu) = \sum_i i \mathbf{n}_i$ can be arbitrarily big. On the other hand, theorem 3.1.1-(c) showed that the same character is given by the polynomial $\mathbf{X}_{\mathbf{E}_\nu^m}$ of $\mathbb{Q}[X_1, \dots, X_{m-1}]$, of weight $\mathbf{w}(\mathbf{E}_\nu^m) < m$.

This disagreement is due to the fact that, while \mathbf{E}_ν expresses a priori all the characters in the family $\{\chi_{\mathbf{E}_\nu^m}\}_{m \geq N}$ *simultaneously*, the Frobenius polynomial $\mathbf{X}_{\mathbf{E}_\nu^m}$ expresses a priori only one of them, viz. $\chi_{\mathbf{E}_\nu^m}$.

Heuristically speaking, the elements in $\ker(\rho_m)$ allow the reduction of the weight of the polynomial \mathbf{E}_ν in order to reach the required bound m . This distinction will become meaningful in the next section.

- It is also worth noting that while the explicit writing of \mathbf{E}_ν is very simple, thanks to formula 2.3.4-(a), the writing of $\mathbf{X}_{\mathbf{E}_\nu^m}$ can be quite complex insofar as it entails knowing the set of multiplicities \mathbf{n}_λ of the irreducible factors V_λ in the decomposition $\mathbf{E}_\nu^m = \bigoplus_{\lambda \vdash m} V_\lambda^{\mathbf{n}_\lambda}$, and also in the explicit description of each \mathbf{X}_λ , for $\mathbf{n}_\lambda \neq 0$.

The following is a useful corollary to theorem 3.1.1-(c)

3.1.3. Corollary. *Let \mathcal{W} be an FB-module which is $(\text{PC})_{m \geq N}$ and has polynomial character $P_{\mathcal{W}}$.*

- a) $P_{\mathcal{W}} = \mathbf{X}_{W_m}$, for all $m \geq \max\{2 \deg_{\mathbf{w}}(P_{\mathcal{W}}), N\}$.
- b) $\mathbf{w}(W_m) = \deg_{\mathbf{w}}(P_{\mathcal{W}})$, for all $m \geq \max\{2 \deg_{\mathbf{w}}(P_{\mathcal{W}}), N\}$.
- c) $\mathbf{w}(\mathcal{W}_{\geq N}) = \deg_{\mathbf{w}}(P_{\mathcal{W}})$.

Proof. (a,b) Immediate after 3.1.1-(c).

(c) For all $m \geq N$, we have $\rho_m(P_{\mathcal{W}}) = \chi_{W_m}$ so that $\mathbf{w}(W_m) \leq \deg_{\mathbf{w}}(P_{\mathcal{W}})$, again after 3.1.1-(c). Therefore, $\sup_{m \geq N} \{\mathbf{w}(W_m)\} = \deg_{\mathbf{w}}(P_{\mathcal{W}})$, after (b). \square

4. (PC) versus (RS)

4.1. Equivalence of the stability conditions

We show the equivalence, for an FB-module \mathcal{W} , between the properties (RS) and (PC). In this, two more elements will deserve special attention. First, the ranks of validity of the properties, and, second, the weight of the polynomial $P_{\mathcal{W}}$. The complete statement is the following.

4.1.1. Theorem. *Let \mathcal{W} be an FB-module.*

- a) *If \mathcal{W} is (RS), then \mathcal{W} is (PC) and $\text{rank}_{\text{PC}}(\mathcal{W}) \leq \text{rank}_{\text{RS}}(\mathcal{W})$.*
- b) *If \mathcal{W} is (PC) with polynomial character $P_{\mathcal{W}}$, then \mathcal{W} is (RS) and*

$$\text{rank}_{\text{RS}}(\mathcal{W}) \leq \max\{\text{rank}_{\text{PC}}(\mathcal{W}), 2 \deg_{\mathbf{w}}(P_{\mathcal{W}})\}.$$

Proof. (a) is corollary 2.2.3. (b) Set $d := \deg_{\mathbf{w}}(P_{\mathcal{W}})$. After 3.1.3-(a), we know that $P_{\mathcal{W}} = \mathbf{X}_{W_m}$ for all $m \geq 2d$, so that, if we decompose W_{2d} in its simple $\mathbb{Q}[\mathcal{S}_{2d}]$ -submodules:

$$W_{2d} = \bigoplus_{|\lambda| \leq d} (V_{\lambda[2d]})^{n_{\lambda}},$$

and if we denote $\lambda' := (\lambda_1 \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell})$ for $\lambda := (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell})$, then the FI-module

$$\mathcal{W}' := \bigoplus_{|\lambda| \leq d} (\mathcal{V}_{\lambda'})^{n_{\lambda}},$$

which is clearly $(\text{RS})_{m \geq 2d}$, will be such that $P_{\mathcal{W}'} = P_{\mathcal{W}}$, by construction. As a consequence, there exists an isomorphism of FB-modules:

$$\mathcal{W}_{\geq \max\{2d, N\}} \sim \mathcal{W}'_{\geq \max\{2d, N\}}, \quad (\diamond)$$

and \mathcal{W} is (RS) for $m \geq \max\{2d, N\}$, as stated. \square

4.1.2. Remark. The last proof shows, in (\diamond) , that an FB-module that is (PC) is asymptotically isomorphic, as FB-module, to an (RS) FI-module.

5. Addendum on the weight of a tensor product

5.1. Reinterpretation of the weight of a representation

The theorem 3.1.1-(c) gives an alternative definition of the weight of a representation W_m of \mathcal{S}_m as the lowest possible weight of the polynomials expressing its character χ_{W_m} .

5.2. On the weight of the tensor product of representations

5.2.1. Proposition

a) If $W_{1,m}$ and $W_{2,m}$ are representations of \mathcal{S}_m , then

$$\mathbf{w}(W_{1,m} \otimes W_{2,m}) \leq \mathbf{w}(W_{1,m}) + \mathbf{w}(W_{2,m}).$$

b) If \mathcal{W}_1 and \mathcal{W}_2 are FB-modules which are $(\text{PC})_{m \geq N}$, then

$$\mathbf{w}(W_{1,m} \otimes W_{2,m}) = \mathbf{w}(W_{1,m}) + \mathbf{w}(W_{2,m}),$$

for all $m \geq 2(\mathbf{w}(W_{1,m}) + \mathbf{w}(W_{2,m}))$. In particular,

$$\mathbf{w}((\mathcal{W}_1 \otimes \mathcal{W}_2)_{\geq N}) = \mathbf{w}((\mathcal{W}_1)_{\geq N}) + \mathbf{w}((\mathcal{W}_2)_{\geq N})$$

c) $\mathbf{w}(V_{\lambda[m]} \otimes V_{\mu[m]}) = |\lambda| + |\mu|, \quad \forall m \geq 2(|\lambda| + |\mu|)$.

Proof. (a) As $\chi_{W_1 \otimes W_2} = \chi_{W_1} \chi_{W_2}$, we get

$$\mathbf{w}(W_1 \otimes W_2) \underset{(1)}{\leq} \deg_{\mathbf{w}}(\mathbf{X}_{W_1} \mathbf{X}_{W_2}) = \deg_{\mathbf{w}}(\mathbf{X}_{W_1}) + \deg_{\mathbf{w}}(\mathbf{X}_{W_2}) \underset{(2)}{=} \mathbf{w}(W_1) + \mathbf{w}(W_2),$$

inequality ' $\leq_{(1)}$ ' after 3.1.1-(b-iii), and equality ' $=_{(2)}$ ' after 3.1.1-(c).

(b) Let $d_i := \deg_{\mathbf{w}}(P_{\mathcal{W}_i})$. We know after corollary 3.1.3-(a), that $P_{\mathcal{W}_i} = \mathbf{X}_{W_{i,m}}$, for $m \geq 2 \max\{d_1, d_2\}$. Hence, $\mathbf{X}_{W_{1,m}} \mathbf{X}_{W_{2,m}} = \mathbf{X}_{W_{1,m} \otimes W_{2,m}}$, for $m \geq 2(d_1 + d_2)$, after the same corollary. Therefore,

$$\mathbf{w}(W_{1,m} \otimes W_{2,m}) = \mathbf{w}(W_{1,m}) + \mathbf{w}(W_{2,m}),$$

for $m \geq 2(d_1 + d_2)$, again after 3.1.1-(c).

(c) Particular case of (b). □

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