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# Isospectrality and Transplantations

– Monograph –

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# Chapter 1
# Background

This work is about *Sunada's method*, developed in the 1990s, which led to the discovery of the first known different shapes of drumhead membranes giving out identical sounds. But before we start, some background about its origins and discovery will help appreciate the true worth of this technique.

The chapter starts with a review of the origins of *spectroscopy*, its beginnings, in Newton's study of the light spectrum, and its evolution and development, marked by amazing astronomical discoveries that, over the course of several centuries, brought scientists to place great hopes in the spectral analysis of all kinds of undulatory phenomena. Of these, the problem that interests us of finding drumhead membranes with the same natural vibrations —the same *sound spectra*—, came much later. The mathematical modeling of membrane vibrations by Euler and Lagrange in the 18th century led to the equivalent problem of finding two different *domains* in the 2-dimensional Euclidean space $\mathbb{R}^2$, such that the corresponding vector spaces of eigenfunctions of the Laplacian vanishing at the boundary are of equal dimension, in short, domains of *equal (Dirichlet) spectrum* or simply *isospectrals*.

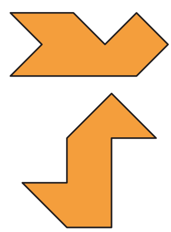
Formulated in this way, the question applies to any Riemannian manifold with boundary $\boldsymbol{M}$ regardless of its dimension, opening the way to the vast field of research of *spectral geometry*. In dimension 1, they are *vibrating strings* for which the pioneering work of Jean le Rond d'Alembert showed, in 1747, that their shape: circular as a ring or straight as a line segment with 0, 1, or 2 fixed ends, may be inferred from its spectrum. Much later, in 1911, Hermann Weyl will prove that the same is true in all dimensions for the *volume* of $\boldsymbol{M}$, something that, at the time, encouraged the widespread hope of determining the whole *shape* of $\boldsymbol{M}$ from its spectrum. But in 1964 this hope was suddenly dashed when John Milnor exhibited two different isospectral 16-dimensional tori. Subsequently, other counterexamples were to follow, but the case of flat surfaces (2-dimensional manifolds with an Euclidean metric), as are the *drumhead membranes*, was to resist for the next thirty years.

In the 1980s, Toshikazu Sunada, searching for a common denominator, set about analyzing the counterexamples that were known at the time, mainly those of Milnor ([Mi], 1964), Vignéras ([V], 1980) and Ikeda ([I], 1983), and he developed what is now known as the *Sunada's method*.

Based on the concept of *almost-conjugated groups*, Sunada established in 1985 that if two such groups $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ of isometries of $\boldsymbol{M}$ act freely on $\boldsymbol{M}$, the quotient spaces $\boldsymbol{M}_i := \boldsymbol{M}/\boldsymbol{H}_i$ are isospectral *a priori*, that is: without any need of explicit determination of the spectrum —which is still now generally out of reach ! It was an unexpectedly hopeful step toward the systematic construction of counterexamples, the beginning of a new paradigm of research. But, for the method to succeed, it was necessary to avoid the case where the $\boldsymbol{H}_i$'s are *conjugate*, case in which the $\boldsymbol{M}$'s would be isometric, and so not interesting. At the time there were known *abstract* groups containing *nonconjugate* almost-conjugate subgroups, such as the linear group $\mathrm{Gl}(3,\mathbb{F}_2)$, but for the aim of constructing drumhead membranes, there were no known *flat surfaces* with such isometry groups. This begged the question of whether it was possible to construct such surfaces with a prescribed isometric group. Peter Buser's method of *thickening* the Cayley graph of a group showed that it was.

Buser implemented these ideas for the group $\mathrm{Gl}(3,\mathbb{F}_2)$ in 1986 and delivered the first known example of isospectral surfaces. But, although 2-dimensional, they were still not domains of $I\!R^2$; and it was not until 1988, when Pierre Bérard extended Sunada's method, that Gordon, Webb and Wolpert (GWW) were able to develop Buser's work further and, in 1992, finally gave a negative answer to Mark Kac's 1966 question "*Can you hear the shape of a drum?*"

This conclusion is notable in a number of ways. On the one hand, it relies solely on the theory of representation of finite groups, avoiding the use of sophisticated techniques of spectral geometry. On the other hand, the construction of the counterexample of GWW highlights a certain 7×7-matrix with coefficients ±1 or 0 which describes how to *transplant* a wave from one membrane to the other, providing an elementary *a posteriori* procedure for the verification of isospectrality. A procedure that overstepped the limits of Sunada's method and led to new examples of isospectral membranes such as the "*arrow*" and the "*hen*" illustrated in the picture to the right. We are in 1992, and this was the debut of the *transplantation method*.

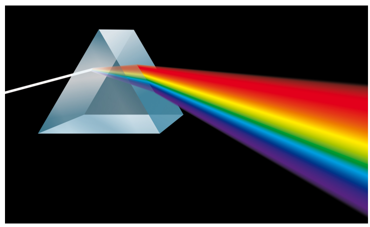


A careful reading of the works of Sunada, Buser, GWW and Bérard shows that a full understanding of Sunada's method is achievable with a moderate level of mathematics. One needs only to master, in algebra: the basics of finite group theory and finite group representation theory, and in Riemannian geometry: flat surfaces with polygonal boundaries, that is, surfaces that one may construct by cutting and pasting pieces of paper. This makes it possible to present a text, accessible to third year university students, that both provides an introduction to the more profound disciplines of later years and describes a complete and elegant solution to a long-standing mathematical problem. This is our purpose.

# 1 Spectral Geometry

Our knowledge of the objects that surround us derives from our perception of the signals they give out, such as the music from a musical instrument or the light emitted by a star. While the signals differ fundamentally from the objects emitting them, they carry lot of information about the objects themselves, but much of it is coded as "*fingerprints*" that we need to find and learn how to interpret. For example, how much can we tell about objects, such as stars, strings, drums, etc., from the signals they emit? Can one determine the chemical composition of the atmosphere of a star simply from its *light spectrum*, the length of a string or the area of a drumhead membrane from its *sound spectrum*?

**1.1. Visible Light Spectroscopy.** The word *spectrum* appears in science in 1665 in Isaac Newton's writings on the dispersion of a light beam passing through a prism. In this experiment, Newton realized that it was not, as was generally thought at the time, the prism that colored the light, but that the prism spreads out the monochrome rays that *compose* the light beam in directions according to their colors. The projection of these rays onto a screen then shows as a multicolored strip recalling the rainbow that Newton called *the light spectrum*.



A century and a half later (1814), Joseph von Fraunhofer will discover dark lines in the Solar spectrum. It was clear that they represented missing color rays, but the causes of the phenomenon remained unexplained until the 1860s when Gustav Kirchhoff realizes that they were the components of the light *absorbed* by the gases of the Solar atmosphere.

Solar spectrum

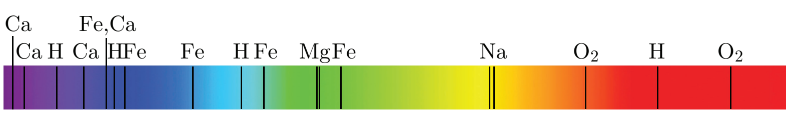
Continuous spectrum

Absorption spectrum of Sodium (Na)

Absorption spectrum of Iron (Fe)

Absorption spectrum of Oxigen ($O_2$)

Indeed, when a gas is traversed by a beam of light, its atoms interact at certain frequencies, absorbing corresponding light components. The set of these frequencies, the *absorption spectrum of the gas*, appears as a collection of dark lines in the light spectrum recalling barcodes familiar to us today. By cataloguing, for all known gases, these barcodes, the composition of an unknown gas can then be determined from its absorption spectrum.

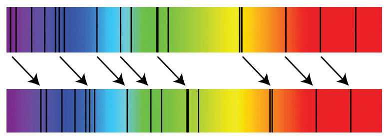



chemical composition of the Solar gaz atmosphere

This meant that the chemical composition of stars' atmospheres could be determined from their light spectra, which until then had been unimaginable.

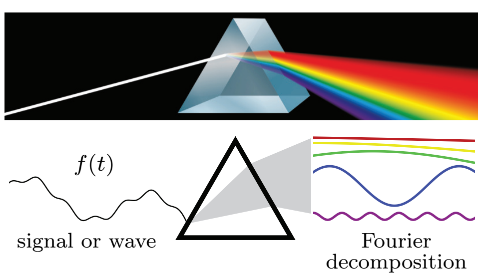
Rudimentary at first, Fraunhofer line detection methods were refined to the point where they even anticipated discoveries: in 1868, Jules Janssen and Joseph Lockyer spotted an unidentified line in the Solar spectrum and conjectured the existence of a new element. Called *Helium* after Helios, the greek god of the Sun, it was finally identified on earth thirty years later!

Spectroscopy rapidly became a standard tool in astrophysics, where it led the astronomer Edwin Hubble to observe in the 1930s the *redshift* phenomenon: the further a star was from the Earth, the more the Fraunhofer lines of its spectrum shifted toward red. The discovery, explained as a Doppler effect, gave a spectacular and immediate experimental confirmation of the general relativity conjecture of Georges Lemaître (1927) stating that the universe was expanding, and even the rate at which it was doing so —in turn providing unexpected support for the Big Bang model...



These examples, which neatly illustrate the wealth of information nested in the light spectrum, elevated *spectroscopy* to a discipline in its own right, experimental and theoretical, whose importance has not ceased to affirm itself.

**1.2 From Light To Sound.** For Newton, the nature of light was *corpuscular* and his immense reputation ensured that his point of view prevailed for more than a century, retarding the development of the *wave theory of light* initiated by Christiaan Huygens in 1690. During the 19th century, the latter gradually gained ground, notably thanks to Augustin Fresnel who used it to explain the phenomenon of polarization of light in 1821, which could not be explained by Newton's theory. At about the same time, a fundamental discovery took place in mathematics when, in 1822, Joseph Fourier found that a wave signal decomposes as a sum of trigonometric (sines and cosines) functions of various amplitudes and frequencies, by now known as the *Fourier decomposition*. Applied to light, then considered as a wave, it gave a new and deeper interpretation of how the optical prism operates, and conversely made of the Fourier decomposition the mathematical analog of the prism for *any* wave phenomenon.



According to Fourier, a wave function of time $t$, periodic of period $T$, may be expressed in a *unique* way as the sum of *harmonics*, *i.e.* *trigonometric functions* of frequencies multiples of the *fundamental frequency* $F_0 = 1/T$,

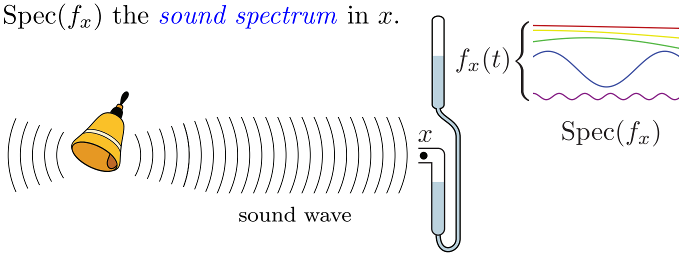
$$f(t) = \sum_{n\geq 0} a_n \sin\bigl(2\pi n F_0 t\bigr) + b_n \cos\bigl(2\pi n F_0 t\bigr), \tag{1}$$

and, if this equality can be considered as the analog of the decomposition of a beam of light, the functions $a_n \sin(2\pi n F_0 t) + b_n(2\pi n F_0 t)$ are the analogs of the monochrome component rays. It is therefore natural to define the *spectrum* of the function $f$ as the set of all *frequencies* $nF_0$ in (1) for which at least one of the coefficients $a_n$, $b_n$, doesn't vanish, *i.e.* :

$$\mathrm{Spec}(f) := \bigl\{ nF_0 \bigm| (a_n,b_n) \neq (0,0) \bigr\}. \tag{2}$$

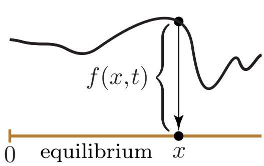
The Fourier decomposition thus prompted the use of spectroscopy methods for all sorts of undulatory phenomena, including sound.

Denote by $f_x(t)$ the air pressure at the point $x$ in the atmosphere at time $t$. The function $f_x$ represents the *sound at $x$*, the equality (1) its *harmonic* decomposition, and $\mathrm{Spec}(f_x)$ the *sound spectrum* in $x$.




**1.3 The Wave Equations.** The second half of the 18th century was marked by an outstanding collective effort, largely federated by the French school, researching differential equations describing all sorts of physical wave phenomena. For sound, the most relevant contributions (interesting us) came from Jean le Rond d'Alembert in 1750, for vibrating strings, Joseph-Louis Lagrange for the propagation of sound in air, and Leonard Euler in 1760 for vibrating membranes.

In the case of strings and small-amplitude vibrations, denote by $f(x,t)$ the height at the instant $t$ of a point of the string whose (vertical) projection on the equilibrium position is $x$. The *d'Alembert wave equation* describing the movements of the points of the string, is the celebrated partial differential equation:

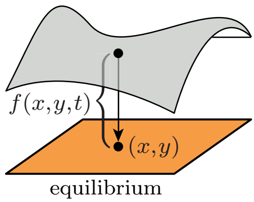



$$\partial_t^2 f(x,t) = c^2\, \partial_x^2 f(x,t), \tag{3}$$

where $c$ is a constant condensing physical characteristics of the string: density, elasticity, tension …

Similarly, for small-amplitude vibrations of *membranes*, the *Euler-Lagrange wave equation*

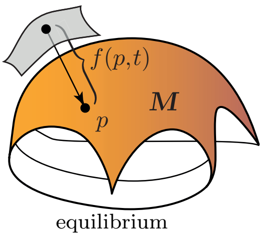
$$\partial_t^2 f(x,y,t) = c^2 \Delta_{\mathbb{R}^2}(f)(x,y,t), \tag{4}$$




applies, where $\Delta_{\mathbb{R}^2} := \partial_x^2 + \partial_y^2$ is the 2-dimensional version of the general *Laplace operator* on the Euclidean $n$-dimensional space:

$$\Delta_{\mathbb{R}^n} = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2 .$$

This operator was to be generalized at the end of the 19th century by Eugenio Beltrami, who gave it meaning for much more general spaces: the *Riemannian manifolds* $\boldsymbol{M}$ of arbitrary dimension. The new operator, denoted by $\Delta_{\boldsymbol{M}}$, is the *Laplace-Beltrami operator*, in short *Laplacian*, which made it possible to define by analogy to membranes, "*a sound of* $\boldsymbol{M}$" as a solution of the differential equation which naturally generalizes (4), the *wave equation for* $\boldsymbol{M}$:




$$\boxed{\partial_t^2 f(p,t) = c^2 \Delta_{\boldsymbol{M}} f(p,t)} \tag{5}$$

**1.4 Harmonics and Eigenfunctions of $-\boldsymbol{\Delta_M}$.** By analogy with music, a *harmonic of frequency $F$ at the point $p \in \boldsymbol{M}$* is a solution of (5) of the form

$$f(p,t) = A_F(p)\sin(2\pi F t) + B_F(p)\cos(2\pi F t). \tag{6}$$

As the coefficients $A_F(p)$ and $B_F(p)$ are real numbers varying with $p \in \boldsymbol{M}$, they define functions $A_F, B_F : \boldsymbol{M} \to \mathbb{R}$. Formal elementary computations using (5) and (6) then show that these $A_F$ and $B_F$, belong to the family of functions $\phi : \boldsymbol{M} \to \mathbb{R}$ verifying the equality:

$$\boxed{(2\pi F/c)^2 \phi = -\Delta_{\boldsymbol{M}}(\phi)} \tag{7}$$

which presupposes some kind of differentiability of $\phi$, since $\Delta_{\boldsymbol{M}}$ is a second order differential operator. Notice that after (6) and (7), the existence of a harmonic of frequency $F$ is equivalent to the existence of an *eigenfunction* of the *minus-Laplacian* $-\Delta_{\boldsymbol{M}}$ of *positive eigenvalue* $(2\pi F/c)^2$.

**1.5 The Spectrum of a Riemannian Manifold.** In order to go into more detail, we will need to be precise about the concepts of *differentiability* of functions and of the Laplacian $\Delta_{\boldsymbol{M}}$ on a Riemannian manifold $\boldsymbol{M}$ with boundary $\partial\boldsymbol{M}$, with corners $\partial^2\boldsymbol{M}$, ... These will be natural generalizations of the classical concepts for the Euclidean spaces $\mathbb{R}^n$. We will come to these on chapter **6**, but, for now, let us suppose the concepts already defined and continue to set some notations.

Denote by $\mathcal{C}^\infty(\boldsymbol{M})$ the real vector space of differentiable real functions $f : \boldsymbol{M} \to I\!R$ and by $\mathcal{C}^\infty_{\mathcal{D}}(\boldsymbol{M})$ the subspace of those functions satisfying the vanishing condition $f|_{\partial \boldsymbol{M}} = 0$, called the *the Dirichlet condition*. Notice that this is the natural constraint that must obey the function $\phi : \boldsymbol{M} \to I\!R$ in 1.4 when $\boldsymbol{M}$ is a drumhead membrane, as its boundary is fixed and stationary.

**1.5.1. Eigenfunctions, Eigenvalues and Multiplicities.** Let us recall some standard terminology of linear algebra.

For $\lambda \in I\!R$, *the eigenspace of the minus-Laplacian associated with $\lambda$* is the vector space $\mathcal{H}_\lambda(\boldsymbol{M})$ of functions $\phi \in \mathcal{C}^\infty(\boldsymbol{M})$ s.t. $\lambda\phi = -\Delta_{\boldsymbol{M}}(\phi)$. A *nonzero* function in $\mathcal{H}_\lambda(\boldsymbol{M})$ is an *eigenfunction of eigenvalue $\lambda$*. The *multiplicity $\mu(\lambda,\boldsymbol{M})$ of $\lambda$* is the number $\mu(\lambda,\boldsymbol{M}) := \dim_{I\!R}\big(\mathcal{H}_\lambda(\boldsymbol{M})\big)$, finite or not. In the presence of the Dirichlet condition, we shall also denote :

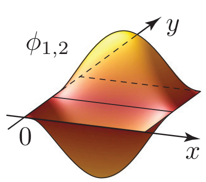
$$\mathcal{H}_{\lambda,\mathcal{D}}(\boldsymbol{M}) := \mathcal{H}_\lambda(\boldsymbol{M}) \cap \mathcal{C}^\infty_{\mathcal{D}}(\boldsymbol{M}) \quad \text{and} \quad \mu_{\mathcal{D}}(\lambda,\boldsymbol{M}) := \dim_{I\!R}(\mathcal{H}_{\lambda,\mathcal{D}}(\boldsymbol{M})).$$

**1.5.2. Spectrum and Eigenvalues.** Pursuing the analogy between the solutions of the wave equation (5) and sound vibrations, we might be tempted to define *the set of harmonic frequencies of $\boldsymbol{M}$* as the set of numbers $F \geq 0$ for which there exists a *nonzero* $\phi \in \mathcal{C}^\infty_{\mathcal{D}}(\boldsymbol{M})$ verifying (7). This set is clearly in bijection, by $F \mapsto (2\pi F/c)^2$, with the set of *positive* eigenvalues of $-\Delta_{\boldsymbol{M}}$ admitting eigenfunctions belonging to $\mathcal{C}^\infty_{\mathcal{D}}(\boldsymbol{M})$, which is called *the Dirichlet spectrum of $\boldsymbol{M}$* and is denoted $\sigma_{\mathcal{D}}(\boldsymbol{M})$. We therefore define

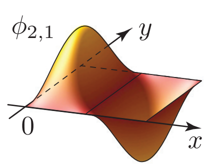
$$\sigma_{\mathcal{D}}(\boldsymbol{M}) := \big\{\lambda \in I\!R_{\geq 0} \;\big|\; \mathcal{H}_{\lambda,\mathcal{D}}(\boldsymbol{M}) \neq 0\big\}.$$

Thus, the study of what might be called the *the sound spectrum of $\boldsymbol{M}$* is equivalent to that of set $\sigma_{\mathcal{D}}(\boldsymbol{M})$ which, as we shall see, only depends on the geometry of $\boldsymbol{M}$.

**1.6 Extended Spectrum and Modes of Vibration.** The eigenspaces of the minus-Laplacian entail a subtlety that can't be perceived in sound as it doesn't appear in the set $\sigma_{\mathcal{D}}(\boldsymbol{M})$. This is the fact that the same sound may be produced by different ways of playing. Different harmonic waves, produced by different *modes of vibration*, may have the same *frequency*. This phenomenon corresponds to the existence of eigenvalues of multiplicity strictly greater than 1. For example, the following two independent functions defined on the square $\boldsymbol{D} := [0,1]^2 \in I\!R^2$ have the same minus-Laplacian eigenvalue $\lambda = 5\pi^2$.




$\phi_{1,2}(x,y) = -\sin(\pi x)\sin(2\pi y)$




$\phi_{2,1}(x,y) = \sin(2\pi x)\sin(\pi y)$

This example might suggest that there exists a relationship between the existence of internal symmetries, here $(x,y) \mapsto (y,1-x)$, and the existence of multiplicities strictly greater than 1. Both properties are in fact independent. We will soon see examples of domains with the same group of symmetries where, in some cases, the multiplicities are all equal to 1, while in others they are arbitrary large (*cf.* 4.4.1-P4-P5). In order to take care of this diversity of vibratory modes, one defines the *extended* Dirichlet spectrum of $\boldsymbol{M}$ by setting

$$\boxed{\mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M}) := \big\{(\lambda,\mu_{\mathcal{D}}(\lambda,\boldsymbol{M})) \;\big|\; \lambda \in \sigma_{\mathcal{D}}(\boldsymbol{M})\big\} \subseteq \mathbb{R}_{\geq 0} \times [\![1,+\infty]\!]} \tag{8}$$

**1.7 Spectrum and Geometry.** The advantage of $\mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M})$ compared to a so-called *sound spectrum of* $\boldsymbol{M}$, is its pure geometric nature. Free of time and other physical considerations such as the constant $c^2$ in (5), its definition involves only the vector space $\mathcal{C}^\infty(\boldsymbol{M})$, the Laplacian $\Delta_{\boldsymbol{M}}$ and the Dirichlet condition. Now, if $\alpha : \boldsymbol{M}_1 \to \boldsymbol{M}_2$ is an isomorphism of Riemannian manifolds, it clearly induces the isomorphism of vector spaces

$$\alpha^* : \mathcal{C}^\infty(\boldsymbol{M}_2) \to \mathcal{C}^\infty(\boldsymbol{M}_1), \quad \phi \mapsto \phi \circ \alpha, \tag{9}$$

which respects the Dirichlet's condition since $\alpha(\partial \boldsymbol{M}_1) = \partial \boldsymbol{M}_2$, but also, and this is really important, it will be compatible with Laplacians, *i.e.* :

$$\Delta_{\boldsymbol{M}_1}(\phi \circ \alpha) = \Delta_{\boldsymbol{M}_2}(\phi) \circ \alpha, \quad \forall \phi \in \mathcal{C}^\infty(\boldsymbol{M}_2),$$

so that (9) is in fact an isomorphism of *vector spaces with linear operators*:

$$\alpha^* : (\mathcal{C}^\infty_{\mathcal{D}}(\boldsymbol{M}_2), \Delta_{\boldsymbol{M}_2}) \underset{\simeq}{\longrightarrow} (\mathcal{C}^\infty_{\mathcal{D}}(\boldsymbol{M}_1), \Delta_{\boldsymbol{M}_1}), \tag{10}$$

and, as such, it induces isomorphisms $\alpha^* : \mathcal{H}_{\lambda,\mathcal{D}}(\boldsymbol{M}_2) \underset{\simeq}{\to} \mathcal{H}_{\lambda,\mathcal{D}}(\boldsymbol{M}_1)$, for all $\lambda \in \mathbb{R}$. In particular,

$$\mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M}_1) = \mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M}_2). \tag{11}$$

The important consequence of this is that the extended Dirichlet spectrum of a Riemannian manifold $\boldsymbol{M}$ (with boundary) is a *geometric invariant* of $\boldsymbol{M}$.

**1.7.1. Remark.** The equality (11) is also valid when $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are solely supposed to be isometric. Indeed, the Myers-Steenrod theorem (1939) states that a bijection between Riemannian manifolds that locally preserves distances is automatically a differentiable map, hence an isomorphism ([1]).

**1.7.2. Comment.** The existence of a linear isomorphism compatible with Laplacians like (10) always implies the equality of extended spectra. In 1.7 it was induced by an isometry $\alpha : \boldsymbol{M}_1 \to \boldsymbol{M}_2$, which is not really interesting. As we will see later, what makes Sunada's method remarkable is its ability to construct such isomorphisms in situations where the manifolds are no longer isometric! These are the *transplantations*.

([1]) This is the analog of the theorem of Euclidean geometry stating that a map between Euclidean spaces *locally* preserving distances is an *affine* (thus differentiable).

**1.8 Direct and Inverse Problems in Spectral Geometry.** The aim of *spectral geometry* is the research of relationships which might, or might not, exist between the geometry of a Riemannian manifold and its extended spectrum. For example, between a musician playing a musical instrument (the geometry) and the music one hears (sound spectrum). There are two major themes in spectral geometry: the "direct" and "inverse" problems.

In the *direct problem*, one seeks properties of its extended spectrum from what is known of the geometry of $\boldsymbol{M}$. For example, the nature and beauty of the music from what is known of the musician and the instrument he plays, or, more abstractly, find geometric properties of $\boldsymbol{M}$ guaranteeing the finiteness of multiplicities $\mu_{\mathcal{D}}(\lambda, \boldsymbol{M})$, ... In all cases, it is worth recalling that complete knowledge $\mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M})$ is almost never possible as, apart from very few exceptions, there is no known general approach to its full determination.

In the *inverse problem*, one seeks geometric properties of $\boldsymbol{M}$ from what is known of $\mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M})$ . For example, from a given music, determine the musical instruments, musicians, director, ... Or, more abstractly, the celebrated *Hermann Weyl's law* (1911) (*cf.* 5.4) which determines the volume of $\boldsymbol{M}$ (a geometric property) from the asymptotic behavior of $\mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M})$ ([2])

The inverse problem can lead to dead-ends, especially when the correspondence $\mathrm{Geometry}(\boldsymbol{M}) \rightsquigarrow \mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M})$ is not injective. It is then necessary to restrict the family of spaces $\boldsymbol{M}$ taken into consideration. The case of drumhead membranes is a typical example. Although we will exhibit isospectral membranes of different shapes (geometry), Steve Zelditch showed in 2000 ([Z]) that the shape is indeed fully determined by its spectrum when restricted to the family of simply connected membranes with analytic boundary and having two orthogonal axes of symmetry ([3]).

At this point, one can't help recalling the famous quotation of Sir Arthur Schuster (1851-1934), a German-British physicist: *To find out the different tunes sent out by a vibrating system is a problem which may or may not be solvable in certain special cases, but it would baffle the most skillful mathematician to solve the inverse problem and to find out the shape of a bell by means of the sounds which it is capable of giving out. And this is the problem which ultimately spectroscopy hopes to solve in the case of light. In the meantime we must welcome with delight even the smallest step in the right direction (1882).*

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([2]) Using Kac's periphrasis, Weyl proved that one can hear the area of a drum. It is said that, when David Hilbert listen the famous physicist H.A. Lorentz ask that question at a conference in Göttingen in 1910, he bet that it would remain unanswered during his lifetime. Soon after, Weyl, a student of his, gave the answer using the theory of integral equations conceived by the same Hilbert himself!

([3]) A great deal of research concerns the question of how to restrict the class of manifolds so that they might be determined by their spectrum. Many articles on this subject, like the one of Zelditch, deal with *smoothness* conditions of the boundary $\partial\boldsymbol{M}$. However, the question still remains largely open, for example it is not yet known if the shape of a convex domain with smooth boundary is, or is not, determined by its extended spectrum.

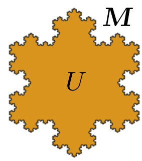
## 2 Isospectrality *and* Isometry

**2.1 Interpreting Kac's Question.** The geometric invariance of $\mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M})$ evoked in 1.7, leads us to interpret Kac's question "*Can one hear the shape of a drum ?*" as whether the correspondence

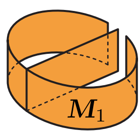
$$\boldsymbol{M}\ (\textit{modulo isometry}) \rightsquigarrow \mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M}) \tag{12}$$

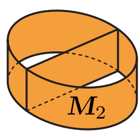
can be inverted. But, as such, the question would be too ambitious, as if one wanted to reconstruct the face of a caveman from his fossilized fingerprints... A more modest (and realistic) question is whether the spectrum would allow to *differentiate* the $\boldsymbol{M}$'s. In other words, is the correspondence (12) *injective*?

**2.2. Boundary and Irreducibility.** A *(Euclidean) domain* is the closure $\boldsymbol{M} := \overline{U}$ of a connected open subset $U$ of the Euclidean space $\mathbb{R}^n$. Its *boundary* is the closed subset $\partial\boldsymbol{M} := \boldsymbol{M} \setminus U$. For $n = 1$, we get closed real intervals with a boundary having at most two points, but in larger dimensions the boundary $\partial\boldsymbol{M}$ may be any closed subset of $\mathbb{R}^n$ with empty interior, and this can be so complex (e.g. fractal) that even a qualitative study of $\mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M})$ may be intractable. For this reason the researches in spectral geometry often incorporate simplifying hypotheses for the boundary, for example, in dimension 2, that of being piecewise submanifolds of $\mathbb{R}^2$ of dimension 1.

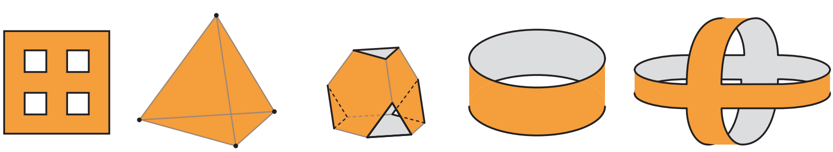



This being said, it is not difficult to build isospectral manifolds. For example, consider the two manifolds $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, to the right, obtained by gluing three rectangular membranes and imposing the Dirichlet condition on the boundary (drawn in black). In both, a vibration is just the assemblage of three vibrations, one on each rectangle fixing its own boundary. Therefore $\mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M}_1) = \mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M}_2)$, while the manifolds are clearly not isometric. To avoid these uninteresting examples, we will ask the $\boldsymbol{M}$'s to be *irreducible*, that is, such that their *interior* $\boldsymbol{M}^\circ := \boldsymbol{M} \setminus \partial\boldsymbol{M}$ consists of just one *path-connected* component, which is not the case in our example where $\boldsymbol{M}_i^\circ$ has 3 components.



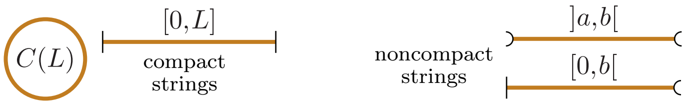

We will see that, even under this restriction, the correspondence (12) is not injective in dimension 2 or greater. Counterexamples will be constructed using very simple Riemannian manifolds: *irreducible compact flat surfaces with polygonal boundary*. These are obtained by gluing together domains of the Euclidean space $\mathbb{R}^2$ with *polygonal boundary*, as in the following examples:



Once the 2-dimensional case is settled, it will be easy to give counterexamples for all higher dimensions.

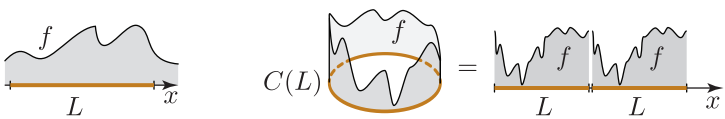
# 3 Strings' Spectra

**3.1 Strings.** These are 1-dimensional Riemannian manifolds $\boldsymbol{S}$. When $\boldsymbol{S}$ is irreducible and compact, it is a bounded closed interval $[0,L] \in I\!\!R$ or a circle $C(L) \in I\!\!R^2$, where $L > 0$ denotes its *length*. When $\boldsymbol{S}$ is irreducible and noncompact, it is an open real interval $]a,b[$ or a semi-open real interval $[0,b[$, with $a,b \in [\infty,+\infty]$.




**3.2 Differentiable Functions on Strings.** The notion of *differentiability* of functions is the usual one on an open interval $I = \,]a,b[$, but if $I \subseteq I\!\!R$ is not open, one must specify its meaning at boundary points. A function $f : I \to I\!\!R$ is said *differentiable* if it is the restriction of a differentiable function $g : \,]a,b[\, \to I\!\!R$, where $]a,b[\, \supseteq I$. In that case, the *n-th differential of f*, denoted by $\partial_x^n(f) : I \to I\!\!R$, is the restriction to $I$ of $\partial_x^n(g) : \,]a,b[\, \to I\!\!R$.

In the case of the circle $C(L)$, a function $f : C(L) \to I\!\!R$ naturally identifies with a function over the real line $\tilde{f} : I\!\!R \to I\!\!R$ which is periodic of period $L$. We then say that $f$ is *differentiable* if $\tilde{f}$ is differentiable in the usual sense, in which case its differential corresponds to the (periodic) function $\partial_x(\tilde{f})$.




Under these identifications, the Laplacian $\Delta_{\boldsymbol{S}}$ appears to be the classical second order derivative operator $\partial_x^2 := (d/dx)^2$.

When $\boldsymbol{S}$ is not connected, a map $f : \boldsymbol{S} \to I\!\!R$ is said *differentiable* if its restriction to each of its connected components are also. We denote by $\mathcal{C}^\infty(\boldsymbol{S})$ the set of such functions endowed with the natural structure of real vector space. The linear action of the Laplacian $\Delta_{\boldsymbol{S}}$ on $\mathcal{C}^\infty(\boldsymbol{S})$ then corresponds to the action of $\partial_x^2$ on each component of $\boldsymbol{S}$. The subspace of functions vanishing on the boundary $\partial\boldsymbol{S}$ is denoted by $\mathcal{C}^\infty_{\mathcal{D}}(\boldsymbol{S})$.

**3.3 The Spectrum of a String.** A well-known theorem of the theory of linear differential equations states that the functions $f \in \mathcal{C}^\infty(I\!\!R)$ satisfying

$$-\partial_x^2 f = \lambda f\,, \tag{13}$$

are the functions

$$\begin{cases} f(x) = ae^{-\sqrt{-\lambda}x} + be^{\sqrt{-\lambda}x}, & \text{for } \lambda < 0, \\ f(x) = ax + b, & \text{for } \lambda = 0, \\ f(x) = a\sin(\sqrt{\lambda}x) + b\cos(\sqrt{\lambda}x), & \text{for } \lambda > 0, \end{cases} \tag{14}$$

where $a,b \in I\!\!R$.

In addition, the theorem says that any solution of (13) defined in the neighborhood of a point $x_0 \in I\!R$ extends in a unique way to a solution defined in the whole real line $I\!R$, and thereby of one of the forms (14).

We deduce the following table of eigenvalues.

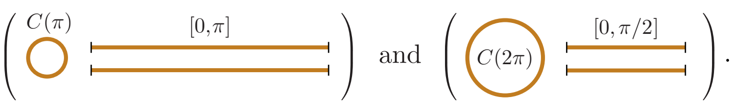
| **Space** | **Boundary Condition** | **Eigenvalues** |
|---|---|---|
| $[0,b[$ | $f(0) = 0$ | $\{\lambda^{[1]} \mid \lambda \in I\!R\}$ |
| $]a,b[$ | | $\{\lambda^{[2]} \mid \lambda \in I\!R\}$ |
| $[0,L]$ | $f(0) = 0$ et $f(L) = 0$ | $\{((\pi n/L)^2)^{[1]} \mid 0 < n \in \mathbb{N}\}$ |
| $C(L)$ | | $\{0^{[1]}, ((2\pi n/L)^2)^{[2]} \mid 0 < n \in \mathbb{N}\}$ |

where '$\lambda^{[i]}$' indicates that $\lambda$ is an eigenvalue of $-\partial_x^2$ of multiplicity $i$.

We notice that the (positive) spectrum clearly differentiates the spaces that are not *homeomorphic*, but also that it *does not see* their length when they are not compact. Moreover, the third column shows that for $L_1 \neq L_2$, the disjoint unions

$$C(2L_1) \sqcup [0,L_2] \sqcup [0,L_2] \quad \text{and} \quad C(2L_2) \sqcup [0,L_1] \sqcup [0,L_1],$$

have exactly the same extended spectrum, although they are not isometric. For example, if $L_1 = \pi/2$ and $L_2 = \pi$




These phenomena add reasons to restrict the question of isospectrality in 2 to irreducible (hence connected) and *compact* Riemannian manifolds.

The following proposition states the conclusion of these remarks.

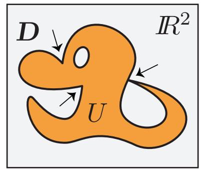
**3.3.1 Proposition.** *Call string a compact connected Riemannian manifold of dimension 1. The correspondence which associates a string with its spectrum is injective modulo isometry.*

**3.3.2 Exercise.** Show that a finite disjoint union $\mathcal{S} := \boldsymbol{S}_1 \sqcup \ldots \sqcup \boldsymbol{S}_r$ of homeomorphic strings (e.g. a harp) is totally determined by its extended spectrum and the homeomorphic type of the strings.

**3.3.3 Exercise.** Let $F$ be a closed subset of a string $\boldsymbol{S}$. Let $\mathrm{Spec}_{\mathcal{D},F}(\boldsymbol{S})$ denote the extended spectrum of the space of the eigenfunctions of $-\Delta_{\boldsymbol{S}}$ vanishing on $F \cup \partial\boldsymbol{S}$. Show that $\mathrm{Spec}_{\mathcal{D},F_1}(\boldsymbol{S}) = \mathrm{Spec}_{\mathcal{D},F_2}(\boldsymbol{S})$ if and only if there exists a bijection $\alpha : (\boldsymbol{S} \smallsetminus F_1) \to (\boldsymbol{S} \smallsetminus F_2)$ which is a local isometry.

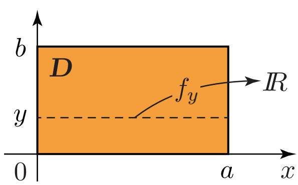
# 4 Membranes' Spectra

**4.1. Domains and Membranes.** In accordance with our discussion in 2.2, we call *domain* the closure $\boldsymbol{D} := \overline{U}$ of a connected bounded open subset $U \in I\!\!R^2$. For simplicity, we restrict to the case where the *boundary* $\partial\boldsymbol{D} := \overline{U} \smallsetminus U$ is a submanifold of dimension 1 of $I\!\!R^2$ except possibly on a finite number of points (arrows in the image to the right). These domains may serve as patterns for drumhead *membranes* so that both terms will be considered as synonymous.




**4.2 Differentiable Functions on Membranes.** A function $f : \boldsymbol{D} \to I\!\!R$ is said to be *differentiable* if it is the restriction to $\boldsymbol{D}$ of a differentiable function $\tilde{f} : I\!\!R^2 \to I\!\!R$ ([4]). We denote by $\mathcal{C}^\infty(\boldsymbol{D})$ the real vector space of these functions. The action of *the Laplacian $\Delta_{\boldsymbol{D}}$* on $f \in \mathcal{C}^\infty(\boldsymbol{D})$ is defined by the action of the Laplace operator $\Delta_{I\!\!R^2} = \partial_x^2 + \partial_y^2$ on $\tilde{f} \in \mathcal{C}^\infty(I\!\!R^2)$. We denote $\mathcal{C}^\infty_{\mathcal{D}}(\boldsymbol{D})$ the vector subspace of $\mathcal{C}^\infty(\boldsymbol{D})$ of the functions, which vanish on $\partial\boldsymbol{D}$.

**4.3 Rectangular Membranes.** ([5]) Let $\boldsymbol{D} \subseteq I\!\!R^2$ be a rectangular domain of width $a$ and height $b$ as shown in the picture below.




Let $f \in \mathcal{C}^\infty_{\mathcal{D}}(\boldsymbol{D})$. For each fixed $y \in [0,b]$, the Fourier decomposition of the function $f_y : [0,a] \to I\!\!R$, $x \mapsto f(x,y)$, is the *Fourier sine series*

$$f_y(x) = \sum_{0<n\in\mathbb{N}} c_n(y)\sin(n\pi x/a), \tag{15}$$

where the coefficient $c_n(y)$ is given by the integral formula

$$c_n(y) = \frac{2}{a}\int_0^a f(x,y)\sin(n\pi x/a)\,dx.$$

The theorem of continuity and differentiation under the integral sign then states that the map $c_n : [0,b] \to I\!\!R$, $y \mapsto c_n(y)$ is continuously derivable, and since $c_n(0) = c_n(b) = 0$, it has a Fourier decomposition

$$c_n(y) = \sum_{0<m\in\mathbb{N}} c_{n,m}\sin(m\pi y/b). \tag{16}$$

As the series (15) and (16) are *normally convergent* on $\boldsymbol{D}$, the same is true for the *double series* of general term $c_{n,m}\sin(n\pi x/a)\sin(m\pi y/b)$.

---

([4]) To be consistent with 3.2, we should have taken $\tilde{f}$ defined on some open neighborhood $V$ of $\boldsymbol{D}$ in $I\!\!R^2$, but, thanks to Urysohn lemma, we can always assume that $V = I\!\!R^2$.

([5]) The classical reference for this subject is §V.5 *The Vibrating Membrane*, p. 297, and §V.5.4 *Rectangular Membrane*, p. 300, in Courant and Hilbert's [CH] book.

By these remarks we have shown that any function $f \in \mathcal{C}^\infty_{\mathcal{D}}(\boldsymbol{D})$ is the sum of a double sine series normally convergent on $\boldsymbol{D}$:

$$f(x,y) = \sum_{0<n,m\in\mathbb{N}} c_{n,m} \sin(n\pi x/a) \sin(m\pi y/b), \quad \forall (x,y) \in \boldsymbol{D}. \tag{17}$$

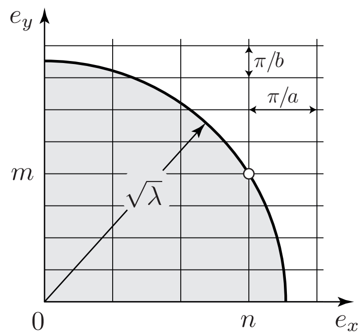
**4.4 The Spectrum of a Rectangular Membrane.** Thanks to (17), one can easily check that the eigenfunctions of $-\Delta_{\boldsymbol{D}}$ on $\mathcal{C}^\infty_{\mathcal{D}}(\boldsymbol{D})$ are the functions

$$\phi_{n,m}(x,y) = \sin(n\pi x/a) \sin(m\pi y/b), \quad \forall n,m \in \mathbb{N} \setminus \{0\}, \tag{18}$$

and that the eigenvalues are the *positive* real numbers

$$\lambda = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 > 0, \tag{19}$$

with multiplicity $\mu_{\mathcal{D}}(\lambda,\boldsymbol{D})$ equal to the cardinal of the set of points of the circle centered at $0 \in I\!R^2$ of radius $\sqrt{\lambda}$ with positive coordinates with respect to $a^{-1}\pi\vec{e}_x$, and $b^{-1}\pi\vec{e}_y$ (exercise!).




(20)

**4.4.1. Some Notable Properties of Spec$_{\mathcal{D}}(\boldsymbol{D})$.** The following properties, with the exception of underlined P4, P5 and P8, are general and go far beyond the example of rectangular membranes and the dimension 2.

P1) *Local/Global Uniqueness of Eignefunctions.*
From (18) we see (exercise!) that an eigenfunction $\phi_{n,m}$ vanishes in the neighborhood of a point $(x,y) \in \boldsymbol{D}$ if and only if $\phi_{n,m} = 0$. Therefore, if two eigenfunctions $\phi$, $\phi'$, coincide in the neighborhood of a point, first, they have the same eigenvalue, and second, they coincide everywhere in $\boldsymbol{D}$ (exercise!).

P2) *The set $\sigma_{\mathcal{D}}(\boldsymbol{D})$ is a countably infinite subset of $I\!R_+$.* Follows from (19).

P3) *Finite Multiplicities.*
Given $\lambda$, the solutions $(n,m)$ of (19) are parametrized by their abscissa, the positive integer $n$, itself bounded above by $\lfloor a\sqrt{\lambda}/\pi \rfloor$.

<u>P4</u>) *Simple Multiplicities.*

$$(b/a)^2 \notin \mathbb{Q} \iff \mu_{\mathcal{D}}(\lambda,\boldsymbol{D}) \le 1,\ \forall \lambda \in I\!R.$$

Indeed, $\mu_{\mathcal{D}}(\lambda,\boldsymbol{D}) > 1$, if and only if, it exists $(n_1,m_1),(n_2,m_2) \in \mathbb{N}^2$ with $n_1 > n_2$ such that $(n_1/a)^2 + (m_1/b)^2 = (n_2/a)^2 + (m_2/b)^2$, in other words, such that

$$b^2(n_1+n_2)(n_1-n_2) = a^2(m_1+m_2)(m_2-m_1) > 0 \tag{21}$$

where $(n_1 + n_2) \equiv_2 (n_1 - n_2)$ and $(m_1 + m_2) \equiv_2 (m_2 - m_1)$.

In particular, $(b/a)^2 \in \mathbb{Q}$. Conversely, if $(b/a)^2$ is a rational number, we write it as a fraction

$$\left(\frac{b}{a}\right)^2 = \frac{A_2 B_2}{A_1 B_1},$$

with $A_i, B_i \in \mathbb{N}$ such that $A_1 \equiv_2 B_1$, $A_2 \equiv_2 B_2$, $A_1 > B_1$, $A_2 > B_2$.

Then, if we set

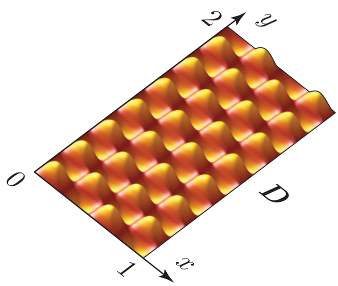
$$\begin{cases} n_1 := \dfrac{A_1 + B_1}{2}, & m_1 := \dfrac{A_2 - B_2}{2}, \\ n_2 := \dfrac{A_1 - B_1}{2}, & m_2 := \dfrac{A_2 + B_2}{2}, \end{cases}$$

we have $(n_1, m_1) \neq (n_2, m_2)$ and the equality (21) is satisfied.

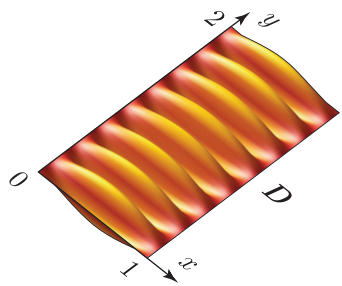
For example, if $b = 2$ and $a = 1$, we write

$$\left(\frac{2}{1}\right)^2 = \frac{30 \times 2}{5 \times 3},$$

in which case, we get $(n_1, m_1) = (4, 14)$, $(n_2, m_2) = (1, 16)$, hence the two following eigenfunctions with same eigenvalue $\lambda = 65\pi^2$,




$\phi_{4,14} = \sin(4\pi x)\sin(7\pi y)$




$\phi_{1,16} = \sin(\pi x)\sin(8\pi y)$

(22)

In a drum, these waves would vibrate vertically at the same frequency $F = \frac{c}{2}\sqrt{65}$, following formula (7). They are two different *modes of vibration* for the same harmonic frequency (*cf.* 1.6).

P5) *Unbounded Multiplicities.*

$$(b/a)^2 \in \mathbb{Q} \iff \{\mu_{\mathcal{D}}(\lambda, \boldsymbol{D}) \mid \lambda \in \mathbb{R}\} \text{ is unbounded above.}$$

We begin with a digression that will remind the reader of the familiar Pythagorean triples, those nonzero positive integers $(n, m, r) \subseteq \mathbb{N}^3$ satisfying the equation $n^2 + m^2 = r^2$.

*Lemma. Given $z \in \mathbb{Z}$, there exists an infinite family of triples of nonzero positive integers $\{(n_i, m_i, r_i)\}_{i \in \mathbb{N}}$ verifying the equation*

$$z n^2 + m^2 = r^2, \tag{23}$$

*and such that the vectors $(n_i, m_i) \in \mathbb{R}^2$ are pairwise noncollinear.*

Proof. Equality (23) is equivalent to $zn^2 = (r-m)(r+m)$. Hence, a factorization $zn^2 = A \cdot B$, where $A < B$ are integers of the same parity, gives the triple $\big(n,(B-A)/2,(B+A)/2\big)$ satisfying (23). Now, if $n := 2k$, with $k \in \mathbb{N}$, the factorization $z(2k)^2 = 2z \cdot 2k^2$ gives

$$\big(n_k,m_k,r_k\big) = (2k,k^2 - z,k^2 + z). \tag{24}$$

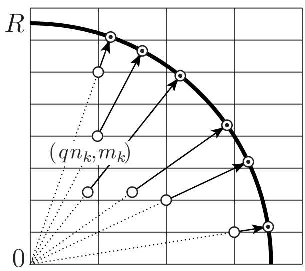
The slope of the couples $(n_k,m_k)$ is $\frac{1}{2}\big(k - \frac{z}{k}\big)$ which is strictly increasing with respect to integers $k > \sqrt{|z|}$. Therefore the family of triples (24) indexed by those integers satisfies the lemma.

Now, let $(b/a)^2 = p/q$, with $p,q \in \mathbb{N}$, and consider, for $s \in \mathbb{N}$ fixed, a family $\{(n_{k_i},m_{k_i},r_{k_i})\}_{i=1,\dots,s}$ of $s$ triples satisfying the lemma for the equation $(pq)n^2 + m^2 = r^2$ which will be written in the equivalent form

$$\Big(\frac{b}{a}\Big)^2 (qn)^2 + m^2 = r^2 .$$

As the slope of $(qn_{k_i},m_{k_i})$ is $\frac{1}{2}\big(k_i/q - p/k_i\big)$, which is strictly increasing with respect to the $k_i$'s, the vectors $(qn_{k_i},m_{k_i})$ are pairwise noncollinear.

For $s \in \mathbb{N}$ arbitrary large, set $R := \mathrm{ppcm}\{r_{k_1},\dots,r_{k_s}\}$. The couples




$$(n'_i,m'_i) := \Big(qnk_i R/r_{k_i},m_{k_i} R/r_{k_i}\Big) \tag{25}$$

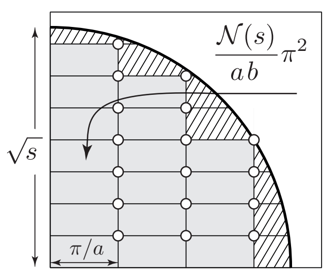
represent the coordinates of $s$ *distinct* points of $(\mathbb{N}\setminus\{0\})^2$ verifying, by construction, the equality:

$$\Big(\frac{n'\pi}{a}\Big)^2 + \Big(\frac{m'\pi}{b}\Big)^2 = \Big(\frac{R\pi}{b}\Big)^2.$$

In this way, the couples (25) determine a family of $s$ linearly independent eigenfunctions of $-\Delta_{\boldsymbol{D}}$ for the same eignevalue $\lambda := (R\pi/b)^2$.

P6) *Discrete Spectrum.*

The *counting function of multiplicities of eigenvalues* is the function that maps $s \in \mathbb{R}$ to the sum of the multiplicities of the eigenvalues bounded above by $s$, that is



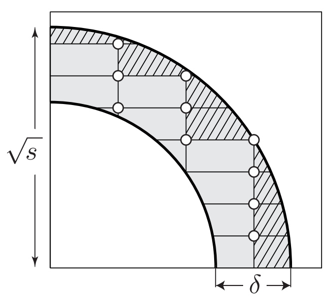

$$\mathcal{N}(s) := \sum_{\lambda \leq s} \mu_{\mathcal{D}}(\lambda,\boldsymbol{D}).$$

The image to the left justifies the following upper bound for the greyed surface

$$\frac{\mathcal{N}(s)}{ab}\pi^2 \leq \frac{1}{4}\pi s, \tag{26}$$

which implies that $\#(\sigma_{\mathcal{D}}(\boldsymbol{D}) \cap [0,s]) < +\infty$, for $s \in \mathbb{R}$. The set $\sigma_{\mathcal{D}}(\boldsymbol{D})$ is thus countable infinite and discrete in $\mathbb{R}$.

P7) *Area and Asymptotic Behavior of the Extended Spectrum.*
The difference between the two members of (26) is bounded above by the area of the quarter of annulus indicated in the image to the left,




where $\delta := \pi\sqrt{b + a/ab}$ is the greatest distance between two points of the rectangle of width $\pi/a$ and height $\pi/b$. We thus have

$$0 \leq \frac{1}{4}\pi s - \frac{\mathcal{N}(s)}{ab}\pi^2 \leq \frac{\pi\delta}{4}\Big(2\sqrt{s} - \delta\Big),$$

which implies the asymptotic equality

$$\boxed{\lim_{s\mapsto+\infty}\frac{\mathcal{N}(s)}{s} = \frac{ab}{4\pi} = \frac{\mathrm{Area}(\boldsymbol{D})}{4\pi}} \tag{27}$$

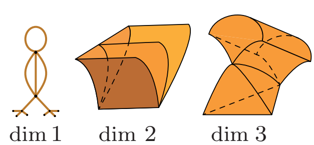
This is *Weyl's law* for the rectangular membrane $\boldsymbol{D}$. It expresses how the area of $\boldsymbol{D}$ can be determined by $\mathrm{Spec}_{\mathcal{D}}(\boldsymbol{D})$ (*cf.* 5.4).

P8) *Equivalence Isospectrality/Isometry.*
If we suppose $a \leq b$, equation (19) immediately shows that the first two eignevalues in $\sigma_{\mathcal{D}}(\boldsymbol{D})$ are: $(\pi/a)^2 + (\pi/b)^2$ and $(\pi/a)^2 + (2\pi/b)^2$, from which $a$ and $b$ are uniquely determined. We see therefore that two rectangular membranes are isospectral if and only if they are isometric.

## 5 Some General Results of Spectral Geometry

We now recall some classical results of spectral geometry, which, although they will not be used here, are fundamental in *spectral analysis* and allow a better perspective on phenomena observed in rectangular membranes. The techniques used to justify them are very different from those used in this book and for this reason, rather than revisit them, we direct interested readers to the book by Isaac Chavel [Cha] or the notes of Yaiza Canzani [Ca].

**5.1. Regularity Conditions of the Boundary.** This paragraph is a continuation of note 2.2 about boundaries. We often find in the statements of spectral geometry the sentence: *let $\boldsymbol{M}$ be a manifold with sufficiently regular boundary* without explaining what that means. For us, it will be synonymous with *Riemannian manifold with piecewise smooth boundary*. That is, any space $\boldsymbol{M}$ obtained by glueing polyhedra (not necessarily convex) of some affine space $(\mathbb{R}^n, g)$ —where $g$ denotes certain *Riemannian metric* on $\mathbb{R}^n$. A point $x \in \boldsymbol{M}$ is then said to be *interior* if it admits an open neighborhood $V_x$ which is isometric to an open ball of $(\mathbb{R}^n, g)$. The set $\boldsymbol{M}^\circ$ of interior points of $\boldsymbol{M}$ is an open *dense* subset of $\boldsymbol{M}$, and its complement $\partial\boldsymbol{M} := \boldsymbol{M} \smallsetminus \boldsymbol{M}^\circ$ is the *boundary of $\boldsymbol{M}$*. When $\partial\boldsymbol{M}$ is not empty, it is in turn a Riemannian manifold with piecewise smooth boundary of dimension strictly smaller than that of $\boldsymbol{M}$.



**5.2 Boundary Conditions.** These are constraints on the behavior of a function $f \in \mathcal{C}^\infty(\boldsymbol{M})$ on the boundary $\partial\boldsymbol{M}$ of $\boldsymbol{M}$. The following two conditions are fundamental in spectral geometry.

- $(\mathcal{D})$ *The Dirichlet Condition*. Already introduced in (1.5), this requires the vanishing of the function on the boundary (*cf.* 1.5):

$$\boxed{f|_{\partial\boldsymbol{M}} = 0} \tag{28}$$

The Dirichlet condition is a classical constraint for drumhead membranes, as their boundary are tightly fixed to the drumcase.

- $(\mathcal{N})$ *The Neumann Condition*. This requires the vanishing of the derivative of the function in the direction normal to the boundary:

$$\boxed{\partial_{\vec{n}}(f)(x) = 0, \quad \forall x \in \partial\boldsymbol{M}} \tag{29}$$

Here $\partial_{\vec{n}}$ denotes the *normal derivative* —the partial differential perpendicular to the boundary going out from $\boldsymbol{M}$ (*cf.* **7**♦6.1). This is a classical constraint in hi-hat cymbals, as they are clamped at a central point and their edge is free to vibrate remaining radially parallel to the horizontal plane.

The notations introduced in 1.5 that refer to the Dirichlet condition by the letter '$\mathcal{D}$' have their counterparts for the Neumann condition. They will be indicated by the letter '$\mathcal{N}$'. We thus denote by $\mathcal{C}^\infty_{\mathcal{N}}(\boldsymbol{M})$ the vector space of functions verifying (29), and by analogy to 1.5.1 we set

$$\mathcal{H}_{\lambda,\mathcal{N}}(\boldsymbol{M}) := \mathcal{H}_\lambda(\boldsymbol{M}) \cap \mathcal{C}^\infty_{\mathcal{N}}(\boldsymbol{M}), \quad \mu_{\mathcal{N}}(\lambda,\boldsymbol{M}) := \dim_{I\!R}(\mathcal{H}_{\lambda,\mathcal{N}}(\boldsymbol{M})),$$

$$\sigma_{\mathcal{N}}(\boldsymbol{M}) := \big\{\lambda \in I\!R \;\big|\; \mathcal{H}_{\lambda,\mathcal{N}}(\boldsymbol{M}) \neq 0\big\},$$

and define the *extended Neumann spectrum of $\boldsymbol{M}$* as:

$$\mathrm{Spec}_{\mathcal{N}}(\boldsymbol{M}) := \big\{(\lambda,\mu_{\mathcal{N}}(\lambda,\boldsymbol{M})) \;\big|\; \lambda \in \sigma_{\mathcal{N}}(\boldsymbol{M}))\big\}.$$

**5.3 On Eigenfunctions and Eignevalues.** Very interesting properties of eigenfunctions appear when the manifold $\boldsymbol{M}$ is *compact and orientable*. The following theorem gives two general properties that are simultaneously valid for Dirichlet and Neumann boundary conditions.

**5.3.1 Theorem.** *Let $\boldsymbol{M}$ be a compact oriented Riemannian manifold with sufficiently regular boundary and denote by $\beta$ any of the conditions $\mathcal{D}$ or $\mathcal{N}$. Then,*

a) *(Positivity). If $\mathcal{H}_{\lambda,\beta}(\boldsymbol{M}) \neq 0$, then $\lambda \geq 0$. (Heuristically, this is saying that eigenfunctions always generate* sounds *(cf. 1.4).)*

b) *(Discreteness). The sum $\mathcal{N}_\beta(s,\boldsymbol{M}) := \sum_{\lambda \leq s} \mu_\beta(\lambda,\boldsymbol{M})$ is finite $\forall s \in I\!R$.*

*Hints.* (a) results from the Green identity ([6]) stating that for $f,g \in \mathcal{C}^\infty(\boldsymbol{M})$, one has the equality

$$\int_{\boldsymbol{M}} -\Delta(f) g d_{\boldsymbol{M}} = \int_{\boldsymbol{M}} \big(\overrightarrow{\mathrm{grad}} f, \overrightarrow{\mathrm{grad}} g\big)\, d_{\boldsymbol{M}} - \int_{\partial \boldsymbol{M}} \partial_{\vec{n}}(f) g d_{\boldsymbol{M}}.$$

Now, if $\phi = f = g \in \mathcal{H}_{\lambda,\beta}(\boldsymbol{M})$, the product $\partial_{\vec{n}}(f)g$ must vanish on $\partial \boldsymbol{M}$, in which case we immediately get the positivity condition:

$$\lambda \int_{\boldsymbol{M}} \phi^2 \, d_{\boldsymbol{M}} = \int_{\boldsymbol{M}} \big(\overrightarrow{\mathrm{grad}} \phi, \overrightarrow{\mathrm{grad}} \phi\big)\, d_{\boldsymbol{M}} > 0.$$

Therefore $\lambda \geq 0$, and, moreover, $\lambda = 0$, if and only if $\overrightarrow{\mathrm{grad}} \phi = 0$, or equivalently, if and only if $\phi$ is locally constant.

(d) is a much deeper result whose justification needs more sophisticated techniques. The result in this form is discussed in [Bé$_1$] Th. 18, p. 53. □

**5.4 The Weyl's Law. .** The spectrum of rectangular membranes was originally determined in the 1910s by the physicist Peter Debye, who conjectured that the same phenomenon (27) should occur in any bounded domain in $\mathbb{R}^2$. In 1911 Hermann Weyl proved that conjecture and, even better, established his famous asymptotic formula for *any* bounded domain $\boldsymbol{M}$ with sufficiently regular boundary in $\mathbb{R}^n$ for *any* dimension $n \in \mathbb{N}$. He showed the equality

$$\lim_{s \mapsto +\infty} \frac{\mathcal{N}_\beta(s,\boldsymbol{M})}{s^{n/2}} = \frac{\mathrm{Vol}(\mathbb{B}^n)\,\mathrm{Vol}(\boldsymbol{M})}{(2\pi)^n},$$

where $\mathbb{B}^n$ denotes the unit ball of $\mathbb{R}^n$. The formula has been generalized to any compact Riemannian manifold with sufficiently regular boundary.

**5.5 Local/Global Uniqueness of Eigenfunction.** The results of the previous paragraphs concern the eigenspaces of the minus-Laplacian $-\Delta_{\boldsymbol{M}}$ but they do not say anything about the nature of the eigenfunctions. What do they look like? In the example of the rectangular membranes (4.3), we found products of sine functions, but what can be said for other membrane shapes. This is an important subject in the literature on EDP and more specifically on what are called *solutions of elliptic differential operators* (which include the Laplace-Beltrami operator). One of the most important actors in this subject was Lipman Bers ([7]), author of an important generalization ([Ber], 1955) of the "*Holmgren's uniqueness theorem*" ([Ho], 1901) which states the uniqueness and the analyticity of the eigenfunctions of elliptic operators on a domain of $\mathbb{R}^n$ (see also Hadamard [Ha], page 348, 1903). In the 2-dimensional case of membranes, it may be stated as follows.

([6]) Chapter I of the Isaac Chavel's book [Cha], formula (46), page 8.

([7]) Lipman "*Lipa*" Bers (1914-1993) American mathematician born in Riga, Latvia. Founder of the theory of pseudoanalytic functions.

**5.5.1 Theorem (Bers).** *Let $B(\vec{0},\epsilon)$ be the open ball in the Euclidean space $\mathbb{R}^2$ centered at the origin $\vec{0} \in \mathbb{R}^2$ and radius $\epsilon > 0$. Let $\phi \in \mathcal{C}^\infty(B(\vec{0},\epsilon))$ be an eigenfunction of $-\Delta_{\mathbb{R}^2}$ such that $\phi(\vec{0}) = 0$. Then, either $\phi(x,y) = 0$ for all $(x,y) \in B(\vec{0},\epsilon)$, or there exists a* ***nonzero*** *polynomial $P(x,y)$, homogeneous in $\{x,y\}$, such that*

$$\phi(x,y) = P(x,y) + o(\|(x,y)\|^{\deg P+1}). \tag{$*$}$$

We easily deduce the following corollary, weakened version of a theorem of Cheng ([Che], 1976).

**5.5.2 Corollary (Cheng).** *Let $\boldsymbol{M}$ be an irreducible flat surface and let $\phi$ be an eigenfunction of $-\Delta_{\boldsymbol{M}}$.*

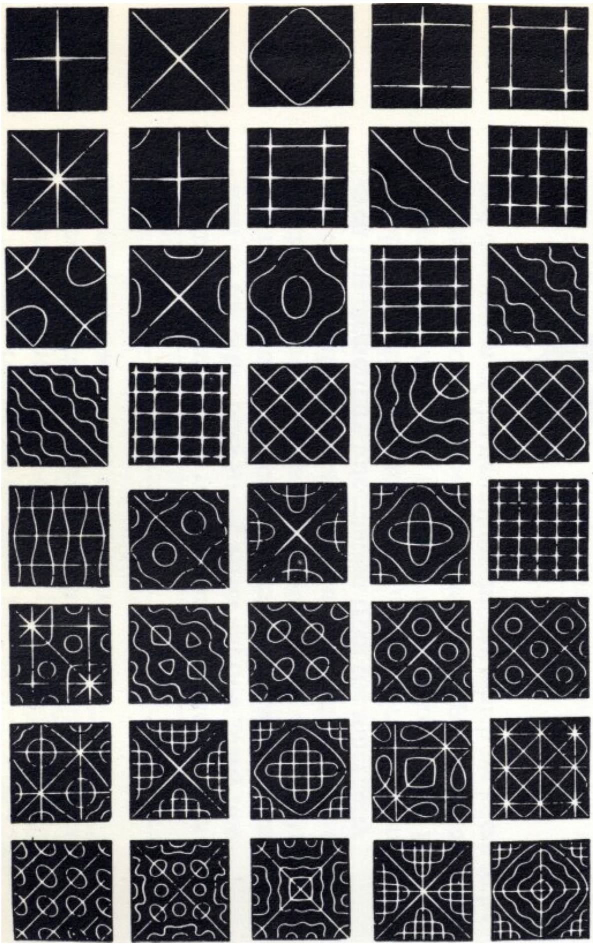
a) *The set $\phi^{-1}(0)$, called nodal set of $\phi$, has empty interior.*
b) *Let $\phi$ et $\phi'$ be two eigenfunctions of $-\Delta_{\boldsymbol{M}}$ (not necessarily of the same eigenvalue), then $\phi = \phi'$ on the whole $\boldsymbol{M}$, if and only if, $\phi = \phi'$ at the neighborhood of some point in $\boldsymbol{M}$.*

*Hint.* (a) Let $\phi \in \mathcal{C}^\infty(\boldsymbol{M})$ be an eigenfunction of $-\Delta_{\boldsymbol{M}}$. Denote by $\mathcal{Z}$ the set of points $z \in \boldsymbol{M}$ such that $\phi$ vanishes in an open neighborhood $W_z$ of $z$. The set $\mathcal{Z}$ is open because it is the union of the open subsets $W_z$. Now, if $p$ belongs to the closure $\overline{\mathcal{Z}}$ of $\mathcal{Z}$, it is the limit of a sequence of points $z_n$ where $z_n \in \mathcal{Z}$. Arguing by continuity, we see that $\partial_1^{m_1}\partial_2^{m_2}(\phi)(p) = \lim_n \partial_1^{m_1}\partial_2^{m_2}(\phi)(z_n) = 0$ for all $m_1,m_2 \in \mathbb{N}$. In that case, the Taylor expansion of $\phi$ at $p$ is zero. But then $\phi$ must vanish in some neighborhood of $p$ following Bers theorem. Hence $p \in \mathcal{Z}$, which proves that $\overline{\mathcal{Z}} = \mathcal{Z}$, and, as $\boldsymbol{M}$ is connected, that either $\mathcal{Z} = \emptyset$, either $\mathcal{Z} = \boldsymbol{M}$. But $\mathcal{Z}$ cannot be equal to $\boldsymbol{M}$ because this would imply that $\phi = 0$, which is not the case, therefore $\mathcal{Z}$, which is the interior of $\phi^{-1}(0)$ by definition, must be empty. Claim (a) is thus proved.

(b) Suppose that $\phi = \phi'$ on an open subset $U$. Then $\phi|_U = \phi'|_U$ is not zero following (a), and $\phi|_U \in \mathcal{H}_\lambda(U) \cap \mathcal{H}_{\lambda'}(U)$ in which case $\lambda = \lambda'$ (5.3.1-a). But then $\phi - \phi'$ is an eigenfunction on $\boldsymbol{M}$, and because it vanishes on $U$, one must have $\phi - \phi' = 0$ on the whole $\boldsymbol{M}$ again after (a). $\square$

**5.6. Nodal Sets.** When one makes vibrate a square plate sprinkled with flour, sand or sugar, the particles jump around and end up grouping together in patterns presenting often very surprising symmetries. These are called *Chladni Acoustic Patterns*, after Ernst Chladni (8), who was fascinated by them and studied them for years.

(8) *Ernst Florens Friedrich Chladni (1756-1827), German physicist, founder of modern acoustics, author of the Treatise on acoustics in 1802.*



*Images taken in 1869 by the Irish physicist John Tyndall (1820-1893).*

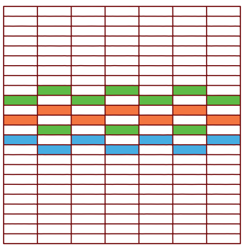
The explanation of the phenomenon is quite simple. As we saw in 1.4-(7), when a plate is made vibrate at a fixed frequency $F$, the vertical movement of a point $p \in \boldsymbol{M}$ is given by the trigonometric function:

$$f(p,t) = A(p)\sin(2\pi Ft) + B(p)\cos(2\pi Ft),$$

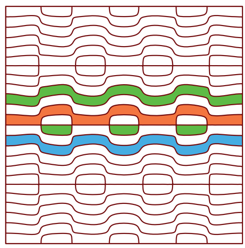
where $A,B : \boldsymbol{M} \to I\!R$ are eigenfunctions of eigenvalue $\lambda = (2\pi F/c)^2$. When $p \in \boldsymbol{M}$ is not a simultaneous zero of $A$ and $B$, the function $t \mapsto f(p,t)$ is a vibration of positive amplitude, and any particle close to $p$ will be shaken. Thus, particles quickly migrate from vibrating areas to the points $p \in \boldsymbol{M}$ which are not moving, *i.e.* such that $A(p) = B(p) = 0$. The Chladni pattern therefore visually represents the set of common zeros of $A$ and $B$.

When the multiplicity of $\lambda$ is one, the eigenfunctions $A$ and $B$ are proportional to the same eigenfunction $\phi \in \mathcal{H}_\lambda(\boldsymbol{M})$. The zeros of $A$ and $B$ coincide with those of $\phi$ and the Chladni pattern appearing when one makes the plate vibrate at frequency $F = c\sqrt{\lambda}/2\pi$ shows the nodal set of $\phi$. Otherwise, when the multiplicity is strictly greater than 1, to see a nodal set of an eigenfunction in $\mathcal{H}_\lambda(\boldsymbol{M})$, one must try different ways of making the plate vibrate at the frequency $F$ until one succeeds to "awake" only two collinear functions $A,B \in \mathcal{H}_\lambda(\boldsymbol{M})$.

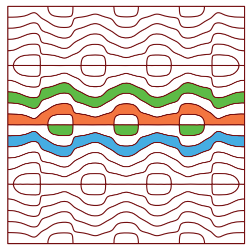
The following images show the nodal sets computed by Maple of six consecutive eigenfunctions lying on the interval $[\phi_{7,24},\phi_{15,20}] \subseteq \mathcal{H}_{625,\mathcal{D}}(\boldsymbol{D})$ for the square membrane $\boldsymbol{D} = [0,1]^2$ and the Dirichet condition. We have artificially colored some regions to draw attention on the amusing "*continuous*" deformation of a $7 \times 25$ grid to another of $15 \times 20$. ([9])



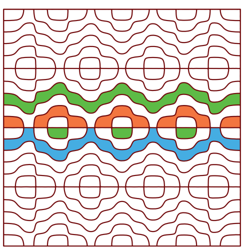
$\phi_{7,24}=0$



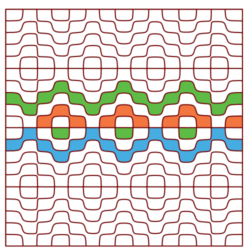
$\frac{4}{5}\phi_{7,24}+\frac{1}{5}\phi_{15,20}=0$



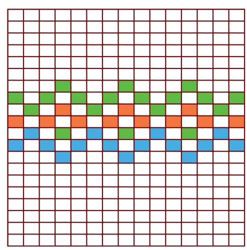
$\frac{3}{5}\phi_{7,24}+\frac{2}{5}\phi_{15,20}=0$



$\frac{2}{5}\phi_{7,24}+\frac{3}{5}\phi_{15,20}=0$



$\frac{5}{5}\phi_{7,24}+\frac{1}{4}\phi_{15,20}=0$



$\phi_{15,20}=0$

While Cheng's theorem (**7**♦9.3.3) tells us that the interior of nodal sets are empty, the reader may have noticed that in the previous images they appear as collections of intercrossing curves. This is not just a coincidence, it is a theorem (*cf.* [Che]), which is also corollary of Bers' 5.5.1.

**5.7 Nodal Components of Eigenfunctions.** This is the name given to the *connected components* of the complementary set in $\boldsymbol{M}$ of the nodal set of an eigenfunction $\phi$ of $-\Delta_{\boldsymbol{M}}$ (colored regions in the previous images). If $\Omega$ is one of these, the function $\phi$ doesn't vanish and keeps a constant sign for

---

([9]) Maple code:

```
>with(plots):
>f1:=sin(07*Pi*x)*sin(24*Pi*y): f2:=sin(15*Pi*x)*sin(20*Pi*y):
>for i from 0 to 5 do
   implicitplot(((5-i)/5)*f1+(i/5)*f2=0,x=0..1,y=0..1,numpoints=10000);
 end do;
```

any $p \in \Omega$ (**3**♦8.1.2-(d)). The next *Courant's theorem for nodal components* (*cf.* [Cha], p. 19) is a famous result in spectral theory.

**5.7.1 Theorem (R. Courant).** *Let $\boldsymbol{M}$ be a compact connected orientable Riemannian manifold with a sufficiently regular boundary. Following proposition 5.3.1, let $\{\lambda_1(\beta,\boldsymbol{M}) \leq \lambda_2(\beta,\boldsymbol{M}) \leq \cdots \leq \lambda_k(\beta,\boldsymbol{M}) \leq \ldots\}$ be the complete ordered list of eigenvalues in $\sigma_\beta(\boldsymbol{M})$, $\beta \in \{\mathcal{D},\mathcal{N}\}$, each repeated as many times as its multiplicity. Next, choose a sequence $\{\phi_1,\phi_2,\ldots,\phi_k,\ldots\}$ of linearly independent eigenfunctions such that $\phi_k \in \mathcal{H}_{\lambda_k,\beta}(\boldsymbol{M})$. Then, for all $k = 1,2,\ldots$, the number of nodal components of $\phi_k$ does not exceed $k$.*

**5.7.2 Corollary.** a) *The eigenfunction $\phi_1$ has constant sign.*

b) *The eigenvalue $\lambda_1(\beta,\boldsymbol{M})$ has multiplicity 1, hence $\lambda_1(\beta,\boldsymbol{M}) < \lambda_2(\beta,\boldsymbol{M})$.*

c) *The eigenfunction $\phi_2$ has exacty two nodal domains.*

*Hint.* (a) Follows from Courant's theorem. (b) If $\phi$ and $\phi'$ belong to $\mathcal{H}_{\lambda_1,\beta}(\boldsymbol{M})$, *every* linear combination $a\phi + b\phi'$, belonging to $\mathcal{H}_{\lambda_1,\beta}(\boldsymbol{M})$, must have a constant sign, which is possible only if $\phi_1$ and $\phi_2$ are collinear (exercice!). (c) After the following Green identity ([10])

$$\int_{\boldsymbol{M}} \{\phi_1 \Delta(\phi_2) - \Delta(\phi_1)\phi_2\}\, d_{\boldsymbol{M}} = \int_{\partial \boldsymbol{M}} \{\phi_1 \partial_{\vec{n}}(\phi_2) - \partial_{\vec{n}}(\phi_1)\phi_2\}\, d_{\partial \boldsymbol{M}}, \qquad (*)$$

if the $\phi_i$'s satisfy the same boundary condition $\beta$, the second member in $(*)$ is zero. In that case, if $\phi_i \in \mathcal{H}_{\lambda_i,\beta}(\boldsymbol{M})$, we obtain the *orthogonality relation*

$$\int_{\boldsymbol{M}} \phi_1(p)\phi_2(p)\, d_{\boldsymbol{M}} = 0,$$

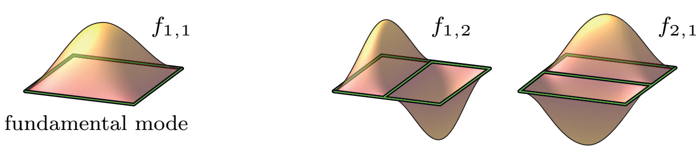
because $\lambda_1 \neq \lambda_2$. But, such a consequence would be impossible if both functions keep their signs constant. Therefore $\phi_2$ has at least two nodal functions and, following Courant's theorem, exactly two. □

**5.8 Example of the Two First Eigenvalues.** In the rectangular membrane 4.3 with $a = b = 1$, and for the Dirichlet condition, we have

$$\lambda_1 = (1^2 + 1^2)\pi^2 = 2\pi^2, \qquad \lambda_2 = (1^2 + 2^2)\pi^2 = 5\pi^2,$$

and the eigenfunctions

- $f_{1,1} := \sin(\pi x)\sin(\pi y) \in \mathcal{H}_{\lambda_1,D}$
- $\left\{ \begin{array}{l} f_{1,2} := \sin(\pi x)\sin(2\pi y) \\ f_{2,1} := \sin(2\pi x)\sin(\pi y) \end{array} \right\} \in \mathcal{H}_{\lambda_2,D}$




where the nodal sets are indicated in green.

([10]) Chapter I in Isaac Chavel's book [Cha], formula (40), page 7.

## 6 Construction of Isospectral Flat Surfaces

Let us now return to our main subject which is, as explained in 2, to show the noninjectivity of the correspondence

$$\boldsymbol{M} \text{ (modulo isometry)} \rightsquigarrow \mathrm{Spec}_{\mathcal{D}}(\boldsymbol{M}) \tag{30}$$

where $\boldsymbol{M}$ denotes an irreducible compact Riemannian manifold with a sufficiently regular boundary. We focus on dimension 2, where we review in some detail the construction of the first known pairs of isospectral flat surfaces $(\boldsymbol{D}_1,\boldsymbol{D}_2)$. Drums with such membranes will be indistinguishable by their sound.


These constructions will lead to *the transplantation method* which, in turn, will show the isospectrality of the spaces $\boldsymbol{D}_1 \times [0,1]^n$ and $\boldsymbol{D}_2 \times [0,1]^n$. The non-injectivity of (30) will thus be established in all dimensions $\geq 2$.

**6.1 Sunada's method.** In 1985 Sunada publish his general method for building isospectral varieties, where the key idea is based on the trace formula for representations of finite groups and the notion of almost-conjugate subgroups. The latter notion, introduced long time before for other needs by Gassmann ([Ga], 1926), was explicitly used for the first time in spectral geometry in the work of M.-F. Vignéras ([V], 1980).

**6.1.1. Almost-Conjugate Subgroups and Gassmann triples.** Two subgroups $\Gamma_1$ and $\Gamma_2$ of a finite group $\boldsymbol{G}$ are said to be *almost-conjugate in $\boldsymbol{G}$*, or that $(\boldsymbol{G},\Gamma_1,\Gamma_2)$ is a *Gassmann (or Sunada) triple*, if for every conjugacy class $C$ of $\boldsymbol{G}$, one has $\mathrm{Card}(\Gamma_1 \cap C) = \mathrm{Card}(\Gamma_2 \cap C)$.

The trace formula then shows that $\Gamma_i$ are almost-conjugate if and only if the $\boldsymbol{G}$-modules $\mathbb{Q}[\boldsymbol{G}/\Gamma_i]$ are isomorphic. As a corollary, one has the following fundamental characteristic property of Gassmann's triples:

**Proposition.** *Let $(\boldsymbol{G},\Gamma_1,\Gamma_2)$ be a Gassmann triple. Given a linear representation of $\boldsymbol{G}$ on a vector space $V$ of arbitrary dimension over a field of characteristic zero, denote by $V^{\Gamma_i}$ the subspace of $\Gamma_i$-invariants of $V$,* i.e. *of elements $v \in V$ such that $h(v) = v$, for all $h \in \Gamma_i$. Then there exist linear combinations $\tau = \sum_{g\in \boldsymbol{G}} x_g g$, with $x_g \in \mathbb{Q}$, depending only on $(\boldsymbol{G},\Gamma_1,\Gamma_2)$, such that $\tau(V^{\Gamma_1}) = V^{\Gamma_2}$ and such that the map*

$$\tau : V^{\Gamma_1} \to V^{\Gamma_2}, \quad v \mapsto \tau(v) = \sum_{g\in \boldsymbol{G}} x_g g(v),$$

*is an isomorphism.*

The linear combinations $\tau \in \sum_{g\in \boldsymbol{G}} x_g g$, not being related to any particular representation of $\boldsymbol{G}$, must be seen as elements of the *group algebra* $\mathbb{Q}[\boldsymbol{G}]$. They are the *transplantation* elements of $(\boldsymbol{G},\Gamma_1,\Gamma_2)$. The reason of this appellation may, at this point, seem dubious, but it will be clear when we will see these elements act on *tessellated* Buser's surfaces.

**6.1.2. Sunada's Theorem.** Let $(\boldsymbol{G},\Gamma_1,\Gamma_2)$ be a Gassmann triple where $\boldsymbol{G}$ is a group of isometries of a Riemannian manifold $\boldsymbol{M}$. We noticed in 1.7 that the action of an isometry on $\mathcal{C}^\infty(\boldsymbol{M})$ is compatible with the Laplacian, which implies that $\mathcal{C}^\infty(\boldsymbol{M})^{\Gamma_i}$ is stable under $\Delta_{\boldsymbol{M}}$. In particular, a transplantation, being a linear combination of elements of $\boldsymbol{G}$, induces an isomorphism of vector spaces $\tau : (\mathcal{C}^\infty(\boldsymbol{M})^{\Gamma_1},\Delta_{\boldsymbol{M}}) \to (\mathcal{C}^\infty(\boldsymbol{M})^{\Gamma_2},\Delta_{\boldsymbol{M}})$ which is also compatible with Laplacians. If in addition, the $\Gamma_i$'s act freely on $\boldsymbol{M}$, the quotient spaces $\boldsymbol{M}_i := \Gamma_i\backslash\boldsymbol{M}$ are again a Riemannian manifold and the vector spaces with linear operators $(\mathcal{C}^\infty(\boldsymbol{M}_i),\Delta_{\boldsymbol{M}_i})$ and $(\mathcal{C}^\infty(\boldsymbol{M})^{\Gamma_i},\Delta_{\boldsymbol{M}})$ are canonically isomorphic. It then follows that the composition

$$(\mathcal{C}^\infty(\boldsymbol{M}_1),\Delta_{\boldsymbol{M}_1}) \cong (\mathcal{C}^\infty(\boldsymbol{M})^{\Gamma_1},\Delta_{\boldsymbol{M}}) \xrightarrow[\cong]{\tau} (\mathcal{C}^\infty(\boldsymbol{M})^{\Gamma_2},\Delta_{\boldsymbol{M}}) \cong (\mathcal{C}^\infty(\boldsymbol{M}_2),\Delta_{\boldsymbol{M}_2})$$

is an isomorphism of vector spaces with linear operators, hence inducing isomorphisms between corresponding eigenspaces. This is key idea of the proof of the following theorem, which is the essence of "*Sunada's method*".

**Theorem ([S], 1985).** *Let $\boldsymbol{M}$ be a Riemannian manifold (without boundary). If $(\boldsymbol{G},\Gamma_1,\Gamma_2)$ is a Gassmann triple where $\boldsymbol{G}$ is a group of isometries of $\boldsymbol{M}$ such that the subgroups $\Gamma_i$ acts freely on $\boldsymbol{M}$. Then the quotients $\boldsymbol{M}_1 := \Gamma_1\backslash\boldsymbol{M}$ and $\boldsymbol{M}_2 := \Gamma_2\backslash\boldsymbol{M}$ are Riemannian manifolds and a transplantation element $\tau$ of $(\boldsymbol{G},\Gamma_1,\Gamma_2)$ induces an isomorphism of vector spaces with Laplacians $\tau : (\mathcal{C}^\infty(\boldsymbol{M}_1),\Delta_{\boldsymbol{M}_1}) \to (\mathcal{C}^\infty(\boldsymbol{M}_2),\Delta_{\boldsymbol{M}_2})$. In particular,*

$$\mathrm{Spec}(\boldsymbol{M}_1) = \mathrm{Spec}(\boldsymbol{M}_2).$$

**6.1.3. Comment.** *This theorem is notable in that it gives a linear isomorphism between $\mathcal{C}^\infty(\boldsymbol{M}_1)$ and $\mathcal{C}^\infty(\boldsymbol{M}_2)$ compatible with Laplacians, avoiding the much more delicate question of the explicit determination of Laplacian eigenvalues and eigenspaces. In particular, the final statement about spectra is independent of discreteness of spectra or finiteness of multiplicities; no deep spectral geometry result is needed. By contrast, the theorem gives no reason for these varieties not to be isometric, which must be established a posteriori, although this is generally not difficult to check.*

**6.1.4. Bérard's Theorem.** Pierre Bérard later lifted two important limitations from Sunada's theorem: the fact that the manifold $\boldsymbol{M}$ had to be with no boundaries, and that the subgroups $\Gamma_i$ had to act freely on $\boldsymbol{M}$. The following version of his theorem will suffice for our purposes.

**6.1.5 Theorem ([Bé$_3$], 1992).** *Let $(\boldsymbol{G},\Gamma_1,\Gamma_2)$ is a Gassmann triple where $\boldsymbol{G}$ is a group of isometries of a Riemannian manifold $\boldsymbol{M}$. Then, the quotients $\boldsymbol{M}_1 := \Gamma_1\backslash\boldsymbol{M}$ and $\boldsymbol{M}_2 := \Gamma_2\backslash\boldsymbol{M}$ are Riemannian manifolds, moreover with sufficiently smooth boundaries if such is $\boldsymbol{M}$, and a transplantation element $\tau$ of $(\boldsymbol{G},\Gamma_1,\Gamma_2)$ induces an isomorphism of vector spaces with operators $\tau : (\mathcal{C}^\infty(\boldsymbol{M}_1),\Delta_{\boldsymbol{M}_1}) \to (\mathcal{C}^\infty(\boldsymbol{M}_2),\Delta_{\boldsymbol{M}_2})$ compatible with Dirichlet and Neumann boundary conditions. In particular:*

$$\mathrm{Spec}_\beta(\boldsymbol{M}_1) = \mathrm{Spec}_\beta(\boldsymbol{M}_2), \quad \beta \in \{\mathcal{D},\mathcal{N}\}.$$

**6.2 Construction of Isospectral Surfaces.** Once the Sunada (and Bérard) theorem became known, a general strategy for building isospectral manifolds rapidly emerged.

S1) Find Gassmann triples $(\boldsymbol{G},\Gamma_1,\Gamma_2)$.

S2) Find manifolds $\boldsymbol{M}$ admitting $\boldsymbol{G}$ as group of isometries.

S3) (For Kac's question) Find a *flat surface* $\boldsymbol{M}$ admitting $\boldsymbol{G}$ as group of isometries such that the spaces $\Gamma_i\backslash\boldsymbol{M}$ are *contractible* domains of the Euclidean space $\mathbb{R}^2$ (as are the drumhead membranes).

S4) Verify that the manifolds $\Gamma_i\backslash\boldsymbol{M}$ are not isometric of each other.

In the following praragraphs we briefly review each of these items.

**6.3 Gasssmann triples examples.** The notion of almost-conjugate subgroups first appeared in 1925 in the appendix of an article by Adolf Hurwitz, written by Fritz Gassmann ([Ga], 1926), that aimed to give a condition for the arithmetic equivalence two numbers fields. The appendix contained the first known example of nonconjugate almost-conjugate subgroups inside the permutation group $\boldsymbol{G} = \mathfrak{S}_6$. The subgroups are

$$\begin{cases} \Gamma_1 = \{\mathrm{id};(1,2)(3,4);(1,3)(2,4);(1,4)(2,3)\} \\ \Gamma_2 = \{\mathrm{id};(1,2)(3,4);(1,2)(5,6);(3,4)(5,6)\}. \end{cases}$$

Notice that in this example, the $\Gamma_i$'s are of *index* $|\boldsymbol{G}|/|\Gamma_i| = 180$.

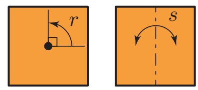
Robert Perlis took the subject further, exploring in the direction of group structures ([P], 1977), highlighting the example of the Gassmann triple $(\mathrm{Sl}(3,\mathbb{F}_2),\Gamma_1,\Gamma_2)$ where $\mathrm{Sl}(3,\mathbb{F}_2)$ is the group of $3{\times}3$ matrices with coefficients in the 2-elements field $\mathbb{F}_2 := \{0,1\}$, and determinant equal to 1. The subgroups $\Gamma_i$ are made of matrices of the following forms

$$\Gamma_1 := \left\{ \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\}, \quad \Gamma_2 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \right\},$$

and have index $|\boldsymbol{G}|/|\Gamma_i| = 7$, which is the smallest possible for a Gassmann triple having nonconjugate $\Gamma_i$'s. However, at the time, interest in such triples had not yet been sparked in spectral geometry. This would happen only in Vignéras (1980), and obviously in Sunada (1986) inspired by the latter.

**6.4 Buser and Cayley Graphs.** At the time of Sunada's theorem, its usefulness to answer Kac's question was moderated by the difficulty of finding flat surfaces $\boldsymbol{M}$ on which the group $\boldsymbol{G}$ of known Gassmann triples acts freely by isometries. Peter Buser soon proposed ([$\mathrm{Bu}_2$], 1988) the idea of using the "*Cayley graph*" of $\boldsymbol{G}$ to construct, by a procedure of enlarging edges, a flat surface $\boldsymbol{M}(\boldsymbol{G})$ where $\boldsymbol{G}$ tautologically and faithfully acts by isometries.

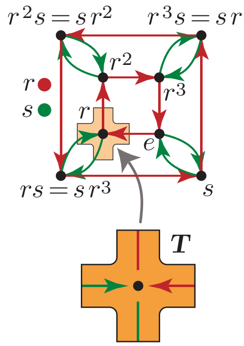
As an example, consider the dihedral group $\mathbb{D}_4$ of symmetries of a square. It has 8 elements and is generated by the set $\boldsymbol{S} = \{r,s\}$, where $r$ is the anti-clockwise rotation of angle $\pi/2$ and $s$ is the horizontal reflection. These two symmetries verify the relations $\mathcal{R}$ to the right, which determine the group structure of $\mathbb{D}_4$.




$$\mathcal{R} = \begin{cases} r^4 = s^2 = 1, \\ srs = r^{-1}, \end{cases}$$

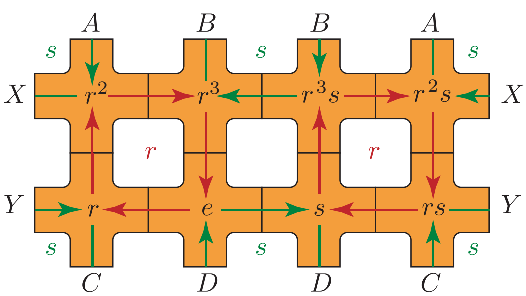
The *Cayley graph of* $\mathbb{D}_4$ *relative to* $\boldsymbol{S}$, denoted by $\mathrm{Cay}(\mathbb{D}_4,\boldsymbol{S})$ consists of *vertices* and *arrows*. The vertices are the elements of $\mathbb{D}_4$, and the arrows are the pairs $\{(x,xs) \mid x \in \boldsymbol{G} \text{ and } s \in \boldsymbol{S}\}$. The image to the right shows the Cayley graph where the arrows are colored to differentiate the elements of $\boldsymbol{S}$.

$\mathrm{Cay}(\mathbb{D}_4,\boldsymbol{S})$

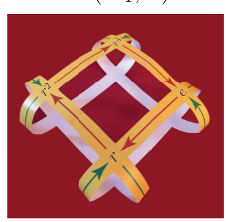



Buser then proposed to enlarge, or to *thicken*, the graph by placing at each arrowed vertex a *tile* $\boldsymbol{T}$ on which the tails and heads of the arrows are drawn, as illustrated in the detail of this image.

By patching the tiles in a way that respects tails, heads and colors of the arrows in the Cayley graph, one gets a surface denoted by $\boldsymbol{M}(\mathbb{D}_4,\boldsymbol{S})$ which comes out endowed with a *tesselation* of tile $\boldsymbol{T}$.




$\boldsymbol{M}(\mathbb{D}_4,\boldsymbol{S})$

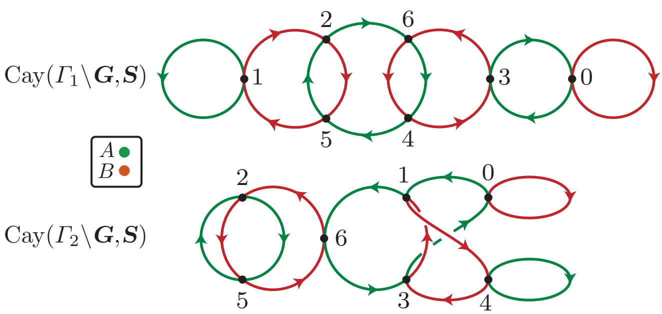


The group $\mathbb{D}_4$ naturally acts on the set of vertices of $\mathrm{Cay}(\mathbb{D}_4,\boldsymbol{S})$ by left multiplication, *i.e.* $g{\cdot}x = gx$, and on the set of arrows by $g{\cdot}(x,xs) = (gx,gxs)$. The key feature in Buser's thickening trick is that this action uniquely extends to a group action of $\mathbb{D}_4$ on the whole Buser's flat surface $\boldsymbol{M}(\mathbb{D}_4,\boldsymbol{S})$ *by isometries preserving tessellation*. This is the fundamental property of Buser's thickenings, it allows to prove that any abstract finite (or countable) group is the group of isometries of a flat surface with boundary! Notice that by doing this, Buser fully answered the question 6.2-(S2).

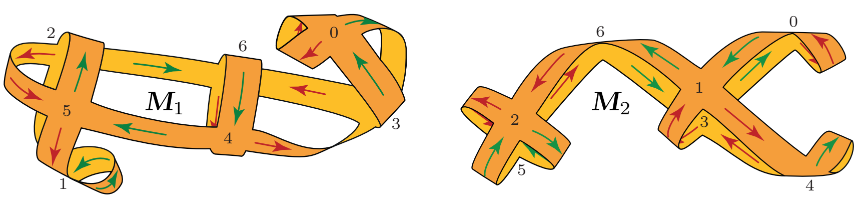
**6.5 Buser Flat Surfaces.** In the Gassmann triple $(\boldsymbol{G} := \mathrm{Sl}(3,\mathbb{F}_2), \varGamma_1, \varGamma_2)$, the group $\boldsymbol{G}$ is generated by the set $\boldsymbol{S} = \{A, B\}$ with

$$A := \begin{bmatrix} 0\,1\,1 \\ 0\,1\,0 \\ 1\,0\,0 \end{bmatrix} \text{ and } B := \begin{bmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,1 \end{bmatrix},$$

and the Buser surface $\boldsymbol{M} := \boldsymbol{M}(\boldsymbol{G},\boldsymbol{S})$ is build up of $|\boldsymbol{G}| = 168$ tiles $\boldsymbol{T}$. Now, if the effective construction of this surface would clearly be very tedious, it fortunately is of no interest since the conclusion of Sunada's theorem concerns only the quotients $\boldsymbol{M}_i := \varGamma_i \backslash \boldsymbol{M}$, and these may be also constructed enlarging the Cayley graphs of the sets of *left cosets* $\varGamma_i \backslash \boldsymbol{G}$. Indeed, Cayley graphs also make sense for sets $\boldsymbol{E}$ endowed with a *right* action of $\boldsymbol{G}$. Denote by $\mathrm{Cay}(\boldsymbol{E},\boldsymbol{S})$ the graph where the *vertices* are the elements of $\boldsymbol{E}$, and where the *arrows* are, for each $s \in \boldsymbol{S}$, the *arrowed handles* going from $x$ to $xs$, for every $x \in \boldsymbol{E}$. It is then straightforward to check that the $\boldsymbol{M}_i$'s coincide with the Buser's thickenings $\boldsymbol{M}(\varGamma_i \backslash \boldsymbol{G},\boldsymbol{S})$ of the Cayley graphs $\mathrm{Cay}(\varGamma_i \backslash \boldsymbol{G},\boldsymbol{S})$. Finally, notice that these graphs have only 7 vertices, which represents a huge simplification compared to the 168 vertices of $\mathrm{Cay}(\boldsymbol{G},\boldsymbol{S})$. The graphs are




and their thickening, the Buser's flat surfaces $\boldsymbol{M}_i = \boldsymbol{M}(\varGamma_i \backslash \boldsymbol{G},\boldsymbol{S})$ :

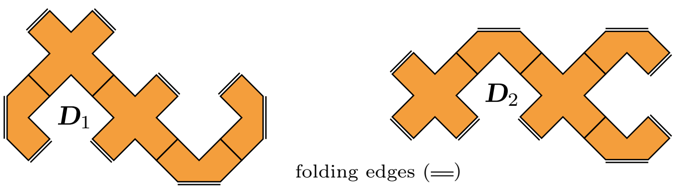



These constitute the first known examples of isospectral flat surfaces (with boundary) which are not isometric, as is visibly obvious from these pictures.

**6.6 The Gordon-Webb-Wolpert Surfaces.** In Buser's examples, one has a certain flat surface $\boldsymbol{M}$ and isometry groups $\varGamma_i$ acting freely on $\boldsymbol{M}$. The natural projections $\boldsymbol{M} \to \boldsymbol{M}_i := \varGamma_i \backslash \boldsymbol{M}$ are then *topological coverings*. This fact prevents the method from producing contractible spaces $\boldsymbol{M}_i$, as this would imply that $\boldsymbol{M}$ is not connected, which is never the case.

To overcome this difficulty and try to catch contractible isospectral drumhead membranes with Sunada's method, one must accept the possibility that the groups $\Gamma_i$ do not act freely.

Regarding this point, Gordon-Webb-Wolpert noticed the existence in Buser's surfaces $\boldsymbol{M}_i$ of a particular isometry $\sigma$, which appears to be the reflection by a plane easy to discern in the previous images. The quotient spaces $\boldsymbol{D}_i := \langle\sigma\rangle\backslash\boldsymbol{M}_i$ are visibly nonisometric and also *contractible domains* with polygonal boundaries of $\mathbb{R}^2$ :




which is very good, as it answers question 6.2-(S3). But, because $\sigma$ has fixed points (the folding edges), it does not act freely on the $\boldsymbol{M}_i$'s and Sunada's theorem cannot be used to conclude that the surfaces are isospectral.

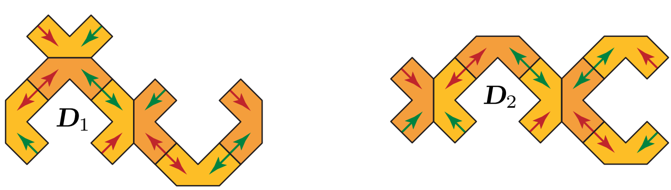
Gordon-Webb-Wolpert, moreover, noticed that $\sigma$ arises from a certain *interior automorphism* $I_\sigma$ of $\boldsymbol{G} := \mathrm{Sl}(3,\mathbb{F}_2)$ stabilizing the $\Gamma_i$'s, and that the triple of extensions

$$(\boldsymbol{G},\Gamma_1,\Gamma_2) \rtimes \langle I_\sigma\rangle = (\boldsymbol{G} \rtimes \langle I_\sigma\rangle, \Gamma_1 \rtimes \langle I_\sigma\rangle, \Gamma_2 \rtimes \langle I_\sigma\rangle)$$

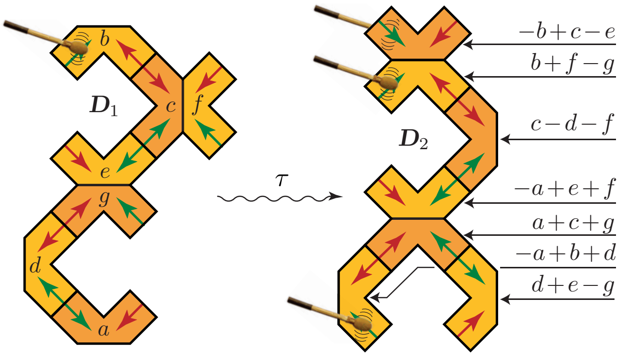
was also a Gassmann triple acting on $\boldsymbol{M}$ and giving rise to the domains $\boldsymbol{D}_i$ by the same Sunada's method. But obviously, still the group $\boldsymbol{G} \rtimes \langle I_\sigma\rangle$ did not act freely on $\boldsymbol{M}$.

It is at this point that Bérard's extension **15♦**3.3.2 of Sunada's theorem became unvaluable because of its insensitivity to the question of freeness. This allowed Gordon-Webb-Wolpert to go one step further and finally prove that the surfaces $\boldsymbol{D}_i$ were isospectrals, and even better, for both the Dirichlet and Neumann conditions ([GWW], 1992). The surfaces $\boldsymbol{D}_i$, above, constitute the first known examples of isospectral non-isometric contractible domains of the Euclidean plane $\mathbb{R}^2$.

**6.7 Transplantations.** When making explicit the action of a transplantation $\tau : \mathcal{C}^\infty(\boldsymbol{D}_1) \to \mathcal{C}^\infty(\boldsymbol{D}_2)$, the membranes $\boldsymbol{D}_i$, due to the overlap of arrows in opposite direction by the reflexion $\sigma$, get a new tiling with two *oriented tiles* (inverted colored arrows) as shown in the figure.



If we now denote by the seven letters $a,\dots,g$ the tiles of $\boldsymbol{D}_1$, and use the same to denote the restrictions of a given function $f : \boldsymbol{D}_1 \to I\!R$ to the corresponding tiles, then the figure below explains how the transplantation $\tau$ acts translating and recomposing these restrictions from $\boldsymbol{D}_1$ to $\boldsymbol{D}_2$.




It is quite amazing that this rather mysterious recombination of the partial functions of a differentiable function (verifying in addition boundary conditions) remains differentiable (and satisfying the same boundary conditions!). Notice also that to obtain on $\boldsymbol{D}_2$ the same sound effect as a stick hit on $\boldsymbol{D}_1$, it is necessary to hit simultaneously three different points on $\boldsymbol{D}_2$, two in one direction and another in the opposite direction. It's quite magic!

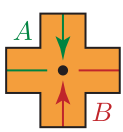
The relation between the input and the output of the transplantation reveals the *transplantation matrix*, in this case the $7 \times 7$ matrix in the equality

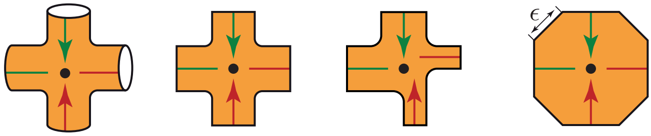
$$\begin{bmatrix} -b+c-e \\ b+f-g \\ c-d-f \\ -a+e+f \\ a+c+g \\ -a+b+d \\ d+e-g \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix},$$

where a minus sign appears when the orientation of the tile in $\boldsymbol{D}_1$ supporting a given restriction gets inverted in $\boldsymbol{D}_2$.

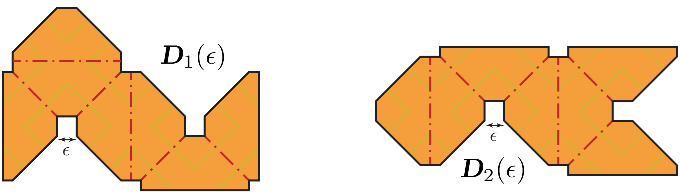
As Bérard said, the above matrix constitutes, by itself, a complete proof of the isospectrality of the two surfaces. But, of course, it is practically impossible to lay hands on it without all the theoretical work that comes before it.

**6.7.1. Buser-Bérard's Surfaces.** The shapes of Buser's flat surfaces are clearly related to the shape of the tile $\boldsymbol{T}$ used to thicken Cayley graphs. In the previous paragraphs, for the group $\mathrm{Sl}(3,\mathbb{F}_2)$ and its generators $A$, $B$, we have used the tile

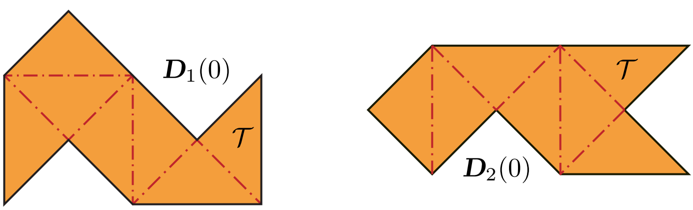
$\boldsymbol{T} :=$  , but other shapes are equally possible, for example



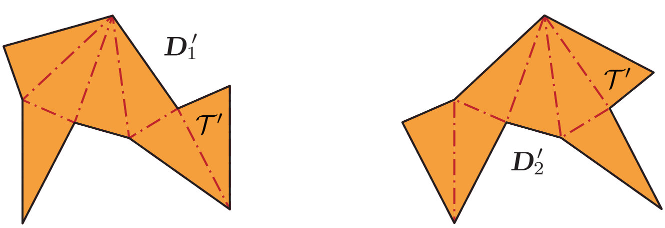

The first one produces smooth (but not flat) surfaces *without* boundary. The second and third produce flat surfaces with *smooth* boundaries, the third, having no internal symmetries, makes the group of isometries of the thickening of the Cayley graph of a group $\boldsymbol{G}$ to *coincide* with the group $\boldsymbol{G}$ itself. The forth produces *families* of flat surfaces, parametrized by $\epsilon > 0$:




On these, the method of transplantation remains valid with exactly the same transplantation matrix. The happy surprise here is that, while the underlying Sunada's theory ceases to make sense when $\epsilon = 0$, the transplantation continues to transplant functions efficiently (!), proving the isospectrality of the surfaces $\boldsymbol{D}_i(0)$ below, familiarly known under the names of the "*hen*" and the "*arrow*".




At this point, it is easy to understand that one can further modify the membranes by modifying the shape of the triangle $\mathcal{T}$ to some other shape $\mathcal{T}'$ and still be able to prove isospectrality using the same transplantation matrix, for example:



## 7 General References

The bibliography for Sunada's method in spectral geometry is profuse, both in books and in articles, with the main actors in this adventure and key references being Sunada, Buser, Gordon, Webb, Wolpert and Bérard. Some of the most important works include:

- The original Toshikazu Sunada's article ([S]), *Riemannian coverings and isospectral manifolds* (1985). It was published in Annals of Mathematics, and was awarded the Iyanaga Prize (1987) and the Publication Prize (2013), both by the Japan Mathematical Society.
- The reference book on the subject, Peter Buser's book ([$\mathrm{Bu}_3$]): *Geometry and Spectra of Compact Riemann Surfaces* (1992).
- The original article by Carolyn Gordon, David Webb and Scott Wolpert (1992). Published by Inventiones Mathematicae, it marked the end point of thirty years of research to answer Mark Kac's question, and in it the picture of isospectral and nonisometric contractible domains of the Euclidean plane appeared for the first time.
- The popularised article of Carolyn Gordon and David Webb ([GW]), *You can't hear the shape of a drum* (1996). Published by the American Scientist, it was awarded the Chauvenet Prize of the American Mathematical Association in 2001.
- The *Exposé n° 705* of the Bourbaki Seminar (1988-89) by Pierre Bérard ([Bé]), *Non-isometric Isospectral Riemannian Variables*, of which Section VIII gives a list of open problems.

## 8 The Plan of the Book

The aim of this book is to provide the details necessary to fully understand the digest of the material described from paragraph 6 of this chapter onwards. With a large readership in mind, we adopt a basic level so that the chapters may also serve as elementary introductions to the important fields of Riemannian Geometry, Differential Geometry, Group Theory and Group Representation Theory.

Chap. 2.— *Building Spaces.* The chapter begins with an informal discussion on different common ways to build complex objects by glueing or assembling together simple ones. Apart from the fact that this will produce all possible vibrating surfaces, the main motivation is to unveil the concepts of *space* and *structure*. The discussion leads to the fundamental concept of *category of spaces* and the operations of *quotient* and *assemblage*, which formalize the intuitive idea of glueing structures.

Chap. 3.— *Quotients and Assemblages of Topological Spaces.* We recall basics and terminology of topology. We define the *category of topological spaces* and prove the existence and uniqueness of quotients and assemblages. The chapter ends with a discussion of two basic concepts:

the connectedness and separateness of topological spaces, especially for quotients and assemblages of topological spaces.

Chap. 4.— *Path-distances on Pseudometric spaces.*

Chap. 5.— *Quotients and Assemblages of Pathmetric Spaces.* The aim is the same as in chapter 3, but for path-metric spaces intead of general topological spaces. The main subject is the study of quotients and assemblages in various categories of path-metric spaces.

Chap. 6.— *Flat Riemannian Surfaces.* After a short introduction to the *differentiable* structure of Euclidean affine spaces, we define the category of *flat Riemannian manifolds*. This is a subcategory of the category of metric spaces consisting on spaces on which the metric structure gives rise to a differentiable structure where the *Laplace* differential operator, and thereby the *Euler-Lagrange equation of vibrating membranes*, makes sense. Heuristically speaking, these spaces can emit sounds, and, as such, can resonate at certain characteristic frequencies, the set of which corresponds, as for strings and membranes, to the set of eigenvalues of the Laplacian: the *spectrum* of the space.

Chap. 7.— *Sunada and Bérard theorems.* Introduction to the theory of linear representations of finite groups. Character of a representation. The trace formula for the $\boldsymbol{G}$-modules $\mathbb{Q}[\boldsymbol{G}/\varGamma]$. Almost-conjugate subgroups. The fundamental theorem for almost-conjugate groups. Transplantations.

Chap. 8.— *Buser's flat surfaces associated with Cayley Graphs.* The Cayley graph of a $\boldsymbol{G}$-set. The tessellated flat surface $\boldsymbol{M}(\varGamma\backslash\boldsymbol{G})$. Canonical representation of $\boldsymbol{G}$ as group of symmetries of tessellated space of $\boldsymbol{M}(\varGamma\backslash\boldsymbol{G})$.

Chap. 9.— *Buser's isospectral surfaces* $\boldsymbol{M}_i$. The group $\mathrm{Sl}(n,\mathbb{F}_2)$. The Gassmann triple $(\mathrm{Sl}(n,k),\varGamma_1,\varGamma_2)$.

Chap. 10.— *Gordon-Webb-Wolpert's isospectral surfaces* $\boldsymbol{D}_i$. The Gassmann triple $(\mathrm{Sl}(2,\mathbb{F}_2)^*,\varGamma_1^*,\varGamma_2^*)$.

Chap. 11.— *Transplantations.* The cases of Buser and GWW's surfaces.

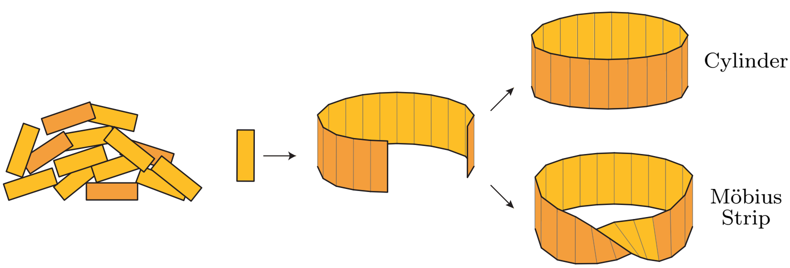
## Chapter Contents

# Chapter 2

# Building Spaces

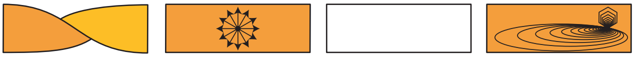
In many important mathematical theories objects of study are introduced in a two-stage process. In the first step, *simple* objects, which are fairly well understood, and for which relevant properties are known, are defined. The second step introduces *compound* objects as *assemblages* of the first, using specific *glueing rules*. Simple objects may have structures that one might want to preserve throughout the assembling process, such as topological spaces, differentiable or Riemannian manifolds, endowed with particular differential operators, etc. To accommodate this, the glueing rules need to include *ad hoc* instructions on how to endow the compound objects with the same structure, without modifying that of the constituent objects. This process of assembling structured objects, is known as "*the glueing of structures*".

Just as ordinary bricks and mortar are still used to create an amazing variety of buildings, the compound objects may be surprising, and at times even paradoxical, with properties very different from those of the simple objects they are made of. For example, consider the following two classic surfaces made up of copies of the same rectangular piece and the glueing rule that consists to assemble them by their longest edge.

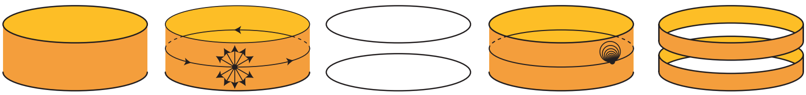



There is always some place in the Möbius strip where the "back" becomes the "front", which is typical of surfaces with just one face: *If one paints the surface without ever raising the brush, one ends up painting the whole surface, back and front, of the same color.*

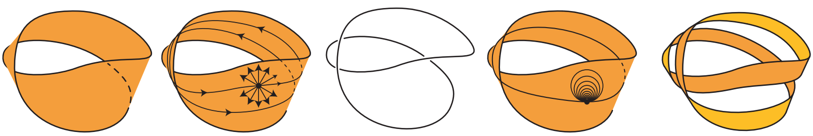
- *The Simple Object*: has two faces; is limited in all directions; has a connected boundary; any loop may be shrunk to a point.



- *The Cylinder*: has two faces; is unlimited in some directions; has a boundary with two connected components; some loops cannot be shrunk to a point; cutting it at the Equator gives two identical cylinders.



- *The Möbius Strip*: has only one face; is unlimited in some directions; has a connected boundary; some loops cannot be shrunk to a point; cut it at the Equator gives a cylinder (recovering thus a second face) (*cf.* **5**♦8.5.1).

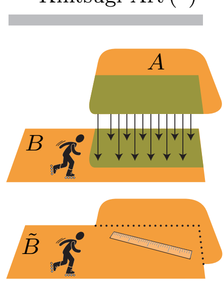


We will return later to these examples, but for now we will set them aside and move on to the main subject of this chapter.

## 1 Building Spaces

**1.1. Assembling spaces.** In mathematics, the methods for assembling spaces are analogous to those commonly used to repair broken objects by glueing together matching pieces. These pieces, scattered to start with, combine to form a single object, while themselves *remaining unaltered*. To see how, consider the two pieces of paper, $A$ and $B$ to the right. Overlap and glue them as shown *without making folds or rips* and denote by $A \cup B$ the result. Now, if one asks a 2-dimensional being of $A \cup B$ to restrain its movements to the area $\tilde{B}$ that has become $B$, it would find no difference from moving in the original $B$. Thus we say that $B$ remains *unaltered*. Moreover, the *glueing rule* that forbids folding or tearing ensures that the distances between points in the overlapping area coincide whether measured in $A$, $B$ or $A \cup B$. As with the matching pieces, the metrics of A and B match, allowing them to be glued together in the same metric within $A \cup B$.

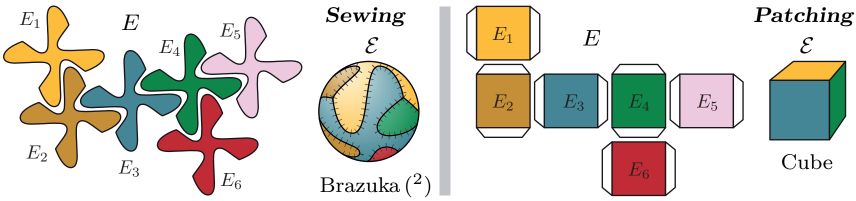
Kintsugi Art [1]




$A \cup B$

([1]) Japanese art of glueing broken pottery with lacquer mixed with precious metals. As a philosophy, it considers breakings as traits to be highlighted, rather than to be disguised.

In later chapters we will see that the different ways of joining things together: sewing, pasting, patching, etc, have their counterpart in the assemblage of spaces, according to the structures to be glued.




**1.2 The Identification Map.** In the above examples, we start with a family of spaces $\{E_i\}_{i\in\mathfrak{I}}$, or, what amounts to the same thing, with its disjoint union $E := \sqcup_{i\in\mathfrak{I}} E_i$. We then apply certain rules for identifying points in $E$, the *glueing rules*, and we obtain a new space $\mathcal{E}$ whose points all come from $E$. The *identification map* is the surjective map

$$\nu : E \twoheadrightarrow \mathcal{E}, \tag{1}$$

which sends a point in $E$ to the point that it becomes in $\mathcal{E}$. In section 5.1, we will give the conditions that (1) must satisfy in the case of *assemblages of spaces* but before we do, let us look more closely at the identification map.

The glueing rules tell how to join together points in $E$, but they need not be exhaustive. They may say *glue $x_1$ to $x_2$* and *glue $x_2$ to $x_3$*, without asking that $x_1$ be glued to $x_3$. In that case, we will have $\nu(x_1) = \nu(x_2)$ and $\nu(x_2) = \nu(x_3)$, but then it logically follows that $\nu(x_1) = \nu(x_3)$. We now discuss this difference between the rules: (*glue-$x$-to-$y$*) and the identifications: $(\nu(x) = \nu(y))$.

**1.3 Binary Relations and Equivalences.** Let $E$ be a set. A *binary relation* on $E$ is a collection $\mathcal{R}$ of ordered pairs of elements of $E$, *i.e.* a subset of the cartesian product $E \times E$. When a pair $(x,y)$ belongs to $\mathcal{R}$, we say that "*$(x,y)$ satisfies $\mathcal{R}$*", and we write $(x\ \mathcal{R}\ y)$.

**1.3.1 Definition.** The relation $\mathcal{R}$ is called "*equivalence*", if it satisfies the following three properties

$$\begin{cases} \textit{reflexivity} & \forall x \in E, \quad (x\ \mathcal{R}\ x) \\ \textit{symmetry} & \forall x,y \in E, \quad (x\ \mathcal{R}\ y) \Rightarrow (y\ \mathcal{R}\ x) \\ \textit{transitivity}\ (^3) & \forall x,y,z \in E, \quad (x\ \mathcal{R}\ y)\ \&\ (y\ \mathcal{R}\ z) \Rightarrow (x\ \mathcal{R}\ z) \end{cases} \tag{2}$$

Two elements $x,y \in E$ such that $(x\ \mathcal{R}\ y)$ are then said *$\mathcal{R}$-equivalent*.

---

(2) The *Adidas Brazuca* was the official match ball of the FIFA World Cup held in Brazil in 2014. Made up of six flat pieces of leather, it is, heuristically speaking, the deformation of a cube along its edges preserving the faces flatness, *i.e.* avoiding foldings or stretching.

(3) Another form of transitivity was explicitly stated in Euclid's Elements: "*Things which equal the same thing also equal one another*".

**1.3.2 Exercise.** The relation "*to be born in the same country*" is an equivalence relation in humankind, but because people may have several citizenships, the relation "*to have a common citizenship*" is not. Explain which of the properties (2) could fail. Give a counterexample.

**1.3.3 Exercise.** Given a set $E$, denote by $\mathrm{Bin}(E)$ the set of binary relations on $E$, and by $\mathscr{P}(E\times E)$ the power set of $E\times E$ ((4)).

1) The *graph map* $\mathrm{Gr}:\mathrm{Bin}(E)\to\mathcal{P}(E\times E)$ associates with a relation $Q$ its *graph* $\mathrm{Gr}(\mathcal{Q}):=\big\{(x,y)\in E\times E \;\big|\; (x\;\mathcal{Q}\;y)\big\}$. Show that Gr is a bijection.
2) Define in $\mathrm{Bin}(E)$ the operations of inclusion, union, intersection, and complement of relations in such a way that Gr is compatible with them.
3) Show that the intersection of a family of equivalences is an equivalence. Deduce that for any $\mathcal{R}\in\mathrm{Bin}(E)$, there is a finest equivalence relation $\tilde{\mathcal{R}}$ such that $\mathcal{R}\subseteq\tilde{\mathcal{R}}$. This is *the equivalence generated by $\mathcal{R}$.*
4) Let $\mathcal{R}\in\mathrm{Bin}(E)$. For $a,b\in E$ write $(a\;\underline{\mathcal{R}}\;b)$ if there exists a finite family $\{x_1,\dots,x_m\}\subseteq E$, with $m\geq 1$, such that $a=x_1$, $b=x_m$, and either $(x_i\;\mathcal{R}\;x_{i+1})$ or $(x_{i+1}\;\mathcal{R}\;x_i)$ for all $1\leq i<m$. Show that $\tilde{\mathcal{R}}=\underline{\mathcal{R}}$.
5) Conclude that in 1.2 the relation $(\nu(x)=\nu(y))$ is the equivalence generated by the set of glueing rules (glue-$x$-to-$y$).

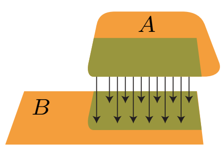
**1.4 Quotient Sets.** For $x\in E$, "*the equivalence class of $x$ (modulo $\mathcal{R}$)*" is the set $\mathcal{R}\,(x)$ of elements $\mathcal{R}$-equivalent to $x$, *i.e.*

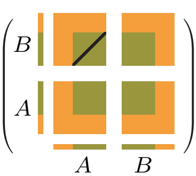
$$\mathcal{R}\,(x):=\big\{y\in E\;\big|\;(x\;\mathcal{R}\;y)\big\}.$$

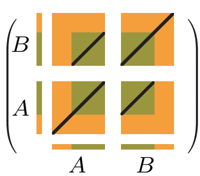
By properties (2), two equivalence classes are either equal or disjoint (exercise!), so that they define a *partition* of the set $E$. This partition is called *the quotient set of $E$ by $\mathcal{R}$* and is denoted by $E/\mathcal{R}$. The *canonical surjection* is the *identification map* that associates with $x\in E$ its equivalence class $\mathcal{R}\,(x)$:

$$\nu_{\mathcal{R}}:E\twoheadrightarrow E/\mathcal{R},\quad x\mapsto\mathcal{R}\,(x).$$

**1.4.1 Example.** In 1.1, endow $E:=A\sqcup B$ with the equivalence relation $\mathcal{R}:=$(?-coincides-in-$A\cup B$-to-?). The equivalence class $\mathcal{R}\,(x)$ is the singleton $\{x\}$ if $x$ is not in the overlapping area, otherwise, it is the set $\{x,y\}$ where $y$ is in the vertical projection of $x$ in the image to the right. The graphs of the glueing rules (glue-?-to-?) and of the equivalence it generates are:




Gr(glue-?-to-?) : 


Gr($\mathcal{R}$) : 


(4) The *power set* of a set $S$ is the set $\mathscr{P}(S)$ of subsets of $S$.

**1.5 Universal Property of Quotient Sets.** A map $f : E \to Y$ between sets, defines a relation $\mathcal{R}_f \in \mathrm{Bin}(E)$, called the *relation induced by f*:

$$(x\ \mathcal{R}_f\ y) \Leftrightarrow_{\mathrm{def}} (f(x) = f(y)). \tag{3}$$

**1.5.1 Exercise.** 1) Show that $\mathcal{R}_f$ is an equivalence relation on $E$.

2) A function $f : E \to Y$ is said to be *compatible with* a relation $\mathcal{R} \in \mathrm{Bin}(E)$, *$\mathcal{R}$-compatible* for short, if two elements $\mathcal{R}$-related have the same image under $f$. Show that $f$ is $\mathcal{R}$-compatible if and only if $\tilde{\mathcal{R}} \subseteq \mathcal{R}_f$.

The next theorem formulates the fact that $E/\mathcal{R}$ is *the most concise* set that can be built from $E$ by joining $\mathcal{R}$-related points, the one that identifies as few points as possible. This is called the *universal property of the quotient*.

**1.5.2 Theorem.** *The map $f : E \to Y$ is compatible with $\mathcal{R} \in \mathrm{Bin}(E)$ if and only if it factors through the identification map $\nu_{\tilde{\mathcal{R}}} : E \twoheadrightarrow E/\tilde{\mathcal{R}}$,* i.e. *there exists $\overline{f} : E/\tilde{\mathcal{R}} \to Y$ such that the following diagram is commutative.*

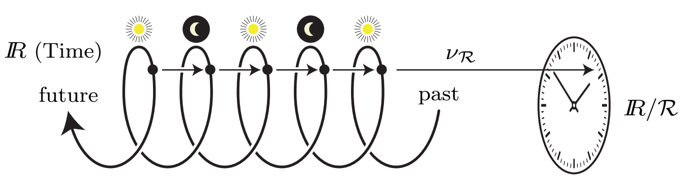
$$\begin{tikzcd} E \arrow[r, two heads, "\nu_{\tilde{\mathcal{R}}}"] \arrow[d, "f"'] & E/\tilde{\mathcal{R}} \arrow[dl, dotted, "\exists!\overline{f}"] \\ Y & \end{tikzcd} \qquad \overline{f}\big(C_{\tilde{\mathcal{R}}}(x)\big) = f(x),\ \forall x \in E. \tag{4}$$

*In that case, $\overline{f}$ is unique. It is injective if and only if $\tilde{\mathcal{R}} = \mathcal{R}_f$.*

*Proof.* The map $f$, being $\mathcal{R}$-compatible, sends the elements of an equivalence class modulo $\tilde{\mathcal{R}}$ to the same element in $Y$. The existence of $\overline{f} : E/\tilde{\mathcal{R}} \to Y$ is then clear, as is its definition $\overline{f}(C_{\tilde{\mathcal{R}}}(x)) := f(x)$, hence its uniqueness. The converse is obvious and the claim about injectivity is left to the reader. □

**1.5.3 Corollary.** *In 1.2, giving the identification map $\nu : E \twoheadrightarrow \mathcal{E}$, where $E := \bigsqcup_i E_i$, and giving the canonical surjection $\nu_{\tilde{\mathcal{R}}} : E \twoheadrightarrow E/\tilde{\mathcal{R}}$, where $\mathcal{R}$ is the set of glueing rules in $E$, are equivalent things.*

**1.5.4 Exercise.** Show that in the time line, identified as the real affine space $\mathbb{R}^1$, the relation $(t_1\ \mathcal{R}\ t_2) \Leftrightarrow_{\mathrm{def}} |t_1 - t_2| = 12h$, is an equivalence. Show that an *analog clock* is a possible realization of the quotient $\mathbb{R}/\mathcal{R}$.




Describe what could be the relation $\mathcal{R}$, to the quotient set $\mathbb{R}/\tilde{\mathcal{R}}$, to be realized by a 24-hour *digital* clock where the smallest measure is a minute.

## 2 Spaces

In sections 1.2–1.5, we focused on the elementary problem of giving an equivalence relation $\mathcal{R}$ on a set $E$ and defining the quotient *set* $\mathcal{E} := E/\mathcal{R}$ identifying only $\mathcal{R}$-equivalent points. The universal property 1.5.2 states that such a set *exists* and, moreover, is *unique* up to canonical isomorphism.

Such elementary setting is not without its uses as it gives a guideline of how to generalize the concept of quotient to the context of *spaces*, which is what we're concerned with here. We shall see that answers to the questions of *conciseness*, *existence* and *uniqueness* of quotient spaces may vary according the the structures we consider. Needless to say, in the context of spaces, we still have to give a mathematical meaning to the essential feature of assemblages, that which ensures that each individual space $E_i$ remains *unaltered* in the building process. So now is a good time to approach the concept of *space*.

In this book, the word 'space' means *a set endowed with some specific structure*: vector, affine, topological, differential, Riemannian, etc. Although we will define these precisely, we will begin by recalling the general framework for dealing with collections of objects that share a common structure.

**2.1 Categories.** A *category* is a collection $\mathfrak{C}$ of objects defined by some unifying set of characteristics, some *structure*, that one seeks to explore. The idea of category emerged in mathematics in the 1940s when Eilenberg and MacLane provided the appropriate language for formalizing the proven method for investigating structures, which entails collecting all possible objects endowed with the structure and examining how they *interact* with each other. This is done through the concept of *morphism*, which is a shortcut for *interaction compatible with the structure*.

| Name | Objects | Morphisms |
|---|---|---|
| **Set** | sets | maps between sets |
| **Top** | topological spaces | continuous maps |
| **Man** | manifolds | differentiable maps |
| **Met** | metric spaces | continuous maps |
| $\mathbf{Met}_\sigma$ | metric spaces | local isometries |
| $\mathbf{Met}_{\mathrm{Iso}}$ | metric spaces | global isometries |
| $\mathrm{Vec}(I\!\!K)$ | vector spaces over a field $I\!\!K$ | $I\!\!K$-linear maps |
| $\mathbf{Rep}(\boldsymbol{G}, I\!\!K)$ | $I\!\!K$-linear representation of a group $\boldsymbol{G}$ | intertwining operators |
| **Grp** | groups | homomorphisms |

*Table .1. Some of the categories on which we will work*

If $X$ and $Y$ are objects in $\mathfrak{C}$ the *morphisms from $X$ to $Y$*, usually represented by arrows '$f : X \to Y$', constitute a *set*, which will be denoted by $\mathrm{Mor}_{\mathfrak{C}}(X,Y)$.

Each category includes a specific *law of composition* for morphisms

$$\circ : \mathrm{Mor}_{\mathfrak{C}}(Y,Z) \times \mathrm{Mor}_{\mathfrak{C}}(X,Y) \to \mathrm{Mor}_{\mathfrak{C}}(X,Z).$$

In the above table, morphisms are maps between the underlying sets and the law is the usual composition of maps. In the general setting, the law is required only to verify the two following properties.

$\mathfrak{C}$-1) *Associativity.* Given $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f. \qquad \begin{tikzcd} W \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & X \arrow[r, "g"] \arrow[rr, bend right, "h\circ g"'] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

$\mathfrak{C}$-2) *Unit.* For every object $X$ there exists a morphism $\mathbf{1}_X \in \mathrm{Mor}_{\mathfrak{C}}(X,X)$ such that $f \circ \mathbf{1}_X = f$, for every $f \in \mathrm{Mor}_{\mathfrak{C}}(X,Y)$, and $\mathbf{1}_X \circ g = g$, for every $g \in \mathrm{Mor}_{\mathfrak{C}}(W,X)$. This morphism is the *identity morphism of $X$*.

**2.2 Functors.** These allow categories to be related by associating objects and arrows from one to the other in a consistent way. More precisely, given two categories $\mathfrak{C}$ and $\mathfrak{D}$, a (covariant) *functor* from $\mathfrak{C}$ to $\mathfrak{D}$ is a correspondence $F : \mathfrak{C} \rightsquigarrow \mathfrak{D}$, which consists of giving:

F-1) for each object $X \in \mathfrak{C}$, an object $F(X) \in \mathfrak{D}$;

F-2) for each morphism $f : X \to Y$ in $\mathfrak{C}$, a morphism $F(f) : F(X) \to F(Y)$ in $\mathfrak{D}$;

such that

$$F(\mathbf{1}_X) = \mathbf{1}_{F(X)} \quad \text{and} \quad F(f \circ g) = F(g) \circ F(g).$$

**2.2.1 Exercise.** Let $\mathfrak{N}$ be the collection of natural numbers $1,2,\ldots$, and define $\mathrm{Mor}_{\mathfrak{N}}(\ell,m) := \mathscr{P}([\![1,\ell]\!] \times [\![1,m]\!])$, the power set of the product of intervals of natural numbers $[\![1,\ell]\!] \times [\![1,m]\!]$ ([5]).

- For $\ell \in \mathfrak{N}$, define $\mathbf{1}_\ell \in \mathrm{Mor}_{\mathfrak{N}}(\ell,\ell)$ as $\mathbf{1}_\ell := \{(x,x) \mid x \in [1,\ell]\} \subseteq [1,\ell]^2$.
- For $f : \ell \to m$ and $g : m \to n$, define the composition $g \circ f : \ell \to n$ by

$$g \circ f := \big\{(x,z) \bigm| \exists y \in Y \text{ s.t. } \big((x,y) \in f\big) \text{ and } \big((y,z) \in g\big)\big\}.$$

1) Show that $\mathfrak{N}$ is a category.
2) Let $\sigma : \mathfrak{N} \rightsquigarrow \mathbf{Set}$ associate with $\ell \in \mathfrak{N}$, the set $\sigma(\ell) := \{\text{subsets of } [\![1,\ell]\!]\}$, and with the morphism $f \in \mathrm{Mor}_{\mathfrak{N}}(\ell,m)$ the map

$$\sigma(f) : \sigma(\ell) \to \sigma(m), \quad [1,\ell] \supseteq I \mapsto \big\{j \in [\![1,m]\!] \bigm| \exists i \in I \text{ s.t. } (i,j) \in f\big\}.$$

Show that $\sigma : \mathfrak{N} \rightsquigarrow \mathbf{Set}$ is a functor and that

$$\mathrm{Mor}_{\mathfrak{N}}(n,m) \subset \mathrm{Mor}_{\mathbf{Set}}(\sigma(n),\sigma(m)).$$

([5]) The elements of $\mathrm{Mor}_{\mathfrak{N}}(\ell,m)$ are classically called *correspondences* from $[\![1,\ell]\!]$ to $[\![1,m]\!]$. A map $f : [\![1,\ell]\!] \to [\![1,m]\!]$ defines a correspondence by its graph $\mathrm{Gr}(f) \subseteq [\![1,\ell]\!] \times [\![1,m]\!]$. The composition of correspondences is the obvious generalization of the composition of maps.

**2.3. The Forgetful Functor.** In the table .1, some categories are richer than others in the sense that they incorporate more structure. For example, a representation of a group $\boldsymbol{G}$ is a vector space *plus* a linear action of $\boldsymbol{G}$, and a vector space is a set *plus* algebraic operations, and similarly for morphisms. In the diagram to the right, the arrows indicate the direction of decreasing complexity, and '$o$' denotes the *forgetful functor* that disregards extra structure.

$$\begin{tikzcd} \mathbf{Rep} \arrow[d, "o"'] & \mathbf{Man} \arrow[dr, "o"'] & & \mathbf{Met} \arrow[dl, "o"] \\ \mathrm{Vec} \arrow[dr, "o"'] & & \mathbf{Top} \arrow[dl, "o"] & \\ & \mathbf{Set} & & \end{tikzcd}$$

For example, if $\mathfrak{C}$ is any of these categories, the functor $o : \mathfrak{C} \rightsquigarrow \mathbf{Set}$ associates with a space its underlying set, and with a morphism, the map it defines between the underlying sets. Notice that in the process of pauperizing a structure, while the underlying set does not change, the set of morphisms usually grows, as they have fewer constraints to respect. For example, in Table 2.1 the set of real numbers $I\!R$ appears everywhere with canonical structures for which one can easily check the *strict* inclusions:

$$\mathrm{Mor}_{\mathrm{Vec}}(I\!R,I\!R) \subset \mathrm{Mor}_{\mathbf{Man}}(I\!R,I\!R) \subset \mathrm{Mor}_{\mathbf{Top}}(I\!R,I\!R) \subset \mathrm{Mor}_{\mathbf{Set}}(I\!R,I\!R).$$

**2.4 Categorical definition of Spaces.** The forgetful functor pinpoints the nature of the link that exists between a space and its underlying set and, conversely, it shows how to categorize the concept of space.

A *category of spaces* consists of a category $\mathfrak{C}$ and an functor $\sigma : \mathfrak{C} \rightsquigarrow \mathbf{Set}$, such that for all $X,Y \in \mathfrak{C}$, the map

$$\sigma : \mathrm{Mor}_{\mathfrak{C}}(X,Y) \to \mathrm{Mor}_{\mathbf{Set}}(\sigma X,\sigma Y) \tag{5}$$

is *injective*. The set $\sigma X$ is then called *the underlying set* of $X \in \mathfrak{C}$, and the map $\sigma f : \sigma X \to \sigma Y$ the *underlying map* of the morphism $f : X \to Y$.

**2.4.1 Remarks.** 1) The injectivity of (5) accounts for the fact that a morphism between spaces is determined by its action on *points*.

2) Often the definition of a category does not make clear the spatial nature of its objects. For example, the category $\mathfrak{C}$ in exercise 2.2.1, whose purpose was precisely to reveal this nature, which is rather hidden in that case.

3) **Exercise.** Let $\mathsf{A} := (A,\leq)$ be a partially order set. Call *objects of* $\mathfrak{C}(\mathsf{A})$ the elements of $A$, and call *morphisms of* $\mathfrak{C}(\mathcal{A})$ the arrows $a \to b$ with $a \leq b \in A$. Next, associate with $a \in A$, the set $\sigma a := \{x \in A \mid x \leq a\}$, and with $a \to b$, the inclusion map $\sigma a \subseteq \sigma b$. Show that $(\mathfrak{C}(\mathsf{A}),\sigma)$ is a category of spaces.

**2.4.2. Some Special Morphisms.** A morphism $f : X \to Y$, is called:

- a *monomorphism*, if it is left cancellable, *i.e.*

$$\big(f \circ g = f \circ h\big) \Rightarrow (g = h), \quad \forall g,h : W \to X.$$

- an *epimorphism*, if it is right cancellable, *i.e.*

$$\big(g \circ f = h \circ f\big) \Rightarrow (g = h), \quad \forall g,h : Y \to Z.$$

- an *isomorphism*, if there exists $g : Y \to X$ s.t. $g \circ f = \mathbf{1}_X$ and $f \circ g = \mathbf{1}_Y$.

$$\begin{tikzcd} X \arrow[loop left, "\mathbf{1}_X"] \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \arrow[loop right, "\mathbf{1}_Y"] \end{tikzcd}$$

Then, $g$ is unique (exercise!). It is the *inverse of $f$*, and is denoted by $f^{-1}$.

**2.4.3 Exercises.** 1) Show that in a category of spaces, $f$ is a monomorphism (resp. epimorphism) if $\sigma f$ is an injective (resp. surjective) map.

2) Show that a functor conserves isomorphy.

3) Show that in $\mathfrak{C}(\mathbb{N},\leq)$ (*cf.* ex. 2.4.1-(3)) every morphism $f_{b,a}$ is right and left cancellable but that they are never isomorphisms if $a \neq b$.

4) (**) In table .1, show that the claim "*$f$ is an isomorphism if and only if $\sigma f$ is a bijection*", is false only in **Top**, and **Met** (*cf.* ex. **37**.1.1-(3)).

**Conventions.** In order not to overload the notation, we will often not indicate the functor $\sigma$. Instead, we will use bold letters $\boldsymbol{X}, \boldsymbol{Y}, \ldots$ for spaces and light ones $X, Y, \ldots$ for underlying sets. Moreover, if $f : \boldsymbol{X} \to \boldsymbol{Y}$ is a morphism, expressions such as '$f$ is injective', '$f(\boldsymbol{X}) \subseteq \boldsymbol{Y}$', '$x \in \boldsymbol{X}$', '$\mathcal{R}$ is a relation on $\boldsymbol{X}$', can have only one significant meaning, namely '$\sigma f$ is injective', '$(\sigma f)(X) \subseteq Y$', '$x \in X$' and '$\mathcal{R}$ is a relation on $X$', which they will obviously replace.

# 3 Subspaces

**3.1 Categorical Definition of Subspaces.** A Given a space $\boldsymbol{X}$, a subset $X' \subseteq \sigma\boldsymbol{X}$ *admits a structure of subspace*, if there exists an *injective* morphism $\iota : \boldsymbol{W} \to \boldsymbol{X}$, such that $\iota(W) = X'$, and such that:
*Every morphism $f : \boldsymbol{Y} \to \boldsymbol{X}$ verifying $f(Y) \subseteq X'$, factors (in a unique way) through $\iota : \boldsymbol{W} \to \boldsymbol{X}$,* i.e. *there exists $f' : \boldsymbol{Y} \to \boldsymbol{W}$ (unique) s.t. $f = \iota \circ f'$:*

$$\begin{tikzcd} & Y \arrow[dl] \arrow[d, "f"] \\ W \arrow[r, tail, "\iota"] & X \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & \boldsymbol{Y} \arrow[dl, dotted, "\exists! f'"'] \arrow[d, "f"] \\ \boldsymbol{W} \arrow[r, "\iota"] & \boldsymbol{X} \end{tikzcd} \tag{6}$$

In other words, the map

$$\iota_* : \mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{Y}, \boldsymbol{W}) \to \mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{Y}, \boldsymbol{X}) \cap \iota_*(\mathrm{Maps}(Y, W)),$$

where $\iota_*(\_) := \iota \circ (\_)$, is bijective.

**3.1.1 Proposition.** *There is at most one subspace structure on a subset $W \subseteq \boldsymbol{X}$. In other words, if $\iota_i : \boldsymbol{W}_i \to \boldsymbol{X}$, $i = 1,2$, are subspaces such that $\iota_i(W_1) = \iota_i(W_2)$, then there exist unique isomorphisms $\varphi_{ij} : \boldsymbol{W}_i \to \boldsymbol{W}_j$ such that $\iota_j \circ \varphi_{j,i} = \iota_i$, and such that $\varphi_{j,i} \circ \varphi_{i,j} = \mathbf{1}_{\boldsymbol{W}_i}$.*

**Terminology.** When a subset $X' \subseteq \boldsymbol{X}$ admits a structure of subspace, we will call it shortly *a subspace of $\boldsymbol{X}$* and we will simply write '$\boldsymbol{X}' \subseteq \boldsymbol{X}$'.

*Proof.* Left to the reader. □

**3.1.2 Exercises.** 1) Explain why the expressions in parenthesis in the definition of *subspace* 3.1 are superfluous.

2) Show that in the category $\mathfrak{N}$ (2.2.1), a necessary and sufficient condition for a morphism $f : \ell \to m$ to define a subspace of $m$ is that $f$ be the graph of an injective map from $[\![1,\ell]\!]$ to $[\![1,m]\!]$.

## 4 Quotient Spaces

The theorem 1.5.2, which involves the three properties: *concisness*, *existence* and *uniqueness* of quotients of sets, leads us to the concept of *quotient*.

**4.1 Categorical Definition of Quotient Spaces.** Given a space $\boldsymbol{X}$ and an equivalence relation $\mathcal{R}$ on $X$, a *quotient of* $\boldsymbol{X}$ *by* $\mathcal{R}$ is a *surjective* morphism $\nu_{\mathbf{W}} : \boldsymbol{X} \twoheadrightarrow \boldsymbol{W}$, $\mathcal{R}$-compatible, satisfying the following property:

*A morphism* $f : \boldsymbol{X} \to \boldsymbol{Y}$, *compatible with* $\mathcal{R}$, *factors (in a unique way) through* $\nu_{\mathbf{W}} : \boldsymbol{X} \twoheadrightarrow \boldsymbol{W}$, i.e. *there exists* $\overline{f} : \boldsymbol{W} \to \boldsymbol{Y}$ *(unique) s.t.* $f = \overline{f} \circ \nu_{\mathbf{W}}$*:*

$$\begin{tikzcd} X \arrow[r, two heads, "\nu_{\mathcal{R}}"] \arrow[d, "\sigma f"'] & X/\mathcal{R} \arrow[dl] \\ Y & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \boldsymbol{X} \arrow[r, two heads, "\nu_{\mathbf{W}}"] \arrow[d, "f"'] & \boldsymbol{W} \arrow[dl, dotted, "\exists!\overline{f}"] \\ \boldsymbol{Y} & \end{tikzcd} \tag{7}$$

*In other words, if one sets for* $\boldsymbol{Y} \in \mathfrak{C}$

$$\mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{X},\boldsymbol{Y})^{\mathcal{R}} := \big\{ f \in \mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{X},\boldsymbol{Y}) \;\big|\; f \text{ is compatible with } \mathcal{R} \big)\big\},$$

*one has*

$$\nu_{\mathbf{W}}^{*}(\mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{W},\boldsymbol{Y})) = \mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{X},\boldsymbol{Y})^{\mathcal{R}}, \text{ where } \nu_{\mathbf{W}}^{*}(f) := f \circ \nu_{\mathbf{W}}. \tag{8}$$

**4.1.1 Theorem.** *Let* $\mathcal{R}$ *be an equivalence relation on a space* $\boldsymbol{X} \in \mathfrak{C}$.

a) Uniqueness. *Quotients of* $\boldsymbol{X}$ *by* $\mathcal{R}$ *in* $\mathfrak{C}$*, if they exist, are unique up to canonical isomorphism.*

*In other words, if* $\nu_1 : \boldsymbol{X} \twoheadrightarrow \boldsymbol{W}_1$ *and* $\nu_2 : \boldsymbol{X} \twoheadrightarrow \boldsymbol{W}_2$ *are quotients of* $\boldsymbol{X}$ *by* $\mathcal{R}$*, then there exist unique isomorphisms* $f_{ji} : \boldsymbol{W}_i \to \boldsymbol{W}_j$*, inverse to one another, such that* $\nu_j = f_{ji} \circ \nu_i$.

$$\begin{tikzcd} & \boldsymbol{X} \arrow[dl, "\nu_1"'] \arrow[dr, "\nu_2"] & \\ \boldsymbol{W}_1 \arrow[rr, shift left, "f_{21}"] & & \boldsymbol{W}_2 \arrow[ll, shift left, "f_{12}"] \end{tikzcd} \tag{‡}$$

b) *If* $\nu_{\mathbf{W}} : \boldsymbol{X} \twoheadrightarrow \boldsymbol{W}$ *is a quotient of* $\boldsymbol{X}$ *by* $\mathcal{R}$*, then a morphism* $f : \boldsymbol{X} \to \boldsymbol{Y}$ *is compatible with* $\mathcal{R}$ *if and only if it is compatible with the equivalence* $\mathcal{R}_{\nu_{\mathbf{W}}}$ *induced by* $\nu_{\mathbf{W}}$ *(1.5). One has* $\mathcal{R} \subseteq \mathcal{R}_{\nu_{\mathbf{W}}}$*, and a surjective map*

$$X/\mathcal{R} \twoheadrightarrow W = X/\mathcal{R}_{\nu_{\mathbf{W}}}. \tag{††}$$

**Definition.** The category $\mathfrak{C}$ is *a category with quotients*, if for every objet $\boldsymbol{X} \in \boldsymbol{C}$ and every equivalence $\mathcal{R}$ on $\boldsymbol{X}$, the quotient $\boldsymbol{X}/\mathcal{R}$ exists in $\mathfrak{C}$.

*Proof.* (a) Because $\nu_i$ is surjective, a map $f_{ii} : \boldsymbol{W}_i \to \boldsymbol{W}_i$ such that $f_{ii} \circ \nu_i = \nu_i$ can only be $\mathbf{1}_{\boldsymbol{W}_i}$. But, $f_{12} \circ f_{21} : \boldsymbol{W}_1 \to \boldsymbol{W}_1$ clearly has this property, so that $f_{12} \circ f_{21} = f_{11} = \mathbf{1}_{\boldsymbol{W}_1}$. Exchanging indices, the converse is also true, and (a) follows immediately. (b) is left to the reader. □

**4.1.2 Comments.** a) *The underlying set of the quotient space and the quotient of the underlying set may differ!* When the quotient space of $\boldsymbol{X}$ by $\mathcal{R}$ exists, care must be taken with the notation $\mathbf{X}/\mathcal{R}$ as it can be misleading. Indeed, the surjective map $X/\mathcal{R} \twoheadrightarrow \sigma(\mathbf{X}/\mathcal{R})$ in (††), may fail to be bijective. We can have $X/\mathcal{R} \neq \sigma(\mathbf{X}/\mathcal{R})$, which means that building a quotient *space* can require more identifications than those implicit in the relation $\mathcal{R}$ (*cf.* 5.3.3-(4)). For example, in the category $\mathbf{Top}_{\text{sep}}$ of *separated spaces* ([6]), the equality $X/\mathcal{R} = \sigma(\mathbf{X}/\mathcal{R})$ will be verified only when $\mathcal{R}$ is a closed subspace of $X \times X$ (see **3**♦9.1.2).

b) In some categories, quotient spaces may not exist at all. For example in the category $\mathbf{Met}_{\text{Iso}}$ of metric spaces and global isometries, the identification map $\nu : X \twoheadrightarrow X/\mathcal{R}$, being a morphism, is necessarily injective, and this which reduces $\mathcal{R}$ to the identity relation, which is of no interest.

c) The notions of *subspace* and *quotient space* are dual to each other. This is notable particularly when $\mathcal{R} = \mathcal{R}_{\nu_W}$, in which case diagrams (15) and (6) differ only in the arrows, the directions of which are reversed.

## 5 Assemblages of Spaces

**5.1 The Assembling condition for Spaces.** Given a family $\{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ of spaces in $\mathfrak{C}$ of disjoint union $E := \sqcup_{i\in\mathfrak{I}} E_i$, the presentation in 1.2 applies, on condition that the assemblage $\mathcal{E}$ belongs to $\mathfrak{C}$, as well as the restrictions $\nu_i : \boldsymbol{E}_i \to \mathcal{E}$ of the identification map $\nu : E \to \mathcal{E}$. For $i,j \in \mathfrak{I}$, thus we have a diagram where the solid arrows are morphisms in $\mathfrak{C}$, while the others are just maps between sets.

$$\begin{tikzcd}
\boldsymbol{E}_i \arrow[dr, dashed, "\subseteq"'] \arrow[drr, bend left, "\nu_i"] & & \\
& E \arrow[r, dashed, "\nu"] & \mathcal{E}. \\
\boldsymbol{E}_j \arrow[ur, dashed, "\subseteq"] \arrow[urr, bend right, "\nu_j"'] & &
\end{tikzcd} \tag{9}$$

Now, to say that the spaces $\boldsymbol{E}_i$'s remain *unaltered* through the assembling process means several things. First, that the maps $\sigma\nu_i : \boldsymbol{E}_i \to \sigma\mathcal{E}$ are bijections onto their images, the sets $\operatorname{im}(\sigma\nu_i)$. Second, that these images have a space structure *induced* from the ambient space $\mathcal{E}$, being thereby the underlying sets of *subspaces* $\tilde{\boldsymbol{E}}_i$'s of $\mathcal{E}$ (unique after 3.1.1). Third, that the $\nu_i : \boldsymbol{E}_i \to \tilde{\boldsymbol{E}}_i$ are bijections *compatible with the structures*, *i.e.* are *isomorphism* in $\mathfrak{C}$.

([6]) Different points have disjoint neighborhoods, see **3**♦9.1.

**5.1.1. Transition Maps.** In the preceding, for $i, j \in \mathfrak{I}$, set $\tilde{\boldsymbol{E}}_{i,j} := \tilde{\boldsymbol{E}}_i \cap \tilde{\boldsymbol{E}}_j$ and consider the commutative diagram, *a priori*, of bijections:

$$
\begin{tikzcd}
\boldsymbol{E}_i \arrow[rrd, bend left, "\nu_i", "\simeq"'] & & \\
\boldsymbol{E}_i^j \arrow[u, phantom, sloped, "\subseteq"] \arrow[rd, bend left, "\nu_i^j", "\simeq"'] \arrow[dd, dotted, "\varphi_{j,i}"', "\simeq"] & & \tilde{\boldsymbol{E}}_i \\
\arrow[r, phantom, "\oplus"] & \tilde{\boldsymbol{E}}_{i,j} = \tilde{\boldsymbol{E}}_i \cap \tilde{\boldsymbol{E}}_j \arrow[ru, phantom, sloped, "\subseteq"] \arrow[rd, phantom, sloped, "\supseteq"] & \\
\boldsymbol{E}_j^i \arrow[ru, bend right, "\nu_j^i"', "\simeq"] & & \tilde{\boldsymbol{E}}_j \\
\boldsymbol{E}_j \arrow[u, phantom, sloped, "\supseteq"] \arrow[rru, bend right, "\nu_j"', "\simeq"] & &
\end{tikzcd}
\tag{10}
$$

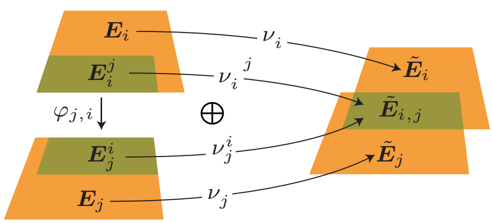
where $\nu_i^j$ is the restriction of $\nu_i$ to $\boldsymbol{E}_i^j := \nu_i^{-1}(\tilde{\boldsymbol{E}}_{i,j})$, and similarly for $\nu_j^i$.

Now, when glueing structures, the $\boldsymbol{E}_i^j$'s will be *subspaces* of the $\boldsymbol{E}_i$'s, and, given $i, j \in \mathfrak{I}$, the bijection $\nu_i^j$, being a restriction of an isomorphism, will be an isomorphism too (exercise!). Consequently, the composition

$$\boxed{\varphi_{j,i} := (\nu_j^i)^{-1} \circ \nu_i^j : \boldsymbol{E}_i^j \to \boldsymbol{E}_j^i,} \tag{11}$$

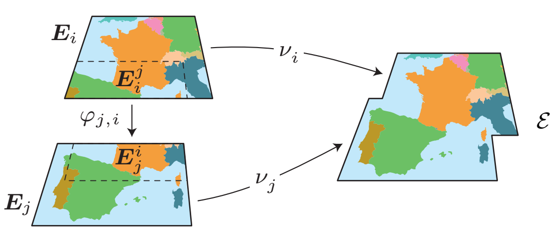
is an isomorphism of spaces. It is *the transition morphism from $\boldsymbol{E}_i$ to $\boldsymbol{E}_j$*.

The following picture, where all the arrows are isomorphisms, summarizes what we have so far discussed.




(12)

**5.2 Atlases.** Up to now, we have focussed on the question of building a space $\mathcal{E}$ from a given family of spaces $\{\boldsymbol{E}_i\}_{i \in \mathfrak{I}}$. In the case of assemblages, because each $\boldsymbol{E}_i$ is unaltered, a complementary point of view appears, where one may consider the $\boldsymbol{E}_i$'s as "*local views*" of $\mathcal{E}$. As in atlases, the $\boldsymbol{E}_i$'s are the *charts* of $\mathcal{E}$ and the $\boldsymbol{E}_i^j$'s the *chart overlaps*, so the last diagram could have been:




(13)

The following definition derives from this analogy.

**5.2.1 Definition.** Let $\nu := \sqcup_{i\in\mathfrak{I}} \boldsymbol{E}_i \twoheadrightarrow \mathcal{E}$ be a *surjection* with *injective* restrictions $\nu_i : \boldsymbol{E}_i \rightarrowtail \mathcal{E}$. The *atlas of $\mathcal{E}$ associated with $\nu$*, denoted by $\mathcal{A}(\nu)$, consists of the families $\{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ of spaces, and $\{\varphi_{j,i} : \boldsymbol{E}_i^j \to \boldsymbol{E}_j^i\}_{i,j\in\mathfrak{I}}$ of transition morphisms, where $\boldsymbol{E}_i^j := \nu_i^{-1}(\nu_j(\boldsymbol{E}_j))$. The maps $\nu_i : \boldsymbol{E}_i \to \mathcal{E}$ are called *the charts* of the atlas.

**5.2.2 Remark.** Notice that, despite the notation, the atlas $\mathcal{A}(\nu)$ does not rely completely on $\nu$. Indeed, knowing $\mathcal{E}$ is superfluous as $\mathcal{A}(\nu)$ consists only of the families of $\boldsymbol{E}_i$'s and of $\varphi_{j,i} : \boldsymbol{E}_i^j \to \boldsymbol{E}_j^i$'s. We now give an intrinsic definition of atlases, *i.e.* independent of any identification map $\nu$ or any space $\mathcal{E}$.

**5.2.3 Theorem.** *Given a family of spaces $\{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ of $\mathfrak{C}$, let $\mathcal{A}$ be a set of data consisting of all the $\boldsymbol{E}_i$'s and, for each pair of indices $i,j \in \mathfrak{I}$, in subspaces $\boldsymbol{E}_i^j \subseteq \boldsymbol{E}_i$ and $\boldsymbol{E}_j^i \subseteq \boldsymbol{E}_j$ and an isomorphism $\varphi_{j,i} : \boldsymbol{E}_i^j \to \boldsymbol{E}_j^i$. Let $E := \sqcup_{i\in\mathfrak{I}} E_i$ (as a set). Then the following are equivalent statements.*

a) *$\mathcal{A} = \mathcal{A}(\nu)$ for some surjective map $\nu : E \twoheadrightarrow \mathcal{E}$.*

b) *For $i,j,k \in \mathfrak{I}$, one has*

$$\text{(i) } \varphi_{i,i} = \mathrm{id}_{\boldsymbol{E}_i}, \quad \text{(ii) } \varphi_{i,j} = \varphi_{j,i}^{-1}, \quad \text{(iii) } \begin{cases} \boldsymbol{E}_i^j \cap \varphi_{i,j}(\boldsymbol{E}_j^k) \subseteq \boldsymbol{E}_i^k \ \textit{and} \\ \qquad \varphi_{k,i} = \varphi_{k,j} \circ \varphi_{j,i}, \end{cases}$$

*Moreover, when* (b) *is satisfied,*

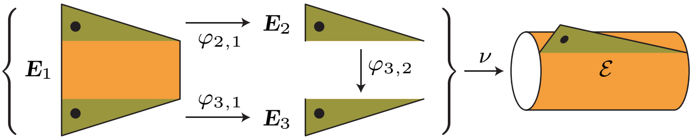
b-1) *In $E$, the relation $(x_i \ \mathcal{R}_\mathcal{A}\ x_j) := \big(x_j = \varphi_{j,i}(x_i)\big)$, for all $x_i \in \boldsymbol{E}_i$ and $x_j \in \boldsymbol{E}_j$, is an equivalence relation.*

b-2) *The restrictions $\nu_i : \boldsymbol{E}_i \to E/\mathcal{R}_\mathcal{A}$ of the identification map $\nu : E \twoheadrightarrow E/\mathcal{R}_\mathcal{A}$, are injective.*

*Proof.* (a)⇒(b). (b-i,ii) are left to the reader. (b-iii) If $x_i \in \boldsymbol{E}_i^j \cap \varphi_{i,j}(\boldsymbol{E}_j^k)$, the element $x_j := \varphi_{j,i}(x_i)$ belongs to $\varphi_{j,i}(\varphi_{i,j}(\boldsymbol{E}_j^k)) = \boldsymbol{E}_j^k$, after (ii). But then, $\exists x_k \in \boldsymbol{E}_k^j$ s.t. $\varphi_{j,i}(x_i) = x_j = \varphi_{j,k}(x_k)$, in which case $\nu(x_i) = \nu(x_j) = \nu(x_k)$. This implies that $x_i \in \boldsymbol{E}_i^k$. Then $\varphi_{k,i}(x_i) = x_k$, because $\nu$ is injective in $\boldsymbol{E}_k$.
(b-1). The relation $\mathcal{R}_\mathcal{A}$ is clearly reflexive and symmetric. To check transitivity, let $x_i,x_j,x_k \in E$, with $x_i \in \boldsymbol{E}_i$, $x_j \in \boldsymbol{E}_j$, $x_k \in \boldsymbol{E}_k$, such that $(x_i \ \mathcal{R}_\mathcal{A}\ x_j)$ and $(x_j \ \mathcal{R}_\mathcal{A}\ x_k)$. This means that $\varphi_{j,i} : x_i \mapsto x_j$ and $\varphi_{k,j} : x_j \mapsto x_k$, so that $x_i$ belongs to $\boldsymbol{E}_{i,j} \cap \varphi_j^i(\boldsymbol{E}_j^k)$ which is included in $\boldsymbol{E}_i^k$ of $\varphi_{k,i}$, after (b-iii). We can then compare $\varphi_{k,i}(x_i)$ and $x_k$. And, again thanks to (b-iii), we have $\varphi_{k,i}(x_i) = \varphi_{k,j}(\varphi_{j,i}(x_i)) = \varphi_{k,j}(x_j) = x_k$, which means that $(x_i \ \mathcal{R}_\mathcal{A}\ x_k)$.
(b-2) follows from (b-1) and the fact that the equivalence $\mathcal{R}_\mathcal{A}$, when restricted to the same $\boldsymbol{E}_i$, is the identity relation thanks to (b-i).
(b)⇒(a) is tautological after (b-1-2), setting $\mathcal{E} := E/\mathcal{R}_\mathcal{A}$. □

**5.2.4 Definition.** We call *atlas in $\mathfrak{C}$* a set of data, denoted $\mathcal{A} = \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ for short, satisfying 5.2.3-(b).

**5.2.5 Remarks.** 1) The transitivity condition (iii) in 5.2.3-(b) is essential to an assemblage. For example, in the following data:




where the transition morphisms are the obvious isometries between the triangles, set $\varphi_{i,i} = \mathrm{id}_{\boldsymbol{E}_i}$ and $\varphi_{i,j} = \varphi_{j,i}^{-1}$. The conditions (i) and (ii) are satisfied, while (iii) is not. The black dots become the same in $\mathcal{E}$ and the restriction $\nu_1 : \boldsymbol{E}_1 \to \mathcal{E}$ of the identification map is not an injective map.

2) In the last theorem, $\mathcal{E}$ and $E/\mathcal{R}_{\mathcal{A}}$ are simply *sets*. Even if the data in $\mathcal{A}$ belong to a category of spaces $\mathfrak{C}$, the theorem gives only a *necessary* condition for $\mathcal{A}$ to be an atlas of some space, but it does not state the existence of anyone in $\mathfrak{C}$. The same remark applies to the last definition, where atlases may not correspond to any existing space.

**5.3 Categorical definition of Assemblages of Spaces.** We follow the same idea of definition as for the quotient spaces (4), keeping the notations of theorem 5.2.3: $\mathcal{A}$ is an atlas in $\mathfrak{C}$ defining an equivalence relation $\mathcal{R}_{\mathcal{A}}$ in the disjoint union of sets $E := \sqcup_{i\in\mathfrak{I}} E_i$. A map between *sets* $f : E \to \sigma\boldsymbol{Y}$ is *a family of morphisms*, denoted $f : E \to \boldsymbol{Y}$, if $\boldsymbol{Y} \in \mathfrak{C}$ and if the restrictions $f_i : \boldsymbol{E}_i \to \boldsymbol{Y}$ are morphisms in $\mathfrak{C}$. The set of these families is denoted by

$$\mathrm{Mor}_{\mathfrak{C}}(E;\boldsymbol{Y}) := \prod\nolimits_{i\in\mathfrak{I}} \mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{E}_i,\boldsymbol{Y}). \tag{14}$$

**5.3.1 Definition.** An *assemblage $\mathcal{E}$ of $\mathcal{A}$ in $\mathfrak{C}$* is a surjective family of morphisms $\nu : E \twoheadrightarrow \mathcal{E}$, compatible with $\mathcal{R}_{\mathcal{A}}$, satisfying the following properties:

$\mathcal{A}$-1) *For all $i \in \mathfrak{I}$, the restriction $\nu_i : \boldsymbol{E}_i \to \mathcal{E}$ of $\nu$ is a subspace of $\mathcal{E}$.*

$\mathcal{A}$-2) *Every family of morphisms $f : E \to \boldsymbol{Y}$, compatible with $\mathcal{R}_{\mathcal{A}}$, factors in $\mathfrak{C}$ (in a unique way) through $\nu : E \twoheadrightarrow \mathcal{E}$,* i.e. *there exists $\overline{f} \in \mathrm{Mor}_{\mathfrak{C}}(\mathcal{E},\boldsymbol{Y})$ (unique) such that $f = \overline{f} \circ \nu$:*

$$\begin{tikzcd} E \arrow[r, "\nu_{\mathcal{R}_{\mathcal{A}}}"] \arrow[d, "\sigma f"'] & E/\mathcal{R}_{\mathcal{A}} \arrow[dl] \\ \sigma\boldsymbol{Y} & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} E \arrow[r, "\nu"] \arrow[d, "f"'] & \mathcal{E} \arrow[dl, dotted, "\exists!\overline{f}"] \\ \boldsymbol{Y} & \end{tikzcd} \tag{15}$$

In other words, if one sets for all $\boldsymbol{Y} \in \mathfrak{C}$

$$\mathrm{Mor}_{\mathfrak{C}}(E,\boldsymbol{Y})^{\mathcal{R}_{\mathcal{A}}} := \left\{ f \in \mathrm{Mor}_{\mathfrak{C}}(E;\boldsymbol{Y}) \;\middle|\; \begin{array}{c} f \text{ and satisfies } (\mathcal{A}\text{-}2) \text{ and} \\ \text{is compatible with } \mathcal{R}_{\mathcal{A}} \end{array} \right\},$$

one has

$$\nu^*(\mathrm{Mor}_{\mathfrak{C}}(\mathcal{E},\boldsymbol{Y})) = \mathrm{Mor}_{\mathfrak{C}}(E,\boldsymbol{Y})^{\mathcal{R}_{\mathcal{A}}}, \text{ where } \nu^*(f) := f \circ \nu. \tag{16}$$

The next theorem is the exact analog of theorem 4.1.1 for quotient spaces. The proof is essentially the same and is left to the reader.

**5.3.2 Theorem.** *Let $\mathcal{A} = \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ be an atlas in a category of spaces $\mathfrak{C}$.*

a) Uniqueness. *Assemblages of $\mathcal{A}$ in $\mathfrak{C}$, if they exist, are unique up to canonical isomorphisms. Because of this, the assemblage is denoted by:*

$$\nu_{\mathcal{A}} : E(\mathcal{A}) \to \mathcal{E}(\mathcal{A}),$$

*where $E(\mathcal{A}) := \coprod_i E_i$ and $\nu_{\mathcal{A}}$ is the identification map.*

b) *If $\nu_{\mathcal{A}} : E(\mathcal{A}) \twoheadrightarrow \mathcal{E}(\mathcal{A})$ is the assemblage of $\mathcal{A}$ in $\mathfrak{C}$, then, a family of morphisms $f : E(\mathcal{A}) \to \boldsymbol{Y}$, satisfying ($\mathcal{A}$-2), is compatible with $\mathcal{R}_{\mathcal{A}}$ if and only if it is compatible with the equivalence $\mathcal{R}_{\nu_{\mathcal{A}}}$. One has $\mathcal{R}_{\mathcal{A}} \subseteq \mathcal{R}_{\nu_{\mathcal{A}}}$, and a surjective map:*

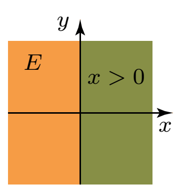
$$E(\mathcal{A})/\mathcal{R}_{\mathcal{A}} \twoheadrightarrow \sigma\mathcal{E}(\mathcal{A}) = E(\mathcal{A})/\mathcal{R}_{\nu_{\mathcal{A}}}.$$

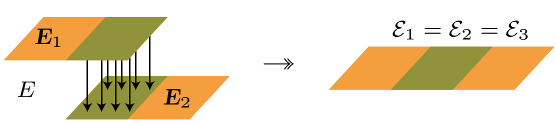
**5.3.3 Comments.** 1) Beyond the comments 4.1.2, which also apply to assemblages of spaces, it should be mentioned that, unlike quotients, some categories can have assemblages that are not unique and so do not comply fully with the categorical definition. An example of this is the category $\mathbf{Met}_{\text{iso}}$ of metric spaces and global isometries. If $\boldsymbol{X} := (X, d_{\boldsymbol{X}})$ and $\boldsymbol{Y} := (Y, d_{\boldsymbol{Y}})$ are metric spaces, let $Z := X \sqcup Y$. For any real number $\epsilon > 0$, the map $d_\epsilon : Z \times Z \to I\!\!R_+$:

$$\begin{cases} d_\epsilon(x,y) := d_X(x,y), \text{ if } x,y \in X, \\ d_\epsilon(x,y) := d_Y(x,y), \text{ if } x,y \in Y, \\ d_\epsilon(x,y) := \epsilon, \text{ otherwise,} \end{cases}$$

is a distance extending those of $\boldsymbol{X}$ and $\boldsymbol{Y}$. Thus, when the glueing rules are void, the metric space $(Z, d_\epsilon)$ is an assemblage of $\boldsymbol{X}$ and $\boldsymbol{Y}$, however as soon as $\epsilon_1 \neq \epsilon_2$, the spaces $(Z, d_{\epsilon_1})$ and $(Z, d_{\epsilon_1})$ are no longer isomorphic.

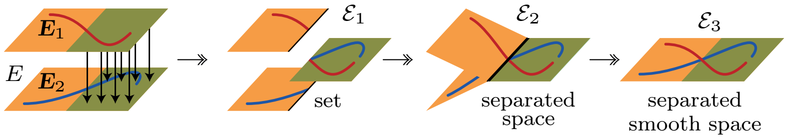
2) Let $E$ be the square $]-1,1[^2 \subseteq I\!\!R^2$. The green area in the figure to the right, indicates the *open* half ($x > 0$). The set $E$ will be successively considered as belonging to the categories **Set**, $\mathbf{Top}_{\text{sep}}$ of separated spaces, and **Man** of smooth separated spaces. We will be more explicit about these categories later, but for now we want only to illustrate the differences that might appear when assembling spaces in different categories. In the case of the rule that glues $(x,y) \in \boldsymbol{E}_1$ to $(1-x,y) \in \boldsymbol{E}_2$, for $x > 0$, as in the figure,



the assemblages: $\mathcal{E}_1$ as a set, $\mathcal{E}_2$ as a separated space and $\mathcal{E}_3$ as a smooth space, all have the same underlying set, *i.e.* $\sigma\mathcal{E}_1 = \sigma\mathcal{E}_2 = \sigma\mathcal{E}_3$.

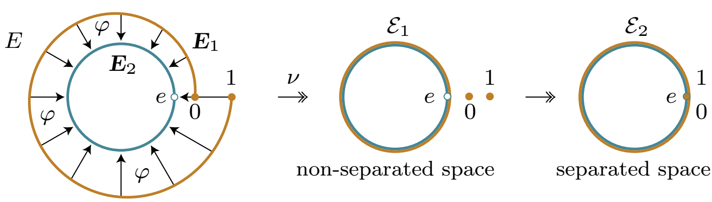
In the case of the rule that glues $(x,y) \in \boldsymbol{E}_1$ to $(x,y) \in \boldsymbol{E}_2$, for all $x > 0$,




the assemblage $\mathcal{E}_1$ is also the underlying set of a *topological quotient*, but it is not a *separated* space (*cf.* **3♦**9). In it, the blue and red curves *do not* intersect, even though their limits do meet while $x$ tends to 0 in the green area. In the category of *separated* spaces, the assemblage is the space $\mathcal{E}_2$, where the identification map also joins together the points on the edge $(x = 0)$. Thus, in this space, the curves do intersect, but the space is no longer smooth. We will see that to get an assemblage that is both separated and smooth the identification map has to identify $(x,y) \in \boldsymbol{E}_1$ with $(x,y) \in \boldsymbol{E}_2$, for *all* $x \in [-1,1]$, in this case resulting in the space $\mathcal{E}_3$.

3) In the statement 5.3.2-(b), one must bear in mind that enlarging identification rules is often necessary to glue structures and that this may compromise the principal feature of an assemblage: that of preserving the structure of each individual component.

To illustrate, consider the closed real segment $\boldsymbol{E}_1 = [0,1]$ (in brown) wrapped around the circle $\boldsymbol{E}_2 := \mathbb{S}^1$ (in blue) in such a way that its endpoints lie on the same line passing through the center of $\boldsymbol{E}_2$, as shown in the image below. For $x \in \boldsymbol{E}_1$, denote by $\varphi(x) \in \boldsymbol{E}_2$ the central projection of $x$ on $\boldsymbol{E}_2$. Denote $e := \varphi(0) = \varphi(1)$. Let the glueing rule be to join together $x \in \boldsymbol{E}_1$ and $\varphi(x) \in \boldsymbol{E}_2$, but only for the *inner* points of $\boldsymbol{E}_1$, *i.e.* for $0 < x < 1$.




Let $\nu : E \twoheadrightarrow \mathcal{E}_1$ denote the identification map of the assemblage of the $\boldsymbol{E}_i$'s in the category of sets. We can see that the three points $\nu(0)$, $\nu(1)$ and $\nu(e)$ are pairwise different *but* infinitely close to each other. Indeed, the inner points of $\boldsymbol{E}_1$ close to 0 or to 1 are sent by $\varphi$ to points close to $e \in \boldsymbol{E}_2$, so that in $\mathcal{E}_1$ the three points are as close to each other as desired. This reflects the fact that $\mathcal{E}_1$, which is also, as we shall see, the topological assemblage of the $\boldsymbol{E}_i$'s, is a *non-separated* space. The separated *quotient* of $E$ is the space $\mathcal{E}_2$. There, the three points in question become one and the same, in which case $\nu_1 : \boldsymbol{E}_1 \to \mathcal{E}_2$ is no longer injective. As a consequence, the $\boldsymbol{E}_i$'s cannot be assembled, by the glueing rules considered, in the category of separated spaces (*cf.* theorem 6.4.1).

4) The point in these examples is that the more structure one needs to glue, the more identifications of points may be required. These extra identifications are not implicit in the glueing rules, they come *a posteriori* as constraints imposed by the structure !

# 6 Assemblages in Categories with Coproducts

**6.1 Products and Coproducts in Categories.** In the presentations of assemblages 1.2 and 5.3, the disjoint union $E := \sqcup_{i\in\mathfrak{I}} E_i$ was a convenient placeholder for the families of morphisms $\{f_i : \boldsymbol{E}_i \to \boldsymbol{Y}\}_{i\in\mathfrak{I}}$ (*cf.* 5.3-(14)), but being merely a *set* it is rather useless. The question arises of whether such an object may have a categorical definition. The answer is given by the concept of *coproduct* or *categorical sum*. In this, the prefix 'co' comes from the fact that, in category theory, the *coproduct* is dual to the *product*, which means that the diagrams in their definitions are the same except that arrows are reversed. For this reason, while the product is denoted by '$\prod$', the coproduct is denoted by '$\coprod$'. This is also why, before addressing the categorical concept of *coproduct*, it will be instructive to first recall that of *product*.

**6.2 Products.** In a category $\mathfrak{C}$, a *product* of a family of objects $\{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ is an object $\boldsymbol{P} \in \mathfrak{C}$ *and* a family of morphisms $\{\pi_i : \boldsymbol{P} \to \boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ satisfying the following universal property.

*Given a family of morphisms* $\{f_i : \boldsymbol{Y} \to \boldsymbol{E}_i\}_{i\in\mathfrak{I}}$, *there exists a unique morphism* $f : \boldsymbol{Y} \to \boldsymbol{P}$ *such that* $f_i = \pi_i \circ f$, *for all* $i \in \mathfrak{I}$, i.e. *rendering the diagram to the right commutative.* In other words, the correspondence $f \mapsto \pi_i \circ f$ identifies:

$$\begin{tikzcd} & & \boldsymbol{E}_i \\ \boldsymbol{Y} \arrow[urr, "f_i"] \arrow[drr, "f_j"'] \arrow[r, dotted, "\exists! f"] & \boldsymbol{P} \arrow[ur, "\pi_i"] \arrow[dr, "\pi_j"'] & \\ & & \boldsymbol{E}_j \end{tikzcd}$$

$$\mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{Y},\boldsymbol{P}) = \prod\nolimits_{i\in\mathfrak{I}} \mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{Y},\boldsymbol{E}_i). \tag{17}$$

**6.2.1 Proposition.** *Let* $\mathcal{F} := \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ *be a family of objects in* $\mathfrak{C}$ *for which there exists a product* $\{\pi_i : \boldsymbol{P} \to \boldsymbol{E}_i\}_{i\in\mathfrak{I}}$.

a) *If* $\{\pi'_i : \boldsymbol{P}' \to \boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ *is another product for the family* $\mathcal{F}$, *there exists one and only one isomorphism* $\varphi : \boldsymbol{P} \to \boldsymbol{P}'$ *such that* $\pi'_i = \varphi \circ \pi_i$ *for all* $i \in \mathfrak{I}$. *In other words, products, when they exist, are unique up to canonical isomorphism, in which case, they are denoted by :*

$$\pi_i : \prod \mathcal{F} \to \boldsymbol{E}_i.$$

b) *Products exist in the category of sets.*

**Definition.** The category $\mathfrak{C}$ is *a category with (finite) products* if for each (finite) family of objets $\mathcal{F} := \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$, a product $\prod\mathcal{F}$ exists in $\mathfrak{C}$.

*Proof.* (a) Left to the reader. (b) In the Zermelo-Fraenkel set theory (which is assumed) the *axiom of union* warrants the existence of the *disjoint union* $\bigsqcup\mathcal{F}$, after which the *cartesian product* $\bigtimes\mathcal{F}$ is well defined as the set of all maps $\varphi : \mathfrak{I} \to \bigsqcup\mathcal{F}$ such that $\varphi(i) \in \boldsymbol{E}_i$, for all $i \in \mathfrak{I}$. Let $\pi_i : \bigtimes\mathcal{F} \to \boldsymbol{E}_i$ be the *canonical projection* map $\pi_i : \varphi \mapsto \varphi(i)$. Now, if a family of maps $\{f_i : \boldsymbol{Y} \to \boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ is given, the map $f : \boldsymbol{Y} \to \bigtimes\mathcal{F}$, $y \mapsto (i \mapsto f_i(y))$ is well defined and verifies $\pi_i(f(y)) = f_i(y)$, for all $y \in \boldsymbol{Y}$, which is the universal property of a categorical product for the family $\{\pi_i : \bigtimes\mathcal{F} \to \boldsymbol{E}_i\}_{i\in\mathfrak{I}}$. $\square$

**6.2.2 Comments.** The existence of products of sets is independent of the *axiom of choice*. Indeed, this axiom is equivalent to the nonemptyness of the cartesian product of infinite families of nonempty sets, something we do not need to assume, as it is implicit in the universal property of products.

In many useful categories infinite products may not exist, such as the categories of metric spaces **Met** and manifolds **Man** in Table .1.

We now introduce the categorical concept of *coproducts* by dualizing the previous definition and proposition.

**6.3 Coproducts.** In a category $\mathfrak{C}$, a *coproduct*, or *categorical sum*, of a family of objects $\{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$, is an object $\boldsymbol{E} \in \mathfrak{C}$ *and* a family of morphisms $\{\iota_i : \boldsymbol{E}_i \to \boldsymbol{E}\}_{i\in\mathfrak{I}}$ satisfying the following universal property.

*Given a family of morphisms $f_i : \boldsymbol{E}_i \to \boldsymbol{Y}$, there exists a unique morphism $f : \boldsymbol{E} \to \boldsymbol{Y}$ such that $f \circ \iota_i = f_i$, for all $i \in \mathfrak{I}$,* i.e. *rendering the following diagram commutative.* In other words, the correspondence $f \mapsto f \circ \iota_i$ identifies:

$$\begin{tikzcd}
\boldsymbol{E}_i \arrow[dr, "\iota_i"] \arrow[drr, "f_i"] & & \\
& \boldsymbol{E} \arrow[r, dotted, "\exists! f"] & \boldsymbol{Y} \\
\boldsymbol{E}_j \arrow[ur, "\iota_j"] \arrow[urr, "f_j"'] & &
\end{tikzcd}$$

$$\mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{E}, \boldsymbol{Y}) = \prod\nolimits_{i\in\mathfrak{I}} \mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{E}_i, \boldsymbol{Y}). \tag{18}$$

**6.3.1 Proposition.** *Let $\mathcal{F} := \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ be a family of objects in $\mathfrak{C}$ for which there exists a coproduct $\iota_i : \boldsymbol{E}_i \to \boldsymbol{E}$.*

a) *If $\iota_i' : \boldsymbol{E}_i \to \boldsymbol{E}'$ is another coproduct for $\mathcal{F}$, there exists one and only one isomorphism $\varphi : \boldsymbol{E} \to \boldsymbol{E}'$ such that $\iota_i' = \varphi\circ\iota_i$ for all $i \in \mathfrak{I}$. In other words, coproducts, when they exist, are unique up to canonical isomorphism, in which case they are denoted by :*

$$\iota_i : \boldsymbol{E}_i \to \coprod\mathcal{F}.$$

b) *Coproducts exist in the category of sets.*

**Terminology.** The category $\mathfrak{C}$ is *a category with (finite) coproducts* if for each (finite) family of objets $\mathcal{F} := \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$, a coproduct $\coprod\mathcal{F}$ exists in $\mathfrak{C}$.

*Proof.* (a) Left to the reader. (b) The disjoint union $\bigsqcup\mathcal{F}$ and the family of *canonical injections* or *inclusion maps* $\{\iota_i : \boldsymbol{E}_i \hookrightarrow \bigsqcup\mathcal{F}\}_{i\in\mathfrak{I}}$ is a coproduct. $\square$

**6.3.2 Exercise.** Show that if $\mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{E}_i,\boldsymbol{E}_j) \neq \emptyset$, for all $i,j \in \mathfrak{I}$, then each $\iota_i : \boldsymbol{E}_i \to \coprod_{i\in\mathfrak{I}} \boldsymbol{E}_i$ admits a left inverse and is a monomorphism.

**6.4 Assemblages as Quotient Spaces.** We now may state a theorem showing that assemblages are always quotients in categories with *coproducts* and *quotients*. The main difference with theorem 5.3.2 is that the *set* $E(\mathcal{A}) := \bigsqcup_i \boldsymbol{E}_i$ has become the *space* $\boldsymbol{E}(\mathcal{A}) := \coprod_i \boldsymbol{E}_i$.

**6.4.1 Theorem.** *Let $\mathfrak{C}$ be a category of spaces with coproducts and quotients. Let $\mathcal{A}$ be an atlas in $\mathfrak{C}$. The following are equivalent statements in $\mathfrak{C}$.*

- *An assemblage $\nu_{\mathcal{A}} : \boldsymbol{E}(\mathcal{A}) \to \mathcal{E}(\mathcal{A})$ of $\mathcal{A}$ exists.*
- *The quotient $\nu_{\mathcal{R}_{\mathcal{A}}} : \boldsymbol{E}(\mathcal{A}) \to \boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}}$ is the assemblage of $\mathcal{A}$.*
- *The $\boldsymbol{E}_i$'s are subspaces (through $\nu_{\mathcal{R}_{\mathcal{A}}}$) of the quotient space $\boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}}$.*

*In such cases, for all $\boldsymbol{Y} \in \mathfrak{C}$, the pullback*

$$\nu_{\mathcal{A}}^* : \mathrm{Mor}_{\mathfrak{C}}(\mathcal{E}(\mathcal{A}),\boldsymbol{Y}) \to \mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{E}(\mathcal{A}),\boldsymbol{Y}), \qquad f \mapsto f \circ \nu_{\mathcal{A}},$$

*is injective with image*

$$\boxed{\nu_{\mathcal{A}}^*(\mathrm{Mor}_{\mathfrak{C}}(\mathcal{E}(\mathcal{A}),\boldsymbol{Y})) = \mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{E}(\mathcal{A}),\boldsymbol{Y})^{\mathcal{R}_{\mathcal{A}}}}$$

*where*

$$\mathrm{Mor}_{\mathfrak{C}}(\boldsymbol{E}(\mathcal{A}),\boldsymbol{Y})^{\mathcal{R}_{\mathcal{A}}} := \Big\{ f : \boldsymbol{E} \to \boldsymbol{Y} \;\Big|\; (\forall i,j \in \mathfrak{I})\ f_i|_{\boldsymbol{E}_i^j} = f_j|_{\boldsymbol{E}_j^i} \circ \varphi_{j,i} \Big\}, \quad (19)$$

*where $f_i : \boldsymbol{E}_i \to \boldsymbol{Y}$ is the restriction of $f : \boldsymbol{E}(\mathcal{A}) \to \boldsymbol{Y}$ to $\boldsymbol{E}_i$.*

*Proof.* The map $\nu_{\mathcal{A}} : \boldsymbol{E}(\mathcal{A}) \to \mathcal{E}(\mathcal{A})$, being $\mathcal{R}_{\mathcal{A}}$ compatible, factors through the quotient in a map $\phi : \boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}} \to \mathcal{E}(\mathcal{A})$, which is surjective, because $\nu_{\mathcal{A}}$ is so. We have the commutative diagram of surjections:

$$\begin{tikzcd}
 & \boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}} \arrow[d, two heads, "\phi"] \\
\boldsymbol{E}(\mathcal{A}) \arrow[ur, two heads, "\nu_{\mathcal{R}_{\mathcal{A}}}"] \arrow[r, two heads, "\nu_{\mathcal{A}}"'] & \mathcal{E}(\mathcal{A})
\end{tikzcd}$$

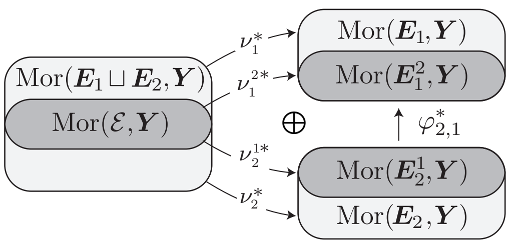
If we restrict this diagram to $\boldsymbol{E}_i$, we get the commutative sub-diagram of *bijective* maps (I), where $F_i := \nu_{\mathcal{R}_{\mathcal{A}}}(E_i)$.

$$\begin{tikzcd}
 & & F_i \arrow[dd, tail, two heads, "\phi"] \arrow[r, phantom, "\subseteq"] & \boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}} \arrow[dd, two heads] \\
\boldsymbol{Z} \arrow[r, dotted, "f'"] \arrow[urr, bend left, tail, "f"] & \boldsymbol{E}_i \arrow[ur, two heads, "\nu_{\mathcal{R}_{\mathcal{A}}}"] \arrow[dr, two heads, "\nu_{\mathcal{A}}"'] \arrow[r, phantom, "(\mathrm{I})"] & {} & \\
 & & \tilde{\boldsymbol{E}}_i \arrow[r, phantom, "\subseteq"] & \mathcal{E}(\mathcal{A})
\end{tikzcd}$$

Now, if the image of the map $f : \boldsymbol{Z} \to \boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}}$ is contained in $F_i$, the image of $\phi \circ f$ is contained in $\tilde{E}_i$, and, because $\nu_{\mathcal{A}} : \boldsymbol{E}_i \to \mathcal{E}(\mathcal{A})$ is a subspace, there exists a unique $f' : \boldsymbol{Z} \to \boldsymbol{E}_i$ rendering the diagram commutative.

The equality $f = \nu_{\mathcal{R}_{\mathcal{A}}} \circ f'$ follows, after 2.4.1-(1), by its validity at the *set* level, since $\phi \circ f = \nu_{\mathcal{A}} \circ f' = \phi \circ \nu_{\mathcal{R}_{\mathcal{A}}} \circ f'$, and that $\phi : F_i \to \mathcal{E}(\mathcal{A})$ is injective. The condition 5.3.1-($\mathcal{A}$-1) for assemblage is thus proved for $\nu_{\mathcal{R}_{\mathcal{A}}}$, and ($\mathcal{A}$-2) is automatic for a quotient. The quotient space $\boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}}$ is therefore an assemblage of $\mathcal{A}$. The rest of the theorem is left to the reader. $\square$

**6.4.2 Comment.** It is instructive to observe that the action of *disassembling morphisms* in $\mathrm{Mor}_{\mathfrak{C}}(\mathcal{E},\boldsymbol{Y})$, *i.e.* the correspondence $f \mapsto \{f_i\}_{i\in\mathfrak{I}}$, is *dual* to that of *assembling spaces*. To be convinced of this, one needs only consider the following diagram which schematizes the elements in the last theorem in the case of the family $\{\boldsymbol{E}_1,\boldsymbol{E}_2\}$. The reader will recognize the diagram (12), here with the arrows reversed representing dual, or *pullback*, maps ([7])




(20)

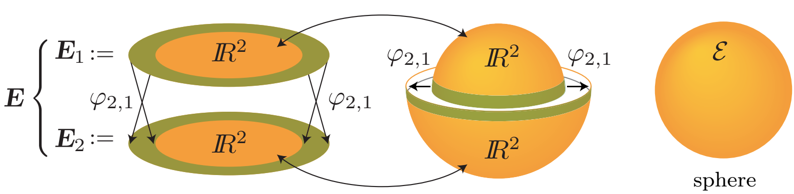
The commutativity of the diagram is due to the fact that one has the equality

$$(\nu_i^j)^* = (\varphi_{j,i})^* \circ (\nu_j^i)^* = (\nu_j^i \circ \varphi_{i,j})^*,$$

which is dual to the equality $\nu_i^j = \nu_j^i \circ \varphi_{i,j}$, coming from formula (11).

**6.4.3 Comment.** As a rationale for the next chapters and to show how useful the categorical concept of assemblages is, let us show how it allows manifolds, flat Riemannian surfaces and Buser surfaces to be defined.

$\mathcal{A}$-1) *The Real Manifolds of dimension $n$* are the assemblages defined by atlases $\mathcal{A}$ where the charts are copies $\boldsymbol{E}_i$ of the real affine space $\mathbb{R}^n$, endowed with its canonical structure of *differentiable* space, and where the transition morphisms $\varphi_{j,i} : \boldsymbol{E}_i^j \to \boldsymbol{E}_j^i$ are *differentiable isomorphisms*, also called *diffeomorphisms*, between open subspaces.

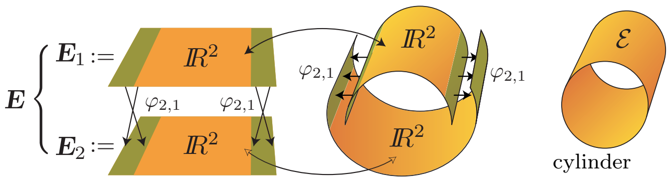



([7]) The *pullback* of $\alpha : \boldsymbol{X} \to \boldsymbol{Y}$, is the map $\alpha^* : \mathrm{Mor}(\boldsymbol{Y},\boldsymbol{Z}) \to \mathrm{Mor}(\boldsymbol{X},\boldsymbol{Z})$, $f \mapsto f \circ \alpha$.

$\mathcal{A}$-2) *The Flat Riemannian Manifolds of dimension $n$* are the assemblages defined by atlases $\mathcal{A}$ where the charts are copies $\boldsymbol{E}_i$ of the real affine *Euclidean* space $I\!R^n$, and where the transition morphisms $\varphi_{j,i} : \boldsymbol{E}_i^j \to \boldsymbol{E}_j^i$ are *local isometries* between open subspaces.

Notice that the sphere is not an a *flat* Riemannian space. The transition map $\varphi_{2,1} : I\!R^2 \to I\!R^2$ does not preserve the Euclidean distance, as it sends circles near infinity to circles near a circle at finite distance.

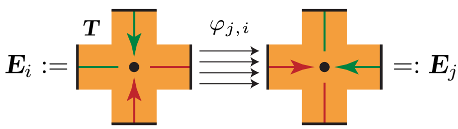
A classic example of a flat Riemannian manifold is the cylinder:




In this assemblage, the transition map preserves the Euclidean distance.

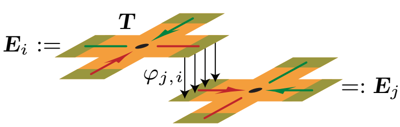
$\mathcal{A}$-3) *The Buser surfaces* are the assemblages defined by atlases $\mathcal{A} = \{\mathcal{E}_i\}_{i\in\mathfrak{I}}$ where the charts are copies $\boldsymbol{E}_i$ of some tile $\boldsymbol{T}$ endowed with the metric induced by the Euclidean space $I\!R^2$, and where the transition morphisms $\varphi_{j,i} : \boldsymbol{E}_i^j \to \boldsymbol{E}_j^i$ are *isometries*:

- between *overlapping edges* of $\boldsymbol{T}$ (in black in the figure).

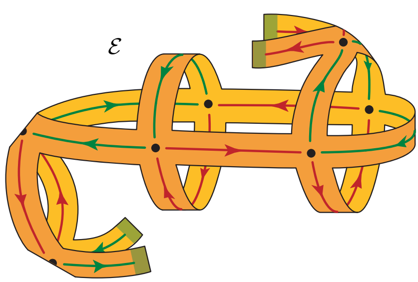



  when simply glueing topologies,

- or between *overlapping surfaces* (in green in the figure)




  when glueing differentiable structures and Laplace operators.




A Buser surface

## Chapter Contents

# Chapter 3

# Quotients and Assemblages of Topological Spaces

In this chapter, we recall concepts and terminology of *general topology*, with the aim of putting them within the framework of category theory as introduced in chapter **2**. Our emphasis will be on clearly defining the concepts of *products*, *coproducts*, *subspaces*, *quotients* and *assemblages* of topological spaces following the lines of the categorical definitions given in that chapter. In the interests of self-sufficiency, we will need to recall, from time to time, elementary facts in topology. Many readers will of course already be familiar with this material, so we ask their forebearance.

## 1 Topological Spaces

**1.1 Topology.** A *topological space* is a couple $\boldsymbol{X} := (X, \mathcal{T}_{\mathbf{X}})$ where $X$ is a set and $\mathcal{T}_{\mathbf{X}}$ is a *topology on $X$*, *i.e.* a collection of subsets of $X$, called the *open subsets of the topology*, that satisfy the following conditions:
$T$-1) $X$ and $\emptyset$ belong to $\mathcal{T}_{\mathbf{X}}$.
$T$-2) The union of any family of elements in $\mathcal{T}_{\mathbf{X}}$ belongs to $\mathcal{T}_{\mathbf{X}}$.
$T$-3) The intersection of any *finite* family of elements in $\mathcal{T}_{\mathbf{X}}$ belongs to $\mathcal{T}_{\mathbf{X}}$.

### 1.1.1. Some Terminology

- *Neighborhoods.* Given $A \subseteq X$, a subset $V_x \subseteq X$ is a *neighborhood* of $A$ if there exists an open subset $U \in \mathcal{T}_{\mathbf{X}}$ such that $A \subseteq U \subseteq V_x$. A subset is open in $\boldsymbol{X}$ if and only if it is a neighborhood of all of its points (exercise !).
- *Closed Subsets.* The complement in $X$ of an open subset is *a closed subset*. As a topology $\mathcal{T}_{\mathbf{X}}$ is clearly equivalent to the collection of its closed subsets, any collection $\mathcal{T}'_{\mathbf{X}}$ of subsets of $\boldsymbol{X}$ satisfying the following complementary conditions to the (T-*)'s, is the collection of closed subsets of a topology:
  $T'$-1) $\emptyset$ and $\boldsymbol{X}$ belong to $\mathcal{T}'_{\mathbf{X}}$.
  $T'$-2) The *intersection* of any family of elements in $\mathcal{T}'_{\mathbf{X}}$ belongs to $\mathcal{T}'_{\mathbf{X}}$.
  $T'$-3) The *union* of any *finite* family of elements in $\mathcal{T}'_{\mathbf{X}}$ belongs to $\mathcal{T}'_{\mathbf{X}}$.

- *Interior and Closure of a Subset.* Let $A$ be a subset of $\boldsymbol{X}$.
  - The union of all the open subsets $V \in \mathcal{T}_{\mathbf{X}}$ such that $V \subseteq A$ is an open subset of $\boldsymbol{X}$ after ($T$-2). It is the biggest open subset within $A$. It is called *the interior of $A$* and is denoted by $A^\circ$. Clearly, the subset $A$ is open in $\boldsymbol{X}$ if and only if $A = A^\circ$.
  - The intersection of all the closed subsets $F \in \mathcal{T}'_{\mathbf{X}}$ such that $A \subseteq F$ is a closed subset of $\boldsymbol{X}$ after ($T'$-2). It is the smallest closed subset containing $A$ and is called *the closure of $A$*, denoted by $\overline{A}$. Clearly, $A$ is closed if and only if $A = \overline{A}$.

**Convention.** When the topology of $\boldsymbol{X} := (X, \mathcal{T}_{\mathbf{X}})$ is understood, the term '$\mathcal{T}_{\mathbf{X}}$' will be omitted and we will say simply "*the (topological) space $\boldsymbol{X}$*".

**1.2 Generated Topology.** Given a set $X$, let $(\mathsf{Top}(X), \subseteq)$ be the set of all possible topologies on $X$, partially ordered by the inclusion relation.

If $\mathcal{T}_1, \mathcal{T}_2 \in \mathsf{Top}(X)$ verify $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$, or that $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$. The finer the topology is, the more open subsets there are, the better the topology discerns points. At the extremes we have:

- *The Coarse Topology.* This is the coarsest possible topology, only $\emptyset$ and $X$ are open. This topology never *separates* distinct elements.
- *The Discrete Topology.* The finest topology, every subset of $X$ is an open subset. This topology is able to *separate* (1) disjoint subsets.

It is straightforward to verify that the set $\mathsf{Top}(X)$ is stable under the intersection operation, *i.e.* for any family $\{\mathcal{T}_i\}_{i \in \mathfrak{I}}$ of topologies on $X$, the intersection $\bigcap_{i \in \mathfrak{I}} \mathcal{T}_i$ is again a topology on $X$. We can thus give a meaning to the *topology generated by a family of subsets $\mathcal{S} \subseteq \mathscr{P}(X)$* It will be the coarsest topology $\mathcal{T}\langle\mathcal{S}\rangle$ containing $\mathcal{S}$, *i.e.* the topology:

$$\mathcal{T}\langle\mathcal{S}\rangle := \bigcap \big\{ \mathcal{U} \in \mathsf{Top}(X) \;\big|\; \mathcal{U} \supseteq \mathcal{S} \big\}.$$

**1.2.1 Proposition.** *Given $\mathcal{S} \subseteq \mathscr{P}(X)$, the open sets in $\mathcal{T}\langle\mathcal{S}\rangle$ are the arbitrary unions of finite intersections of elements in $\mathcal{S}$.*

*Proof.* Left to the reader. □

**1.2.2 Exercise.** Let $(A, \leq)$ be a totally ordered set, for example any subset of the set of real numbers endowed with the natural order $(\mathbb{R}, \leq)$. For $a, b \in A$, there are four families of different types of intervals:

$\mathcal{C}(\mathcal{A})$: the family of *closed* intervals $[a,b] := \{x \in A \mid a \leq x \leq b\}$,
$\mathcal{CO}(A)$: left-closed intervals $[a,b) := \{x \in A \mid a \leq x < b\}$, and
$\mathcal{OC}(A)$: right-closed intervals $(a,b] := \{x \in A \mid a < x \leq b\}$.
$\mathcal{O}(\mathcal{A})$: the family of *open* intervals $(a,b) := \{x \in A \mid a < x < b\}$,

(1) This means that, given two disjoint subsets $A, B \subseteq \boldsymbol{X}$, there exist disjoint open subsets $V, W \in \mathcal{T}_{\mathbf{X}}$, s.t. $A \subseteq V$ and $B \subseteq W$.

Show that these families are stable under finite intersections, in other words: the intersection of a finite number of intervals is still an interval of the same type. Using 1.2.1, describe the topologies that these families generate and denote them respectively by $\mathcal{T}_{[]}$, $\mathcal{T}_{[)}$, $\mathcal{T}_{(]}$ and $\mathcal{T}_{()}$.

i) Show that $\mathcal{T}_{[]}$ is the discrete topology.

ii) Show that $\mathcal{T}_{()}$ is the coarsest of the four and that every *singleton* $\{a\} \subseteq A$ is a closed set for $\mathcal{T}_{()}$.

iii) The topology $\mathcal{T}_{()}$ is called *the order topology*.

The standard topologies on the sets of real numbers $\mathbb{R}$, rational numbers $\mathbb{Q}$, integers $\mathbb{Z}$, and natural numbers $\mathbb{N}$ are the order topologies. *In the sequel, these sets will be assumed to be endowed with this topology, unless otherwise specified.*

**1.3 Pullback Topology.** A map between sets $f : X \to Y$, induces two maps between the power sets $\mathscr{P}(X)$ and $\mathscr{P}(Y)$:

- The *direct image* $f : \mathscr{P}(X) \to \mathscr{P}(Y)$, associates with $S \subseteq X$, the subset
$$f(S) := \{y \in Y \mid \exists s \in S \text{ s.t. } y = f(\boldsymbol{s})\} \subseteq Y .$$
The subset $f(X)$ is called *the image of $f$*, and is usually denoted by $\mathrm{im}(f)$.
- The *inverse image* $f^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X)$, associates with $S \subseteq Y$ the subset
$$f^{-1}(S) := \big\{x \in X \bigm| f(x) \in S\big\} \subseteq X .$$

**1.3.1 Lemma.** a) *The direct image is compatible with the inclusion order and with union of families, but not necessarily with the operations of intersection and of complement. We have*

- $(S \subseteq S') \Rightarrow (f(S) \subseteq f(S'))$;
- $f(\bigcup_{i\in\mathfrak{I}} S_i) = \bigcup_{i\in\mathfrak{I}} f(S_i)$;
- $f(\bigcap_{i\in\mathfrak{I}} S_i) \subseteq \bigcap_{i\in\mathfrak{I}} f(S_i)$.

b) *The inverse image is compatible with the inclusion order and with all the operations on subsets, it is a homomorphism of Boolean algebras*
$$f^{-1} : \big(\mathscr{P}(Y), \emptyset, Y, \cap, \cup, \complement_Y, \subseteq\big) \to \big(\mathscr{P}(X), \emptyset, X, \cap, \cup, \complement_X, \subseteq\big)$$
*moreover, it is an injection if and only if $f$ is a surjective map.*

The following proposition is an immediate corollary of 1.3.1-(b).

**1.3.2 Proposition (Pullback Topology).** *Given $f : X \to \mathbf{Y} := (Y, \mathcal{T}_{\mathbf{Y}})$, the set $f^{-1}(\mathcal{T}_{\mathbf{Y}})$ is a topology on $X$, it is the pullback topology on $X$ by $f$.*

*If $X$ is a subset of $\mathbf{Y}$ and $\iota : X \hookrightarrow \mathbf{Y}$ is the inclusion map, the pullback topology by $\iota$ is the induced topology by $\mathbf{Y}$. In this, the open (resp. closed) sets are the traces $X \cap U$, of open (resp. closed) subsets $U \subseteq \mathbf{Y}$.*

**1.3.3 Exercises.** Let $W$ be a subset of $\boldsymbol{X} := (X,\mathcal{T}_{\mathbf{X}})$ and denote by $\boldsymbol{W}$ the set $W$ endowed with the induced topology by $\boldsymbol{X}$.

1) If $W$ is an open (resp. closed) subset of $\boldsymbol{X}$, then $U \subseteq \boldsymbol{W}$ is open (resp. closed) if and only if $U$ is open (resp. closed) in $\boldsymbol{X}$.
2) Show that the induced topologies on $Z \subseteq W$ by $\boldsymbol{W}$ and $\boldsymbol{X}$ coincide.

**1.4 Continuity of Maps.** A map $f : \boldsymbol{X} \to \boldsymbol{Y}$ between topological spaces is said to be *continuous* if $f^{-1}(\mathcal{T}_{\mathbf{Y}}) \subseteq \mathcal{T}_{\mathbf{X}}$, *i.e.* if for every open (resp. closed) subset $U \subseteq \boldsymbol{Y}$, the *inverse image* $f^{-1}(U)$ is an open (resp. closed) subset of $\boldsymbol{X}$. We will denote by $\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$ the set of those maps.

**1.4.1 Proposition.** *Let $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ be topological spaces. The identity map $\mathrm{id}_{\mathbf{X}} : \boldsymbol{X} \to \boldsymbol{X}$, $x \mapsto x$, and the composition $g \circ f : \boldsymbol{X} \to \boldsymbol{Z}$, $x \mapsto g(f(x))$, of continuous maps $f : \boldsymbol{X} \to \boldsymbol{Y}$ and $g : \boldsymbol{Y} \to \boldsymbol{Z}$, are continuous maps.*

**1.4.2 Exercises.** 1) A map $f : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *continuous at $x \in \boldsymbol{X}$* if for every neighborhood $V_{f(x)}$ of $f(x)$, the inverse image $f^{-1}(V_{f(x)})$ is a neighborhood of $x$. Show that $f$ is continuous if and only if it is continuous at all the points in $\boldsymbol{X}$.

2) A map $c : \boldsymbol{X} \to \boldsymbol{Y}$ is *constant* if $c(x) = c(x')$, for all $x,x' \in \boldsymbol{X}$. Show that constant maps between topological spaces are always continuous. Conclude that if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are nonempty, then $\mathcal{C}(\boldsymbol{X},\boldsymbol{Y}) \neq \emptyset$.

3) Given $f : X \to \boldsymbol{Y} := (Y,\mathcal{T}_{\mathbf{Y}})$, show that $f^{-1}(\mathcal{T}_{\mathbf{Y}})$ is the coarsest topology on $\boldsymbol{X}$ for which $f$ is continuous.

4) Endow $\boldsymbol{S} := \{0,1\}$ with the discrete topology. Let $f : \boldsymbol{X} \to \boldsymbol{S}$ be a continuous map. Show that if $f$ is constant on a subset $A \subseteq \boldsymbol{X}$, it is also constant on its closure $\overline{A}$. Discuss the case when $\boldsymbol{S}$ is endowed with the coarse topology, and when it is the Sierpinski space (*cf.* **4**♦2.2.6-(1)).

5) Let $\boldsymbol{Y}$ be a space whose topology is not the coarse topology. Show that $\mathcal{C}(\boldsymbol{X},\boldsymbol{Y}) = \mathrm{Maps}(\boldsymbol{X},\boldsymbol{Y})$ if and only if $\boldsymbol{X}$ is a *discrete* space.

6) Given $f : X \to Y$ and $\mathcal{S} \subseteq \mathscr{P}(Y)$, show that $f^{-1}(\mathcal{T}\langle\mathcal{S}\rangle) = \mathcal{T}\langle f^{-1}(\mathcal{S})\rangle$. Conclude that for $f : \boldsymbol{X} \to \boldsymbol{Y} := (Y,\mathcal{T}\langle\mathcal{S}\rangle)$ to be continuous, it is necessary and sufficient that $f^{-1}(U)$ is open, for all $U \in \mathcal{S}$ (solely).

7) *The continuity condition relative to covers of the domain of a map.*
   a) Let $\{U_i\}_{i\in\mathfrak{I}}$ be *open cover* of $\boldsymbol{X}$, *i.e.* a family of open sets of $\boldsymbol{X}$ such that $\boldsymbol{X} = \bigcup_{i\in\mathfrak{I}} U_i$. Show that $f : \boldsymbol{X} \to \boldsymbol{Y}$ is continuous, if and only if the restrictions $f|_{U_i} : U_i \to \boldsymbol{Y}$, $x \mapsto f(x)$, are continuous.
   b) Let $\{F_i\}_{i\in\mathfrak{I}}$ be a *closed cover* of $\boldsymbol{X}$, *i.e.* a family of closed sets of $\boldsymbol{X}$ such that $\boldsymbol{X} = \bigcup_{i\in\mathfrak{I}} F_i$. Assume $\mathfrak{I}$ finite. Show that $f : \boldsymbol{X} \to \boldsymbol{Y}$ is continuous, if and only if the restrictions $f|_{F_i} : F_i \to \boldsymbol{Y}$ are continuous.

## 2 The Category of Topological Spaces

The collection **Top** of topological spaces and continuous maps is a category, thanks to 1.4.1. The forgetful functor $o : \mathbf{Top} \rightsquigarrow \mathbf{Set}$, which associates with a space its underlying set, makes it a category of spaces.

In the following sections we show that **Top** is a category of spaces with *products*, *coproducts*, *subspaces*, *quotients* and *assemblages*.

## 3 Products of Topological Spaces

**3.1 The product Topology.** Let $\mathcal{F} := \{\boldsymbol{X}_i\}_{i\in\mathfrak{I}}$ be a family of spaces. Denote by $\prod\mathcal{F}$ the product of underlying sets $\Pi_i X_i$ (**2**♦6.2.1) endowed with the *coarsest* topology rendering all the canonical projections:

$$\pi_i : \prod\mathcal{F} \to \boldsymbol{X}_i, \tag{$*$}$$

continuous, *i.e.* with the topology generated by the union of induced topologies $\pi_i^{-1}(\mathcal{T}_{\mathbf{X}_i})$. This is called the *product topology*.

**3.1.1 Exercises.** 1) Let $\mathcal{F} := \{\boldsymbol{X}_i\}_{i\in\mathfrak{I}}$ be a *finite* family of spaces. Show that if $W_i$ is open (resp. closed) in $\boldsymbol{X}_i$, then $\prod W_i$ is an open (resp. closed) subset of $\prod\boldsymbol{X}_i$. Explain what remains true when $\mathfrak{I}$ is infinite.

2) Show that $\{f_i : \boldsymbol{X}_i \to \boldsymbol{Y}_i\}_{i\in\mathfrak{I}}$ is a family of continuous maps, if and only if the map $\prod f : \prod\boldsymbol{X}_i \to \prod\boldsymbol{Y}_i$, $(x_i) \mapsto (f_i(x_i))$, is continuous.

**3.1.2 Theorem.** *Let $\mathcal{F} := \{\boldsymbol{X}_i\}_{i\in\mathfrak{I}}$ be a family of spaces. The space*

$$\boxed{\prod\mathcal{F} := \big(\Pi_i X_i, \langle\text{product topology}\rangle\big)}$$

*with the canonical projections $\pi_i : \Pi_i X_i \to \boldsymbol{X}_i$, is a product of $\mathcal{F}$ in* **Top**.

*Proof.* Given a family of continuous maps $f_i : \boldsymbol{Y} \to \boldsymbol{X}_i$, $i \in \mathfrak{I}$, we have $\mathcal{T}_{\mathbf{Y}} \supseteq f_i^{-1}(\mathcal{T}_{\mathbf{X}_i})$, for all $i$, but on the other hand $f_i^{-1}(\mathcal{T}_{\mathbf{X}_i}) = f^{-1}(\pi_i^{-1}(\mathcal{T}_{\mathbf{X}_i}))$, where $f : \boldsymbol{Y} \to \prod\mathcal{F}$ is the map such that $f_i = \pi_i \circ f$ (universal property of products of sets (**2**♦6.2)). Then, thanks to 1.4.2-(6), we get $\mathcal{T}_{\mathbf{Y}} \supseteq f^{-1}(\mathcal{T}_{\prod\mathcal{F}})$, which means that $f$ is continuous, which proves the theorem. □

## 4 Coproducts of Topological Spaces

**4.1 The Coproduct Topology.** Let $\mathcal{F} := \{\boldsymbol{X}_i\}_{i\in\mathfrak{I}}$ be a family of spaces. Denote by $\coprod\mathcal{F}$ the disjoint union of underlying sets $\bigsqcup_i X_i$ (**2**♦6.3.1) endowed with the *finest* topology rendering all the canonical injections:

$$\iota_i : \boldsymbol{X}_i \hookrightarrow \coprod\mathcal{F}, \tag{$*$}$$

continuous, *i.e.* a subset $U \subseteq \bigsqcup_i X_i$ open if and only if $U \cap X_i$ is open in $\boldsymbol{X}_i$, for all $i \in \mathfrak{I}$. This is called the *disjoint union topology*.

**4.1.1 Exercises.** 1) Let $\{f_i : \boldsymbol{X}_i \to \boldsymbol{Y}_i\}_{i\in\mathfrak{I}}$ be a family of continuous maps. Show that the map $\coprod f : \coprod \boldsymbol{X}_i \to \coprod \boldsymbol{Y}_i$, $x_i \mapsto f_i(y_i)$, is continuous.

2) Let $\mathcal{F} := \{\boldsymbol{X}_i\}_{i\in\mathfrak{I}}$ be an *arbitrary* family of spaces. Show that each $X_i$ is open and closed in $\coprod\mathcal{F}$, and therefore that the open (resp. closed) subsets of this topology are the disjoint unions $\bigsqcup_i W_i$, where $W_i$ is open (resp. closed) in $\boldsymbol{X}_i$, for all $i \in \mathfrak{I}$.

**4.1.2 Theorem.** *Let $\mathcal{F} := \{\boldsymbol{X}_i\}_{i\in\mathfrak{I}}$ be a family of spaces. The space*

$$\boxed{\coprod\mathcal{F} := \big(\bigsqcup_i \boldsymbol{X}_i, \langle\text{disjoint union topology}\rangle\big)}$$

*with the inclusion maps $\iota_i : \boldsymbol{X}_i \hookrightarrow \coprod\mathcal{F}$, is a coproduct of $\mathcal{F}$ in* **Top**.

*Proof.* Given a family of continuous maps $\{f_i : \boldsymbol{X}_i \to \boldsymbol{Y}\}$, let $f : \bigsqcup_i X_i \to Y$ be the map s.t. $f \circ \iota_i = f_i$ (universal property of coproducts of sets (**2**♦6.3)). If $U$ is open in $\boldsymbol{Y}$, we have $f^{-1}(U) = \bigsqcup_i f_i^{-1}(U)$ which is open following exercise 4.1.1-(2). Hence, $f$ is continuous, and the theorem is proved. ☐

## 5 Subspaces of Topological Spaces

In **2**♦3.1 we noted that in a general category of spaces there is *at most one* subspace structure on a given subset of a space, but we did not discuss its existence, because it generally do not. Exceptions are the categories of topological or metric spaces, when it always exists.

**5.1 Induced Topology.** This is the topology that a topological space $\boldsymbol{X}$ induces on any of its subsets $W \subseteq \boldsymbol{X}$. Introduced in proposition 1.3.2, the open subsets of $\boldsymbol{W}$ are the traces on $W$ of the open subsets of $\boldsymbol{X}$, *i.e.* the subsets $W \cap V$ where $V \in \mathcal{T}_{\mathbf{X}}$.

**5.1.1 Theorem.** *Let $W$ be a subset of the space $\boldsymbol{X} := (X, \mathcal{T}_{\mathbf{X}})$. The space*

$$\boxed{\boldsymbol{W} := \big(W, \langle\text{induced topology}\rangle\big)}$$

*with the inclusion map $\iota : \boldsymbol{W} \hookrightarrow \boldsymbol{X}$, is a subspace of $\boldsymbol{X}$ in* **Top**.

*Proof.* A map $f : \boldsymbol{Y} \to \boldsymbol{X}$ such that $\mathrm{im}(f) \subseteq W$ factors, in the category of sets, as $f = f' \circ \iota$, where $\iota : W \hookrightarrow X$ is the inclusion map. Then $(f')^{-1}(\mathcal{T}_{\mathbf{W}}) = (f')^{-1}(\iota^{-1}(\mathcal{T}_{\mathbf{X}})) = f^{-1}(\mathcal{T}_{\mathbf{X}}) \subseteq \mathcal{T}_{\mathbf{W}}$ by definition of induced topology and continuity of $f$. Hence, $f' : \boldsymbol{Y} \to \boldsymbol{W}$ is continuous and the theorem is proved. ☐

$$\begin{tikzcd} & \boldsymbol{Y} \arrow[dl, dotted, "\exists! f'"'] \arrow[d, "f"] \\ \boldsymbol{W} \arrow[r, "\iota"'] & \boldsymbol{X} \end{tikzcd}$$

**5.1.2 Exercises.** Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two topological spaces.

1) Show that for all $y \in \boldsymbol{Y}$, the map $\iota_y : \boldsymbol{X} \to \boldsymbol{X}\times\boldsymbol{Y}$, $x \mapsto (x,y)$, is a homeomorphism onto the subspace $\boldsymbol{X}\times\{y\} \subseteq \boldsymbol{X}\times\boldsymbol{Y}$.

2) The *diagonal* of $\boldsymbol{X}\times\boldsymbol{X}$ is the subspace $\Delta_X := \{(x,x)\}_{x\in X}$. Show that the map $\delta_X : \boldsymbol{X} \to \boldsymbol{\Delta_X}$, $xx \mapsto (x,x)$ is a homeomorphism.

## 6 Quotients of Topological Spaces

**6.1 The Quotient Topology.** Given a space $\boldsymbol{X} := (X,\mathcal{T}_{\boldsymbol{X}})$ and an equivalence relation $\mathcal{R}$ on $X$, we defined in **2**♦1.4 the quotient set $X/\mathcal{R}$ and the identification map $\nu_{\mathcal{R}} : X \to X/\mathcal{R}$. Now, let $\boldsymbol{X}/\mathcal{R}$ denote the set $X/\mathcal{R}$ endowed with the *finest* topology, making this map continuous, *i.e.* such that $U \subseteq X/\mathcal{R}$ is open (resp. closed) if and only if $\nu_{\mathcal{R}}^{-1}(U)$ is open (resp. closed) in $\boldsymbol{X}$. This is called the *quotient topology* on $X/\mathcal{R}$.

**6.1.1 Theorem.** *Let $\mathcal{R}$ be an equivalence on $\boldsymbol{X} := (X,\mathcal{T}_{\boldsymbol{X}})$. The space*

$$\boxed{\boldsymbol{X}/\mathcal{R} := \big(X/\mathcal{R},\langle\text{quotient topology}\rangle\big)}$$

*with the identification map $\nu_{\mathcal{R}}\!:\!\boldsymbol{X}\!\to\!\boldsymbol{X}/\mathcal{R}$, is a quotient in* **Top**. *Therefore, the pullback map $\nu_{\mathcal{R}}^* : \mathcal{C}(\boldsymbol{X}/\mathcal{R},\boldsymbol{Y}) \to \mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$ is injective with image;*

$$\boxed{\nu_{\mathcal{R}}^*(\mathcal{C}(\boldsymbol{X}/\mathcal{R},\boldsymbol{Y})) = \mathcal{C}(\boldsymbol{X},\boldsymbol{Y})^{\mathcal{R}}}$$

*where $\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})^{\mathcal{R}}$ is the set of continuous maps compatible with $\mathcal{R}$.*

*Proof.* If $f : \boldsymbol{X} \to \boldsymbol{Y}$ is continuous and $\mathcal{R}$-compatible, then for any open $U \subseteq \boldsymbol{Y}$, we have $f^{-1}(U) = \nu_{\mathcal{R}}^{-1}\big(\overline{f}^{-1}(U)\big)$ (†) where $\overline{f} : X/\mathcal{R} \to \boldsymbol{Y}$ is the map such that $f = \overline{f} \circ \nu_{\mathcal{R}}$ (universal property of the quotient set (**2**♦1.5)). But then the equality (†) states that $\overline{f}^{-1}(U)$ is an open set of the quotient topology. Hence, $\overline{f}$ is continuous, and the theorem is proved by **2**♦6.4.1. □

**6.1.2 Comment.** In **2**♦4.1.2, we draw attention to the fact that the natural surjective map $\sigma(\boldsymbol{X})/\mathcal{R} \twoheadrightarrow \sigma(\boldsymbol{X}/\mathcal{R})$ is not necessarily bijective. The previous theorem shows that, as for the subspaces, everything happens at best with topological spaces, and that the equality $\sigma(\boldsymbol{X})/\mathcal{R} = \sigma(\boldsymbol{X}/\mathcal{R})$ is satisfied whatever the topological space $\boldsymbol{X}$ or the relation $\mathcal{R}$ are.

## 7 Assemblages of Topological Spaces

Because the category **Top** has coproducts and quotients, we know after theorem **2**♦6.4.1, that the assemblage of an atlas $\mathcal{A} := \{\boldsymbol{E}_i\}$, exists and is the quotient $\nu_{\mathcal{R}(\mathcal{A})} : \boldsymbol{E}(\mathcal{A}) \to \boldsymbol{E}(\mathcal{A})/\mathcal{R}(\mathcal{A})$, if and only if the $\boldsymbol{E}_i$'s are subspaces of $\boldsymbol{E}(\mathcal{A})/\mathcal{R}(\mathcal{A})$. In this section we will focus attention on two particular types of atlases that arrive frequently in topology, for which this question admits a simple answer. They are the *open* and the *closed* atlases, tightly related to open and closed covers.

### 7.1 Open maps, Closed maps and Homeomorphisms

A map $f:\boldsymbol{X}\to\boldsymbol{Y}$, not necessarily continuous, between topological spaces is:

- *open (resp. closed)*, if the image $f(W)$ of every open (resp. closed) set $W\subseteq\boldsymbol{X}$, is open (resp. closed) in $\boldsymbol{Y}$.
- *bi-continuous* or a *homeomorphism*, if it is a continuous bijection such that $f^{-1}:\boldsymbol{Y}\to\boldsymbol{X}$ is also continuous, in other words, if $f:\boldsymbol{X}\to\boldsymbol{Y}$ is an isomorphism of **Top**. A *topological invariant* of $\boldsymbol{X}$ is any property shared by all the spaces homeomorphic to $\boldsymbol{X}$. The separateness (9.1), the (local, path) connectedness (8.1), and the (local) contractibility (8.4) are classic examples of these.
- *local homeomorphism*, if for every $x\in\boldsymbol{X}$ there exists an open neighborhood $V_x$ such that $f(V_x)$ is open in $\boldsymbol{Y}$, and $f|_{V_x}:V_x\to f(V_x)$ is a homeomorphism.

**7.1.1 Exercises.** 1) Show that if $\mathcal{T}_{\mathbf{X}}$ is generated by $\mathcal{S}\subseteq\mathscr{P}(\boldsymbol{X})$, the map $f:\boldsymbol{X}\to\boldsymbol{Y}$ is open if and only if $f(W)$ is open in $\boldsymbol{Y}$, for all $W\in\mathcal{S}$.

2) Let $\mathcal{F}:=\{\boldsymbol{E}_i\}_i\in\mathfrak{I}$ be a family of spaces.

a) Show that the canonical injections $\iota_i:\boldsymbol{E}_i\to\coprod\mathcal{F}$ are open and closed.

b) Show that $f:\coprod\mathcal{F}\to\boldsymbol{Y}$ is open if and only if $f_i:=f\circ\iota_i:\boldsymbol{E}_i\to\boldsymbol{Y}$ is open for all $i\in\mathfrak{I}$.

c) Show that (2-b) remains true when 'open' is replaced by 'closed', on condition that $\mathfrak{I}$ is finite. Give a counterexample with $\mathfrak{I}$ infinite. *Hint. Let $\boldsymbol{E}_i:=I\!R$, for $i=0,1,2,\ldots$, and let $f_i=\mathrm{id}_{I\!R}$. The subset $W\subseteq\bigsqcup_i\boldsymbol{E}_i$, verifying $W\cap\boldsymbol{E}_0=\emptyset$ and $W\cap\boldsymbol{E}_i=\{1/i\}$, for $i>0$, is closed but its image $f(W)\subseteq I\!R$ is not so.*

d) Show that the projections $\pi_i:\prod\mathcal{F}\to\boldsymbol{E}_i$ are open but not necessarily closed. *Hint. For the openness use (1). For the nonclosedness, show that in $I\!R\times I\!R$, the subset $W:=\{(n,1/n)\mid n=1,2,\ldots\}$ is closed, but its $\pi_2$-projection, the subset $\{1/n\mid n=1,2,\ldots\}\subseteq I\!R$, is not so.*

3) Show that a continuous bijection is a homeomorphism if and only if it is open (resp. closed).

4) Show that if $f$ is open (resp. closed), then for any open (resp. closed) subspace $\boldsymbol{W}\subseteq\boldsymbol{X}$, the restriction $f|_{\boldsymbol{W}}:\boldsymbol{W}\to\boldsymbol{Y}$ is open (resp. closed). Show that if $\boldsymbol{W}$ is not open (resp. closed) the conclusion is false, even for the surjective map $f|_{\boldsymbol{W}}:\boldsymbol{W}\twoheadrightarrow f(\boldsymbol{W})$.

*Hint. The map $f:\boldsymbol{X}:=I\!R\sqcup I\!R\to I\!R$, whose restriction to each component is the identity map $\mathrm{id}_{I\!R}:I\!R\to I\!R$, is an open and closed map. In the subspace $\boldsymbol{W}:=[0,2)\sqcup[1,3]$, which is neither open nor closed in $\boldsymbol{X}$, the subsets $W_1:=[0,2)\sqcup\emptyset$ and $W_2:=\emptyset\sqcup[1,3]$ are respectively closed and open (4.1.1-(2)), but their images in $f(\boldsymbol{W})=[0,3]$ are not so.*

5) Let $\nu_{\mathcal{R}}:\boldsymbol{X}\to\boldsymbol{X}/\mathcal{R}$ be the identification map of the quotient $\boldsymbol{X}/\mathcal{R}$ space. Show that, given $W\subseteq\boldsymbol{X}$, the set $\nu_{\mathcal{R}}(W)$ is open (resp. closed), if

and only if the *$\mathcal{R}$-closure* of $W$, *i.e.* the set:

$$\overline{W}_{\mathcal{R}} := \{x \in \boldsymbol{X} \mid \exists w \in w \text{ s.t. } (x \,\mathcal{R}\, w)\},$$

is itself an open (resp. closed) subset of $\boldsymbol{X}$.

6) Show that if a map $f : \boldsymbol{X} \to \boldsymbol{Y}$, not necessarily continuous, is open (resp. closed), the induced map $\overline{f} : \boldsymbol{X}/\mathcal{R}_f \to \boldsymbol{Y}$ is open (resp. closed). Deduce that if $f$ is surjective, continuous and open (resp. closed), the induced map $\overline{f}$ is a homeomorphism.

**7.2 Open and Closed Atlases.** An atlas $\mathcal{A} = \{\boldsymbol{E}_i\}_{i \in \mathfrak{I}}$ in **Top** is said to be *open* (resp. *closed*), if the overlapping spaces $\boldsymbol{E}_i^j$ are open (resp. closed) in their ambient space $\boldsymbol{E}_i$. The notations are those of theorem **2**♦5.3.2.

**7.2.1 Existence Theorem.** *If an atlas $\mathcal{A} = \{\boldsymbol{E}_i\}_{i \in \mathfrak{I}}$ is open (resp. closed), the quotient space $\boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}}$ is an assemblage of $\mathcal{A}$. In other words, the restrictions of the identification map $\nu_{\mathcal{A}} : \boldsymbol{E}(\mathcal{A}) \twoheadrightarrow \boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}}$ to each $\boldsymbol{E}_i$,* i.e. *the maps*

$$\nu_{\mathcal{A}}|_{\boldsymbol{E}_i} : \boldsymbol{E}_i \to \tilde{\boldsymbol{E}}_i := \nu(\boldsymbol{E}_i), \tag{$*$}$$

*are homeomorphisms. Moreover, each $\tilde{\boldsymbol{E}}_i$ is open (resp. closed) in $\boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}}$.*

*Proof.* After 7.1.1-(3), it is sufficient to show that the restrictions $(*)$ are open (resp. closed), or, after 7.1.1-(5), that if $W \subseteq \boldsymbol{E}_i$ is open (resp. closed) the $\mathcal{R}$-closure $\overline{W}_{\mathcal{R}}$ is itself open (resp. closed) in $\boldsymbol{E}(\mathcal{A})$. Now, in the case of assemblages, the $\mathcal{R}$-closure of $W \subseteq \boldsymbol{E}_i$ is simply the disjoint union

$$\overline{W}_{\mathcal{R}} := \bigsqcup\nolimits_{j \in \mathfrak{I}} \varphi_{j,i}(W \cap \boldsymbol{E}_i^j). \tag{$\dagger$}$$

We then see that if $W$ is open in $\boldsymbol{E}_i$, the intersection $W \cap \boldsymbol{E}_i^j$ is open in $\boldsymbol{E}_i^j$ with the induced topology. Then, because $\varphi_{j,i} : \boldsymbol{E}_i^j \to \boldsymbol{E}_j^i$ is a homeomorphism, the set $\varphi_{j,i}(W \cap \boldsymbol{E}_i^j)$ is open in $\boldsymbol{E}_j^i$, which is itself open in $\boldsymbol{E}_j$, as $\mathcal{A}$ is an open atlas. Using 1.3.3-(1), we conclude that $\varphi_{j,i}(W \cap \boldsymbol{E}_i^j)$ is open in $\boldsymbol{E}_j$. The set $\overline{W}_{\mathcal{R}}$ is then the disjoint union $(\dagger)$ of open subsets, and is then open in $\boldsymbol{E}(\mathcal{A})$ after 4.1.1-(2). The restriction $(\dagger)$ is therefore a homeomorphism.

The last paragraph remains valid word-for-word when replacing 'open' by 'closed', which ends the proof of the theorem. □

## 7.3 Covers

**7.3.1. The Atlas of a Cover.** *A cover of a space* $\boldsymbol{X}$ is a family of subspaces $\mathcal{W} := \{\nu_i : \mathbf{W}_i \rightarrowtail \boldsymbol{X}\}_{i \in \mathfrak{I}}$ such that $X = \bigcup_i \tilde{W}_i$, where $\tilde{W}_i := \nu_i(W_i)$, *i.e.* satisfying the condition ($\mathcal{A}$-1) of the definition of assemblages in **2**♦5.3.1.

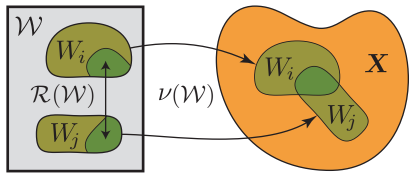
We call *atlas of the cover $\mathcal{W}$*, the atlas obtained from $\mathcal{W}$ by defining as overlapping subspaces to be $\boldsymbol{W}_i^j := \nu_i^{-1}(\tilde{W}_i \cap \tilde{W}_j)$, and as transition maps $\varphi_{j,i} : \boldsymbol{W}_i^j \to \boldsymbol{W}_j^i$, the composition $\nu_j^{-1} \circ \nu_i$.

The family of maps $\{\nu_i : \boldsymbol{W}_i \hookrightarrow \boldsymbol{X}\}_{i\in\mathfrak{I}}$ defines a continuous identification map

$$\nu(\mathcal{W}) : \boldsymbol{E}(\mathcal{W}) \to \boldsymbol{X}, \tag{$\diamond$}$$

clearly compatible with the relation $\mathcal{R}(\mathcal{W})$ generated by the transition maps. We have the factorization

$$\begin{tikzcd} \boldsymbol{E}(\mathcal{W}) \arrow[r, "\nu(\mathcal{W})"] \arrow[d, "\nu_{\mathcal{R}(\mathcal{W})}"'] & \boldsymbol{X} \\ \boldsymbol{E}(\mathcal{W})/\mathcal{R}(\mathcal{W}) \arrow[ur, dashed, "\overline{\nu(\mathcal{W})}"'] & \end{tikzcd}$$




and, as it is clear that $\nu(\mathcal{W})$ and $\nu_{\mathcal{R}(\mathcal{W})}$ are assemblages of $\boldsymbol{X}$ in the category of sets, the question arises if they are so in the category of topological spaces.

**7.3.2 Comments.** a) It is worth noting that the maps $\nu(\mathcal{W})$ and $\nu_{\mathcal{R}(\mathcal{W})}$ both satisfy the condition ($\mathcal{A}$-1) in **2**♦5.3, by which the spaces $\boldsymbol{W}_i$ are homeomorphic to their images. In particular, the transition maps are homeomorphisms, and the atlas defined by the cover is an atlas in **Top**. It therefore follows, after **2**♦6.4.1, that those maps are assemblages if and only if the bijective map $\overline{\nu(\mathcal{W})}$ is a homeomorphism, or, equivalently, *if it is an open map*.

b) The assemblage of a cover of a space $\boldsymbol{X}$ can be different of $\boldsymbol{X}$. For example, if the cover $\mathcal{W} := \{\boldsymbol{W}_i \subseteq \boldsymbol{X}\}_i$ is a partition of $\boldsymbol{X}$, then $\mathcal{R}(\mathcal{W})$ is the identity relation, and each $\boldsymbol{W}_i$ is an $\mathcal{R}(\mathcal{W})$-saturated open subspace of $\boldsymbol{E}(\mathcal{W})$. Hence, after (a), *$\boldsymbol{X}$ is the assemblage of $\mathcal{W}$ if and only if $\mathcal{W}$ is an open partition of $\boldsymbol{X}$*.

The next proposition summarizes these remarks.

**Terminology.** A cover $\mathcal{W} := \{\nu_i : \boldsymbol{W}_i \to \boldsymbol{X}\}_{i\in\mathfrak{I}}$ is called *open (resp. closed)* if the subspaces $\nu_i(\boldsymbol{W}_i) \subseteq \boldsymbol{X}$ are open (resp. closed), it is *finite* if $\mathcal{W}$ is finite.

**7.3.3 Proposition.** *Let $\mathcal{W}$ be the atlas of a cover of a topological space $\boldsymbol{X}$.*

a) *The identification map ($\diamond$):*

$$\nu(\mathcal{W}) : \boldsymbol{E}(\mathcal{W}) \to \boldsymbol{X},$$

*is always an assemblage in the category* **Set**, *and it is also an assemblage in the category* **Top** *if and only if the image $\nu(\mathcal{W})(U)$, of an $\mathcal{R}(W)$-saturated open subset $U \subseteq \boldsymbol{E}(\mathcal{W})$, is open in $\boldsymbol{X}$.*

b) *If $\mathcal{W}$ is an open (resp. closed and finite) cover of $\boldsymbol{X}$, then:*

i) *the maps*

$$\nu(\mathcal{W}) : \boldsymbol{E}(\mathcal{W}) \to \boldsymbol{X} \quad \textit{and} \quad \nu_{\mathcal{R}(\mathcal{W})} : \boldsymbol{E}(\mathcal{W}) \to \boldsymbol{E}(\mathcal{W})/\mathcal{R}(\mathcal{W})$$

*are both open (resp. closed) assemblages of $\mathcal{W}$;*

ii) $f : \boldsymbol{Z} \to \boldsymbol{X}$ *is continuous if and only if the sets* $Z_i := f^{-1}(W_i)$ *are open (resp. closed) and the restrictions* $f|_{Z_i} : Z_i \to W_i$ *are continuous.*

*Proof.* (a) Follows by the comment 7.3.2-(a). (b-i) By 4.1.1-(2) and 7.1.1-(6), both maps are continuous and open (resp. closed). The induced map $\overline{\nu(\mathcal{W})} : \boldsymbol{E}(\mathcal{W})/\mathcal{R}(\mathcal{W}) \to \boldsymbol{X}$ is therefore open (resp. closed), hence a homeomorphism. (b-ii) Left to the reader. □

**7.3.4 Exercise.** 1) Compare 7.3.3-(b) with the continuity conditions for of $\phi : \boldsymbol{X} \to \boldsymbol{Y}$, relative to covers of $\boldsymbol{X}$, of exercise 1.4.2-(7).

2) Show that the finiteness condition in the 'closed' case is necessary.
*Hint. The space* $I\!R$ *is the disjoint union of its singletons, which are closed subsets, but the standard topology of* $I\!R$ *is not the discrete one.*

**7.3.5. Sub-Atlases.** Given an atlas $\mathcal{A} := \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ in **Top**, with identification map $\nu_{\mathcal{A}} : \boldsymbol{E}(\mathcal{A}) \twoheadrightarrow \boldsymbol{E}(\mathcal{A})/\mathcal{R}(\mathcal{A})$, we call *sub-atlas* $\mathcal{B}$ *of* $\mathcal{A}$, and write $\mathcal{B} \subseteq \mathcal{A}$, any sub-family $\mathcal{B} := \{\boldsymbol{U}_i\}_{i\in\mathfrak{I}'}$ where $\mathfrak{I}'$ is a subset of $\mathfrak{I}$, where the transition maps are the same as in $\mathcal{A}$, and such that the restriction of $\nu_{\mathcal{A}}$ to $\boldsymbol{E}(\mathcal{B})$ remains surjective onto $\boldsymbol{E}(\mathcal{A})/\mathcal{R}(\mathcal{A})$. We have the commutative diagram:

$$\begin{tikzcd}
\boldsymbol{E}(\mathcal{B}) \arrow[r, hook] \arrow[d, two heads, "\nu_{\mathcal{B}}"'] \arrow[dr, phantom, "\oplus"] & \boldsymbol{E}(\mathcal{A}) \arrow[d, two heads, "\nu_{\mathcal{A}}"] \\
\boldsymbol{E}(\mathcal{B})/\mathcal{R}(\mathcal{B}) \arrow[r, two heads, "\overline{\nu}"] & \boldsymbol{E}(\mathcal{A})/\mathcal{R}(\mathcal{A})
\end{tikzcd}$$

If $\nu_{\mathcal{A}}$ is a cover, the restriction of $\nu_{\mathcal{A}}$ to each $\boldsymbol{E}_i$ is a homeomorphism, and, if $i \in \mathfrak{I}'$, the same is true for the restriction of $\nu_{\mathcal{B}}$ to the same $\boldsymbol{E}_i$ (exercice!), and $\nu_{\mathcal{B}}$ and $\overline{\nu} \circ \nu_{\mathcal{A}}$ are also covers.

The induced map $\overline{\nu}$ is a continuous bijection, but it is not always a homeomorphism. Indeed if $\mathcal{B} = \{\{t\} \hookrightarrow I\!R \mid t \in I\!R\}$, and if $\mathcal{A}$ is obtained by adding $\mathrm{id} : I\!R \hookrightarrow I\!R$ to $\mathcal{B}$, then $\boldsymbol{E}(\mathcal{B})/\mathcal{R}(\mathcal{B})$ is the real line $I\!R$ endowed with the discrete topology, wile $\boldsymbol{E}(\mathcal{A})/\mathcal{R}(\mathcal{A})$ is $I\!R$ endowed with the standard topology. There are nevertheless two useful cases where $\overline{\nu}$ is a homeomoprhism.

**7.3.6 Proposition.** *Let* $\mathcal{A}$ *be an open (resp. closed and finite) atlas in* **Top**. *If* $\nu(\mathcal{A}) : \boldsymbol{E}(\mathcal{A}) \to \boldsymbol{X}$ *is an assemblage in* **Top**, *then for any sub-atlas* $\mathcal{B} \subseteq \mathcal{A}$, *the restriction*

$$\nu(\mathcal{A})|_{\boldsymbol{E}(\mathcal{B})} : \boldsymbol{E}(\mathcal{B}) \to \boldsymbol{X}$$

*is an assemblage in* **Top**.

*Proof.* Immediate after 7.3.3-(b). □

**7.4 Continuous Maps between Assemblages.** We extend the continuity criterion for maps $f : \mathcal{E}(\mathcal{A}) \to \boldsymbol{Y}$ given in theorem **2**♦6.4.1, to maps $f : \mathcal{E}(\mathcal{A}) \to \mathcal{E}(\mathcal{B})$, where $\mathcal{B}$ is an open or a closed and finite atlas.

**7.4.1 Proposition.** *Let $\nu(\mathcal{A}) : \boldsymbol{E}(\mathcal{A}) \to \boldsymbol{X}$ and $\nu(\mathcal{B}) : \boldsymbol{E}(\mathcal{B}) \to \boldsymbol{Y}$ be assemblages in* **Top** *of atlases $\mathcal{A} = \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ and $\mathcal{B} = \{\boldsymbol{F}_j\}_{j\in\mathfrak{I}}$. We assume $\mathcal{B}$ to be open (resp. finite closed).*

*Given a map $f:\boldsymbol{X}\to\boldsymbol{Y}$, denote by $f_i:\boldsymbol{E}_i\to\boldsymbol{Y}$ the restriction of $f$ to $\boldsymbol{E}_i$, and by $\tilde{f}_{i,j}:f_i^{-1}(\tilde{\boldsymbol{F}}_j)\to\tilde{\boldsymbol{F}}_j$ the corresponding restriction of $f_i$, where $\tilde{\boldsymbol{F}}_j:=\nu(\mathcal{B})(\boldsymbol{F}_j)$. Then, the map $f$ is continuous if and only if, for all $(i,j)\in\mathfrak{I}\times\mathfrak{I}$,*

- *The set $f_i^{-1}(\tilde{\boldsymbol{F}}_j)$ is open (resp. closed) in $\boldsymbol{E}_i$,*
- *The induced map $f_{i,j}:=\nu(\mathcal{B})^{-1}\circ\tilde{f}_{i,j}:f_i^{-1}(\tilde{\boldsymbol{F}}_j)\to\boldsymbol{F}_j$ is continuous.*

$$\begin{tikzcd}
\boldsymbol{E}(\mathcal{A}) \arrow[dd, "\nu(\mathcal{A})"'] & \boldsymbol{E}_i \arrow[l, hook'] \arrow[dd, hook] \arrow[ddrr, "f_i"] & f_i^{-1}(\boldsymbol{F}_j) \arrow[l, hook'] \arrow[r, dotted, "f_{i,j}"] \arrow[dr, dashed, "\tilde{f}_{i,j}"'] & \boldsymbol{F}_j \arrow[r, hook] & \boldsymbol{E}(\mathcal{B}) \arrow[dd, "\nu(\mathcal{B})"] \\
 & & & \tilde{\boldsymbol{F}}_j \arrow[u, "\nu(\mathcal{B})^{-1}"'] \arrow[d, hook] & \\
\boldsymbol{X} \arrow[r, equal] & \boldsymbol{X} \arrow[rr, "f"] & & \boldsymbol{Y} \arrow[r, equal] & \boldsymbol{Y}
\end{tikzcd}$$

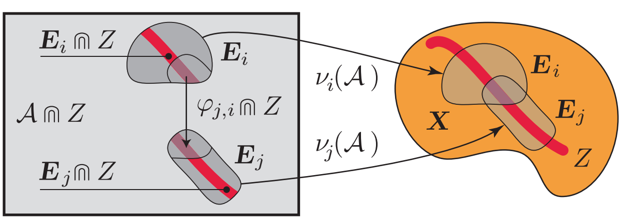
*Proof.* The proofs for the open and closed cases being the same, we discuss only the first. Thanks to **2**♦6.4, the map $f$ is continuous if and only if the maps $f_i$ are continuous. Next, because $\mathcal{B}$ is an open atlas, we can apply theorem 7.2.1 and state that the cover of the assemblage $\mathcal{E}(\mathcal{B}) = \bigcup \boldsymbol{F}_i$ is open. In particular, by 7.3.3-(b-ii), the map $f_i$ is continuous if and only if $f_i^{-1}(\boldsymbol{F}_j)$ is open in $\boldsymbol{E}_i$ and the restriction $f_{i,j} : f_i^{-1}(\boldsymbol{F}_j) \to \boldsymbol{F}_j$, is continuous, which ends the proof. □

## 7.5 Operations on Assemblages

**7.5.1. Restriction.** Let $\mathcal{A} := \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ be an atlas, and $\nu(\mathcal{A}) : \boldsymbol{E}(\mathcal{A}) \twoheadrightarrow \boldsymbol{X}$ an assemblage of $\mathcal{A}$ in **Top**. Denote by $\nu(\mathcal{A})_i : \boldsymbol{E}_i \to \boldsymbol{X}$ the restriction of $\nu(\mathcal{A})$ to $\boldsymbol{E}_i$.

For any subset $Z \subseteq \boldsymbol{X}$, define *the restriction $\mathcal{A} \Cap Z$ of $\mathcal{A}$ to $Z$*, by

- $\mathcal{A} \Cap Z := \big\{\boldsymbol{E}_i \Cap Z\big\}_{i\in\mathfrak{I}}$, where $(\boldsymbol{E}_i \Cap Z) := \nu_i^{-1}(Z)$,
- $\boldsymbol{E}_i^j \Cap Z := \nu^{-1}(\tilde{\boldsymbol{E}}_{i,j} \cap Z)$, and
- $\varphi_{j,i} \Cap Z : \boldsymbol{E}_i^j \Cap Z \to \boldsymbol{E}_j^i \Cap Z$ the restriction of $\varphi_{j,i} : \boldsymbol{E}_i^j \to \boldsymbol{E}_j^i$
- $\nu(\mathcal{A} \Cap Z) : \boldsymbol{E}(\mathcal{A} \Cap Z) \twoheadrightarrow Z$, the restriction of $\nu(\mathcal{A})$ to $\boldsymbol{E}(\mathcal{A} \Cap Z)$.



**7.5.2 Proposition.** *If $\mathcal{A}$ is an atlas (resp. open, finite closed) with assemblage $\nu(\mathcal{A}) : \boldsymbol{E}(\mathcal{A}) \twoheadrightarrow \boldsymbol{X}$ in* **Top**, *the restriction $\mathcal{A} \Cap Z$ is an atlas (resp. open, finite closed), and $\nu(\mathcal{A} \Cap Z) : \boldsymbol{E}(\mathcal{A} \Cap Z) \twoheadrightarrow Z$ is an assemblage of $Z$ (resp. open, closed). Moreover, the natural inclusion $\boldsymbol{E}(\mathcal{A} \Cap Z) \subseteq \boldsymbol{E}(\mathcal{A})$ induces the inclusion map $Z \hookrightarrow \boldsymbol{X}$.*

$$\left\|\;\begin{array}{ccc} \boldsymbol{E}(\mathcal{A} \Cap Z) & \hookrightarrow & \boldsymbol{E}(\mathcal{A}) \\ {\scriptstyle \nu(\mathcal{A} \Cap Z)}\big\downarrow & \bigoplus & \big\downarrow{\scriptstyle \nu(\mathcal{A})} \\ Z & \hookrightarrow & \boldsymbol{X} \end{array}\right.$$

*Proof.* Left to the reader. □

**7.5.3. Sum of Atlases.** Let $\mathcal{A} := \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ and $\mathcal{A}' := \{\boldsymbol{E}_j\}_{j\in\mathfrak{I}'}$ be two atlases, with $\mathfrak{I} \cap \mathfrak{I}' = \emptyset$. Let $\nu(\mathcal{A}) : \boldsymbol{E}(\mathcal{A}) \to \boldsymbol{X}$ and $\nu(\mathcal{A}') : \boldsymbol{E}(\mathcal{A}') \to \boldsymbol{X}$ be assemblages of *same* topological space $\boldsymbol{X}$.

The *sum of $\mathcal{A}_1$ and $\mathcal{A}_2$ over $X$* is the atlas $\mathcal{A} \Cup_X \mathcal{A}' := \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}\sqcup\mathfrak{I}'}$ where the transition maps are given by the relation associated with the coproduct of the $\nu(\mathcal{A}_i)$, *i.e.* by the map

$$\nu(\mathcal{A} \Cup_X \mathcal{A}') : \boldsymbol{E}(\mathcal{A} \Cup_X \mathcal{A}') \to \boldsymbol{X}. \qquad (\dagger)$$

where $\boldsymbol{E}(\mathcal{A} \Cup_X \mathcal{A}') := \boldsymbol{E}(\mathcal{A}) \sqcup \boldsymbol{E}(\mathcal{A}')$ and $\nu(\mathcal{A} \Cup_X \mathcal{A}') := \nu(\mathcal{A}) \sqcup \nu(\mathcal{A}')$.

Notice that the transition maps $\varphi_{j,i}$ when $i,j$ belong both to $\mathfrak{I}$ or both to $\mathfrak{I}'$, are already present in $\mathcal{A}$ and $\mathcal{A}'$, and that we only need to add the transition maps $\varphi_{j,i}$ for mixed indices. It is only for these that we need to refer to the set $X$, in which case they are easy to guess. We leave their precise description to the conscientious reader.

The question now arises if (†) is still an assemblage of $\boldsymbol{X}$. It is.

**7.5.4 Proposition.** *If $\nu(\mathcal{A}) : \boldsymbol{E}(\mathcal{A}) \to \boldsymbol{X}$ and $\nu(\mathcal{A}') : \boldsymbol{E}(\mathcal{A}') \to \boldsymbol{X}$ are assemblages of the* same *topological space $\boldsymbol{X}$, the induced map*

$$\nu(\mathcal{A} \Cup_X \mathcal{A}') : \boldsymbol{E}(\mathcal{A} \Cup_X \mathbf{A}') \to_{\boldsymbol{X}} \boldsymbol{X}.$$

*is also an assemblage of $\boldsymbol{X}$ in both, the category* **Set** *and the category* **Top**.

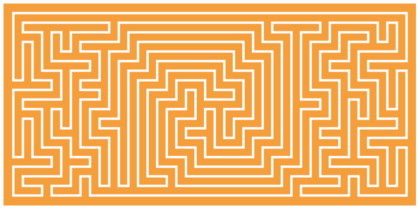
*Proof.* The condition ($\mathcal{A}$-1) in **2♦**5.3 is tautological, and $\nu(\mathcal{A} \Cup_X \mathcal{A}')$ is an assemblage if and only if the image by $\nu(\mathcal{A} \Cup_X \mathcal{A}')$ of an $\mathcal{R}(\mathcal{A} \Cup_X \mathcal{A}')$-saturated open subset of $\boldsymbol{E}(\mathcal{A} \Cup_X \mathcal{A}')$ is open in $\boldsymbol{X}$. Now, if $U$ is such a subset, we can decompose it as $U := (U \cap \boldsymbol{E}(\mathcal{A})) \cup (U \cap \boldsymbol{E}(\mathcal{A}'))$, in which case

$$\nu(\mathcal{A} \Cup_X \mathcal{A}')(U) = \nu(\mathcal{A})(U \cap \boldsymbol{E}(\mathcal{A})) \;\cup\; \nu(\mathcal{A}')(U \cap \boldsymbol{E}(\mathcal{A}')),$$

but, since $U \cap \boldsymbol{E}(\mathcal{A})$ is clearly open and $\mathcal{R}(\mathcal{A})$-saturated in $\boldsymbol{E}(\mathcal{A})$, the image $\nu(\mathcal{A})(U \cap \boldsymbol{E}(\mathcal{A}))$ is open in $\boldsymbol{X}$, and the same for $\nu(\mathcal{A}')(U \cap \boldsymbol{E}(\mathcal{A}'))$. □

# 8 Connected Topological Spaces

In a parenthesis to the main subject, in order to introduce concepts and terminologies that we will need, we now talk about *connectedness*, a mathematical property corresponding to the familiar idea that a space is made of a single indecomposable piece. From this point of view, *connected spaces* are like *atoms*, in which, as we will see, every space decomposes. For example, the figure to the right, representing a domain in $\mathbb{R}^2$ cut out of a rectangle following a labyrinthine outline, has exactly three components. Are you able to distinguish them ?



## 8.1 Topological Connectedness

**8.1.1 Definition.** A subspace $\boldsymbol{W}$ of a topological space $\boldsymbol{X}$ is *connected* if it is not homeomorphic to the coproduct of two nonempty spaces.

The following proposition gives well-known criteria of connectedness.

**8.1.2 Proposition.** *Let $\boldsymbol{X}$ be a topological space. The following conditions are equivalent.*

a) *$\boldsymbol{X}$ is connected.*
b) *$\boldsymbol{X}$ is not the disjoint union of two open (resp. closed) nonempty subsets.*
c) *The only open and closed subsets of $\boldsymbol{X}$ are: the empty set and $\boldsymbol{X}$ itself.*
d) *If $f : \boldsymbol{X} \to \bigsqcup_{i\in\mathfrak{I}} \mathbf{Y}_i$, then $f(\boldsymbol{X}) \subseteq \mathbf{Y}_i$ for some $i \in \mathfrak{I}$.*
e) *Any continuous map $f : \boldsymbol{X} \to \boldsymbol{D}$, where $\boldsymbol{D}$ is discrete, is constant.*

*Proof.* (a)⇔(b)⇔(c) left to the reader. $(a)$ ⇒(e) The singleton $\{y\}$ is both open and closed in $\boldsymbol{D}$, and, since $f$ is continuous, the same is true for $f^{-1}(\{y\})$, which is then empty or equal to $\boldsymbol{X}$ after (c). $(e)$ ⇒(a) If $\boldsymbol{X}$ is a disjoint union of nonempty open subsets $\boldsymbol{X} := V_1 \sqcup V_2$, the map $f : \boldsymbol{X} \to \mathbb{N}$, which associates with $x_i \in V_i$ its index $i$, is continuous and is not constant.□

**8.2 Decomposition in Connected Components.** We introduce the notion of *connected components* and show how these canonically decompose a space as a disjoint union of *maximal* connected subsets.

**8.2.1 Proposition.** *Let $\boldsymbol{X}$ be a topological space.*

a) *If $A,B \subseteq \boldsymbol{X}$ are such that $A \cap B \neq \emptyset$, then the union $A \cup B$ is connected.*
b) *A finite product of connected spaces is connected.*
c) *If $A \subseteq \boldsymbol{X}$ is connected, every $B$ verifying $A \subseteq B \subseteq \overline{A}$ is connected also.*
d) *The binary relation $\mathcal{C}$ on $\boldsymbol{X}$, where $(x \,\mathcal{C}\, y)$ means that $\{x,y\}$ is contained in some connected subset of $\boldsymbol{X}$, is an equivalence relation. The equivalence class of $x$, denoted by $\mathcal{C}(x)$, is the connected component of $x$. The connected components are maximal connected closed subsets of $\boldsymbol{X}$.*

e) *Let* $\boldsymbol{X}$ *be locally connected,* i.e. *such that for* $x \in \boldsymbol{X}$ *and every neighborhood* $V_x \ni x$*, there exists a connected open subset* $U$ *such that* $x \in U \subseteq V_x$*. Then, the connected components of* $\boldsymbol{X}$ *are both open and closed. The space* $\boldsymbol{X}$ *is the coproduct of its connected components, and the canonical surjection* $\nu_{\mathcal{C}} : \boldsymbol{X} \twoheadrightarrow \boldsymbol{X}/\mathcal{C}$ *is continuous onto the discrete space* $\boldsymbol{X}/\mathcal{C}$*.*

*Proof.* (a,b,c) These are immediate consequences of the criterion 8.1.2-(e).

(a) If $A$ and $B$ are connected, every continuous map $f : \boldsymbol{A}\cup\boldsymbol{B} \to \mathbb{N}$ is constant over $A$ and $B$, and if furthermore $x \in A \cap B$, then $f(A) = f(x) = f(B)$.

(b) We need only consider the case of the product of two connected spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $f : \boldsymbol{X}\times\boldsymbol{Y} \to \mathbb{N}$ be continuous. For all $x,x' \in X$ and $y,y' \in Y$, the couples $(x,y)$ and $(x,y')$ belong to the subspace $\{x\}\times\boldsymbol{Y}$, which is *connected* as it is homeomorphic to $\boldsymbol{Y}$ (5.1.2-(1)), therefore $f(x,y) = f(x,y')$. The same argument based in the connectedness of $\boldsymbol{X}$, shows that $f(x,y') = f(x',y')$. The map $f$ is then constant and we can conclude that $\boldsymbol{X}\times\boldsymbol{Y}$ is connected.

(c) We can consider $\boldsymbol{A}$ as subspace of the subspace $\boldsymbol{B} \subseteq \boldsymbol{X}$ (1.3.3-(2)). As such, $\boldsymbol{A}$ is always a connected space, but its topological closure is $\overline{\boldsymbol{A}} = \boldsymbol{B}$.

Now, if $f : \boldsymbol{B} \to \mathbb{N}$ is continuous, it is constant on $A$, and therefore also on $\overline{\boldsymbol{A}}$ after 1.4.2-(4). We may thus conclude that $\boldsymbol{B}$ is connected.

(d) Immediate consequence of (a) and (c).

(e) As $\boldsymbol{X}$ is locally connected, every $x \in \boldsymbol{X}$ admits a connected open neighborhood. Then, thanks to (d), any connected component $C$ is a neighborhood of all its elements, hence it is open. It is also closed, as the complement $\boldsymbol{X}\backslash C$, being the union of connected components of $\boldsymbol{X}$, is also open. □

**8.2.2 Exercises.** 1) Show that a subset of the space of real numbers $\mathbb{R}$ is connected if and only if it is an interval.

2) Let $f : \boldsymbol{X} \to \boldsymbol{Y}$ be continuous. Show that if $C \subseteq \boldsymbol{X}$ is connected in $\boldsymbol{X}$, then $f(C)$ is connected in $\boldsymbol{Y}$.

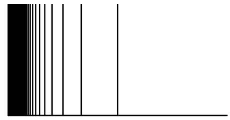
3) Show that any continuous map $f : \boldsymbol{X} \to \boldsymbol{Y}$ is $\mathcal{C}$*-compatible* in the sense that if $(x \;\mathcal{C}\; x') \Rightarrow (f(x) \;\mathcal{C}\; f(x'))$. Conclude that there exists a unique map $\overline{f} : \boldsymbol{X}/\mathcal{C} \to \boldsymbol{Y}/\mathcal{C}$ such that the diagram to the right is commutative. Show that $|\mathcal{C}(\boldsymbol{X},\{0,1\})| = |\mathrm{Maps}(\boldsymbol{X}/\mathcal{C},\{0,1\})| = 2^{|\boldsymbol{X}/\mathcal{C}|}$.

$$\begin{tikzcd} \boldsymbol{X} \arrow[r, "f"] \arrow[d, twoheadrightarrow, "\nu_{\mathcal{C}}"'] & \boldsymbol{Y} \arrow[d, twoheadrightarrow, "\nu_{\mathcal{C}}"] \\ \boldsymbol{X}/\mathcal{C} \arrow[r, "\overline{f}"] & \boldsymbol{Y}/\mathcal{C} \end{tikzcd}$$

4) Show that an interval $\boldsymbol{I} \subseteq \mathbb{R}$ and the unit circle $\mathbb{C}_u \subseteq \mathbb{C}$ are not homeomorphic. *Hint: show that* $\boldsymbol{I}$ *and* $\mathbb{C}_u$ *are connected spaces, and that, for all* $x \in \mathbb{C}_u$*, the subspace* $\mathbb{C}_u \setminus \{x\}$ *remains connected. Show that, on the other hand, if* $\boldsymbol{I}$ *is not a singleton, there exists* $t \in \boldsymbol{I}$ *such that* $\boldsymbol{I} \setminus \{t\}$ *is no longer connected. Conclude.*

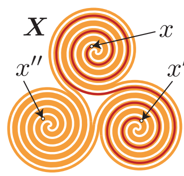
5) Show that if $\boldsymbol{C} \neq \emptyset$ is connected and $f : \boldsymbol{C} \to (\boldsymbol{X} \amalg \boldsymbol{Y})$ is continue, then the image of $f$ is contained either in $\boldsymbol{X}$ or in $\boldsymbol{Y}$. In particular, if $f : \boldsymbol{C} \to \mathbb{R}$ is continue and $f^{-1}(0) = \emptyset$, then either $f > 0$ or $f < 0$.

6) The *comb* space is the subspace of $I\!R^2$ defined as the union of $[0,2]\times\{0\}$, $\{0\}\times[0,1]$ and $\{1/n\}\times[0,1]$ for all $n \in \mathbb{N}_{>0}$, which is an example of a connected space that is not locally connected. Explain why.



7) (**) Generalize 8.2.1-(b) proving that any product, finite or not, of connected spaces is connected.

**8.3. Path-Connectedness.** We now come to the most common idea of connectedness, which is that, given any two points $x$,$x'$ in a space $\boldsymbol{X}$, we can draw a *path* joining them. For example, in the *triskelion* ([2]) domain $\boldsymbol{X}$ to the right, there is a path joining $x$ and $x'$, and none between $x$ and $x''$.




**8.3.1 Definitions.** Let $\boldsymbol{X}$ be a topological space.

a) *Paths.* A path in $\boldsymbol{X}$ is any *continuous* map $\gamma : [0,1] \to \boldsymbol{X}$. The point $x_0 := \gamma(0)$ is the *initial* point and $x_1 := \gamma(1)$ is the *terminal* point. One says that *the path $\gamma$ joins $x_0$ to $x_1$* or that it *goes from $x_0$ to $x_1$*.

b) *Contatenation of Paths.* Given $\gamma_i : [0,1] \to \boldsymbol{X}$ such that $\gamma_1(1) = \gamma_2(0)$, the *concatenation* $\gamma_1 \veebar \gamma_2 : [0,1] \to \boldsymbol{X}$ is the path defined by $t \mapsto \gamma_1(2t)$ if $t \in [0,1/2]$, and $t \mapsto \gamma_2(2t-1)$ if $t \in [1/2,1]$.

c) The space $\boldsymbol{X}$ is said to be *path-connected* if for every pair of points $w,w' \in W$, there exists a path joining them.

d) The space $\boldsymbol{X}$ is said to be *locally path-connected* if for every $x \in \boldsymbol{X}$ and every neighborhood $V_x \ni x$ there exists an open path-connected subset $U$ such that $x \in U \subseteq V_x$.

**8.3.2 Proposition.** a) *A path-connected space is connected.*

b) *The binary relation $\mathcal{P}$ on $\boldsymbol{X}$, where $(x \mathcal{P} y)$ means that $x$ and $y$ may be joined by a path in $\boldsymbol{X}$, is an equivalence relation. The equivalence class of $x$, denoted by $\mathcal{P}(x)$, is the path-connected component of $x$.*

c) *For a locally path-connected space, the properties of connectedness and path-connectedness are equivalent.*

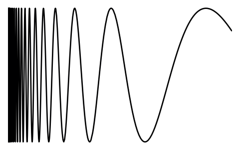
*Proof.* (a) If $\boldsymbol{X}$ were not connected, it would be a disjoint union of two nonempty open subsets $\boldsymbol{X} = U_0 \amalg U_1$. Now, because the image of a continuous map $\gamma : [0,1] \to \boldsymbol{X}$ is connected (8.2.2-(1,2)), we either have $\gamma(C) \subseteq U_0$ or $\gamma(C) \subseteq U_1$. But then, joining a point in $U_0$ to a point in $U_1$ would be impossible, which contradicts the path-connectedness of $\boldsymbol{X}$.

([2]) Neolithic symbol (around 4000 BC), from the greek word"τρισκέλιον" : three-legged.

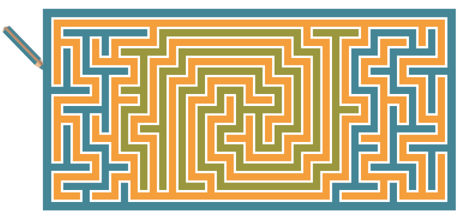
(b) In short, if one has a path from $x_0$ to $x_1$, and another from $x_1$ to $x_2$, the concatenation of both is a path from $x_0$ to $x_2$. Details are left to the reader.

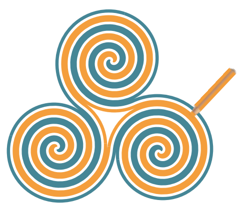
(c) We have only to show that a connected and locally path-connected space is path-connected. This is clear after (b), because, by locally path-connectedness, for all $x \in \boldsymbol{X}$, there is an open path-connected subset $U_x$ containing $x$, in which case $\mathcal{P}(x) \supseteq U_x$. One immediately concludes that the path-connected components are open, and then also closed, because the complement of any such $\mathcal{P}$-equivalence class is the union of the (open) others. Hence, if $\boldsymbol{X}$ is connected, $\boldsymbol{X} = \mathcal{P}(x)$ for any $x \in \boldsymbol{X}$ after 8.1.2-(c). □

**8.3.3 Exercise.** The comb space (8.2.2-(6)) is a path-connected space which is not locally path-connected. The *topologist sine curve* to the right, is the subset of $\mathbb{R}^2$ defined as the union of $\{(x,\sin(1/x)) | x \in (0,1]\}$ and $\{0\}\times[-1,1]$, and is a classic example of a connected space which is not path-connected. Explain why.



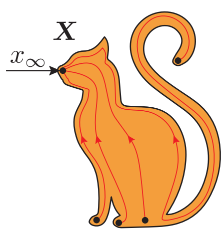
**8.3.4 Example.** The property 8.3.2-(c) allows us to reveal the connected components of locally path-connected spaces as, for example, the two maze shaped domains considered in the previous pages. With colored pencils, gradually color a region without ever raising your hand. From a mathematical point of view, such a plot is a continuous path that, joining adjacent points of a region, eventually covers the whole connected component.





**8.4. Contractibility.** A topological space $\boldsymbol{X}$ is said to be "*contractible*", if there exists a continuous map

$$h : [0,1] \times \boldsymbol{X} \to \boldsymbol{X}$$




such that $h(0,\_) : \boldsymbol{X} \to \boldsymbol{X}$ is the identity map, and $h(1,\_) : \boldsymbol{X} \to \boldsymbol{X}$ is constant to some point $x_\infty \in \boldsymbol{X}$.

Such a map is called *homotopy (of $\boldsymbol{X}$ to $x_\infty$)*. If we fix the point $x \in \boldsymbol{X}$, the restriction of the homotopy to the map $t \mapsto h(t,x)$ appears as a path joining $x$ to $x_\infty$. Thus, the homotopy $h$ continuously *retracts* $\boldsymbol{X}$ up to the single point $x_\infty$, which explains way this kind of space is called 'contractible'.

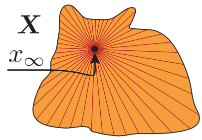
**8.4.1 Exercises.** 1) Show that the image of a contractible space under a continuous map may not be contractible.

2) Show that ‘contractible’ implies ‘path-connected’, hence ‘connected’.

3) Show that the comb space (8.2.2-(6)) is contractible, contrary to the topologist sine curve (8.3.3) which is not.

**8.4.2. Examples**

1) A “*star-shaped*” subset in the affine space $\mathbb{R}^n$ is a subset $\boldsymbol{X}$ containing a point $x_\infty$ such that for all $x \in \boldsymbol{X}$, the *line segment* $[x_\infty,x]$ is contained in $\boldsymbol{X}$, for example, the domain of $\mathbb{R}^2$ in the figure to the right. A star-shaped subspace $\boldsymbol{X} \subseteq \mathbb{R}^n$ is always contractible, with homotopy $(t,x) \mapsto h(t,x) := tx_\infty + (1-t)x$.



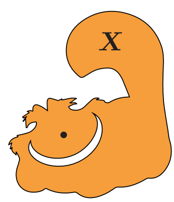

The subset $\boldsymbol{X}$ is said to be *convex* if it is star-shaped with respect to any of its points, in other words, for all $x,y \in \boldsymbol{X}$, one has $[x,y] \subseteq \boldsymbol{X}$.

2) The unit circle $\mathbb{C}_u$ is not contractible. Indeed, otherwise, we would have a homotopy $h : [0,1] \times \mathbb{C}_u$ of $\mathbb{C}_u$ to a point, that we can assume to be $1 \in \mathbb{C}_u$. For fixed $t \in [0,1]$, the map $\gamma_t : \mathbb{C}_u \to \mathbb{C}$, $\gamma_t(x) = h(t,x)$, is called a *loop*, as it is the continuous image of the circle. The loop $\gamma_0$ is the identity map, it turns faithfully over $\mathbb{C}_u$ one turn, while the loop $\gamma_1$ is the singleton $\{1\}$. The family of loops $\{\gamma_t\}_{t\in[0,1]}$ is thus a continuous deformation of $\gamma_0$ to $\gamma_1$. These loops are then said to be “*homotopic*”. Moreover, the deformation takes place *within* $\mathbb{C}_u$ itself, thus inside $\mathbb{C}\backslash\{0\}$. Now, the *Cauchy's integral formula* says that the *contour integral* of $\frac{dz}{z}$ remains constant along homotopic loops of $\mathbb{C}\backslash\{0\}$, but then:

$$2\pi i = \oint_{\gamma_0} \frac{dz}{z} = \oint_{\gamma_1} \frac{dz}{z} = 0,$$

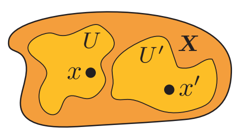
which is impossible. Therefore $\mathbb{C}_u$ is not contractible.

3) Exercise. Using (2), show that the domain of $\mathbb{R}^2$ represented by the figure to the right, is connected and locally path-connected, but is not contractible.




# 9 The Category of Hausdorff Spaces

**9.1. Topological Separateness.** A topological space $\boldsymbol{X}$ is a said to be *separated* or *Hausdorff* (3) whenever the topology differentiates distinct points. More precisely if, for every pair of different points $x \neq x' \in \boldsymbol{X}$, there exist *disjoint* open subsets $U$ and $U'$ such that $x \in U$ and $x' \in U'$. The property of separation of a space is so familiar to

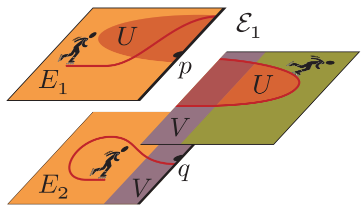



(3) After the German mathematician Felix Hausdorff (1868–1942), who is considered to be one of the founders of modern topology.

us that we may be disturbed if it is missing. Separateness is a rather fragile property, especially when taking quotients and making assemblages. We will shortly give some examples of this, as well as guidelines for preventing it. The separation of the topology is an important requirement in many theories, notably in the presence of *partial differential equations* as it is an essential ingredient to the uniqueness of solutions, a subject that will be especially relevant to us when we introduce the eigenfunctions of Laplacians, which are precisely solutions of second order differential operators.

**9.1.1 Examples.** 1) The discrete topology is always Hausdorff, whereas the coarse topology on a set with at least two points, never is.

2) The space $\mathcal{E}_1$ in **2**♦5.3.3-(2), which assembles two copies, $E_1$ and $E_2$, of the same square $]-1,1[\times]-1,1[\subseteq I\!R^2$, by identifying the *open* halves $\{x>0\}$, is a *nonseparated* space. If $p$ and $q$ are the central points of the $E_i$'s, as shown in the image to the right, any pair of neighborhoods $U \ni p$ and $V \ni q$ always meet on the subspace $\{x>0\}$ (in green).




In the space $\mathcal{E}_1$, a 2-dimensional roller skater will go smoothly from $E_1$ to $E_2$, following the red path, without noticing any gap!

**9.1.2 Proposition.** *Let $\boldsymbol{X}$ be a topological space and $\mathcal{R}$ an equivalence relation on $\boldsymbol{X}$.*

a) *If $\boldsymbol{X}/\mathcal{R}$ is Hausdorff, then the graph* $\mathrm{Gr}(\mathcal{R})$ *is closed in $\boldsymbol{X}\times\boldsymbol{X}$.*

b) *If the canonical map $\nu : \boldsymbol{X} \to \boldsymbol{X}/\mathcal{R}$ is open and the graph* $\mathrm{Gr}(\mathcal{R})$ *is closed in $\boldsymbol{X}\times\boldsymbol{X}$, then $\boldsymbol{X}/\mathcal{R}$ is Hausdorff. In particular, $\boldsymbol{X}$ is Hausdorff if and only if the diagonal $\Delta_{\mathbf{X}}$ is closed in $\boldsymbol{X}\times\boldsymbol{X}$.*

c) *If $\boldsymbol{X}$ is compact, the following properties are equivalent.*
   i) *$\boldsymbol{X}/\mathcal{R}$ is Hausdorff;*
   ii) $\mathrm{Gr}(\mathcal{R})$ *is closed in $\boldsymbol{X}\times\boldsymbol{X}$;*
   iii) *$\nu : \boldsymbol{X} \to \boldsymbol{X}/\mathcal{R}$ is a closed map.*

*Proof.* (a) If the quotient $\boldsymbol{X}/\mathcal{R}$ is Hausdorff, then two classes $\mathcal{R}(x) \neq \mathcal{R}(y)$ are separated by disjoint open $\mathcal{R}$-saturated ([4]) subsets $U(x)$ and $U(y)$ in $\boldsymbol{X}$, or, equivalently, the open subset $U(x)\times U(y)$, does not meet $\mathrm{Gr}(\mathcal{R})$. This easily implies that $\mathrm{Gr}(\mathcal{R})$ is closed in $\boldsymbol{X}\times\boldsymbol{X}$.

(b) Conversely, if $\mathrm{Gr}(\mathcal{R})$ is closed and $\neg(x\,\mathcal{R}\,y)$, then there exist open neighborhoods $V_x \ni x$ and $V_y \ni y$ such that $(V_x\times V_y)\cap\Delta_X=\emptyset$. In oher words, the map $\nu : V_x\cup V_y \to \boldsymbol{X}/\mathcal{R}$ is injective. Now, if moreover $\nu$ is open, the sets

([4]) Given $A \subseteq X$, the *$\mathcal{R}$-saturation of $A$* is the set $\mathcal{R}(A)$ of elements $x \in X$ $\mathcal{R}$-equivalent so some $a \in A$, *i.e.* if $\mathcal{R}(A) := \nu^{-1}(\nu(A))$. We say that $A$ is *$\mathcal{R}$-saturated* if $A = \mathcal{R}(A)$.

$\nu(V_x)$ and $\nu(V_x)$ are disjoint open subsets of $\boldsymbol{X}/\mathcal{R}$ separating $\nu(x)$ and $\nu(y)$. To finish, if $\mathcal{R}$ is the equality relation '$=$', we have $\boldsymbol{X} = \boldsymbol{X}/\mathcal{R}$, the map $\nu$ is the identity, $\mathrm{Gr}(\mathcal{R}) = \Delta_{\boldsymbol{X}}$, and the Hausdorff criterion for $\boldsymbol{X}$ follows.

(c-i)⇒(c-ii) This is (a). (c-ii)⇒(c-iii) The $\mathcal{R}$-saturation of $A \subseteq X$ is the set

$$\mathcal{R}(A) = p_2\big(p_i^{-1}(A) \cap \mathrm{Gr}(\mathcal{R})\big),$$

where we denote by $p_i : \boldsymbol{X}\times\boldsymbol{X} \to \boldsymbol{X}$ the projection map $p_i(x_1,x_2) := x_i$. If $A$ is closed in $\boldsymbol{X}$, the inverse image $p^{-1}(A)$ is closed in $\boldsymbol{X}\times\boldsymbol{X}$, as will also be $\big(p^{-1}(A) \cap \mathrm{Gr}(\mathcal{R})\big)$, after (c-ii). Now, because $\boldsymbol{X}$ is compact, the map $p_2$ is closed and $\mathcal{R}(A)$ is a closed subset of $\boldsymbol{X}$, *i.e.* $\nu(A)$ is closed in $\boldsymbol{X}/\mathcal{R}$.

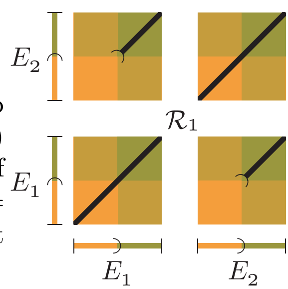
(c-iii)⇒(c-i) Given two equivalent classes $\mathcal{R}(x) \neq \mathcal{R}(y)$ we must show that there exist disjoint open $\mathcal{R}$-saturated neighborhoods $V(x) \supseteq \mathcal{R}(x)$ and $V(y) \supseteq \mathcal{R}(y)$, or equivalently closed $\mathcal{R}$-saturated subsets $F(x)$ and $F(y)$ such that $F(x) \cup F(y) = \boldsymbol{X}$ and $F(x) \cap \mathcal{R}(x) = \emptyset$ and $F(y) \cap \mathcal{R}(y) = \emptyset$. Now, because $\mathcal{R}(y)$ and $\mathcal{R}(y)$ are compact and disjoint, there exist disjoint *open* neighborhoods $\mathcal{R}(x)_\epsilon \supseteq \mathcal{R}(x)$ and $\mathcal{R}(y)_\epsilon \supseteq \mathcal{R}(y)$. And, because $\nu$ is closed, the $\mathcal{R}$-saturation of the *closed* set $\mathcal{R}(x)_\epsilon^{\mathrm{c}}$ is a *closed* subset that we denote by $F(x)$. This verifies, by construction, that $F(x) \cap \mathcal{R}(x) = \emptyset$, because $\mathcal{R}(x)_\epsilon^{\mathrm{c}} \subseteq \mathcal{R}(x)^{\mathrm{c}}$ and that $\mathcal{R}(x)^{\mathrm{c}}$ is $\mathcal{R}$-saturated. We define $F(y)$ *mutatis mutandis*. Then

$$\big(F(x) \cap F(y)\big) \supseteq \big(\mathcal{R}(x)_\epsilon^{\mathrm{c}} \cup \mathcal{R}(y)_\epsilon^{\mathrm{c}}\big) = \big(\mathcal{R}(x)_\epsilon \cap \mathcal{R}(y)_\epsilon\big)^{\mathrm{c}} = \boldsymbol{X},$$

which ends the proof that $\boldsymbol{X}/\mathcal{R}$ is Hausdorff. □

**9.1.3 Examples.** In the example 9.1.1-(2), the space $\mathcal{E}_1$ is the quotient of $\boldsymbol{E} := \boldsymbol{E}_1 \sqcup \boldsymbol{E}_2$, $\boldsymbol{E}_i := [0,1]\times[0,1]$, by the relation

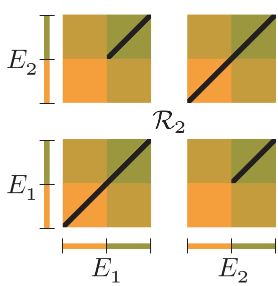
$$\big((x,y)\ \mathcal{R}_1\ (x',y')\big) \Leftrightarrow_{\mathrm{def}} (x = x' > 0)\ \&\ (y = y').$$




The space $\boldsymbol{E}$ is compact and the identification map $\boldsymbol{E} \to \boldsymbol{E}/\mathcal{R}$ is open, so that both criteria 9.1.2-(b,c) apply. The graph $\mathrm{Gr}(\mathcal{R})$ is the subset of $\boldsymbol{E}\times\boldsymbol{E}$ of points $\big((x,y),(x',y')\big)$ where $(x = x' > 0)$ and $(y = y')$, plus the diagonal $\Delta_{\boldsymbol{E}}$. This set is clearly not closed as $\{x = x' = 0\}$ is its boundary.

On the other side, the assemblage $\mathcal{E}_2$ in **2**♦5.3.3-(2) is the quotient of $\boldsymbol{E} := \boldsymbol{E}_1 \sqcup \boldsymbol{E}_2$, by the relation

$$\big((x,y)\ \mathcal{R}_2\ (x',y')\big) \Leftrightarrow_{\mathrm{def}} (x = x' \geq 0)\ \&\ (y = y').$$




The space $\boldsymbol{E}$ is compact but the identification map $\boldsymbol{E} \to \boldsymbol{E}/\mathcal{R}$ is not open, so that only the criterion 9.1.2-(c) applies, where we can easily check that $\nu$ is a closed map, but also that the graph $\mathrm{Gr}(\mathcal{R})$ is now a closed subset of $\boldsymbol{E}\times\boldsymbol{E}$. Thus $\mathcal{E}_2$ is Hausdorff.

**9.2 The Hausdorff Quotient Space.** For $\boldsymbol{X} \in \mathbf{Top}$, denote by $\mathcal{R}_{\boldsymbol{X},\mathrm{sep}}$ the equivalence relation

$$\mathcal{R}_{\boldsymbol{X},\mathrm{sep}} := \bigcap \big\{ \mathcal{R}_g \;\big|\; g : \boldsymbol{X} \to \boldsymbol{Z}_{\mathrm{sep}} \text{ is continuous} \big\}$$

where $g$ ranges over the collection of continuous maps $g$ defined on $\boldsymbol{X}$ with target spaces the Hausdorff spaces $\boldsymbol{Z}_{\mathrm{sep}}$. Notice that, by definition,

$$\mathrm{Gr}(\mathcal{R}_{\boldsymbol{X},\mathrm{sep}}) := \bigcap\nolimits_{f:\boldsymbol{X}\to\boldsymbol{Z}_{\mathrm{sep}}} f^{-1}\Delta_{\boldsymbol{Z}_{\mathrm{sep}}},$$

so that $\mathrm{Gr}(\mathcal{R}_{\boldsymbol{X},\mathrm{sep}})$ is closed in $\boldsymbol{X}\times\boldsymbol{X}$.

**9.2.1 Theorem.** a) *The quotient* $\boldsymbol{X}_{\mathrm{sep}} := \boldsymbol{X}/\mathcal{R}_{\boldsymbol{X},\mathrm{sep}}$ *is Hausdorff. The identification map* $\nu_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}_{\mathrm{sep}}$ *is the Hausdorff quotient of* $\boldsymbol{X}$.

b) *Let* $f : \boldsymbol{X} \to \boldsymbol{Y}$ *be a continuous map between topological spaces. For any equivalence relation* $\mathcal{R}$ *on* $\boldsymbol{Y}$*, denote by* $f^*\mathcal{R}$ *the pullback relation:*

$$(x\ f^*\mathcal{R}\ x') \Leftrightarrow_{\mathrm{def}} (f(x)\ \mathcal{R}\ f(x')).$$

*Then* $\mathcal{R}_{\boldsymbol{X},\mathrm{sep}} \subseteq f^*\mathcal{R}_{\boldsymbol{Y},\mathrm{sep}}$*, and there exists a unique continuous map* $f_{\mathrm{sep}} : \boldsymbol{X}_{\mathrm{sep}} \to \boldsymbol{Y}_{\mathrm{sep}}$ *such that the following diagram is commutative.*

$$\begin{tikzcd} \boldsymbol{X} \arrow[r, "f"] \arrow[d, "\nu_{\boldsymbol{X}}"'] & \boldsymbol{Y} \arrow[d, "\nu_{\boldsymbol{Y}}"] \\ \boldsymbol{X}_{\mathrm{sep}} \arrow[r, "f_{\mathrm{sep}}"] & \boldsymbol{Y}_{\mathrm{sep}} \end{tikzcd}$$

c) *Denote by* $\mathbf{Top}_{\mathrm{sep}}$ *the category of Hausdorff topological spaces and continuous maps. The correspondence*

$$(\_)_{\mathrm{sep}} : \mathbf{Top} \rightsquigarrow \mathbf{Top}_{\mathrm{sep}}$$

*which associates with* $\boldsymbol{X}$ *its Hausdorff quotient* $\boldsymbol{X}_{\mathrm{sep}}$ *and with* $f : \boldsymbol{X} \to \boldsymbol{Y}$ *the induced map* $f_{\mathrm{sep}} : \boldsymbol{X}_{\mathrm{sep}} \to \boldsymbol{Y}_{\mathrm{sep}}$*, is a functor.*

d) *In the category* $\mathbf{Top}_{\mathrm{sep}}$*, the quotient of* $\boldsymbol{X}$ *by* $\mathcal{R}$ *is the composition*

$$\boldsymbol{X} \xrightarrow{\nu_{\mathcal{R}}} \boldsymbol{X}/\mathcal{R} \xrightarrow{\nu_{\boldsymbol{X}/\mathcal{R}}} (\boldsymbol{X}/\mathcal{R})_{\mathrm{sep}}$$

*Proof.* (a) A continuous map $f : \boldsymbol{X} \to \boldsymbol{Z}_{\mathrm{sep}}$ factors through the canonical surjection $\nu : \boldsymbol{X} \to \boldsymbol{X}/\mathcal{R}_{\boldsymbol{X},\mathrm{sep}}$ in a continuous map $\tilde{f} : \boldsymbol{X}/\mathcal{R}_{\boldsymbol{X},\mathrm{sep}} \to \boldsymbol{Z}_{\mathrm{sep}}$. If $a,b \in \boldsymbol{X}$ are such that $\nu(a)$ and $\nu(b)$ cannot be separated by disjoint open subsets, then $f(a) = \tilde{f}(\nu(a)) = \tilde{f}(\nu(b)) = f(b)$, so that $(a\ \mathcal{R}_f\ b)$. As this is true for all $f$, we conclude that $(a\ \mathcal{R}_{\boldsymbol{X},\mathrm{sep}}\ b)$ in which case $\nu(a) = \nu(b)$. This proves that $\boldsymbol{X}/\mathcal{R}_{\boldsymbol{X},\mathrm{sep}}$ is a Hausforff space.

(b,c,d) Left to the reader. $\square$

**9.2.2 Exercises.** 1) Let $\boldsymbol{X}$ be a *compact* topological space and $\mathcal{R}$ an equivalence relation on $\boldsymbol{X}$. Denote by $\tilde{\mathcal{R}}$ the smallest *closed* equivalence relation containing $\mathcal{R}$. Show that the natural map $\boldsymbol{X}/\mathcal{R} \to (\boldsymbol{X}/\mathcal{R})_{\text{sep}}$ factors through the quotient $\boldsymbol{X} \twoheadrightarrow \boldsymbol{X}/\tilde{\mathcal{R}}$ in a canonical homeomorphism $\tilde{f} : \boldsymbol{X}/\tilde{\mathcal{R}} \to (\boldsymbol{X}/\mathcal{R})_{\text{sep}}$

$$\begin{tikzcd} \boldsymbol{X} \arrow[r, "f"] \arrow[d, "\nu"'] & (\boldsymbol{X}/\mathcal{R})_{\text{sep}} \\ \boldsymbol{X}/\tilde{\mathcal{R}} \arrow[ur, tail, two heads, "\tilde{f}"'] & \end{tikzcd}$$

2) Show that the category $\mathbf{Top}_{\text{sep}}$ has products, coproducts, subspaces and quotient spaces.

## Chapter Contents

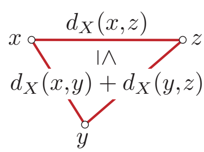
# Chapter 4
# Path-Pseudometric Spaces

The previous chapter introduced topological spaces in a very general and abstract framework, going far beyond our real needs. The main reason in doing that, was that even if we start from relatively simple topological spaces, their quotients and assemblages can easily take us out of an elementary environment, producing very sophisticated spaces. It was to anticipate the properties of these new spaces, that we needed so much generality. Now, in this chapter, we get back close to our concrete needs: *the metric spaces*. These are topological spaces in which the concept of neighborhoodness meets its original meaning, as it is given by the *measure* of the *distance* between two points. If the whole subject is quite elementary, it becomes more interesting when we need to glue together metric spaces, as we soon discover that there is no general canonical way to glue distances. The study of this problem will lead us to the notion of *path-distances*, opening the way to the main subject of the next chapter: *the category of path-metric spaces*, where solutions always exist.

## 1 Pseudometric Spaces

We extend the set $\mathbb{R}_{\geq 0}$ of positive real numbers by adding infinity to it. We will write $\overline{\mathbb{R}}$ for $\mathbb{R} \cup \{+\infty\}$, and consequently extend the usual addition operation by setting $x + \infty := +\infty$, for all $x \in \overline{\mathbb{R}}$.

**1.1 Definitions.** A *pseudometric space* is a couple $(X,d_X)$ consisting of a set $X$ and a *pseudodistance*, *i.e.* a positive function $d_X : X \times X \to \overline{\mathbb{R}}_{\geq 0}$ satisfying, for all $x,y,z \in X$:




*D*-1) $d_X(x,x) = 0$;
*D*-2) $d_X(x,y) = d_X(y,x)$ (*symmetry*);
*D*-3) $d_X(x,z) \leq d_X(x,y) + d_X(y,z)$ (*triangle inequality*).

Notice that $d_X$ can take the value $+\infty$, and also that in ($D$-1), one may have $d_X(x,y) = 0$ while $x \neq y$, which is quite peculiar as it allows the distance between different points to be zero, which is typical of *non-separated* topologies as we will see in **3**♦9.1. At the opposite, *i.e.* when we have

*D*-0) $d_X(x,y) < +\infty$ (*finiteness*),
*D′*-1) $d_X(x,y) = 0$ if and only if $x = y$ (*separation*),

we say that $d_X$ is a *distance*, and that $(X,d_X)$ is a *metric* space.

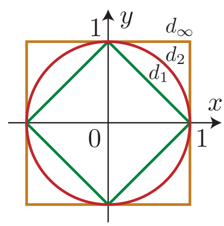
**1.1.1 Comment.** Although our main concern is metric spaces, we need to introduce *pseudo*distances because, as we will see in the next chapter, the most natural way to induce distances on quotients and assemblages of metric spaces will sometimes fail to satisfy the finiteness or the separation axiom, thus giving pseudodistances. We will discuss ways to circumvent this.

**1.1.2 Examples.** The list below is presented without justifications, readers who are unfamiliar with these examples should consider them as exercises.

1) The most basic examples of metric spaces are the set of real numbers $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$, endowed with the *absolute value* distance $d_{\mathrm{abs}}(a,b) := |a-b|$.
2) Given pseudometric spaces $(X,d_X)$ and $(Y,d_Y)$, the cartesian product $X\times Y$ can be endowed with the following three pseudodistances

$$\begin{cases} d_1\big((x,y),(x',y')\big) = d_X(x,x') + d_Y(y,y') & (\textit{city block distance}(^1)) \\ d_2\big((x,y),(x',y')\big) = \sqrt{d_X(x,x')^2 + d_Y(y,y')^2} & (\textit{Euclidean distance}) \\ d_\infty\big((x,y),(x',y')\big) = \max\{d_X(x,x'),d_Y(y,y')\} & (\textit{chessboard distance}(^2)) \end{cases}$$

One has $\qquad d_1 \geq d_2 \geq d_\infty \geq \frac{1}{2}d_1\,. \qquad (1)$




The image to the right shows the circles of radius 1 for these distances in the case $\mathbb{R}^2 := \mathbb{R}\times\mathbb{R}$, where $\mathbb{R}$ is endowed with the absolute value distance. This procedure can be iterated to any finite number of spaces. For example, the *Euclidean distance* of the $n$-dimensional space $\mathbb{R}^n$, given by the well known Pythagoras formula:

$$d_2(\vec{x},\vec{y}) := \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}\,.$$

3) *Induced (Pseudo)metric.* If $\mathbf{Y} := (Y,d_Y)$ is a (pseudo)metric space, for any subset $X \subseteq Y$, the restriction of $d_Y : Y\times Y \to \overline{\mathbb{R}}_{\leq 0}$ to $X\times X$ is a (pseudo)distance $d_X$ on $X$. It is the *induced (pseudo)metric on $X'$*.
4) *Pullback Pseudometric.* Given a (pseudo)metric space $\mathbf{Y} := (Y,d_Y)$, a natural way to define a pseudodistance on a set $X$, is through a map $f : X \to (Y,d_Y)$. One then measures the distance between points of $X$ by looking at their images in $Y$. The "*pullback of $d_Y$ by $f$*" is the pseudodistance $f^*d_Y : X\times X \to \overline{\mathbb{R}}_{\geq 0}$ defined by:

$$f^*d_Y(x,x') := d_Y(f(x),f(x')), \quad \forall x,x' \in X\,. \tag{2}$$

It is finite is $d_Y$ is so, but to be separated it is necessary that $f$ be injective. In particular, it is a distance if $\mathbf{Y}$ is a metric space and $f$ is injective, for example, if $X \subseteq \mathbf{Y}$ and $f : X \hookrightarrow \mathbf{Y}$ is the inclusion, as in (3).

($^1$) Shortest distance a car has to travel between two intersections on a square gridded city.
($^2$) Minimum number of moves a king has to do between two squares on a chessboard.

5) *Sequential Distance.* Positive functions $h : X \times X \to \overline{\mathbb{R}}_{\geq 0}$ satisfying $(D\text{-}1)$ $h(x,x) = 0$, and $(D\text{-}2)$ $h(x,y) = h(y,x)$ are frequent and easy to construct, but to satisfy the triangle inequality $(D\text{-}3)$ is much harder. We now give a natural recipe to deal with this property.

Given a *finite* sequence $\boldsymbol{s} := (x_0,\dots,x_{\ell(\boldsymbol{s})})$ of elements of $X$. Call the number $\ell(\boldsymbol{s})$, the *length* of $\boldsymbol{s}$, and, as with paths, call $x_0$ the *initial point* and $x_{\ell(\boldsymbol{s})}$ the *terminal point* of $\boldsymbol{s}$.

The *inverse* of $\boldsymbol{s} := (x_0,\dots,x_{\ell(\boldsymbol{s})})$ is the sequence $\bar{\boldsymbol{s}} := (y_0,\dots,y_{\ell(\boldsymbol{s})})$ where $y_k = x_{\ell(\boldsymbol{s})-k}$. If $\boldsymbol{s} := (x_0,\dots,x_{\ell(\boldsymbol{s})})$ and $\boldsymbol{t} := (y_0,\dots,y_{\ell(\boldsymbol{t})})$ are such that $x_{\ell(\boldsymbol{s})} = y_0$, their *concatenation* is the sequence

$$\boldsymbol{s} \veebar \boldsymbol{t} = (x_0,\dots,x_{\ell(\boldsymbol{s})}) \veebar (y_0,\dots,y_{\ell(\boldsymbol{t})}) := (x_0,\dots,x_{\ell(\boldsymbol{s})}{=}y_0,\dots,y_{\ell(\boldsymbol{t})}),$$

whose length is $\ell(\boldsymbol{s} \veebar \boldsymbol{t}) = \ell(\boldsymbol{s}) + \ell(\boldsymbol{t})$.

An *admissible set of sequences on $X$* is a set $\mathcal{S}$ of sequences verifying:

$\mathcal{S}$-1) $\mathcal{S}$ is stable under the operations of *inversion* and *concatenation*;

$\mathcal{S}$-2) for all $x \in X$, the one-term sequence $\boldsymbol{s} := (x)$ belongs to $\mathcal{S}$.

Now, given any map $h : X \times X \to \overline{\mathbb{R}}_{\geq 0}$, we extend it to $\mathcal{S}$, by setting:

$$\ell((x),h) := 0 \quad \text{and} \quad \ell(\boldsymbol{s},h) := h(x_0,x_1) + \cdots + h(x_{\ell(\boldsymbol{s})-1},x_{\ell(\boldsymbol{s})}).$$

We then define:

$$\begin{cases} d_{\mathcal{S},h}(a,b) := +\infty, & \text{if } \mathcal{S}(a,b) = \emptyset, \\ d_{\mathcal{S},h}(a,b) := \min\big\{\ell(\boldsymbol{s},h) \;\big|\; \boldsymbol{s} \in \mathcal{S}(a,b)\big\}, & \text{otherwise.} \end{cases}$$

where $\mathcal{S}(a,b)$ is the set of $\boldsymbol{s} \in \mathcal{S}$ with initial point $a$ and terminal point $b$. It is straightforward to check that if the function $h$ satisfies the conditions $(D\text{-}1,D\text{-}2)$, the map $d_{\mathcal{S},h} : X \times X \to \overline{\mathbb{R}}_{\geq 0}$ is a pseudodistance. It is the *sequential pseudodistance on $X$ associated with $(\mathcal{S},h)$*.

*The Little Poucet Distance*

Notice that,

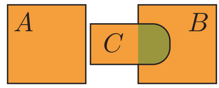
- If $h : X \times X \to \overline{\mathbb{R}}_{\geq 0}$ is a pseudodistance, then $d_{\mathcal{S},h} = h$.
- If $d_{\mathcal{S},h}$ is separated then $h$ *separates points*, *i.e.* $h(x,x') = 0$ if and only if $x = x'$, but the converse is not necessarily true (see 2.2.5-(2,3)).

6) If $(X,d_X)$ is a pseudometric space, let $\mathscr{P}'(X)$ denote the set of all nonempty subsets of $X$. One usually uses the function $d_X$ to define also the distance separating two nonempty sets $A,B \in \mathscr{P}'(X)$, by setting:

$$d_X(A,B) := \min\{d_X(a,b) \mid a \in A,\ b \in B\}.$$

However, this is generally not a distance on $\mathscr{P}'(X)$, as it may not satisfy the triangle inequality. Indeed, in the figure to the right, we see that $d_X(A,B)$ is much bigger than $d_X(A,C) + d_X(C,B)$.

Nevertheless, as $d_X$ satisfies $(D\text{-}1)$ and $(D\text{-}2)$, we may consider the sequential distance $\sigma(d_X)$ it generates relative to the set $\mathcal{S}$ of all sequences on $X$, as in the next example.




7) *Pushforward Pseudometric.* Given a *surjective* map $f : (X,d_X) \twoheadrightarrow Y$, the function $f_* d_X : Y \times Y \to \overline{I\!R}_{\geq 0}$ defined by

$$f_* d_X(y,y') := \sigma(d_X)(f^{-1}(y), f^{-1}(y'))$$

is a pseudodistance in $Y$. We clearly have

$$f_* d_X(f(x), f(x')) \leq d_X(x,x'), \quad \forall x,x' \in X. \tag{3}$$

## 2 Topology Associated with a Pseudodistance

**2.1 Open and Closed Balls.** Let $(X,d_X)$ be a pseudometric space. Given $x \in X$ and $R \in I\!R$, we call respectively "*open and closed balls of center $x$ and radius $R$*", the sets

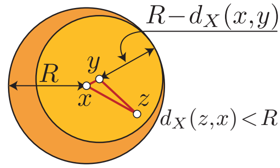
$$\begin{cases} B_{d_X}(x,R) := \{y \in X \mid d_X(x,y) < R\}, \\ \overline{B}_{d_X}(x,R) := \{y \in X \mid d_X(x,y) \leq R\}. \end{cases}$$

**2.2 Topology of a Pseudometric Space.** The topology *associated with the metric $d_X$*, denoted by $\mathcal{T}(d_X)$, is the topology generated by the open balls of $(X,d_X)$ (**3**♦1.2). A pseudometric space will always be considered to be endowed with this topology. The notation $\boldsymbol{X} := (X,d_X)$ will henceforth identify $\boldsymbol{X} = (X,\mathcal{T}(d_X))$ as topological spaces, and a map $f : \boldsymbol{X} \to \boldsymbol{Y}$ will be said *continuous* if it is continuous in the topological sense **3**♦1.4.

**2.2.1 Proposition.** *Let $(X,d_X)$ be a pseudometric space.*

a) *A finite intersection of open balls of $(X,d_X)$ is a union of open balls.*
b) *An open subset of $\mathcal{T}(d_X)$ is the union of the open balls it contains.*
c) *The topology $\mathcal{T}(d_X)$ does not depend on the large size open balls. In other words, given any map $\rho : X \to I\!R_{>0}$, define $\mathcal{T}(d_X,\rho)$ to be the topology generated by the balls $B(x,R)$ with $R \leq \rho(x)$. Then $\mathcal{T}(d_X,\rho) = \mathcal{T}(d_X)$.*

*Proof.* (a) For any open ball $B(x,R)$ and any $y \in B(x,R)$, we have the inclusion $B\big(y,R-d_X(x,y)\big) \subseteq B(x,R)$, by the triangle inequality. Thus, if $\{B(x_i,R_i)\}_{i\in\mathfrak{A}}$ is a family of open balls and $y \in B(x_i,R_i)$, for all $i \in \mathfrak{A}$, we will have $B(y,r(y)) \subseteq \bigcap_i B(x_i,R_i)$, for




$$r(y) = \min\big\{R_i - d_X(x_i,y) \;\big|\; i \in \mathfrak{A}\big\},$$

If, in addition, the set $I$ is finite, then $r(y) \neq 0$ and $y \in B(y,r(y))$.

(b) Following **3**♦1.2.1, an open set of $\mathcal{T}(d_X)$ is a union of finite intersections of balls, but these are already unions of open balls, after (a).

(c) The inequality $\mathcal{T}(d_X) \supseteq \mathcal{T}(d_X,\rho)$ is obvious. The converse follows, like in (a), by observing that if $y \in B(x,R)$, then $B(y,\min\{R-d_X(x,y),\rho(y)\})$ is contained in $B(x,R)$, in which case, any open ball $B(x,R)$ is covered by generators of $\mathcal{T}(d_{X,\epsilon})$. ☐

**2.2.2 Exercise.** *Finiteness and continuity of pseudodistances.*

1) Let $\boldsymbol{X} := (X,d_X)$ be a pseudometric space (resp. separated). Show that for all $x \in X$, the open ball $B(x,\infty)$ is both open and closed. Conclude that $d_X$ is *finite* (resp. *a distance*) on each connected component of $\boldsymbol{X}$.

2) Extend the absolute value distance $d_{\text{abs}}(a,b) := |a-b|$ of $\mathbb{R}$, to $\overline{\mathbb{R}}_{\geq 0}$, by

$$\bullet\ d_{\text{abs}}(a,+\infty) = \infty, \quad \forall a \neq +\infty; \qquad \bullet\ d_{\text{abs}}(+\infty,+\infty) = 0.$$

   a) Show that $(\overline{\mathbb{R}}_{\leq 0},d_{\text{abs}})$ is a separated pseudometric space, homeomorphic to the coproduct of $(\mathbb{R}_{\geq 0},d_{\text{abs}})$ and $\{+\infty\}$. Show that a map $f : \boldsymbol{X} \to (\overline{\mathbb{R}}_{\leq 0},d_{\text{abs}})$ is continuous, if and only if $Z := f^{-1}(+\infty)$ is open and closed in $\boldsymbol{X}$, and $f : \boldsymbol{X} \smallsetminus Z \to \overline{\mathbb{R}}_{\geq}0$ is continuous.

   b) Let $(X,d_X)$, $(Y,d_Y)$ be pseudometric spaces. Show that the following statements are equivalent.

      i) $f : (X,d_X) \to (Y,d_Y)$ is continuous.

      ii) The map
      $$d_f : (X,d_X)\times(Y,d_Y) \to (\overline{\mathbb{R}}_{\geq 0},d_{\text{abs}}), \quad (x,y) \mapsto d_Y\big(f(x),y\big),$$
      is continuous.

      iii) For every $y \in Y$, the map
      $$d_{f,y} : (X,d_X) \to (\overline{\mathbb{R}}_{\geq 0},d_{\text{abs}}), \quad x \mapsto d_Y\big(f(x),y\big),$$
      is continuous.

   c) Denote by $\widehat{\mathbb{R}}_{\geq 0} = [0,+\infty]$ the Alexandroff compactification of $\mathbb{R}_{\geq 0}$. Recall that $\widehat{\mathbb{R}}_{\geq 0}$ is homeomorphic to $[0,1] \subseteq \mathbb{R}$ (e.g. under the map $x \mapsto x/(x+1)$). Show that the identity map $\phi : (\overline{\mathbb{R}}_{\geq 0},d_{\text{abs}}) \to \widehat{\mathbb{R}}_{\geq 0}$, is continuous but not homeomorphic. Then, show that in replacing $d_f$

by $\phi \circ d_f$ in (b), we still have the implications (i)⇒(ii)⇒(iii), but that (iii)⇒(i) is no longer true.

d) Apply (b) to $f := \mathrm{id}_X$ and conclude that

$$d_X : (X,d_X)\times(X,d_X) \to (\overline{\mathbb{R}}_{\geq 0},d_{\text{abs}}), \quad (x,y) \mapsto d_X(x,y),$$

and, for every $x_0 \in X$,

$$d_{X,x_0} : (X,d_X) \to (\overline{\mathbb{R}}_{\geq 0},d_{\text{abs}}), \quad x \mapsto d_X\big(x_0,x\big),$$

are continuous maps.

**2.2.3 Exercise.** Development of example 1.1.2-(2).

1) Let $h : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ be any map (I) *homogeneous,* (II) *monotonically increasing and* (III) *convex*, *i.e.* verifying the following conditions.

$$(\text{H}) \quad \begin{cases} (\text{I}) \ \ h(t\vec{v}) = th(\vec{v}), & \forall t \in [0,\infty[, \\ (\text{II}) \text{ if } \vec{v} \leq \vec{w} \text{ then } h(\vec{v}) \leq h(\vec{w}), & \\ (\text{III}) \, h(t\vec{v} + (1-t)\vec{w}) \leq th(\vec{v}) + (1-t)h(\vec{w}), & \forall t \in [0,1]. \end{cases}$$

where $\vec{v} := (v_1,\ldots,v_n) \leq \vec{w} := (w_1,\ldots,w_n)$ means that $v_i \leq w_i$, for all $i$.

a) Show that if $(X_i,d_i)$, for $1 \leq i \leq n$, are pseudometric spaces, then

$$d_h((x_1,\ldots,x_n),(x'_1,\ldots,x'_n)) := h(d_1(x_1,x'_1),\ldots,d_n(x_n,x'_n))$$

is a pseudodistance on $X_1\times\ldots\times X_n$.

b) Show that if $h(\vec{v}) = 0$ only if $\vec{v} = 0$, then the previous statement (a) is also true for ‘distances’ instead of ‘pseudodistances’.

2) For $s \in \mathbb{R}_{>0}$, define $h_s : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ by the formula:

$$h_s(\vec{z}) := \sqrt[s]{z_1^s + \cdots + z_n^s}.$$

a) (**) Show that $h_s$ satisfies conditions (H) and (b), for all $s \geq 1$.

*Hint: Take advantage of the derivability of $h_s$. For* (II)*, show that $\frac{d}{dt}\big(h_s(\vec{z}+t\vec{w})\big) \geq 0$, for all $\vec{z} \in \mathbb{R}^n$ and $\vec{w} \in \mathbb{R}^n_{\geq 0}$, which is a condition for increasing monotony; for* (III)*, show that $\frac{d^2}{dt^2}\big(h_s(t\vec{z}_1 + \vec{z}_2)\big) \geq 0$, for all $\vec{z}_i \in \mathbb{R}^n$, which is a condition for convexity.*

b) (**) Show that for fixed $\vec{z} \in \mathbb{R}^n_{\geq 0}$, the function $\mathbb{R}_{\geq 0} \ni s \mapsto h_s(\vec{z})$ is monotonically decreasing.

*Hint. As in (a), one can take advantage of the derivability of $h_s$ relatively to $s$. Hence, show that $\frac{d}{ds}h_s(\vec{z}) \leq 0$, for all $\vec{z} \in \mathbb{R}^n_{\geq 0}$.*

3) Let $\{(X_i,d_i)\}_{i=1,\ldots,n}$ be a family of (pseudo)metric spaces. Thanks to the previous questions, we now define for all $s \geq 1$, the (pseudo)distance

$$d_{\Pi,s}\big((x_1,\ldots,x_n),(x'_1,\ldots,x'_n)\big) := \sqrt[s]{\textstyle\sum_{i=1}^n d_i(x_i,x'_i)^s}\ ,$$

on the set $\varPi := X_1 \times \cdots \times X_n$.

a) Show that for all $\overline{x}, \overline{x}' \in \varPi$ and $s_1 \leq s_2 \in [1,\infty[$, we have:

$$d_{\varPi,s_1}(\overline{x},\overline{x}') \geq d_{\varPi,s_2}(\overline{x},\overline{x}') \quad \text{and} \quad \lim_{s \mapsto \infty} d_{\varPi,s}(\overline{x},\overline{x}') = d_{\varPi,\infty}(\overline{x},\overline{x}'),$$

where $d_{\varPi,\infty}$ is the *chessboard distance* introduced in 1.1.2-(2).

b) Show the following inclusions of open balls centered on $\overline{x} \in \varPi$, respectively relative to the distances $d_1$, $d_s$, $d_\infty$, $d_1$:

$$B_1(\overline{x},R) \subseteq B_s(\overline{x},R) \subseteq B_\infty(\overline{x},R) \subseteq B_1(\overline{x},nR).$$

Conclude that the associated topologies with the (pseudo)metric spaces $(\varPi, d_{\varPi,s})$, they all coincide with the *product topology* (**3**♦3.1).

c) Deduce from (b) that any *affine* map $\phi = \mathbb{R}^n \to \mathbb{R}^m$ is continuous for the topology induced by the Euclidean distance.

**2.2.4 Exercises.** As a continuation of the previous exercise, we give some more recipes to construct new distances from others, on a given set $X$.

1) *Bounding pseudodistances*. Given $(X,d_X)$ and $\epsilon > 0$, let

$$d_{X,\epsilon}(x_1,x_2) := \min(d_X(x_1,x_2),\epsilon).$$

Show that $d_{X,\epsilon}$ is a pseudodistance, which is separated if and only if $d_X$ is so. Show that in all cases $\mathcal{T}(d_{X,\epsilon}) = \mathcal{T}(d_X)$.

2) *Positive linear combination of pseudodistances*. If $\{d_i\}_{i=1,\ldots,n}$ is a family of pseudodistances on $X$, show that $d := \sum_i a_i d_i$, where $a_i \geq 0$, is a pseudodistance.

3) Let $\{d_i\}_{i=1,\ldots,n}$ be a family of pseudodistances on the same set $X$. Following exercise 2.2.3-(3), set for all $x,x' \in X$:

$$d_{X,s}(x,x') := \sqrt[s]{\textstyle\sum_{i=1}^n d_i(x,x')^s}\ .$$

a) Show that $d_{X,s}$ is a pseudodistance on $X$, and it is a distance if one of the $d_i$'s is so.

b) Show that for all $s_1 \leq s_2 \in [1,\infty[$:

$$d_{X,s_1}(x,x') \geq d_{X,s_2}(x,x') \quad \text{and} \quad \lim_{s \mapsto \infty} d_{X,s}(x,x') = \max\{d_i(x,x')\},$$

c) Show that the topologies of the spaces $(X,d_{X,s})$ coincide and that the map $\mathrm{id}_X : (X,d_{X,s}) \to (X,d_i)$ is continuous. In addition, show that if the topologies of the $(X,d_i)$'s coincide, this map is a homeomorphism.

*Hint: All are immediate consequence of exercise 2.2.3-(3) and the fact that $d_{X,s}$ is the pullback pseudodistance on $X$, induced by the diagonal embedding $X \hookrightarrow \varPi := X^n$, $x \mapsto (x,\ldots,x)$.*

4) Let $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a *concave function*, namely:

$$h(ta + (1-t)b) \geq th(a) + (1-t)h(b) \quad \big(\forall a,b \in \mathbb{R}_{\geq 0}\big)\big(\forall t \in [0,1]\big),$$

vanishing at the origin, *i.e.* $h(0) = 0$, (in particular, $h$ is monotonically increasing).

a) Show that if $d$ is a pseudodistance $h \circ d$ is also a pseudodistance, and that it is a distance if $d$ is a distance and 0 is the only zero of $h$, in which case, it is necessarily injective.

b) Show that the functions $h(t) := \min(t,\epsilon)$ and $h(t) := t^s$, with $\epsilon > 0$ and with $0 \leq s \leq 1$ verify the stipulated conditions, so that if $d_X$ is a (pseudo)distance, $\min(d_X,\epsilon)$ and $d_X^s$ are also (pseudo)distances.

Relate these observations to exercises 2.2.4-(1) and 2.2.5-(3)

**2.2.5 Exercises.** Nonseparated pseudodistances.

1) Let $\boldsymbol{X} := (X,d_X)$ be a pseudometric space.

a) Show that $\boldsymbol{X}$ is Hausdorff if and only if $d_X$ is separated.
(*Hint:* if $x \neq x'$ and $d \leq d_X(x,x')/2$, then $B(x,d) \cap B(x',d) = \emptyset$.)

b) Show that the relation $(x \,\mathcal{R}_{\mathrm{sep}}\, y) \Leftrightarrow_{\mathrm{def}} d_X(x,y) = 0$ is an equivalence and that any open (resp. closed) subset of $\boldsymbol{X}$ is $\mathcal{R}_{\mathrm{sep}}$-saturated.

c) Show that the canonical surjection $\nu_{\boldsymbol{X},\mathrm{sep}} : \boldsymbol{X} \twoheadrightarrow \boldsymbol{X}/\mathcal{R}_{\mathrm{sep}}$ is both, an open and a closed map.

d) Show that if $x' \in \mathcal{R}_{\mathrm{sep}}(x)$ and $y' \in \mathcal{R}_{\mathrm{sep}}(y)$ then $d(x',y') = d(x,y)$. Conclude that the function $d_{X_{\mathrm{sep}}}(\nu(x),\nu(y)) := d_X(x,y)$ is well defined and that $\boldsymbol{X}_{\mathrm{sep}} := (X/\mathcal{R}_{\mathrm{sep}}, d_{X_{\mathrm{sep}}})$ is a metric space. Then, show that the natural map $\boldsymbol{X}_{\mathrm{sep}} \to \boldsymbol{X}_{\mathrm{sep}}$ (*cf.* **3♦**9.2) is a homeomorphism.

e) Show that for any continuous map $f : \boldsymbol{X} \to \boldsymbol{Y}$, there exists a unique map $f_{\mathrm{sep}} : \boldsymbol{X}_{\mathrm{sep}} \to \boldsymbol{Y}_{\mathrm{sep}}$ such that the diagram

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\;f\;} & \boldsymbol{Y} \\ {\scriptstyle \nu_{\boldsymbol{X},\mathrm{sep}}}\downarrow & & \downarrow{\scriptstyle \nu_{\boldsymbol{Y},\mathrm{sep}}} \\ \boldsymbol{X}_{\mathrm{sep}} & \xrightarrow{\;f_{\mathrm{sep}}\;} & \boldsymbol{Y}_{\mathrm{sep}} \end{array}$$

is commutative. Show also that $f_{\mathrm{sep}}$ is continuous.

f) Show that the correspondence $(\_)_{\mathrm{sep}} : \mathbf{PMet} \rightsquigarrow \mathbf{Met}$, where $\mathbf{PMet}$ (resp. $\mathbf{Met}$) denotes the category of pseudometric (resp. metric) spaces and continuous maps, that associates $\boldsymbol{X} \rightsquigarrow \boldsymbol{X}_{\mathrm{sep}}$ and $f \rightsquigarrow f_{\mathrm{sep}}$, is a functor.

2) In example **3♦**9.1.1-(2), endow each $E_i$ with the Euclidean metric $d_{\mathrm{e}}$, then, for $E := E_1 \sqcup E_2$, define $d_E(x,y) := d_{\mathrm{e}}(x,y)$ if both $x,y$ belong to the same $E_i$, and $d_E(x,y) := 4$, otherwise. Show that $d_E$ is a distance on $E$. Let $\nu : \boldsymbol{E} \twoheadrightarrow \mathcal{E}_1$ be the canonical surjection.

a) Show that the function

$$h : \mathcal{E}_1 \times \mathcal{E}_1 \to I\!R_+, \quad h(a,b) := d_E\big(\nu^{-1}(a), \nu^{-1}(b)\big),$$

*separates points*, *i.e.* $(h(x,x') = 0) \Leftrightarrow (x = x')$, but that the pushforward pseudodistance $d_{\mathcal{E}_1} := \nu_* d_E$ is not separated.

b) Show that the restriction of $\nu : (E, d_E) \to (\mathcal{E}_1, d_{\mathcal{E}_1})$ to each $E_i$ preserves distances, *i.e.* $d_E(x,y) = d_{\mathcal{E}_1}(\nu(x), \nu(y))$, for all $x,y \in E_i$.

3) Let $\boldsymbol{X} := (I\!R^n, d)$ be the Euclidean $n$ dimensional space (1.1.2-(2)). For all $s \in I\!R_{>0}$, define $h_s : X \times X \to I\!R_{\geq 0}$ to be $h_s(x,x') := d(x,x')^s$. Show that the sequential distance $d_{\mathcal{S},h_s}$, where $\mathcal{S}$ is the set of all sequences on $X$, is such that:

$$\begin{cases} d_{\mathcal{S},h_s} = 0, & \text{if } s > 1; \\ d_{\mathcal{S},h_s} = d^s, & \text{if } s \leq 1. \end{cases}$$

Compare with 2.2.4-(4-b)

**2.2.6 Exercises.** Metrizability of topologies.

1) *A topological space $\mathbf{X} := (X, \mathcal{T}_{\mathbf{X}})$ is said to be pseudometrizable, if $\mathcal{T}_{\mathbf{X}} = \mathcal{T}(d_X)$ for some pseudodistance $d_X$.*

a) Show that on a pseudometric space $(X, d_X)$ where $|X| = 2$, any bijection $f : X \to X$ is a homeomorphism.

b) The *Sierpinski space* is the topological space $\boldsymbol{S} := (\{0,1\}, \mathcal{T}_{\mathbf{S}})$ where $\mathcal{T}_{\mathbf{S}} := \{\emptyset, \{0\}, S\}$. Show that the bijection that exchanges 1 and 0 is not continuous.

c) Conclude that not all topological spaces are pseudometrizables.

2) *A separated topological space $\mathbf{X} := (X, \mathcal{T}_{\mathbf{X}})$ is said to be metrizable, if $\mathcal{T}_{\mathbf{X}} = \mathcal{T}(d_X)$ for some distance $d_X$.*

a) The *Sorgenfrey line* $I\!R_l$ is the real line $I\!R$ endowed with the topology generated by all the half-open intervals $[a,b[$.

Show that $I\!R_l$ is not *second-countable*, *i.e.* there is no countable family of open subsets $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ such that any open subset in $I\!R_l$ is union of $U_n$'s.

*Hint: Otherwise, for $t \in I\!R$, the set $[t, \infty[$ is open in $I\!R_l$ and it exists $U_{n(t)} \in \mathcal{U}$ such that $t \in U_{n(t)}$ and $t \leq U_{n(t)}$. Hence $t \mapsto U_{n(t)}$ is injective, which is incompatible with the countability of $\mathcal{U}$.*

b) Conclude that not all separated topological spaces are metrizable.

*Hint: the set of rational numbers $\mathbb{Q}$ is dense in $I\!R_l$. Hence, if $I\!R_l$ were metrizable, it would also be second-countable.*

3) Show that if a topological space $\boldsymbol{X}$ is pseudometrizable, its Hausdorff quotient $\boldsymbol{X}_{\text{sep}}$ (**3**♦9.2) is metrizable.

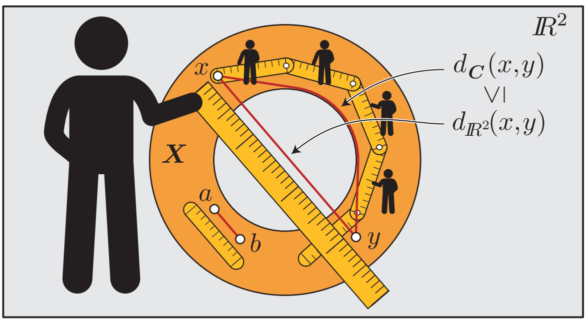
*Hint: Apply exercise 2.2.5-(1).*

## 3 Lenghts of Paths through Local Pseudodistances

That the topology associated with a pseudometric space $(X,d_X)$ depends only on the local behavior of $d_X$ (2.2.1-(b)), leads us to explore more closely the set $\mathcal{D}(X,d_X)$ of all the pseudodistances, *global* or *local* ([3]), which locally coincide with $d_X$. We will see that, under quite natural assumptions, there exists in $\mathcal{D}(X,d_X)$ a distinguished pseudodistance $\widetilde{d}_X$, known as the *path-distance*, or the *intrinsic distance*, which is *universal* in the sense that it is finer (hence greater) than any other pseudodistance in $\mathcal{D}(X,d_X)$, being therefore unique. Furthermore, $\widetilde{d}_X$ may be defined from any given pseudodistance in $\mathcal{D}(X,d_X)$.

**3.1 Locally Equal Distances.** Two pseudodistances $d_1$ and $d_2$ defined on a set $X$, are said to be *locally equal* if, for every $x \in X$, there exists $\epsilon > 0$ such that $B_{d_1}(x,\epsilon) = B_{d_2}(x,\epsilon)$ and that $d_1 = d_2$ on those open balls.

As an illustration, we can consider two very natural distances defined on the annulus $\boldsymbol{X} \subseteq \mathbb{R}^2$ of the figure below.




A distance between $x,y \in \boldsymbol{X}$ can be measured in at least two ways.

- Extrinsically: from the ambient space $\mathbb{R}^2$ and with a sufficiently big ruler, hence obtaining the distance $d_{\mathbb{R}^2}(x,y)$ with one single measure!
- Intrinsically: by beings who live inside the annulus and who, therefore, can only use 'small' rulers whose dimensions do not exceed the limits of $\boldsymbol{X}$. In this case, if $x$ and $y$ are 'far away from each other', *i.e.* beyond the scope of the rulers, many intermediate measures can be needed, each seeking for shorter paths, so that, step by step, one gets an estimation of a distance $d_{\boldsymbol{X}}(x,y)$ which should be lowest bound of such estimations.

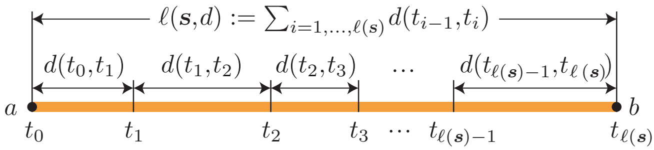
The process clearly leads to the *a priori* inequality: $d_{\boldsymbol{X}}(x,y) \geq d_{\mathbb{R}^2}(x,y)$, which is generally *strict*, unless obviously the two points $a,b \in \boldsymbol{X}$ are 'close enough', and a single measure suffices, in which case $d_{\mathbb{R}^2}(a,b) = d_{\boldsymbol{X}}(a,b)$, and this is what we mean when we say that "*$d_{\mathbb{R}^2}$ and $d_{\boldsymbol{X}}$ are locally equal*".

([3]) The distance is *global* if it is defined on the whole $X \times X$, and *local* when it is only defined for nearby points, *i.e.* in a neighborhood of the diagonal $\Delta_X \subseteq X \times X$.

**3.1.1 Comment.** As a metamathematical reflection, the previous example tells us that if in our universe $(\boldsymbol{U},d_{\boldsymbol{U}})$ there were, in all solar systems, beings who, like us, have the ability to measure distances in their neighborhood, and who, in a collective work would participate in measuring distances between very distant points seeking shortest paths, even combining our efforts, we might be unable to realize that, viewed from the 'outside', these same points might be extremely close to each other. And yet at zero distance, which would reveal that our universe, even if separated for us, would be a subspace of a larger *nonseparated* pseudometric space $\widetilde{\boldsymbol{U}} := (\widetilde{U},d_{\tilde{U}})$.

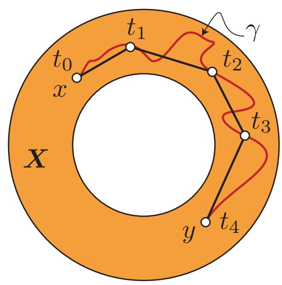
## 3.2 The Length of a Map $\gamma : [a,b] \to (X,d_X)$

For $a \leq b \in \mathbb{R}$, denote by $\mathcal{S}_{\mathbb{R}}(a,b)$ the set of *ordered* sequences of real numbers $\boldsymbol{s} := (a = t_0 \leq t_1 \leq \cdots \leq t_{\ell(\boldsymbol{s})} = b)$. If $d$ is a pseudodistance on $[a,b]$, the *length of $[a,b]$ relative to $d$* is, by definition, the upper bound (finite or not) of sequential lengths $\ell(\boldsymbol{s},d) := \sum_{i=1,\ldots,\ell(\boldsymbol{s})} d(t_{i-1},t_i)$, as $\boldsymbol{s}$ ranges over $\mathcal{S}_{\mathbb{R}}(a,b)$,




*i.e.* , we set:

$$\boxed{\ell([a,b],d) := \max\{\ell(\boldsymbol{s},d) \mid \boldsymbol{s} \in \mathcal{S}_{\mathbb{R}}(a,b)\}}$$




Now, if $\gamma : [a,b] \to (X,d_X)$ is a any *map* (not necessarily continuous), its *length* $\ell(\gamma,d_X)$ is, by definition, the length of $[a,b]$ relative to the pull-back distance $\gamma^* d_X$ (1.1.2-(4)), *i.e.* we set:

$$\boxed{\ell(\gamma,d_X) := \ell\big([a,b],\gamma^* d_X\big)} \qquad (4)$$

This corresponds to the intuitive idea expressed in the above figure to the right, where the length of a path (in red) is the upper bound of the lengths of the polygonal paths (in black) defined by its intermediate points (in white).

**3.2.1 Exercises.** 1) Let $\phi : [c,d] \to [a,b]$ be a bijection between real intervals. Show that $\ell(\gamma,d_X) = \ell(\gamma \circ \phi,d_X)$, for all $\gamma : [a,b] \to (X,d_X)$, if and only if $\phi$ is monotonic (*i.e.* is a homeomorphism). This is usually formulated by: *the length of a path does not depend on its parametrization*.

2) Given a map $\gamma : [0,1] \to (X,d_X)$ and $t \in [0,1]$, show that

$$\ell(\gamma,d_X) = \ell(\gamma|_{[0,t]},d_X) + \ell(\gamma|_{[t,1]},d_X),$$

which is usually formulated by *the length of the concatenation of two paths is the sum of their lengths* (*cf.* **3**♦8.3.1-(b)).

**3.3 The Length of a Path $\gamma : [a,b] \to (X,d_X)$.** An important property that occurs when the map $\gamma : [a,b] \to (X,d_X)$ is continuous, *i.e.* when it is a *path*, is that its length is the same when computed with any two *locally* equal distances, in other words, it depends only on the local behavior of the distance. This is stated in the next proposition.

**3.3.1 Proposition.** *Let $\boldsymbol{X} := (X,d_X)$ be a pseudometric space.*

a) *For any map $\gamma : [0,1] \to \boldsymbol{X}$ (continuous or not), we have:*

$$d_X(\gamma(0),\gamma(1)) \le \ell(\gamma,d_X).$$

b) *Let $\gamma : [0,1] \to \boldsymbol{X}$ be a map of finite length. The map $\gamma$ is continuous if and only if the increasing function $\ell_{\gamma,d_X} : [0,1] \to \mathbb{R}_{\ge 0}$, $t \mapsto \ell(\gamma|_{[0,t]},d_X)$, is continuous.*

c) *If $d'_X : X\times X \to \mathbb{R}_{\ge 0}$ is a pseudodistance locally equal to $d_X$, then a map $\gamma:[0,1]\to X$ is a path in $(X,d_X)$ if and only if it is a path in $(X,d'_X)$. In that case, its length is the same whether computed with $d_X$ or $d'_X$,* i.e.

$$\ell(\gamma,d_X) = \ell(\gamma,d'_X).$$

*Proof.* (a) Obvious, because $d_X(\gamma(0),\gamma(1)) \le d_X(\boldsymbol{s})$, for each $\boldsymbol{s} \in \mathcal{S}_{\mathbb{R}}(0,1)$, by the triangle inequality.

(b) By definition of the length of $\gamma$, we have, for all $t < u \in [0,1]$,

$$\ell_\gamma(u) = \lim_{\epsilon\mapsto 0}\Big(\ell_\gamma(t) + d_X\big(\gamma(t),\gamma(t+\epsilon)\big) + \ell_\gamma(t+\epsilon,u)\Big),$$

where $\ell_\gamma(t+\epsilon,u)$ denotes $\ell(\gamma|_{[t+\epsilon,u]},d_X)$. In particular, if $\ell_\gamma(v) < +\infty$, all the terms in the previous equality are finite and if we subtract $\ell_\gamma(u)$ we get:

$$\begin{aligned}\lim_{\epsilon\mapsto 0} d_X\big(\gamma(t+\epsilon),\gamma(t)\big) &= \lim_{\epsilon\mapsto 0}\big(\ell_\gamma(u) - \ell_\gamma(t+\epsilon,u)\big) - \ell_\gamma(t)\\ &= \lim_{\epsilon\mapsto 0}\ell_\gamma(t+\epsilon) - \ell_\gamma(t),\end{aligned}$$

and (b) follows.

(c) Given a sequence $\boldsymbol{s} := (0 = t_0 \le t_1 \le \dots \le t_{\ell(\boldsymbol{s})} = 1)$ on $[0,1]$, if we denote by $\boldsymbol{s}'$ any sequence obtained from $\boldsymbol{s}$ by adding some intermediate point, for example $\boldsymbol{s}' := (0 = t_0 \le t'_0 \le t_1 \le \cdots 1)$, we have

$$d(\boldsymbol{s}) \le d(\boldsymbol{s}'), \tag{$*$}$$

for any pseudodistance $d$, simply because of the triangle inequality:

$$d(\gamma(t_1),\gamma(t_0)) \le d(\gamma(t_1),\gamma(t'_0)) + d(\gamma(t'_0),\gamma(t_0)).$$

More generally, if $\boldsymbol{s}$ and $\boldsymbol{t}$ are sequences on $[0,1]$, denote by $\boldsymbol{s} \preceq \boldsymbol{t}$ the fact that $\boldsymbol{t}$ is obtained from $\boldsymbol{s}$ by adding a finite number of intermediate points. The inequality $(*)$ then clearly generalizes to show that

$$\big(\boldsymbol{s} \preceq \boldsymbol{t}\big) \Rightarrow \big(d(\boldsymbol{s}) \le d(\boldsymbol{t})\big). \tag{$**$}$$

Now, as $d_X$ and $d'_X$ are locally equal, we may choose, for every $x \in \boldsymbol{X}$, an open subset $W_x \ni x$, where $d_X = d'_X$, and, thanks to the continuity of $\gamma$, the family $\{\gamma^{-1}(W_t)\}_{0\le t\le 1}$ is an open cover of the compact space $[0,1] \subseteq I\!R$. Then, by the Heine-Borel-Lebesgue theorem, there exists $\epsilon > 0$ such that if $|a-b| < \epsilon$, the set $\{\gamma(a),\gamma(b)\}$ is contained in some $W_x$, in which case

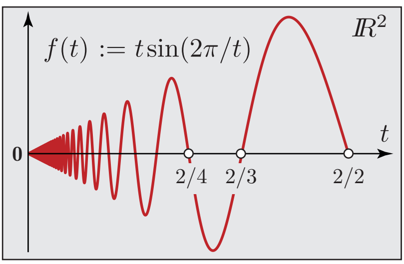
$$d_X(\gamma(a),\gamma(b)) = d'_X(\gamma(a),\gamma(b)). \qquad (\dagger)$$

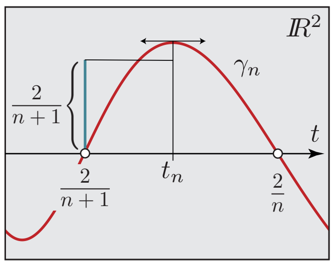
We can now conclude. For any pair of sequences $\boldsymbol{s}$ and $\boldsymbol{t}$, take any sequence $\boldsymbol{u}$ that merges the intermediate terms of $\boldsymbol{s}$ and $\boldsymbol{t}$, and that, moreover, incorporates as many terms as necessary to warrant that two successive terms $u_{i-1}$ and $u_i$ satisfy $|u_{i-1} - u_i| < \epsilon$. Then, after $(**)$ and $(\dagger)$, we have

$$d_X(\boldsymbol{s}) \le d_X(\boldsymbol{u}) = d'_X(\boldsymbol{u}) \ge d'_X(\boldsymbol{t}),$$

from which, $\max\{d_X(\boldsymbol{s}) \mid \boldsymbol{s} \in \mathcal{S}_{I\!R}(0,1)\} = \max\{d'_X(\boldsymbol{t}) \mid \boldsymbol{t} \in \mathcal{S}_{I\!R}(0,1)\}$. □

**3.3.2 Remark.** The finiteness length condition in 3.3.1-(b) is necessary. Indeed, from the classical result that the real function $f(t) := t\sin(2\pi/t)$ is continuous for $t \in [0,1]$, one easily shows that $t \mapsto (t, f(t))$ is a path in $I\!R^2$.




The vanishing points for $f$ are of the form $2/n$ for $n \ge 2$. Denote by $\gamma_n$ the restriction of $\gamma$ to $\boldsymbol{J}_n := [2/(n+1), 2/n]$, and by $t_n$ the point where $\gamma_n$ has its extremum. For some $u_n \in \boldsymbol{J}_n$, we have $\sin(2\pi/u_n) = 1$ and $|f(u_n)| = u_n$ so that

$$\ell(\gamma_n) > |f(t_n)| \ge u_n \ge 2/(n+1),$$

but then

$$\ell(\gamma) = \textstyle\sum_{n\ge 2} \ell(\gamma_n) > 2 \sum_{n\ge 3} \frac{1}{n} = +\infty.$$

The same argument shows that $\ell(\gamma|_{[0,\epsilon]}) = +\infty$ for all $\epsilon > 0$.

The function $\ell_\gamma : [0,1] \to \overline{I\!R}_{\ge 0}$ is therefore not continuous at $t = 0$, because we have $\ell_\gamma(0) = 0$ but $\ell_\gamma(t) = +\infty$, for all $t > 0$. Otherwise, the function is continuous over $(0,1]$, after 3.3.1-(b) and because $\ell(\gamma_n) < \frac{4}{n} + (\frac{2}{n} - \frac{2}{n+1})$ which implies that $\ell(\gamma|_{[\epsilon,1]}) < +\infty$ for all $\epsilon > 0$. (*cf.* exercise 2.2.2-(2-a))

# 4 Path-Pseudodistances

**4.1 The Path-Pseudodistance $\widetilde{d}_X$.** Let $\boldsymbol{X} := (X, d_X)$ be a pseudometric space. The *path-pseudodistance $\widetilde{d}_X$ generated by $d_X$*, also called the *intrinsic pseudodistance of $\boldsymbol{X}$*, is the lower bound of the lengths of paths joining points in $X$. More precisely

$$\widetilde{d}_X(x,y) := \begin{cases} +\infty, & \text{if } \Gamma_{d_X}(x,y) = \emptyset, \\ \min\{\ell(\gamma, d_X) \mid \gamma \in \Gamma_{d_X}(x,y)\}, & \text{otherwise.} \end{cases}$$

where $\Gamma_{d_X}(x,y)$ denotes the set of paths joining $x$ to $y$ in $\boldsymbol{X}$.

The following proposition is elementary and left to the reader.

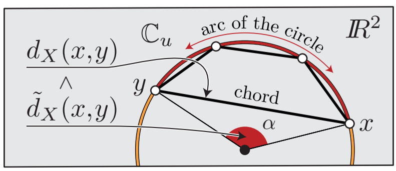
**4.1.1 Proposition.** *The path-pseudodistance $\widetilde{d}_X$ is a pseudodistance on $X$,* i.e. *for all $x,y,z \in X$, one has:*

*E-0)* $\widetilde{d}_X(x,y) \in \mathbb{R}_{\geq 0} \sqcup \{+\infty\}$, *E-1)* $\widetilde{d}_X(x,x) = 0$,

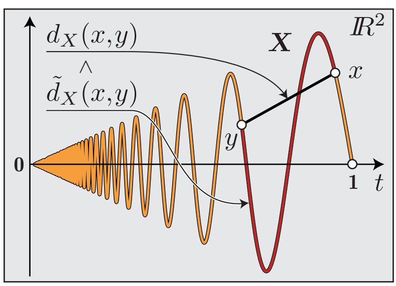
*E-2)* $\widetilde{d}_X(x,y) = \widetilde{d}_X(y,x)$, *E-3)* $\widetilde{d}_X(x,z) \leq \widetilde{d}_X(x,y) + \widetilde{d}_X(y,z)$.

Be aware that the distance $\widetilde{d}_X(x,y)$ can be infinite even if the set $\Gamma_{d_X}(x,y)$ is nonempty (see example 4.1.2-(2)).

**4.1.2 Examples.** 1) In the unit circle $(\mathbb{C}_u, d_e)$, the *chordal distance* is the distance $d_e$ induced by the Euclidean space $\mathbb{R}^2 (= \mathbb{C})$, and the *intrinsic distance* $\widetilde{d}_e$ between $x, y \in \mathbb{C}_u$ is the length of the smallest arc joining $x$ and $y$ (the usual measure of angles in *radians*). Notice that for $\alpha \leq \pi/4$, one has $d_e \leq \widetilde{d}_e \leq \frac{\alpha/2}{\sin(\alpha/2)} d_e$, which implies that the associated topologies with $d_e$ and $\widetilde{d}_e$ are the same.




2) In $\boldsymbol{X} := \{(t, t\sin(4\pi/t)) \mid t \in [0,1]\} \subseteq \mathbb{R}^2$, the *chordal distance* is the distance $d_X$ induced by $\mathbb{R}^2$, and the *intrinsic distance* $\widetilde{d}_X$ is the arc length. Notice that, while the chordal distance is bounded by 2, the intrinsic distance is unbounded, and even infinite between the endpoints $\mathbf{0} := (0,0)$ and $\mathbf{1} := (1,0)$ (*cf.* 3.3.2), and this, even though $\boldsymbol{X}$ is path-connected and *compact*. Furthermore, the topology $\mathcal{T}(\widetilde{d}_X)$ is *strictly* finer than $\mathcal{T}(d_X)$. Indeed, if $x > 0$, the open ball $B_{\widetilde{d}_X}(x,R)$, with $R \leq x$, belongs to $\mathcal{T}(d_X)$, contrary to $B_{\widetilde{d}_X}(\mathbf{0},R)$ which is the singleton $\{\mathbf{0}\}$, for any $R > 0$. This implies that $(X, \widetilde{d}_X)$ is homeomorphic to the coproduct $\{\mathbf{0}\} \sqcup \mathbb{R}$, which is neither connected or compact.



**4.1.3 Exercises.** 1) Work out the details in the two previous examples.

2) This is the analog of exercise 5.1.3-(1). Show that the binary relation $(x\ \mathcal{R}\ y) \Leftrightarrow_{\mathrm{def}} \Gamma_{d_X}(x,y) \neq \emptyset$ is an equivalence whose equivalence classes are open. Thereby, if $\boldsymbol{X}$ is a connected space, $\Gamma_{d_X}(x,y) \neq \emptyset$, for all $x,y$.

**4.1.4 Proposition.** *Let $(X,d_X)$ be a pseudometric space.*

a) $\widetilde{d}_X \geq d_X$.

b) *If $\gamma : [0,1] \to (X,d_X)$ is a* path, *then $\ell(\gamma,d_X) = \ell(\gamma,\widetilde{d}_X)$. And, if furthermore $\ell(\gamma,d_X) < +\infty$, then $\gamma$ is also a path in $(X,\widetilde{d}_X)$.*

c) $\widetilde{\widetilde{d}}_X = \widetilde{d}_X$.

d) *If $d'_X$ is a pseudodistance locally equal to $d_X$, then $\widetilde{d}'_X = \widetilde{d}_X$.*

*Proof.* (a) Results from 3.3.1-(a) which states that $d_X(x,y)$ is a lower bound for all $\ell(\gamma,d_X)$ with $\gamma \in \Gamma_{d_X}(x,y)$, but then

$$d_X(x,y) \leq \min\bigl\{\ell(\gamma,d_X) \bigm| \gamma \in \Gamma_{d_X}(x,y)\bigr\} = \widetilde{d}_X(x,y). \qquad (*)$$

(b) The inequality (a) immediately gives the inequality

$$\ell(\gamma,d_X) \leq \ell(\gamma,\widetilde{d}_X). \qquad (\ddagger)$$

But then,

— if $\ell(\gamma,d_X) = +\infty$, we have $\ell(\gamma,\widetilde{d}_X) = +\infty$, and (b) is satisfied;

— if $\ell(\gamma,d_X) < +\infty$, we have, for any closed subinterval $[t_1,t_2] \subseteq [0,1]$,

$$\widetilde{d}_X(\gamma(t_1),\gamma(t_2)) \leq \ell(\gamma|_{[t_1,t_2]},d_X),$$

because $\gamma|_{[t_1,t_2]}$ is a *path* in $(X,d_X)$ joining $\gamma(t_1)$ to $\gamma(t_2)$. Then, for any sequence $\boldsymbol{s} \in \mathcal{S}_{\mathbb{R}}(0,1)$, we have (*cf.* 3.3):

$$\ell(\boldsymbol{s},\gamma^*\widetilde{d}_X) = \sum\nolimits_i \widetilde{d}_X(\gamma(t_i),\gamma(t_{i+1})) \leq \sum\nolimits_i \ell(\gamma|_{[t_i,t_{i+1}]},d_X) = \ell(\gamma,d_X).$$

Therefore $\ell(\gamma,\widetilde{d}_X) \leq \ell(\gamma,d_X)$, and the equality follows by ($\ddagger$).

For the last claim in (b), we now apply proposition 3.3.1-(b): thanks to the fact that $\ell(\gamma,\widetilde{d}_X) = \ell(\gamma,d_X) < +\infty$, the continuity of $\gamma$ for $\widetilde{d}_X$ is equivalent to the continuity of the function $\ell_{\gamma,\widetilde{d}_X}$. But this function coïncides with $\ell_{\gamma,d_X}$, which is continuous because $\gamma$ is so.

(c) Let $x,y \in X$. If $\widetilde{d}_X(x,y) = +\infty$, then $\widetilde{\widetilde{d}}_X(x,y) = +\infty$, as $\widetilde{\widetilde{d}}_X \geq \widetilde{d}_X$ after (a). If otherwise $\widetilde{d}_X(x,y) < +\infty$, then any path of finite length of $(X,d_X)$ is also a *path* of $(X,\widetilde{d}_X)$ of the same length, after (b). Therefore $\widetilde{\widetilde{d}}_X(x,y) \leq \widetilde{d}_X(x,y)$ and the equality follows again after (a).

(d) After 3.3.1-(c), the spaces $(X,d_X)$ and $(X,d'_X)$ have the same paths and equal lengths, so that the equality $\widetilde{d}_X = \widetilde{d}'_X$ is obvious. □

**4.1.5 Exercises.** Consider the conditions:

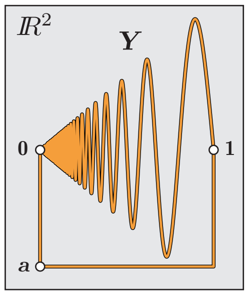
$\widetilde{d}$-i) $\widetilde{d}_X$ is *finite* on each connected component of $\boldsymbol{X}$;

$\widetilde{d}$-ii) the topologies associated with $\widetilde{d}_X$ and $d_X$ coincide.

We already know after exercise 2.2.2-(1) that ($\widetilde{d}$-ii)$\Rightarrow$($\widetilde{d}$-i).

1) Condition ($\widetilde{d}$-ii) says that for all $\epsilon > 0$ there exists $\eta > 0$ such that for all $z \in B_{d_X}(x,\eta)$ there exists a path $\gamma : [0,1] \to \boldsymbol{X}$ joining $x$ to $z$ of length $\ell(\gamma,d_X) < \epsilon$. Show that this is so in 4.1.2-(1), but not in (2).

2) This exercise gives an example where condition ($\widetilde{d}$-i) is satisfied whereas ($\widetilde{d}$-ii) is not. Connect the endpoints in 4.1.2-(2), as shown in the figure to the right, and call $\boldsymbol{Y} := (Y,d_Y)$ the resulting metric space where $d_Y$ is the distance induced by $\mathbb{R}^2$. Show that $\boldsymbol{Y}$ is homeomorphic to the circle $\mathbb{C}_u$. Show that the extended metric $\widetilde{d}_{\boldsymbol{Y}}$ is finite and unbounded. Show that the topology $\mathcal{T}(\widetilde{d}_Y)$ is strictly finer than $\mathcal{T}(d_Y)$. *Hint: the closed-open interval* $[\mathbf{0},\boldsymbol{a}) \subseteq \mathbb{R}^2$ *is the open ball* $B_{\widetilde{d}_Y}(\mathbf{0},|\mathbf{a}|)$. Show that there exists a map $f : (Y,\widetilde{d}_Y) \to \mathbb{R}_{\geq 0}$ preserving distances, hence a homeomorphism. Conclude that the identity map $\mathrm{id} : (Y,\widetilde{d}_Y) \to (Y,d_Y)$, $x \mapsto x$, is continuous but is not a homeomorphism.




## 5 Coherent Families of Local Pseudodistances

**5.1** The proof of proposition 3.3.1-(c) shows that in order to compute the length of a *path* on $(X,d_X)$, we do not need to know the distance $d_X$ for every pair of points $(x,y) \in X \times X$, but only for nearby points, in other words, the local knowledge of $d_X$. On the other hand, we will see that many topological spaces, notably quotients and assemblages of metric spaces, arrive naturally endowed only with local data, and these, although local, suffice to give a meaning to the length of *any* path in $X$. One is thus lead to consider the following extension of the notion of pseudodistance.

**5.1.1 Definitions.** Let $\boldsymbol{X} := (X,\mathcal{T})$ be a topological space.

*LD*-1) A *coherent family of local pseudodistances* for $\boldsymbol{X}$, denoted by $d_{\mathcal{U}}$, consists of the following data and conditions:

i) A cover $\mathcal{U} := \{U_k\}_{k\in\mathfrak{K}}$ of $\boldsymbol{X}$ by open subsets called the *domains* of $d_{\mathcal{U}}$.

ii) For each $k \in \mathfrak{K}$, a pseudodistance $d_{U_k}$ in $U_k$, such that the associated topology $\mathcal{T}(d_{U_k})$ is the induced topology by $\boldsymbol{X}$.

iii) For each $k,l \in \mathfrak{K}$, the restrictions of $d_{U_k}$ and $d_{U_l}$ to $U_k \cap U_l$ coincide.

iv) Denote by $\mathcal{S}_{\mathcal{U}}$ the set of finite sequences $\mathbf{s} := (x_0,\ldots,x_{\ell(\mathbf{s})})$ of elements in $\boldsymbol{X}$ such that every two successive elements $\{x_{i-1},x_i\}$ belong to a same domain $U_?$. The set $\mathcal{S}_{\mathcal{U}}$ is admissible in the sense of 1.1.2-(5). Notice that for $\boldsymbol{s} \in \mathcal{S}_{\mathcal{U}}$, the sum: $\ell(\boldsymbol{s}) := \sum_{i=1,\ldots,\ell(s)} d_{U_?}(x_{i-1},x_i)$ is well defined thanks to (iii). We can thus define the sequential distance associated with $\mathcal{S}_{\mathcal{U}}$, which we denote by the same notation $d_{\mathcal{U}}$, by:

$$d_{\mathcal{U}}(x,y) := \begin{cases} +\infty, & \text{if } \mathcal{S}_{\mathcal{U}}(x,y) = \emptyset, \\ \min\{\ell(\boldsymbol{s}) \mid \boldsymbol{s} \in \mathcal{S}_{\mathcal{U}}(x,y)\}, & \text{otherwise.} \end{cases}$$

*LD*-2) The topology $\mathcal{T}$ is the topology $\mathcal{T}(d_{\mathcal{U}})$.

*LD*-3) The length of a *path* $\gamma : [0,1] \to (\boldsymbol{X},d_{\mathcal{U}})$, is the length $\ell(\gamma,d_{\mathcal{U}})$ defined by the same formula 3.2-(4).

The condition (*LD*-2) is quite subtle. It will be convenient to state its fundamental property in a proposition.

**5.1.2 Proposition.** *The condition (LD-2) is equivalent to the fact that: for $i \in \mathfrak{I}$ and $x \in U_i$, there exists $\epsilon > 0$ such that $B_{d_{\mathcal{U}}}(x,\epsilon) = B_{d_{U_i}}(x,\epsilon) \subseteq U_i$, where, moreover, $d_{\mathcal{U}}$ and $d_{U_i}$ are equal.*

*Proof.* Suppose $\mathcal{T} = \mathcal{T}(d_{\mathcal{U}})$. For all $i \in \mathfrak{I}$ and $z \in U_i$, there exists $\epsilon > 0$ such that $B_{d_{\mathcal{U}}}(z,\epsilon) \subseteq U_i$. For $a,b \in B_{d_{\mathcal{U}}}(z,\epsilon/3)$, $d_{\mathcal{U}}(a,b) \leq 2\epsilon/3$, and, for all $\boldsymbol{s} := (a = x_0,\ldots,x_n = b) \in \mathcal{S}_{\mathcal{U}}$ verifying $\ell(\boldsymbol{s}) = \sum_j d_{U_?}(x_j,x_{j+1}) < 2\epsilon/3$, we have $d_{\mathcal{U}}(a,x_j) < 2\epsilon/3$ and then $d_{\mathcal{U}}(z,x_j) \leq d_{\mathcal{U}}(z,a) + d_{\mathcal{U}}(a,x_j) < \epsilon$. As a consequence, $x_j \in U_i$, for all $j$, and $\ell(\boldsymbol{s}) = \sum_j d_{U_i}(x_j,x_{j+1}) \geq d_{U_i}(a,b)$. Therefore $d_{\mathcal{U}}(a,b) \leq d_{U_i}(a,b) \leq \ell(\boldsymbol{s})$, and $d_{\mathcal{U}}(a,b) = d_{U_i}(a,b)$, because $\ell(\boldsymbol{s})$ can be as close as desired to $d_{\mathcal{U}}(a,b)$. The converse is immediate. $\square$

**5.1.3 Exercises.** 1) Show that in (*LD*-1-iv), the relation $(x \mathcal{R} y) \Leftrightarrow_{\text{def}} \mathcal{S}_{\mathcal{U}}(x,y) \neq \emptyset$ is an equivalence whose equivalence classes are open. Thereby, if $\boldsymbol{X}$ is a connected space, $\mathcal{S}_{\mathcal{U}}(x,y) \neq \emptyset$, for all $x,y$.

2) Let $\boldsymbol{X} := (X,d_X)$ be a pseudometric space, and $\mathcal{U} := \{U_i\}_{i\in\mathfrak{I}}$ an open cover of $\boldsymbol{X}$. Show that, if we endow each $U_i$ with the induced distance $d_{U_i}$, we get a local pseudodistance $d_{\mathcal{U}}$ for $\boldsymbol{X}$. Show that $d_X \leq d_{\mathcal{U}}$, and that this inequality can be strict (*hint:* the annulus example in 3.1). Show that $d_X$ and $d_{\mathcal{U}}$ are nevertheless equal on each $U_i \in \mathcal{U}$.

3) Exercise 2.2.5-(2) gave un example where the distance $d_{\mathcal{U}}$ can fail to be separated while the $d_i$'s are all so. It also showed that all the inclusion maps $\iota_i : (U_i,d_{U_i}) \hookrightarrow (X,d_{\mathcal{U}})$ preserve distances. Give example where the $d_i$'s are separated, $d_{\mathcal{U}}$ is not separated and none of the $\iota_i$ preserve distances (although we know that they do preserve distances *locally*).

The next proposition extends 3.3.1 for coherent families of local distances.

**5.1.4 Proposition.** a) *Let $d_{\mathcal{U}}$ be a local pseudodistance for $\boldsymbol{X}$. For $i \in \mathfrak{I}$ and for each $x \in U_i$, there exists $\epsilon > 0$ such that, $\mathcal{B}_{d_{\mathcal{U}}}(x,\epsilon) \subseteq U_i$, and such that for every path $\gamma : [0,1] \to \boldsymbol{X}$ joining $a$ to $b$ in $B_{d_{\mathcal{U}}}(x,\epsilon/3)$,*

i) $d_{\mathcal{U}}(a,b) \leq \ell(\gamma,d_{\mathcal{U}})$.

ii) *If $\gamma$ is not contained in $B_{d_{\mathcal{U}}}(x,\epsilon)$, then $\ell(\gamma,d_{\mathcal{U}}) \geq 2\epsilon/3$.*

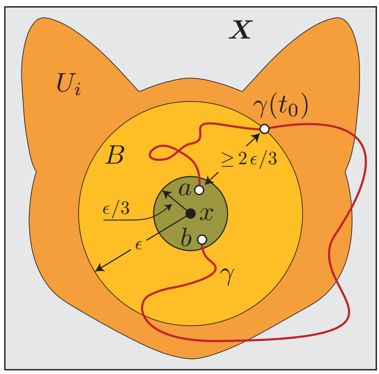
b) *If $d_{\mathcal{U}}$ and $d_{\mathcal{V}}$ are locally equal local pseudodistances for $\boldsymbol{X}$, then for any path $\gamma : [0,1] \to \boldsymbol{X}$, we have:*

$$\ell(\gamma,d_{\mathcal{U}}) = \ell(\gamma,d_{\mathcal{V}}).$$

*Proof.* (a) Given $x \in U_i$, let $R > 0$ such that $B_{d_{\mathcal{U}}}(x, R) = B_{d_{U_i}}(x, R) \subseteq U_i$, and fix any $\epsilon > 0$ verifying $0 < \epsilon < R$. We claim that, if $\gamma : [0,1] \to \boldsymbol{X}$, with initial point $\gamma(0) \in B_{d_{\mathcal{U}}}(x,\epsilon)$, is *not* totally contained in $B_{d_{\mathcal{U}}}(x,\epsilon)$, then there exists a smallest $t \in [0,1]$ such that $d_{\mathcal{U}}(x,\gamma(t)) = \epsilon$.

Indeed, otherwise $\gamma$ would be a path in the disjoint union $A \sqcup B_{d_{\mathcal{U}}}(x,\epsilon)$, where $A$ is the set $\{y \in \boldsymbol{X} \mid d_{\mathcal{U}}(x,y) > \epsilon\}$, which is easily seen to be open in $\boldsymbol{X}$. But then $[0,1]$ would be the disjoint union of open subsets $\gamma^{-1}(A) \sqcup \gamma^{-1}(B_{d_{\mathcal{U}}}(x,\epsilon))$, which is contrary to the connectedness of $[0,1]$.

The set of $t \in [0,1]$ verifying $d_{\mathcal{U}}(x,\gamma(t)) = \epsilon$ is, thus, *closed and nonempty* in the compact interval $[0,1]$, and it has, as such, a lower element $t_0$. Thereby, if, now, $a := \gamma(0)$ belongs to $B_{\mathcal{U}}(x,\epsilon/3)$, we get




$$\begin{aligned}\ell(\gamma,d_{\mathcal{U}}) &\geq \ell(\gamma|_{[0,t_0]},d_{\mathcal{U}})\\ &\geq_1 d_{U_i}(\gamma(t_0),a)\\ &\geq d_{U_i}(\gamma(t_0),x) - d_{U_i}(x,a)\\ &\geq \epsilon - \epsilon/3 = 2\epsilon/3.\end{aligned} \qquad (\dagger)$$

Where '$\geq_1$' comes from the fact that the subpath $\gamma|_{[0,t_0]}$ lies within $B_{d_{\mathcal{U}}}(x,R) \subseteq U_i$, where $d_{\mathcal{U}} = d_{U_i}$. Then 3.3.1-(a) applies and justifies the other inequalities.

Now, if $\gamma : [0,1] \to \boldsymbol{X}$ is a path joining two points $a,b \in B_{d_{\mathcal{U}}}(x,\epsilon/3)$, there are two cases to consider. Either $\gamma$ is contained in $U_i$, in which case we already know, by 3.3.1-(a), that

$$d_{U_i}(a,b) \leq \ell(\gamma,d_{\mathcal{U}}); \qquad (\dagger\dagger)$$

either $(\dagger)$ applies, and $(\dagger\dagger)$ is still verified, because $d_{U_i}(a,b) < 2\epsilon/3 \leq \ell(\gamma,d_{\mathcal{U}})$.

(b) Same as for 3.3.1-(c), details are left to the reader. □

**5.2 The Path-Pseudodistance $\widetilde{d}_{\mathcal{U}}$.** In defining a path-pseudodistance, we only need to know how to compute le length of a path, which can be done by any coherent family of local distances. Therefore, if $d_{\mathcal{U}}$ is such a family of pseudodistances for $\boldsymbol{X}$, the same approach as in 4.1 can be followed. We will denote by $\widetilde{d}_{\mathcal{U}}$ the path-pseudodistance associated with $d_{\mathcal{U}}$.

# 6 Path-Pseudometric Spaces

**6.1 Definition.** A pseudometric space $\boldsymbol{X} := (X,d_X)$ is a *path-pseudometric space*, if the path-pseudodistance $\widetilde{d}_X$ and the pseudodistance $d_X$ are locally equal, *i.e.* if the identity map $\mathrm{id}_X : (X,\widetilde{d}_X) \to (X,d_X)$ is a *local isometry* (hence a homeomorphism).

A topological space $\boldsymbol{X} := (X,\mathcal{T}_{\mathbf{X}})$ is said to be *path-pseudometrizable* if there exists a path-pseudodistance $\widetilde{d}_X$ such that $\mathcal{T}_{\mathbf{X}} = \mathcal{T}(\widetilde{d}_X)$.

The following theorem states fundamental properties of path-pseudometric spaces, notably, in (c,d), the local nature of path-pseudometrics.

**6.1.1 Theorem.** a) *If $(X,d_X)$ is a pseudometric space then $\widetilde{\widetilde{d}}_X = \widetilde{d}_X$ and $(X,\widetilde{d}_X)$ is path-pseudometric.*

b) *A connected path-pseudometric space $\boldsymbol{X} := (X,d_X)$ is path-connected and locally path-connected.*

c) *Let $\mathcal{U} := \{U_i\}_{i\in\mathfrak{I}}$ be an open cover of a topological space $\boldsymbol{X}$, and let $d_{\mathcal{U}}$ be coherent family of local pseudodistances for $\boldsymbol{X}$ (5.1.1). If $(U_i,d_{U_i})$ is a path-pseudometric space, then $\boldsymbol{X}$ is a path-pseudometrizable space. More precisely, $\widetilde{d}_{\mathcal{U}}$ in the only pseudodistance $d_X$ on $\boldsymbol{X}$, such that $\widetilde{d}_X = d_X$, and such that $d_X|_{U_i}$ and $d_{U_i}$ are locally equal for all $i$.*

d) *Let $\boldsymbol{X} := (X,d_X)$ be a pseudometric space.*

   i) *If $\boldsymbol{X}$ is a path-pseudometric space and $\boldsymbol{U} := (U,d_U)$ is an open subset of $\boldsymbol{X}$ with $d_U := d_X|_U$, then $\boldsymbol{U}$ is a path-pseudometric space, and $\widetilde{d}_X|_U$ and $\widetilde{d}_U$ are locally equal.*

   ii) *If there exists an open cover $\mathcal{U} := \{U_i\}_{i\in\mathfrak{I}}$ such that each $(U_i,d_{U_i})$, where $d_{U_i} := d_X|_{U_i}$, is a path-pseudometric space, then $\boldsymbol{X}$ is a path-pseudometric space and $\widetilde{d}_U$ are locally equal.*

*Proof.* (a) Rewording of 4.1.4-(c).

(b) Because $\widetilde{d}_X$ is finite (2.2.2-(1)), the space $\boldsymbol{X}$ is path-connected. Next, since an open subset $U \subseteq (X,d_X)$ is open in $(X,\widetilde{d}_X)$, it can be covered by open balls of finite radius of $(X,\widetilde{d}_X)$. These balls are tautologically path-connected subsets of $(X,d_X)$. As the topologies coincide, the open balls of $(X,\widetilde{d}_X)$ are open subsets of $(X,d_X)$, and this ends the proof that $\boldsymbol{X}$ is locally path-connected.

(c) After 5.1.2, we already know tha $d_{\mathcal{U}}|_{U_i}$ and $d_{U_i}$ are locally equal. Next, thanks to 5.1.4-(a), if $x \in U_i$ and if $\epsilon > 0$ is small enough,

$$B_{\widetilde{d}_{U_i}}(x,\epsilon/3) \subseteq B_{d_{U_i}(x,\epsilon)} = B_{d_{\mathcal{U}}}(x,\epsilon),$$

so that the two distances $\widetilde{d}_{\mathcal{U}}$ and $\widetilde{d}_{U_i}$ coincide on $B_{\widetilde{d}_{U_i}}(x,\epsilon/3)$ because the paths in $\boldsymbol{X}$ joining points in this ball cannot go beyond $B_{d_{U_i}(x,\epsilon)}$. Therefore $B_{\widetilde{d}_{U_i}}(x,\epsilon/3) = B_{\widetilde{d}_{\mathcal{U}}}(x,\epsilon/3)$, where $\widetilde{d}_{\mathcal{U}} = \widetilde{d}_{U_i}$.

To finish, if $d_X$ is locally equal to $\widetilde{d}_{\mathcal{U}}$, then, $\widetilde{d}_X = \widetilde{\widetilde{d}}_{\mathcal{U}} (= \widetilde{d}_{\mathcal{U}})$, after 5.1.4-(b), and $d_X = \widetilde{d}_{\mathcal{U}}$, because $\widetilde{d}_X = d_X$ by hypothesis.

(d-i) The arguments in (c) applied to the cover $\mathcal{U} := \{(X,d_X),(U,d_U)\}$ show already that $\widetilde{d}_X|_U$ and $\widetilde{d}_U$ are locally equal. On the other hand, as $\boldsymbol{X}$ is path-pseudometric, $\widetilde{d}_X$ is locally equal to $d_X$. Hence, $\widetilde{d}_U$ is locally equal to $d_U$, and $(U,d_U)$ is well path-pseudometric. (d-ii) Particular case of (c). □

**6.2 Examples of Path-Metric Spaces.** The following examples are essential for Buser surfaces, the central subject of this book. In these examples $d_X$ is already the length of certain paths which, because of the inequality $d_X \leq \widetilde{d}_X$, will automatically be of minimal length, and even *locally* of minimal length, which is the very definition of a *geodesic*.

**6.2.1 Definition.** Denote by $I\!R^n$ the *affine Euclidean n-dimensional space*. A subset $X \subseteq I\!R^n$ is said to be:

- *star-shaped with center $x_0$*, if the affine interval $[x_0,x] \subseteq I\!R^n$ is contained in $X$, for all $x \in X$; it is *totally locally star-shaped*, if every $x \in X$ is a center a star-shaped neighborhood $V_x \ni x$;
- *locally convex*, if every $x \in X$ has a neighborhood $V_x$ convex in $I\!R^n$.

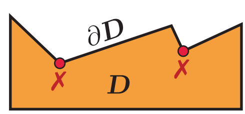
**6.2.2 Theorem.** *For a subset $X \subseteq I\!R^n$, we denote by $\mathrm{d_e}$ the induced distance by the Euclidean distance of $I\!R^n$.*

a) *A totally locally star-shaped subspace $\boldsymbol{X} \subseteq I\!R^n$ is path-metrizable. More precisely, for every $x \in X$ there exists $\epsilon > 0$ such that*

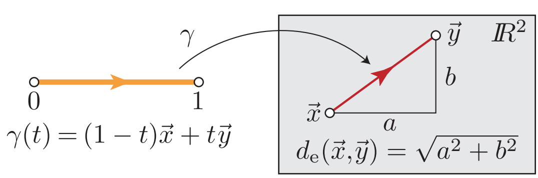
$$B_{\widetilde{\mathrm{d}}_{\mathrm{e}}}(x,\epsilon') = B_{\mathrm{d_e}}(x,\epsilon'), \quad \forall \epsilon' \leq \epsilon,$$

*and the map* $\mathrm{id}_X : (X,\widetilde{\mathrm{d}}_{\mathrm{e}}) \to (X,\mathrm{d_e})$ *is a homeomorphism. If, in addition, $X$ is locally convex in $I\!R^n$, then $\widetilde{d}_X$ and $d_X$ are locally equal,* i.e. $\mathrm{id}_X$ *is a local isometry.*

b) *A domain $\boldsymbol{D} = (D,\mathrm{d_e}) \subseteq I\!R^2$ with polygonal boundary $\partial\boldsymbol{D}$ is path-metrizable by $\widetilde{\mathrm{d}}_{\mathrm{e}}$. The distances $\mathrm{d_e}$ and $\widetilde{\mathrm{d}}_{\mathrm{e}}$ coincide at the neighborhood of every point of $\boldsymbol{D}$ with the exception of vertices in $\partial\boldsymbol{D}$ of reentering angle (red dots in the figure to the right).*




*Proof.* (a) The Euclidean distance between two points $x,y \in I\!R^n$ is defined as $\mathrm{d_e}(x,y) := \|x-y\|$ (1.1.2-(2)). It is the length of the affine segment $[x,y]$ which is also the image of the path $\gamma : [0,1] \to I\!R^n$, $t \mapsto (1-t)x + ty$:



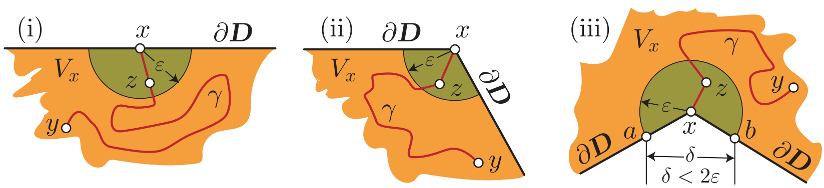
Now, as $d_{\mathrm{e}} \leq \widetilde{d}_{\mathrm{e}}$ (4.1.4), this shows immediately that $d_{\mathrm{e}} = \widetilde{d}_{\mathrm{e}}$, and $I\!R^n$ is a path-metric space. The same holds, for the same reasons, for any convex subspace $\boldsymbol{C} := (C,d_{\mathrm{e}}) \subseteq I\!R^n$, where, once again, $d_{\mathrm{e}}{=}\widetilde{d}_{\mathrm{e}}$.

If $(X,d_{\mathrm{e}})$ is totally star-shaped, for every $x \in X$, there exists $\epsilon > 0$ such that

$$B_{\widetilde{d}_{\mathrm{e}}}(x,\epsilon') = B_{d_{\mathrm{e}}}(x,\epsilon'), \quad \forall \epsilon' \leq \epsilon.$$

In particular, the topologies associated with $d_X$ and $\widetilde{d}_X$ are the same, and the map $\mathrm{id}_X : (X,\widetilde{d}_{\mathrm{e}}) \to (X,d_{\mathrm{e}})$ is a homeomorphism. When, in addition, $X$ is locally convex, we can take $\epsilon$ to be small enough to ensure that $B_{d_{\mathrm{e}}}(x,\epsilon)$ is convex, in which case $\widetilde{d}_{\mathrm{e}}$ and $d_{\mathrm{e}}$ coincide on this $B_{d_{\mathrm{e}}}(x,\epsilon)$, from what we conclude that $\mathrm{id}_X$ is a local isometry.

(b) In **1**♦2.2 we defined an *Euclidean domain* as the closure $\boldsymbol{D} := \overline{U}$ of a *connected* open subset $U \subseteq I\!R^2$. The boundary $\partial\boldsymbol{D}$, is the closed subset $\boldsymbol{D} \smallsetminus U$. As we assumed that $\partial\boldsymbol{D}$ consists of broken lines, the local view $V_x$ of $\boldsymbol{D}$ around a point $x \in \partial\boldsymbol{D}$, has one of the three following shapes:




Hence, $\boldsymbol{D}$ is totally locally star-shaped and we can apply (a).

To finish, observe that for any $x \in \boldsymbol{D}$ such that $B_{d_{\mathrm{e}}}(x,\epsilon)$ is *convex*, the distance $d_{\mathrm{e}}$ coincides with $\widetilde{d}_{\mathrm{e}}$. This is the case for $x \in \boldsymbol{D}$, except when $x \in \partial\boldsymbol{D}$ is a vertex of reentering angle (subimage (iii)). In that case $d_{\mathrm{e}}$ and $\widetilde{d}_{\mathrm{e}}$ are not locally equal at the neighborhood of $x$, which should be clear from the picture where we see that $d_{\mathrm{e}}(a,b) = \delta < 2\epsilon = \widetilde{d}_{\mathrm{e}}(a,b)$. □

### 6.2.3. Path-metrizability of Finite Products

If $\boldsymbol{U}_i \subseteq (I\!R^{d_i},d_{\mathrm{e}})$, $i \in [\![1,n]\!]$, are locally convex subsets, then the product $\prod \boldsymbol{U}_i$ view as subset of $I\!R^{d_1+\cdots+d_n}$, has this same property and we can apply theorem 6.2.2. The space $(\prod \boldsymbol{U}_i,d_{\mathrm{e}})$ is therefore a path-metric space.

Apart from this particularly simple example, there is a general result.

Let $\boldsymbol{X}_i := (X_i,d_i)$, $i \in [\![1,n]\!]$, be pseudometric spaces. In exercise 2.2.3, we considered functions $h_s : I\!R^n_{\geq 0} \to I\!R_{\geq 0}$, $(a_1,\ldots,a_n) \mapsto \sqrt[s]{a_1^s + \cdots + a_n^s}$, and showed that the distances $d_{\Pi X_i,s} := h_s(d_1,\ldots,d_n)$ define the product topology of $\prod \boldsymbol{X}_i$. Now, after **3**♦3.1.1-(2), a path on $\prod \boldsymbol{X}_i$ is simple an $n$-tuple $\gamma = (\gamma_1,\ldots,\gamma_n)$ of paths $\gamma_i : [0,1] \to \boldsymbol{X}_i$, and, thanks to the convexity and homogeneity of $h_s$, we easily see that

$$\ell(\gamma,d_{\Pi X_i,s}) \geq_{(1)} h_s(\ell(\gamma_1,d_1),\ldots,\ell(\gamma_n,d_n)) \geq d_{\Pi X_i,s}(\gamma(0),\gamma(1)),$$

where $\geq_{(1)}$ is always an equality only for $s = 1$. In that case, *i.e.* if we endow $\prod \boldsymbol{X}_i$ with the pseudodistance $d_{\Pi X_i} := \sum_i d_{X_i}$, the space $\prod \boldsymbol{X}_i$ is path-pseudometric space, if and only if each $\boldsymbol{X}_i$ is so (exercise !).

**6.2.4 Comment.** Although not used in this book, it is worth recalling that the theory of Riemannian manifolds shows how, on a differentiable manifold $\boldsymbol{M}{:=}(M,g)$ equipped with an inner product $g_x$ on each tangent space $T_x\boldsymbol{M}$, that varies smoothly with $x \in \boldsymbol{M}$, one can define the length of *differentiable paths*, and then define a pseudodistance $d_M$ using these paths only. The theory shows then that every $x \in \boldsymbol{M}$ has a neighborhood $V_x$ such that, for all $y \in V_x$ there is a *unique* differentiable path $\gamma : [0,1] \to V_x$ joining $x$ to $y$, and such that $d_M(x,y) = \ell(\gamma, d_M)$. Such a path realizes therefore the minimum possible length for any *continuous* path in the metric space $(M,d_M)$. It is a *geodesic* of $\boldsymbol{M}$. The distances $\widetilde{d}_M$ and $d_M$ are locally equal, and $\boldsymbol{M} := (M,d_M)$ is a path-metric space.

## Chapter Contents

## Chapter 5

# Quotients and Assemblages of Pseudometric Spaces

This chapter is the analogous of chapter **3**, but for *pseudometric spaces* instead of topological spaces. With pseudometric spaces, one is naturally lead to introduce different classes of continuous maps, each giving rise to a specific category. We will briefly take a look in each of these categories to the question of the existence of quotients and assemblages, sufficiently enough to show that in many cases they are hard to be established or simply nonexistent. We will then focus on a particular important class of maps, the *local isometries*. These maps, in the presence of *differentiability conditions* (chap. **7**), have the remarkable property of preserving sounds, a key point in sound transplantation methods and isospectrality, as it allows to transfer sounds to quotients and assemblages, but, obviously, provided that they exist. And that is indeed the case in the category of path-pseudometric spaces and local isometries, which is therefore a very suitable framework where to search examples of isospectral spaces. The goal of the present chapter is to give a thorough description of this category.

## 1 Some Classes of Continuous Maps

**1.1** The following are important classes of continuous maps. Each class $\mathcal{C}$, contains the identity maps and the composition of two maps of $\mathcal{C}$. As a consequence, the class $\mathcal{C}$ determines a category. Readers who are unfamiliar with these examples should consider them as exercises.

$\mathcal{C}$-1) *Lipschitz Maps.* A map $f:(X,d_X)\to(Y,d_Y)$ is called *Lipschitz* if there exists $k\in\mathbb{R}_{\geq 0}$, such that $d_Y(f(x),f(x'))\leq k d_X(x,x')$, for all $x,x'\in X$, in which case $f$ is also called *$k$-Lipschitz*.

i) Lipschitz maps are continuous.

ii) The *pullback* and the *pushforward* pseudometrics (**4**♦1.1.2-(4,7)) give 1-Lipschitz maps.

We denote by $\mathcal{C}_{\mathrm{lip}}(\boldsymbol{X},\boldsymbol{Y})$ the subset of $\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$ of Lipschitz maps.

$\mathcal{C}$-2) *(Global) Isometries.* A map $f : (X,d_X) \to (Y,d_Y)$ is called *isometry* if it respects the distances, *i.e.* if

$$d_Y(f(x),f(x')) = d_X(x,x'), \quad \forall x,x' \in X.$$

Notice that $f$ is not necessarily a bijection. For example the inclusions map $\iota_W : (W,d_W) \subseteq (X,d_X)$ where $d_W$ is the induced metric by $d_X$, are isometries. Nonetheless, we frequently find in the definition of isometry the condition that it is a bijection. To avoid confusion, we will call the former *distance preserving* and reserve the name of *isometries*, *global isometries* or *isometric bijections* to the latter.

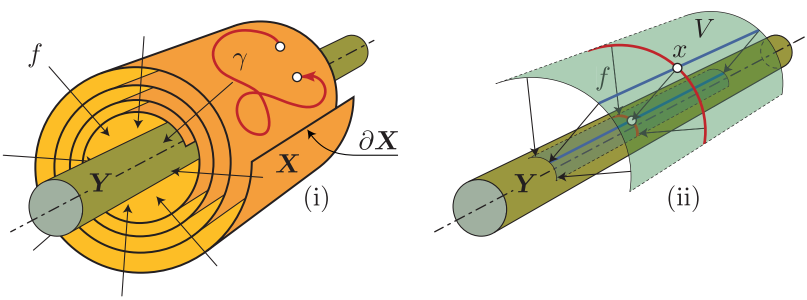
We denote by $\mathcal{C}_{\text{iso}}(\boldsymbol{X},\boldsymbol{Y})$ (also $\text{Iso}(\boldsymbol{X},\boldsymbol{Y})$), resp. $\mathcal{C}_{\text{d}}(\boldsymbol{X},\boldsymbol{Y})$, the subset of isometries (resp. distance preserving maps) in $\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$.

$\mathcal{C}$-3) *Local Distance Preserving map.* A map $f : (X,d_X) \to (Y,d_Y)$ is said to *locally preserve distances*, if for all $x \in X$ there exists an open neighborhood $V_x \subseteq X$ such that the restriction $f|_{V_x} : V_x \to Y$ preserves distances. We denote by $\mathcal{C}_{\ell\text{-d}}(\boldsymbol{X},\boldsymbol{Y})$ the subset of $\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$ of these maps.

$\mathcal{C}$-4) *Local Isometries.* A map $f : (X,d_X) \to (Y,d_Y)$ is said to be a *local isometry* if for all $x \in X$ there exists an open neighborhood $V_x \subseteq X$ such that $f(V_x)$ is *open* in $Y$, and the restriction $f|_{V_x} : V_x \to f(V_x)$ is an isometric bijection.

We denote by $\mathcal{C}_{\ell\text{-iso}}(\boldsymbol{X},\boldsymbol{Y})$ the subset of $\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$ of these maps.

**1.1.1 Example.** In the next figure, the map $f : \boldsymbol{X} \to \boldsymbol{Y}$ is the radial projection of a rectangular domain $\boldsymbol{X}$ (orange) wrapped around a cylinder $\boldsymbol{Y}$ (green), without folds or tears. We denote by $\partial\boldsymbol{X}$ the boundary of $\boldsymbol{X}$, and by $\boldsymbol{X}^\circ := \boldsymbol{X} \smallsetminus \partial\boldsymbol{X}$ the interior of $\boldsymbol{X}$.




All the spaces are embedded in the three dimensional Euclidean space $\mathbb{R}^3$ as shown in the pictures. They are equipped with the induced distances $d_X$ and $d_Y$, generating the path-distances on $\widetilde{d}_X$ and $\widetilde{d}_Y$. Now, because $\boldsymbol{X}$ and $\boldsymbol{Y}$ are flat sheets rolled up, it is clear, (as fot $\mathbb{C}_u \subseteq \mathbb{R}^2$ in **4♦**4.1.2-(1)), that the path-distances are finite and do not modify the topology so that the metric spaces $\boldsymbol{X} := (X,d_X)$ and $\boldsymbol{Y} := (Y,d_Y)$ are path-metric spaces. ([1])

The picture (ii) shows that through radial projection, the length of a path in the axial direction (blue) does not change, while in the radial direction (red) it shrinks by a factor that depends on the ratio of the distance to the axis. As a consequence, for every $x \in \boldsymbol{X}$, there exist $\epsilon(x) > 0$ and $\eta(x) > 0$ such that, for all $a,b \in B_{\widetilde{d}_X}(x,\epsilon(x))$, we have the inequalities:

$$\widetilde{d}_X(a,b) \geq \widetilde{d}_Y(f(a),f(b)) \geq \eta(x) \cdot \widetilde{d}_X(a,b).$$

We conclude that $f$ is globally 1-Lipschitz, and that it is, locally, a homeomorphism onto its image (which is not always open in $\boldsymbol{Y}$). The same ideas work on the open subsets $V \subseteq \boldsymbol{X}$ shown in (ii) (soft-green rectangles) where the image set $f(V \backslash \partial \boldsymbol{X})$ is open in $\boldsymbol{Y}$. In particular, the restriction of $f$ to the interior $\boldsymbol{X}^\circ := \boldsymbol{X} \backslash \partial \boldsymbol{X}$ is a local homeomorphism (**3**♦7.1).

Notice that, if $\gamma : [0,1] \to \boldsymbol{X} := (X,d_X)$ is a path, then we have an inequality of lengths $\ell(f \circ \gamma, d_Y) < \ell(\gamma, d_X)$, and the same for path-distances.

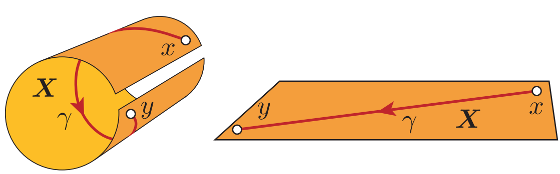
Once we know that $f : \boldsymbol{X} \to \boldsymbol{Y}$ is locally a homeomorphism onto its image, we have a simple way of endowing $\boldsymbol{X}$ with a local distance $d_{\mathcal{U}}$, with the same associated topology as $d_X$ and such that $f$ locally preserves distances. Indeed, let $\mathcal{U}$ be the family of all open sets $V \subseteq \boldsymbol{X}$ such that $f : V \to f(V)$ is a homeomorphism. Endow each such $V$ with the *pullback distance* $d'_V := f^* d_Y$. The family of metric spaces $\mathcal{U} := \{(V, d'_V)\}$ is then clearly a coherent family of local distances for $\boldsymbol{X}$ (**4**♦5.1.1-$LD$-1), and, moreover,

- the map $f : \boldsymbol{X} := (X, d_{\mathcal{U}}) \to (Y, d_Y)$ locally preserves distances;
- the restriction $f : (\boldsymbol{X}^\circ, d_{\mathcal{U}}) \to (\boldsymbol{Y}, d_Y)$, is a local isometry;
- we have an *equality* of lengths of paths $\ell(f \circ \gamma, d_Y) = \ell(\gamma, d_{\mathcal{U}})$.

To finish, $(\boldsymbol{X}, d_{\mathcal{U}})$ is a path-metric space and we can replace in the last three claims $d_{\mathcal{U}}$ and $d_Y$ respectively by $\widetilde{d}_{\mathcal{U}}$ and $\widetilde{d}_Y$, without rendering them false.

**1.1.2 Exercise.** (i) Show that a derivable function $f : [0,1] \to \mathbb{R}$ is $k$-Lipschitz if and only if $|f'| \leq k$. Deduce that the continuous function $f(t) := t \sin(1/t)$ is not Lipschitz on $[0,1]$, but it is on $[\epsilon,1]$, for all $\epsilon > 0$. Give the condition on an interval $\boldsymbol{J} \subseteq [0,1]$ that warrants that the restriction of $f$ to $\boldsymbol{J}$ is $k$-Lipschitz. (ii) Same questions for $f_n(t) := t^n \sin(1/t)$.

([1]) The path-distances are the distances that we can measure with a ruler after unfolding the cylinders. In so doing, we immediately see that geodesics come from line segments.



### 1.1.3. Local and Global Isometries in Path-pseudometric Spaces

A specific property of path-pseudometric spaces is the tight relation between local and global behavior of the path-distance. We end this section summarizing in a proposition some important properties of path-metric spaces that distinguishes them from other metrics spaces that will often be used.

**1.1.4 Proposition.** *Let $f : \boldsymbol{X} := (X,d_X) \to \boldsymbol{Y} := (\boldsymbol{Y},d_Y)$ be a map between pseudometric spaces.*

a) *The map $f$ is locally $k$-Lipschitz if and only if it is globally $k$-Lipschitz for the associated path-distances.*

b) *If $f$ is a surjective local isometry, then $\boldsymbol{X}$ is a path-pseudometric space if and only if $\boldsymbol{Y}$ also is.*

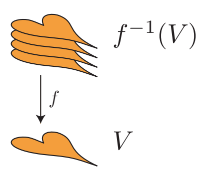
c) *Let $f$ be a bijection. The map $f$ is a local isometry, if and only if it is a global isometry for the associated path-distances.*

*Hint.* (a) Exercise. (b) Apply **4♦**6.1.1-(d-ii). (c) From **4♦**3.3.1-(c), because $\boldsymbol{X}$ and $\boldsymbol{Y}$ share the same paths, and $d_X$ and $f^*d_Y$ are locally equal. □

**1.2 Topological Coverings.** These are an important class of maps, with a substantial role in the construction of quotients and of isospectral surfaces.

**1.2.1 Definition.** The *fiber* of a map $f : \boldsymbol{X} \to \boldsymbol{B}$ at a point $b \in B$, is the set $F_b := f^{-1}(b) = \big\{x \in X \bigm| f(x) = b\big\}$. The map $f$ is a *covering* if, for every $b \in \boldsymbol{B}$ there exists a neighborhood $V_b \ni b$ and a partition of its inverse image $f^{-1}(V_b)$ as disjoint union of neighborhoods $\{V_x\}_{x\in F_b}$, such that the restrictions $f|_{V_x} : V_x \to V_b$ are *homeomorphisms*, for all $x \in F_b$. We thus have the following commutative diagram:

$$\begin{tikzcd}
\boldsymbol{X} \arrow[d, two heads, "f"'] & \supseteq & f^{-1}(V_b) \arrow[r, equal] \arrow[d, two heads, "f"'] & \coprod_{x\in F_b} V_x \arrow[dl, "\sqcup f"'] \arrow[d, "\simeq"', "\phi"] \\
\boldsymbol{B} & \supseteq & V_b & F_b\times V_b \arrow[l, "p_2"']
\end{tikzcd}$$



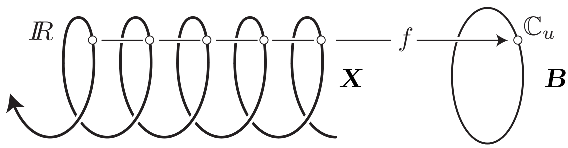

where $\phi(z) = (x,f(z))$, for all $z \in W_x$.

The space $\boldsymbol{X}$ is called the *total space*, $\boldsymbol{B}$ the *base space*, the $V_b$'s the *trivializing* open subsets, and the $V_x$'s the *sheets above $V_b$*, of the covering.

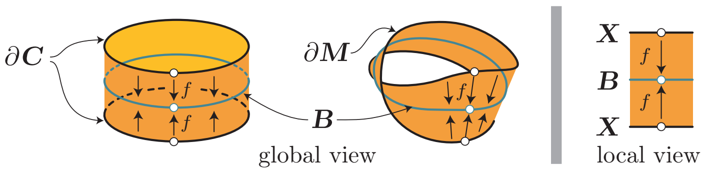
**1.2.2 Examples.** 1) If $\boldsymbol{F}$ is a discrete space, the map $p_2 : \boldsymbol{F}\times\boldsymbol{B} \to \boldsymbol{B}$, $p_2 : (f,b) \mapsto b$ is the simplest example of covering, it is a *trivial covering*. In such cases, the number of (path-)connected components of $\boldsymbol{F}\times\boldsymbol{B}$ is that of $\boldsymbol{B}$ multiplied by $|\boldsymbol{F}|$. Any covering is locally trivial.

2) The exercise **2♦**1.5.4 was about projecting time $t \in \mathbb{R}$ on the circle, which can be realized by the map $f : \mathbb{R} \to \mathbb{C}_u$, $t \mapsto e^{-it}$, which in turn is a covering with countably infinite fibers. Note that it is not a trivial covering

since $I\!R$ is connected.




3) Two classic examples of coverings are the projections of the boundary $\partial\boldsymbol{C}$ of a cylinder $\boldsymbol{C}$, and the boundary $\partial\boldsymbol{M}$ of a Möbius strip $\boldsymbol{M}$, onto the medial (blue) axis $\boldsymbol{B}$, as shown in the image.




Notice that while the base spaces and the local views coincide, the global views are quite different. The total space of the first, $\partial\boldsymbol{C}$, has two connected components, while the other, $\partial\boldsymbol{M}$, is connected.

### 1.2.3. Path Lifting Problem

We denote by $\mathrm{Paths}(\boldsymbol{X})$ the set of paths in $\boldsymbol{X}$, and by $\mathrm{Paths}_x(\boldsymbol{X})$ the subset of paths with initial point $x \in \boldsymbol{X}$. A continuous map $f : \boldsymbol{X} \to \boldsymbol{B}$, then induces a *pushforward* map

$$f_* : \mathrm{Paths}_x(\boldsymbol{X}) \to \mathrm{Paths}_{f(x)}(\boldsymbol{B}), \quad \gamma \mapsto f \circ \gamma. \tag{1}$$

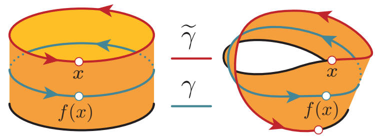
A path $\widetilde{\gamma}$ in $\boldsymbol{X}$, such that $\gamma := f_*(\widetilde{\gamma})$, is a *lift* of $\gamma$. The *path lifting problem* concerns the existence and uniqueness of liftings, more precisely, of the bijectivity of (1).

**1.2.4 Proposition.** *If $f : \boldsymbol{X} \to \boldsymbol{B}$ is a covering, then, for any $x \in \boldsymbol{X}$, the pushforward map $f_* : \mathrm{Paths}_x(\boldsymbol{X}) \to \mathrm{Paths}_{f(x)}(\boldsymbol{B})$ is bijective.*

*Proof. Existence of a lift.* Fix an open cover $\mathcal{V} := \{V_\alpha\}_{\alpha\in\mathfrak{A}}$ of $\boldsymbol{B}$ by trivializing open subsets of the covering $f$. A path $\gamma$ in $\boldsymbol{B}$ defines an open cover $f^{-1}\mathcal{V} := \{f^{-1}(V_\alpha)\}_{\alpha\in\mathfrak{A}}$ of the interval $[0,1] \subseteq I\!R$. By the Heine-Borel-Lebesgue theorem, then there exists a finite sequence $(0 = t_0 < t_1 < \cdots < t_r = 1)$ of points in $[0,1]$, such that $\gamma([t_i,t_{i+1}]) \subseteq V_{\alpha_i}$ for all $i = 1,\ldots,r$ and some $\alpha_i \in \mathfrak{A}$. Denote by $\gamma_i$ the restriction $\gamma|_{[t_i,t_{i+1}]}$. Il is clear that to lift $\gamma$ with initial point $f(x)$ by a path $\widetilde{\gamma}$ with initial point $x$, it suffices to lift $\gamma_0$ to some $\widetilde{\gamma}_0$ with initial point $x$, then $\gamma_1$ to some $\widetilde{\gamma}_1$ with initial point the terminal point of $\widetilde{\gamma}_0$, *i.e.* the point $\widetilde{\gamma}_0(t_1)$, and then $\gamma_2$ to some $\widetilde{\gamma}_2$ with initial point $\widetilde{\gamma}_1(t_2)$, and so on. As each individual lifting is possible because of the local triviality of the covering map $f$ around the subsets $\gamma([t_i,t_{i+1}])$, and as the number of steps needed is always finite, a lift $\widetilde{\gamma}$ always exists.

*Uniqueness.* if $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are two lifts of $\gamma$, the subsets $\{\widetilde{\gamma}_1(t) = \widetilde{\gamma}_2(t)\}$ and $\{\widetilde{\gamma}_1(t) \neq \widetilde{\gamma}_2(t)\}$ of $[0,1]$, are open, disjoint and cover the connected space $[0,1]$. One of them must then be empty and the other the whole interval $[0,1]$. In particular, if the $\widetilde{\gamma}_i$ share the same initial point, they are equal. □

**1.2.5 Example.** The picture to the right shows a lift (in red) of the medial path (in blue) for the examples of coverings 1.2.2-(3).




**1.2.6. Open Atlas Associated with a Covering Space.** Given a surjective covering map $f : \boldsymbol{X} \twoheadrightarrow \boldsymbol{B}$, let $\mathcal{V} := \{V_\alpha\}_{\alpha\in\mathfrak{A}}$ be an open cover of $\boldsymbol{B}$ by trivializing open subsets of $f$. Let $\mathcal{W} := \{W_\beta\}_{\beta\in\mathfrak{B}}$ the set of sheets above the $V_\alpha$'s. Then, let $V_{\alpha(\beta)} := f(W_\beta)$, and let $f_\beta : W_\beta \to V_{\alpha(\beta)}$ be the restriction of $f$ to the sheet $W_\beta$. For $\beta_1,\beta_2 \in \mathfrak{B}$, define

$$W_{\beta_1,\beta_2} := f_{\beta_1}^{-1}(V_{\alpha(\beta_1)} \cap V_{\alpha(\beta_2)}),$$

and introduce the *transition map*

$$\varphi_{\beta_2,\beta_1} := f_{\beta_2}^{-1} \circ f_{\beta_1} : W_{\beta_1,\beta_2} \to W_{\beta_2,\beta_1},$$

which clearly is a homeomorphism.

**1.2.7 Proposition.** *The family of charts* $\mathcal{W} := \{W_\beta\}_{\beta\in\mathfrak{B}}$ *and transition maps* $\{\varphi_{\beta_2,\beta_1}\}_{\beta_i\in\mathfrak{B}}$ *constitute an (open) atlas* $\mathcal{A}(\mathcal{W},f)$ *in the category* **Top** *(2♦5.2.4). The family of open embeddings* $f_\beta : W_\beta \simeq V_{\alpha(\beta)} \subseteq \boldsymbol{B}$ *induces a homeomorphism*

$$\nu : \mathcal{E}(\mathcal{A}(\mathcal{W},f)) \to \boldsymbol{B},$$

*where* $\mathcal{E}(\mathcal{A}(\mathcal{W},f))$ *is the assemblage of* $\mathcal{A}(\mathcal{W},f)$ *in* **Top**.

*Proof.* Left to the reader (*cf.* **2♦**5.3). □

**1.2.8. Pullback Pseudometrics in Coverings.** Assume that the base of a covering $f : \boldsymbol{X} \twoheadrightarrow \boldsymbol{B}$ is a pseudometric space $\boldsymbol{B} := (B,d_B)$. Endow each $W_\beta \in \mathcal{A}(\mathcal{W},f)$ with the pullback pseudodistance $d_{W_\beta} := f_\beta^* d_B$. The family $d_\mathcal{W} := \{(W_\beta,d_{W_\beta})\}$ is then clearly a coherent family of local pseudodistances (**4♦**5.1.1), and, as such, defines the pseudodistance $d_\mathcal{W}$ on $\boldsymbol{X}$, which locally coincides with the distances $d_{W_\beta}$, after prop. **4♦**5.1.2.

**1.2.9 Proposition.** *If the base of a topological covering* $f : \boldsymbol{X} \to \boldsymbol{B}$ *is a pseudometric space* $\boldsymbol{B} := (B,d_B)$*, the space* $\boldsymbol{X}$ *admits a pseudometric space structure* $\boldsymbol{X} := (X,d_X)$ *such that* $f$ *is a local isometric. Moreover, if* $\boldsymbol{B}$ *is a path-pseudometric space, the covering space* $\boldsymbol{X}$ *also is, and*

$$\widetilde{d}_B(b_1,b_2) = \widetilde{d}_X(f^{-1}(b_1), f^{-1}(b_2)).$$

*Hint.* The existence of $d_X$ is justified by the previous remarks. The statements concerning the path-pseudometric structures are obvious, the last equality is an application of the path lifting existence theorem 1.2.4. □

**1.2.10. Pushforward Pseudometrics in Coverings.** We now assume that the total space of a *surjective* covering $f : \boldsymbol{X} \twoheadrightarrow \boldsymbol{B}$ is a pseudometric space $\boldsymbol{X} := (X,d_X)$, but also that the transition maps $\varphi_{\beta_2,\beta_1}$ in some atlas $\mathcal{A}(\mathcal{W},f)$ are *isometries*. This additional condition warrants that on each $V_\alpha := f(W_\beta)$ the pushforward pseudodistance $d_{V_\alpha} := f_{\beta,*}(d_{W_\beta})$ is independent of the choice of the sheet $W_\beta$. The family $d_{\mathcal{V}} := \{(V_\alpha,d_{V_\alpha})\}$ is then a coherent family of local pseudodistances defining the pseudodistance $d_{\mathcal{V}}$ on $\boldsymbol{X}$, which also locally coincides with the distances $d_{V_\alpha}$ after prop. **4**♦5.1.2.

**1.2.11 Proposition.** *If the total space of a surjective topological covering $f : \boldsymbol{X} \twoheadrightarrow \boldsymbol{B}$ is a pseudometric space $\boldsymbol{X} := (X,d_X)$ such that for some atlas $\mathcal{A}(\mathcal{W},f)$ the transition maps are isometries, then the base space $\boldsymbol{B}$ admits a pseudometric space structure $\boldsymbol{B} := (B,d_B)$ such that $f$ is a local isometry. Moreover, if $\boldsymbol{X}$ is a path-pseudometric space, the base space $\boldsymbol{B}$ is also, and*

$$\widetilde{d}_B(b_1,b_2) = \widetilde{d}_X(f^{-1}(b_1),f^{-1}(b_2)).$$

*Proof.* Same as for proposition 1.2.9. □

## 2 The Categories of Pseudometric Spaces

**2.1 Subcategory of a Category.** A *sub*category of a category $\mathfrak{C}$ is a collection $\mathfrak{C}'$ of objects of $\mathfrak{C}$ together with subsets $\mathrm{Mor}_{\mathfrak{C}'}(X,Y) \subseteq \mathrm{Mor}_{\mathfrak{C}}(X,Y)$, for all $X,Y \in \mathfrak{C}'$, such that the conditions **2**♦2.1-$\mathfrak{C}$-(1,2) are satisfied with the same unit elements and composition operation. The category $\mathfrak{C}'$ is said to be *full* if, furthermore, $\mathrm{Mor}_{\mathfrak{C}'}(X,Y) = \mathrm{Mor}_{\mathfrak{C}}(X,Y)$, for all $X,Y \in \mathfrak{C}'$.

**2.1.1 Exercises.** 1) Check that the collections **PMet**, $\mathbf{PMet}_{\mathrm{lip}}$, $\mathbf{PMet}_{\mathrm{d}}$, $\mathbf{PMet}_{\mathrm{iso}}$, $\mathbf{PMet}_{\ell\text{-d}}$ and $\mathbf{PMet}_{\ell\text{-iso}}$ of pseudometric spaces and, respectively, continuous maps, Lipschitz maps, distance preserving maps and isometries, as well and their local versions, are all categories. Draw the graph of inclusions between these categories.

2) Show that the correspondence $(\_)_{\mathrm{top}} : \mathbf{PMet} \rightsquigarrow \mathbf{Top}$ which associates a pseudometric space $(X,d_X)$ with its underlying topological space $(X,\mathcal{T}(d_X))$ (**4**♦2.2), is a functor which is not an inclusion of categories although one has $\mathrm{Mor}_{\mathbf{PMet}}(\boldsymbol{X},\boldsymbol{Y}) = \mathrm{Mor}_{\mathbf{Top}}(\boldsymbol{X}_{\mathrm{top}},\boldsymbol{Y}_{\mathrm{top}})$. ([2])

Now that we have introduced the categories $\mathbf{PMet}_*$, we have the right framework to formulate the question about the existence of products, coproducts, subspaces, quotients and assemblages.

([2]) Functors with this property are called *fully faithful*.

# 3 Products of Pseudometric Spaces

Although we do not need products to define quotients and assemblages of pseudometric spaces, it is instructive to take a brief detour on the question of their existence in the categories $\mathbf{PMet}_*$: $\mathbf{PMet}$, $\mathbf{PMet}_{\ell\text{-d}}$ and $\mathbf{PMet}_{\ell\text{-iso}}$. This is because, as we shall soon see, there are subtle links between products and quotients, and the difficulty one often finds in defining arbitrary quotients can be explained somewhat by the impossibility of defining arbitrary products. This section may be skipped without compromising the understanding of the rest of the chapter.

**3.1. Finite Products in PMet.** It is easy to see that the existence of products of finite families of objects in a category is equivalent to the existence of the product of just two objects. Given two pseudometric spaces $\boldsymbol{X}_i := (X_i, d_{X_i})$, we look for some pseudometric space $\boldsymbol{P} := (P, d_P)$ and continuous maps $\pi_i : \boldsymbol{P} \to \boldsymbol{X}_i$, such that, for all $(\boldsymbol{Y}, d_{\boldsymbol{Z}})$ and continuous $f_i : \boldsymbol{Y} \to \boldsymbol{X}_i$, there exists a unique, continuous $f : \boldsymbol{Y} \to \boldsymbol{P}$, such that the diagram to the right is commutative, *i.e.* such that $f_i = \pi_i \circ f$.

$$\begin{tikzcd} & & \boldsymbol{X}_1 \\ \boldsymbol{Y} \arrow[urr, "f_1"] \arrow[drr, "f_2"'] \arrow[r, dotted, "\exists! f"] & \boldsymbol{P} \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ & & \boldsymbol{X}_2 \end{tikzcd}$$

Now, as the morphisms in **PMet** are simply topologically continuous maps, another way of asking for products of pseudometric spaces is to ask if the product topology, which we already know exists (**3**♦3.1.2), is associated with some pseudodistance on the product of the underlying sets.

And this is indeed the case, as we saw in **4**♦2.2.3, where the associated topologies with the pseudometrics $d_s$ defined on a cartesian product, coincide with the product topology. We can therefore claim that the category **PMet** has finite products.

The products in $\mathbf{PMet}_{\ell\text{-d}}$ and $\mathbf{PMet}_{\ell\text{-iso}}$ are much more constrained and, even for two spaces, they might not exist at all.

**3.1.1 Exercise.** Assume that a product $\{p_i : \boldsymbol{P} \to I\!R \mid i = 1,2\}$ of two copies of $(I\!R, |\_|)$ exists in $\mathbf{PMet}_{\ell\text{-iso}}$ (resp. $\mathbf{PMet}_{\ell\text{-d}}$). Show that there is a unique continuous map $\xi : \boldsymbol{P} \to I\!R \times I\!R$, such that $\pi_i \circ \xi = p_i$. Using isometries like $f_i : I\!R \to I\!R$, $x \mapsto x + a_i$, with $a_i \in I\!R$, show that $\xi$ is bijective. Next, show that, for all $(a_1, a_2) \in I\!R \times I\!R$, the subsets $S_\pm := \{(a_1 + t, a_2 \pm t) \mid t \in I\!R\}$ and $S_+ \cap S_- = \{(a_1, a_2)\}$ are open in $\boldsymbol{P}$. Conclude that $\boldsymbol{P}$ is homeomorphic to $I\!R \times I\!R$ endowed with the discrete topology. Show that this leads to contradictions and that the categorical product in question cannot exist.

**3.2 Countable Products in PMet.** Recall that a set $\mathfrak{I}$ is said to be *countable* if it has the cardinality of a subset of the set of natural numbers $\mathbb{N}$.

**3.2.1 Theorem.** *Let $\mathcal{F} := \{\mathbf{X}_i := (X_i, d_{X_i})\}_{i\in\mathfrak{I}}$ be a countable family of pseudometric spaces.*

a) *For any injection $\phi : \mathfrak{I} \hookrightarrow \mathbb{N}$, the function $d_\phi : (\prod\mathcal{F}) \times (\prod\mathcal{F}) \to I\!R_{\geq 0}$,*

$$d_\phi(\overline{x},\overline{y}) := \sum\nolimits_{i\in\mathfrak{I}} \frac{d'_{X_i}(x_i,y_i)}{2^{\phi(i)}}, \quad \textit{where} \quad d'_i := \max(1, d_{X_i}),$$

*is a pseudodistance with the product topology as associated topology.*

b) *The space*

$$\boxed{\prod\mathcal{F} := (\Pi_i X_i, d_\phi)}$$

*with the canonical projections $\pi_i : \Pi_i X_i \to X_i$, is a product of $\mathcal{F}$ in* **PMet**.

c) *Suppose that $d_{X_i} \neq 0$ for all $i$. A necessary condition for the existence of the categorical product $\prod\mathcal{F}$ in* **PMet**, *is that $\mathfrak{I}$ be countable.*

*Proof.* (a,b) Left to the reader. (c) *Sketch:* Let $\mathfrak{I}$ be uncountable and suppose that the product $\{\pi_i : \prod\mathcal{F} \to \mathbf{X}_i\}$ exists in **PMet**. For each $i \in \mathfrak{I}$, choose an open ball $B(x_i, \epsilon_i) \subsetneqq X_i$, possible because $d_{X_i} \neq 0$. The set $U_i := \pi_i^{-1}(V_{x_i})$ is an open neighborhood of $\overline{x} := (x_i)_{i\in\mathfrak{I}}$ in $\prod\mathcal{F}$, and, therefore, there exists $n_i \in \mathbb{N}$ such that $B(\overline{x}, 1/n_i) \subseteq U_i$. But then, because $\mathfrak{I}$ is uncountable, there exists a countably infinite subset $\mathfrak{N} \subseteq \mathfrak{I}$ such that $B(\overline{x}, 1/N) \subseteq \cap_{\mathfrak{n}\in\mathfrak{N}} U_{\mathfrak{n}}$, for some $N > 0$. On the other hand the subset $\Pi_{\mathfrak{N}} := \left(\prod_{\mathfrak{n}\in\mathfrak{N}} X_{\mathfrak{n}}\right) \times (x_k)_{k\in\mathfrak{I}-\mathfrak{N}}$ endowed with the pseudodistance induced by $\prod\mathcal{F}$, is homeomorphic to categorical product $\prod_{\mathfrak{n}\in\mathfrak{N}} \mathbf{X}_{\mathfrak{n}}$, whose existence has been established in (a). But this is impossible, since $B(\overline{x}, 1/N) \cap \Pi_{\mathfrak{N}}$, being an open subset in the $\prod_{\mathfrak{n}\in\mathfrak{N}} \mathbf{X}_{\mathfrak{n}}$, can have only a *finite* number of 'bounded' coordinates, which is obviously not the case. □

# 4 Coproducts of Pseudometric Spaces

**4.1 Coproduct Distance.** Let $\mathcal{F} := \{\mathbf{X}_i := (X_i, d_{X_i})\}_{i\in\mathfrak{I}}$ be a family of pseudometric spaces. Let

$$d_{\mathcal{F}} : \coprod\mathcal{F} \times \coprod\mathcal{F} \to I\!R_{\geq 0},$$

be defined as:

$$d_{\mathcal{F}}(x,y) := \begin{cases} \max\big(d_{X_i}(x,y), 1\big), & \text{if } x \text{ and } y \text{ belong to the same } X_i, \\ 1, & \text{otherwise.} \end{cases}$$

It is easily seen that $d_{\mathcal{F}}$ is a pseudodistance on $\coprod\mathcal{F}$ extending the $d_{X_i}$'s, so that it is a distance if and only if the $d_{X_i}$'s are so. It is called the *coproduct pseudodistance on* $\coprod\mathcal{F}$. Moreover, the open balls in $(\coprod\mathcal{F}, d_{\mathcal{F}})$ of radius $R < 1$ are the balls of the same radius in the different $\mathbf{X}_i$'s (*cf.* **4**♦2.2.4-(1)). It is thus clear that the topology associated with $d_{\mathcal{F}}$ is the coproduct topology (**3**♦4.1). The next theorem follows.

**Theorem.** *Let* $\mathcal{F} := \{\boldsymbol{X}_i := (X_i, d_{X_i})\}_{i \in \mathfrak{I}}$ *be a family of pseudometric spaces. The space*

$$\boxed{\coprod \mathcal{F} := \left(\bigsqcup_i \boldsymbol{X}_i, d_{\mathcal{F}}\right)}$$

*with the inclusion maps* $\iota_i : \boldsymbol{X}_i \hookrightarrow \coprod \mathcal{F}$, *is a coproduct of* $\mathcal{F}$ *in* $\mathbf{PMet}_*$.

## 5 Subspaces of Pseudometric Spaces

Given $f : X \to (Y, d_Y)$, the pullback (or induced) (pseudo)distance $f^* d_X$ on $X$, that we introduced in **4♦**1.1.2-(4), is such that the inverse image of an open ball is an open ball, more precisely, $f^{-1}(B_{d_Y}(f(x), R) = B_{f^* d_X}(x, R)$. Hence, the topology on $X$ associated with $f^* d_X$ is indeed the pullback topology $f^{-1}(\mathcal{T}(d_Y))$ of **4♦**1.1.2. As a consequence, *any subset* of a pseudometric space is a *pseudometric subspace*.

**Theorem.** *Let* $W$ *be a subset of the pseudometric space* $\boldsymbol{X} := (X, d_X)$. *The space*

$$\boxed{\boldsymbol{W} := \big(W, \langle \text{induced distance} \rangle\big)}$$

*with the inclusion map* $\iota : \boldsymbol{W} \hookrightarrow \boldsymbol{X}$, *is a subspace of* $\boldsymbol{X}$ *in* $\mathbf{PMet}_*$.

*Proof.* Since the topology associated with the induced distance is the induced topology, the space $\boldsymbol{W}$ is a *topological subspace* of $\boldsymbol{X}$, which settles the cases of **PMet** and $\mathbf{PMet}_{\ell\text{-d}}$. Now, if $f : (Z, d_Z) \to (X, d_X)$ is a local isometry such that $f(Z) \subseteq W$, one writes $f = \iota \circ \overline{f}$, and proves that $\overline{f} : Z \to W$ is a local isometry. Indeed, for any $z \in \boldsymbol{Z}$, there exists an open neighborhood $V_z \in \boldsymbol{Z}$, such that, first, $f(V_z)$ is open in $\boldsymbol{X}$, hence open in $\boldsymbol{W}$, and, second, the restriction $f|_{V_z} : V_z \to f(V_z)$ is an isometric bijection for the pseudodistance in $f(V_z)$ induced by $\boldsymbol{X}$, hence induced also by $\boldsymbol{W}$. □

## 6 Quotients of Metric Spaces

Before we start, it is worth noting that the questions of existence of categorical quotients and products are closely related. Indeed, one way of thinking about categorical quotients is in terms of the category $\mathcal{C}(\boldsymbol{X}, \mathcal{R})$ whose objects are the morphisms $(f : \boldsymbol{X} \to \boldsymbol{Z})$ in $\mathbf{Met}_*$ compatible with the equivalence relation $\mathcal{R}$ defined on $\boldsymbol{X}$, and where the morphisms between two such objects $(f_1 : \boldsymbol{X} \to \boldsymbol{Z}_1)$ and $(f_2 : \boldsymbol{X} \to \boldsymbol{Z}_2)$ are the morphisms $(\phi_{21} : \boldsymbol{Z}_1 \to \boldsymbol{Z}_2)$ in $\mathbf{Met}_*$, such that $f_2 = f_{21} \circ f_1$. Thus the quotient $\boldsymbol{X}/\mathcal{R}$ appears as an *initial object* of $\mathcal{C}(\boldsymbol{X}, \mathcal{R})$, *i.e.* un objet from which there is one and only one arrow pointing to any other objet of $\mathcal{C}(\boldsymbol{X}, \mathcal{R})$.

$$\begin{tikzcd}
 & & \boldsymbol{Z}_1 \arrow[dd, "\phi_{21}"] & \mathcal{C}(\boldsymbol{X},\mathcal{R}) \\
\boldsymbol{X} \arrow[urr, bend left, "f_1"] \arrow[r, dotted, "\nu"] \arrow[drr, bend right, "f_2"'] & \boxed{\boldsymbol{X}/\mathcal{R}} \arrow[ur, "\exists! f_1'"] \arrow[dr, "\exists! f_2'"'] & & \\
 & & \boldsymbol{Z}_2 &
\end{tikzcd}$$

This is the same formulation as the one used to define categorical products (*cf.* **2**♦6.2). And, indeed, if we had arbitrary products in the category of metric spaces, we could have considered the space $\boldsymbol{P}(\boldsymbol{X},\mathcal{R}) := \prod_f \boldsymbol{Z}$ indexed by the maps $f : \boldsymbol{X} \to \boldsymbol{Z}$ compatible with $\mathcal{R}$. We then would have the projections $\pi_f : \boldsymbol{P}(\boldsymbol{X},\mathcal{R}) \to \boldsymbol{Z}$ and the canonical map $\nu : \boldsymbol{X} \to \boldsymbol{P}(\boldsymbol{X},\mathcal{R})$, $x \mapsto (f(x))$, and the quotient $\boldsymbol{X}/\mathcal{R}$ would then have been the subspace $\mathrm{im}(\nu)$, and the map $f' : \boldsymbol{X}/\mathcal{R} \to \boldsymbol{Z}$, the restriction of $\pi_f$ to $\mathrm{im}(\nu)$.

It is thus the impossibility of arbitrary products that imposes important constraints on the existence of quotients (*cf.* 3).

**6.1. On the Existence of Quotients in Met.** Since a metric space is a topological space, if for some metric space $\boldsymbol{X} := (X,d_X)$ endowed with an equivalence relation $\mathcal{R}$, we had a quotient $\nu_{\mathcal{R},\mathrm{met}} : \boldsymbol{X} \to (\boldsymbol{X}/\mathcal{R})_{\mathrm{met}}$ in **Met**, we will have a canonical continuous map $\phi : (\boldsymbol{X}/\mathcal{R})_{\mathrm{top}} \to (\boldsymbol{X}/\mathcal{R})_{\mathrm{met}}$, and the commutative diagram to the right, where $\nu_{\mathcal{R},\mathrm{top}} : \boldsymbol{X} \to (\boldsymbol{X}/\mathcal{R})_{\mathrm{top}}$ is the quotient in **Top** introduced in **3**♦6. A priori there is no reason for the map $\phi$ to be either injective or surjective, but, in the applications we have in mind, we want it to be a homeomorphism, in which case, the question about the existence of a quotient in **Met** is equivalent to that of the metrizability of the topological quotient $(\boldsymbol{X}/\mathcal{R})_{\mathrm{top}}$.

$$\begin{tikzcd}
\boldsymbol{X} := (X,d_X) \arrow[r, "\nu_{\mathcal{R},\mathrm{met}}"] \arrow[d, "\nu_{\mathcal{R},\mathrm{top}}"'] & (\boldsymbol{X}/\mathcal{R})_{\mathrm{met}} \\
(\boldsymbol{X}/\mathcal{R})_{\mathrm{top}} \arrow[ur, dotted, "\exists! \phi"'] &
\end{tikzcd}$$

The following celebrated theorem gives conditions for this metrizability, which are satisfied by all the spaces that interest us, especially manifolds with boundary and corners (*cf.* [Mu], §34, p. 215). Notice that the theorem does not exhibit any particular distance, it just states that there is one.

**Theorem (Urysohn's Metrizability Theorem).** *If $\boldsymbol{X}$ be a normal topological space with a countable basis, then $\boldsymbol{X}$ is metrizable. In particular, any Hausdorff locally compact space with a countable basis is metrizable.* (3)

(3) • A topological space $\boldsymbol{X}$ is *normal*, if for any closed subset $Y \subseteq \boldsymbol{X}$ and any $x \notin Y$, there exist disjoint neighborhoods $V \supseteq Y$ and $W \ni x$. • The space $\boldsymbol{X}$ *has countable basis* if its topology $\mathcal{T}_{\boldsymbol{X}}$ is generated by a countable family of open subsets. • The space $\boldsymbol{X}$ is *locally compact* if every point has a compact neighborhood.

Accordingly, the following existence theorem for quotients in the category **Met** of metric spaces and continuous maps is an easy corollary:

**6.1.1 Theorem.** *Let $\boldsymbol{X} := (X,d_X)$ be a metric space with a countable basis, and let $\mathcal{R}$ be an equivalence relation on $\boldsymbol{X}$ such that* $\mathrm{Gr}(\mathcal{R})$ *is closed in $\boldsymbol{X}\times\boldsymbol{X}$. Then, in the following cases, the categorical quotient $(\boldsymbol{X}/\mathcal{R})_{\mathrm{met}}$ exists in* **Met**. (a) *If $\boldsymbol{X}$ is compact.* (b) *If $\boldsymbol{X}$ is locally compact and $\nu : \boldsymbol{X} \to \boldsymbol{X}/\mathcal{R}$ is an open map. In those cases, the identity map* $\mathrm{id} : (\boldsymbol{X}/\mathcal{R})_{\mathrm{top}} \to (\boldsymbol{X}/\mathcal{R})_{\mathrm{met}}$ *is homeomorphic.*

*Proof.* In both cases we will apply Urysohn's theorem and the separateness criteria **3♦**9.1.2. (a) The space $\boldsymbol{X}/\mathcal{R}$ is Hausdorff, and, as it is the continuous image of a compact space, it is compact, hence normal. Showing that $\boldsymbol{X}/\mathcal{R}$ has a countable basis is equivalent to showing that there exists a countable family $\{Z_n\}_{n\in\mathbb{N}}$ of closed subsets of $\boldsymbol{X}/\mathcal{R}$ such that any closed subset $Z \subseteq \boldsymbol{X}/\mathcal{R}$ is the intersection of the $Z_n$'s. Now, if $Z \subseteq \boldsymbol{X}/\mathcal{R}$ is closed, the inverse image $\nu^{-1}(Y)$ is closed and $\mathcal{R}$-saturated. Let $\{Y_n\}_{n\in\mathbb{N}}$ be the family of the complements of the elements of a countable basis of open subsets of $\boldsymbol{X}$. We know that $\nu^{-1}(Z) = \bigcap_{i\in\mathbb{L}} Y_i$ for some $\mathbb{L} \subseteq \mathbb{N}$. On the other hand, because the $\mathcal{R}$-saturation of an intersection is the intersection of the $\mathcal{R}$-saturations (exercise !), we see that $\nu^{-1}(Z) = \bigcap_{i\in\mathbb{L}} \mathcal{R}(Y_i)$. Finally,

$$Z = \nu(\nu^{-1}(Z)) = \bigcup\nolimits_{i\in\mathbb{L}} \nu(\mathcal{R}(Y_i)) = \bigcup\nolimits_{i\in\mathbb{L}} \nu(Y_i),$$

where $\nu(Y_i)$ is closed after **3♦**9.1.2-(c-iii). The family $\{\nu(Y_i)\}^{\mathrm{c}}_{i\in\mathbb{N}}$ is thus a countable basis for $\boldsymbol{X}/\mathcal{R}$. The theorem now follows by Urysohn's theorem.
(b) The space $\boldsymbol{X}/\mathcal{R}$ is Hausdorff and, as $\nu$ is open, it is locally compact and has a countable basis, and we apply Urysohn's theorem. □

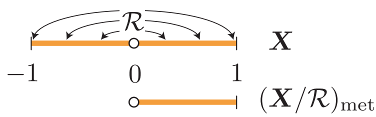
**6.1.2 Exercise.** Let $\mathbf{PMet}_{\mathrm{lip}}$ denote the category of *pseudometric* spaces and Lipschitz maps. Let $\boldsymbol{X} := (X,d_X)$ be a pseudometric space, and $\mathcal{R}$ an equivalence relation on $X$. Denote by $(\boldsymbol{X}/\mathcal{R})_{\mathrm{lip}}$ the quotient *set* $X/\mathcal{R}$ endowed with the pushforward pseudodistance $\nu_* d_X$ (**4♦**1.1.2-(7)).

1) Show that $\nu : \boldsymbol{X} \to (\boldsymbol{X}/\mathcal{R})_{\mathrm{lip}}$ is a categorical quotient in $\mathbf{PMet}_{\mathrm{lip}}$.
2) Let $\boldsymbol{X} := (\mathbb{R}^2\backslash\{0\},d_{\mathrm{e}})$ and $(x\ \mathcal{R}\ y) \Leftrightarrow_{\mathrm{def}} (\exists\lambda \in \mathbb{R}^*_+)(x = \lambda{\cdot}y)$. Show that $(\boldsymbol{X}/\mathcal{R})_{\mathrm{top}}$ is homeomorphic to the unit circle $\mathbb{C}_u \subseteq (\mathbb{R}^2,d_{\mathrm{e}})$, and that the topology associated with $(\boldsymbol{X}/\mathcal{R})_{\mathrm{lip}}$ is the coarse topology. Conclude that the identity map $\mathrm{id} : (\boldsymbol{X}/\mathcal{R})_{\mathrm{top}} \to (\boldsymbol{X}/\mathcal{R})_{\mathrm{lip}}$ is not homeomorphic.

**6.2 On the Existence of Quotients in $\mathbf{Met}_{\ell\text{-d}}$.** In the previous section we were interested in continuous maps between metric spaces, maps for which only the topology is concerned. This is why we did not need to give any explicit definition of the distance in the quotient space: showing that the *topological* quotient was metrizable was enough. But this will not be sufficient in the categories $\mathbf{Met}_{\ell\text{-d}}$ and $\mathbf{Met}_{\ell\text{-iso}}$. There, we need some specific distance on a quotient such that the canonical surjection $\nu : \boldsymbol{X} \to (\boldsymbol{X}/\mathcal{R})_{\ell\text{-d}}$ locally preserves the distances. This additional constraint imposes severe restrictions on the relation $\mathcal{R}$ for the quotient $(\boldsymbol{X}/\mathcal{R})_{\ell\text{-d}}$ to exist. This is better seen in the category of *metric*. We explain this in the next paragraph.

**6.2.1. Local Triviality of Equivalence Relations.** Since the maps in $\mathbf{Met}_{\ell\text{-d}}$ locally preserve distances, and since two points in a metric space are different if and only if the distance between them is not zero, the maps in $\mathbf{Met}_{\ell\text{-d}}$ are always *locally injective*, which implies that quotients for arbitrary equivalence relations might well not exist at all in this category. Indeed, let $\mathcal{R}$ be the relation on $\boldsymbol{X} := [-1,1] \subseteq I\!R$ that identifies $x$ and $-x$. The quotient in $\mathbf{Met}$ is easily seen to be $(\boldsymbol{X}/\mathcal{R})_{\mathrm{met}} = [0,1] \subseteq I\!R$, and $\nu_{\mathrm{met}} : \boldsymbol{X} \to (\boldsymbol{X}/\mathcal{R})_{\mathrm{met}}$ is $x \mapsto x^2$, which is not locally injective at 0. Now, since $\mathbf{Met}_{\ell\text{-d}}$ is a subcategory of $\mathbf{Met}$, the quotients in $\mathbf{Met}_{\ell\text{-d}}$ factors through quotients in $\mathbf{Met}$:

$$\begin{tikzcd}
\boldsymbol{X} \arrow[r, "\nu_{\ell\text{-d}}"] \arrow[d, "\nu_{\mathrm{met}}"'] & (\boldsymbol{X}/\mathcal{R})_{\ell\text{-d}} \\
(\boldsymbol{X}/\mathcal{R})_{\mathrm{met}} \arrow[ur, dashed] &
\end{tikzcd}$$




so that $\nu_{\ell\text{-d}}$ would not be locally injective at 0 either. The quotient $(\boldsymbol{X}/\mathcal{R})_{\ell\text{-d}}$ therefore does not exist, as $\mathcal{R} = \mathcal{R}_{\nu_{\mathrm{met}}}$ is not locally trivial at 0.

**6.2.2 Definition.** An equivalence relation $\mathcal{R}$ on a topological space $\boldsymbol{X}$ is said to be *locally trivial*, if every $x \in \boldsymbol{X}$ has a neighborhood $V_x$ where $\mathcal{R}$ is the *identity relation*.

**6.3 Two Existence Theorems for Quotients in $\mathbf{Met}_{\ell\text{-d}}$.** Rather than giving very general (and elaborate) existence theorems for quotients, we focus on two situations that will occur naturally when constructing Buser and isospectral surfaces. These situations concern: glueing subspaces (in 6.4), and making locally homeomorphic quotients (in 6.5).

**6.4 Glueing Subspaces.** In this situation two subspaces $\boldsymbol{A}$ and $\boldsymbol{B}$ of a pseudometric space $\boldsymbol{X}$, are glued together by pasting a point $a \in \boldsymbol{A}$ with a point $\varphi(a) \in \boldsymbol{B}$, where $\varphi : \boldsymbol{A} \to \boldsymbol{B}$ is a given isometry. The *equivalence relation generated by $\varphi$* is, by definition, the equivalence relation $\mathcal{R}_\varphi$ generated by *the graph of $\varphi$*, *i.e.* by the set $\mathrm{Gr}(\varphi) := \{(a,\varphi(a)) \mid a \in \boldsymbol{A}\} \subseteq \boldsymbol{X} \times \boldsymbol{X}$.

**6.4.1 Theorem.** *Let $\varphi : \boldsymbol{A} \to \boldsymbol{B}$ be an isometry between two disjoint subspaces of a pseudometric space $\boldsymbol{X} := (X,d_X)$. Let $\nu : X \to X/\mathcal{R}_\varphi$ be the canonical surjection, and $d_{X/\mathcal{R}_\varphi} := \nu_* d_X$ the pushforward metric.*

a) *Let $(\tilde{\ })  : \boldsymbol{A} \sqcup \boldsymbol{B} \to \boldsymbol{A} \sqcup \boldsymbol{B}$ be the isometry defined as $\tilde{z} := \varphi^{\pm 1}(z)$ according to whether $z \in \boldsymbol{A}$ or $z \in \boldsymbol{B}$. Then, for all $x,y \in \boldsymbol{X}$,*

$$d_{X/\mathcal{R}_\varphi}(\nu(x),\nu(y)) = \min_{z \in \boldsymbol{A} \sqcup \boldsymbol{B}} \big\{ d_X(\mathcal{R}_\varphi(x),\mathcal{R}_\varphi(y)),\ d_X(x,z) + d_X(\tilde{z},y) \big\}.$$

*Assume now that $\mathcal{R}_\varphi$ is locally trivial (automatic if $\boldsymbol{A}$ and $\boldsymbol{B}$ are closed).*

b) *Let $(\boldsymbol{X}/\mathcal{R}_\varphi)_{\mathrm{met}} := (X/\mathcal{R}_\varphi, d_{X/\mathcal{R}_\varphi})$. The map*

$$\nu_{\mathrm{met}} : \boldsymbol{X} \twoheadrightarrow (\boldsymbol{X}/\mathcal{R}_\varphi)_{\mathrm{met}}, \quad x \mapsto \nu(x), \tag{‡}$$

*locally preserves distances.*

c) i) *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are closed, the maps $\nu_{\mathrm{top}}$ and $\nu_{\mathrm{met}}$ are closed maps. The map $\nu_{\mathrm{met}}$ is a quotient in* **PMet** *and in* $\mathbf{PMet}_{\ell\text{-d}}$.

ii) *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are open, if $\boldsymbol{X}\backslash(\boldsymbol{A}\sqcup\boldsymbol{B})$ is complete* (4)*, and if* $\mathrm{Gr}(\mathcal{R}_\varphi)$ *is closed in $\boldsymbol{X}\times\boldsymbol{X}$, then, the map $\nu_{\mathrm{top}}$ is a local homeomorphism and the map $\nu_{\mathrm{met}}$ is a local isometry. In particular, $\nu_{\mathrm{met}}$ is a quotient in* **PMet** *and also in* $\mathbf{PMet}_{\ell\text{-d}}$ *and* $\mathbf{PMet}_{\ell\text{-iso}}$.

*In both cases, the identity map* $\mathrm{id} : (\boldsymbol{X}/\mathcal{R}_\varphi)_{\mathrm{top}} \to (\boldsymbol{X}/\mathcal{R}_\varphi)_{\mathrm{met}}$ *is a homeomorphism. Moreover, if $\boldsymbol{X}$ is a metric space, then $(\boldsymbol{X}/\mathcal{R}_\varphi)_{\mathrm{top}}$ also is,* i.e. *it is a Hausdorff space.*

*Proof.* To simplify notations we will write $\mathcal{R}$ for $\mathcal{R}_\varphi$.
(a) The definition of the pushforward distance, asks for the lower bound of the finite sums

$$d_X(s_0,s_1) + d_X(s'_1,s_2) + d_X(s'_2,s_3) + \cdots + d_X(s'_{r-1},s_r) \tag{$*$}$$

where $s_0 \in \mathcal{R}(x)$, $s_r \in \mathcal{R}(y)$ et $s'_i \in \mathcal{R}(s_i)$.
There are two ways to reduce the number of terms in $(*)$ while searching for a smallest sum.

- If $s'_i = s_i$ for some $i = 1,\dots,r-1$, then

$$d_X(s'_{i-1},s_i) + d_X(s_i,s_{i+1}) \geq d_X(s'_{i-1},s_{i+1})$$

  by the triangle inequality, and the two successive terms in the left can be replaced by the one at the right. By iterating this idea, we can assume that all intermediate terms satisfy $s_i \neq s'_i$, which implies that the $s_i$'s, for $i = 1,\dots,r-1$, all belong to $\boldsymbol{A} \sqcup \boldsymbol{B}$ and, thereby, verify that $s'_i = \tilde{s}_i$.

- If $s'_i = \tilde{s}_i$ for some $i = 1,\dots,r-1$, then :

$$d_X(s_{i-1},s_i) + d_X(\tilde{s}_i,s_{i+1}) = d_X(s_{i-1},s_i) + d_X(s_i,\tilde{s}_{i+1}) \geq d_X(s_{i-1},\tilde{s}_{i+1}),$$

  because $(\tilde{\ })$ is an isometry, and once again the number $r$ of terms in the sums $(*)$ may be reduced.

---

(4) In a pseudometric space $\boldsymbol{X} := (X,d_X)$, a sequence $(x_n)_{n\in\mathbb{N}}$ is a *Cauchy sequence*, if for all $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that $d_X(x_l,x_m) < \epsilon$, for all $l,m \geq N(\epsilon)$. Every convergent sequence is a Cauchy sequence but the converse is not always true. When it is, the space $\boldsymbol{X}$ is said to be *complete*. Classical examples of complete spaces are the compact metric spaces and the closed subspaces of affine Euclidean spaces $(\mathbb{R}^n, d_{\mathrm{e}})$.

These simplifications apply as long as $r > 2$. As a consequence, the smallest value for the sums $(*)$ is given by the sum of just 1 or 2 terms, in which case it must be the infimum between $d_X(\mathcal{R}(x),\mathcal{R}(y))$ and the lower bound of the 2-term sums $d_X(s_0,z) + d_X(\tilde{z},s_2)$, where $s_0 \in \mathcal{R}(x)$, $s_2 \in \mathcal{R}(y)$ and $z \in \boldsymbol{A} \sqcup \boldsymbol{B}$. In these sums, if we had $x \in \boldsymbol{A} \sqcup \boldsymbol{B}$ and $s_0 = \tilde{x}$, then

$$d_X(\tilde{x},z) + d_X(\tilde{z},s_2) = d_X(x,\tilde{z}) + d_X(\tilde{z},s_2) \geq d_X(\mathcal{R}(x),\mathcal{R}(y)),$$

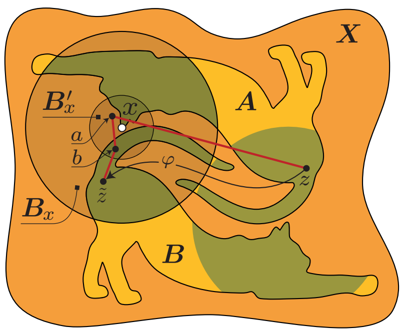
and the same if $s_2 = \tilde{y}$. Therefore, the only 2-term sums that could be smaller than $d_X(\mathcal{R}(x),\mathcal{R}(y))$ are those in which $s_0 = x$ and $s_2 = y$.

(b) We show that, given $x \in \boldsymbol{X}$, there exists $\eta > 0$ such that the restriction of $\nu_{\text{met}}$ to $B_{d_X}(x,\eta)$ is an isometry. There are two opposite cases to consider.

• $x \notin \overline{\boldsymbol{A} \sqcup \boldsymbol{B}}$. Take any $\eta > 0$ such that $B_{d_X}(x,\eta) \cap (\boldsymbol{A} \sqcup \boldsymbol{B}) = \emptyset$. Then if $a,b \in B_{d_X}(x,\eta)$, we clearly have $d_X(a,z) + d_X(\tilde{z},b) > 2\eta$, while $d_X(a,b) < 2\eta$. Hence $d_X(a,b) = d_{X/\mathcal{R}}(\nu(a),\nu(b))$, after (a).

• $x \in \overline{\boldsymbol{A} \sqcup \boldsymbol{B}}$. Let $\epsilon > 0$ be such that $\mathcal{R}$ is trivial in the ball $\mathbb{B}_x := B_{d_X}(x,\epsilon)$. Hence, $d_X(x,\tilde{z}) > \epsilon$, for all $z \in \mathbb{B}_x \cap (\boldsymbol{A} \sqcup \boldsymbol{B})$. Let $\eta := \epsilon/3$ and set $\mathbb{B}'_x := B_{d_X}(x,\eta)$. Then, for all $a,b \in \mathbb{B}'_x$,

$$d_X(\mathcal{R}(a),\mathcal{R}(b)) = d_X(a,b). \qquad (\dagger)$$




Indeed, if $a \in \boldsymbol{A} \sqcup \boldsymbol{B}$, then $d_X$(a,b) is smaller than $d_X(\tilde{a},b)$ by virtue of the triangle inequality:

$$\begin{aligned} d_X(\tilde{a},b) &\geq d_X(\tilde{a},x) - d_X(x,b) \\ &\geq 3\eta - \eta = 2\eta > d_X(a,b). \end{aligned}$$

The same happens if $b \in \boldsymbol{A} \sqcup \boldsymbol{B}$, and when both $a,b \in \mathbb{B}'_x$, we have $d_X(\tilde{a},\tilde{b}) = d_X(a,b)$. In any case, $(\dagger)$ follows.

Suppose now that we had $d_X(a,z) + d_X(\tilde{z},b) < d_X(a,b) \leq 2\eta$ $(\ddagger)$ for some $z \in \boldsymbol{A} \sqcup \boldsymbol{B}$. Then, $d_X(a,z) < 2\eta$ and $d_X(x,z) \leq d_X(x,a) + d_X(a,z) < \epsilon$, in which case $d_X(b,\tilde{z}) \geq d_X(x,\tilde{z}) - d_X(x,b) > 3\eta - \eta = 2\eta$, contrary to $(\ddagger)$. We can hence conclude that $d_X(a,b) = d_{X/\mathcal{R}}(\nu(a),\nu(b))$, after (a) and $(\dagger)$.

(c) The $\mathcal{R}$-saturation of a subset $S$ of $\boldsymbol{X}$, is the set

$$\mathcal{R}(S) = S \cup \varphi(S \cap A) \cup \varphi^{-1}(S \cap B). \qquad (*)$$

When $A$, $B$ and $S$ are closed (resp. open), each term of the right hand side of $(*)$ will be so, and, hence, their union $\mathcal{R}(S)$, which justifies the map $\nu_{\text{top}}$ being closed (resp. open).

Now, as $\nu_{\text{met}}$ is $\mathcal{R}$-compatible, we have $\nu_{\text{met}}(S) = \nu_{\text{met}}(\mathcal{R}(S))$, and, once we know that $\nu_{\text{top}}$ is closed (resp. open), we can conclude that $\nu_{\text{met}}$ is also, if and only if, it preserves the closedness (resp. openness) of *$\mathcal{R}$-saturated* sets. Furthermore, as $\nu(\complement_X S) = \complement_{X/\mathcal{R}}(\nu(S))$ for an $\mathcal{R}$-saturated set $S$, we see that the only thing to check in either the closed or open case, is that an $\mathcal{R}$-saturated *open* subset of $\boldsymbol{X}$ can be covered by sets of the form $\nu^{-1}(B_{d_{X/R}}(\nu(x),\epsilon(x))$, where $\epsilon(x) > 0$ depends on $x$ and is small enough. For this, we need the more precise description of these sets given in the next lemma.

*Lemma. For all $x \in \boldsymbol{X}$ and all $\eta > 0$, define the virtual ball*

$$B^{\varphi}_{d_X}(x,\eta) := \bigcup\nolimits_{z \in A \sqcup B} B_{d_X}\big(\tilde{z},\eta - d_X(x,z)\big).$$

*Then*

$$\nu^{-1}\big(B_{d_{X/\mathcal{R}}}(\nu(x),\eta)\big) = B_{d_X}(\mathcal{R}(x),\eta) \cup B^{\varphi}_{d_X}(x,\eta). \qquad (\diamond)$$

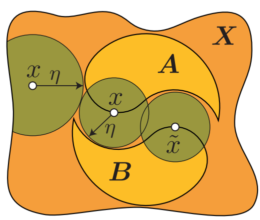
Proof. The inclusion '$\supseteq$' is clear because $\nu_{\mathrm{met}}$ is 1-Lipschitz (1.1-($\mathcal{C}$-1)). The converse follows from the description of the pushforward distance $\nu_* d_X$ given in (a), which shows that $d_{X/\mathcal{R}}(\nu(x),\nu(y)) < \eta$ is equivalent to either

i) $d_X(\mathcal{R}(x),\mathcal{R}(y)) < \eta$; or

ii) $y \in B_{d_X}(\tilde{z},\eta - d_X(x,z))$, for some $z \in (A \sqcup B)$.

When $x \in A \sqcup B$, or $\{x,y\} \cap (A \sqcup B) = \emptyset$, (i) is equivalent to $y \in B_{d_X}(\mathcal{R}(x),\eta)$. When $x \notin A \sqcup B$ but $y \in (A \cup B)$, then $y$ satisfies (ii). In all cases $y$ belongs to the right hand side of $(\diamond)$. $\boxminus$

When the sets $A$ and $B$ are both closed or both open, it is easy to see that the set $B^{\varphi}_{d_X}(x,\eta)$ in $(\diamond)$, is empty if one of the following conditions is verified:




($\dagger_1$) $x \notin \overline{A \sqcup B}$ and $B_{d_X}(x,\eta) \cap (A \sqcup B) = \emptyset$,

($\dagger_2$) $x \in A$ and $B_{d_X}(x,\eta) \cap B = \emptyset$,

($\dagger_3$) $x \in B$ and $B_{d_X}(x,\eta) \cap A = \emptyset$.

In which cases the equality $(\diamond)$ simplifies and gives:

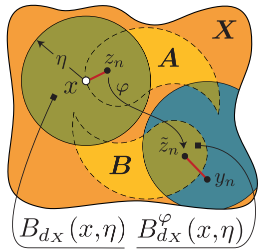
$$\nu^{-1}\big(B_{d_{X/\mathcal{R}}}(\nu(x),\epsilon)\big) = B_{d_X}(\mathcal{R}(x),\epsilon), \quad (\forall \epsilon)(\epsilon \leq \eta).$$

Now, making use of the ideas that precede the lemma, we go on to prove the remaining claims in (c), showing that:

—*An open $\mathcal{R}$-saturated subset $U \subseteq \boldsymbol{X}$ can be covered by the inverse images of open balls in $(\boldsymbol{X}/\mathcal{R})_{\mathrm{met}}$.*

(c-i) As $A$ and $B$ are closed, one has $\overline{A \sqcup B} = A \sqcup B$, so that the conditions ($\dagger$) cover all the logical possibilities. The claim (c-i) then follows from the obvious fact that if $x \in U$, then $B_{d_X}(\mathcal{R}(x),\epsilon) \subseteq U$, for $\epsilon > 0$ small enough.

(c-ii) In this case, the three conditions (†) are no more exhaustive, as it may also happen that $x \in \overline{(A \sqcup B)} \backslash (A \sqcup B)$. In the figure to the right, the $\mathcal{R}$-saturation of the open ball $B_{d_X}(x,\eta)$ is shown in green, and the virtual ball $B^{\varphi}_{d_X}(x,\eta)$ may contain points outside (in blue).




Suppose this happens for all $\eta$. Then, for any decreasing sequence $(\eta_n)_{n\in\mathbb{N}}$ converging to 0, choose, for each $n$, a point $z_n$ in $B_{d_X}(x,\eta_n) \cap (A \sqcup B)$ and a point $y_n \in B_{d_X}(\tilde{z}_n, \eta_n - d_X(x,z_n)) \backslash (A \sqcup B)$. From this we have three sequences of points:

- $(z_n)_{n\in\mathbb{N}}$, converging to $x$, hence a Cauchy sequence,
- $(\tilde{z}_n)_{n\in\mathbb{N}}$, a Cauchy sequence because $d_X(z_n,z_{n+1}) = d_X(\tilde{z}_n,\tilde{z}_{n+1})$,
- $(y_n)_{n\in\mathbb{N}}$, a Cauchy sequence of the closed subset $F := \boldsymbol{X} \backslash (A \sqcup B)$, because the sequence is adjacent to $(\tilde{z}_n)_{n\in\mathbb{N}}$.

Now, under the hypothesis of completeness of $F$, we can conclude that there exists $y \in F$ such that $\lim_n y_n = y$, in which case $(x,y) = \lim_n(z_n,y_n) = \lim_n(z_n,\tilde{z}_n)$, where $(z_n,\tilde{z}_n) \in \mathrm{Gr}(\mathcal{R})$. Hence, $(x,y) \in \mathrm{Gr}(\mathcal{R})$, because $\mathrm{Gr}(\mathcal{R})$ is closed in $\boldsymbol{X} \times \boldsymbol{X}$, and so $x = y$, since $x \notin A \sqcup B$. But the pairs $(z_n,\tilde{z}_n)$ are of different $\mathcal{R}$-equivalent points, which now appear to be as close as desired to $x$, which contradicts the assumption of local triviality of $\mathcal{R}$. This contradiction occurs because we assumed that the virtual ball $B^{\varphi}_{d_X}(x,\eta)$ and the complement of $A \sqcup B$ are not disjoint, for all $\eta > 0$. We therefore can conclude that this is not the case, and that there exists $\eta > 0$ such that

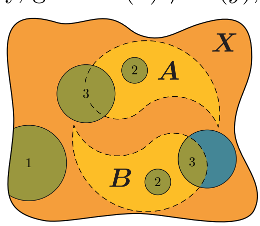
$$\nu^{-1}\big(B_{d_{X/\mathcal{R}}}(\nu(x),\epsilon)\big) = B_{d_X}(x,\epsilon) \cup B^{\varphi}_{d_X}(x,\epsilon) \subseteq \mathcal{R}(B_{d_X}(x,\epsilon)),\ (\forall \epsilon)(\epsilon \leq \eta).$$

It then follows that an $\mathcal{R}$-saturated open subset $U \subseteq \boldsymbol{X}$ is covered by sets of the form $\nu^{-1}\big(B_{d_{X/\mathcal{R}}}(\nu(x),\epsilon)\big)$.

The claims in (c) about quotients, are now straightforward since the identity map $\mathrm{id} : (\boldsymbol{X}/\mathcal{R})_{\mathrm{top}} \to (\boldsymbol{X}/\mathcal{R})_{\mathrm{met}}$ being continuous and closed (resp. open), is a homeomorphism.

It remains only to justify that if $\boldsymbol{X}$ is a metric space the quotient $(\boldsymbol{X}/\mathcal{R})_{\mathrm{met}}$ is also a metric space, and this follows from the previous claims simply by showing that $(\boldsymbol{X}/\mathcal{R})_{\mathrm{top}}$ is Hausdorff.

• When $A$ and $B$ are closed, $\nu^{-1}(B_{d_{X/\mathcal{R}}}(\nu(x),\epsilon(x))$ is equal to $B_{d_X}(\mathcal{R}(x),\epsilon(x))$, for all $x \in \boldsymbol{X}$ and $\epsilon(x) > 0$ small enough. It is then easy, given $\mathcal{R}(x) \neq \mathcal{R}(y)$, to prove that there exist $\epsilon(x) > 0$ and $\epsilon(y) > 0$ such that $B_{d_X}(\mathcal{R}(x),\epsilon(x)) \cap B_{d_X}(\mathcal{R}(y),\epsilon(y)) = \emptyset$.




• When $A$ and $B$ are open, the image to the right shows the three kinds of sets $\nu^{-1}(B_{d_{X/\mathcal{R}}}(\nu(x),\epsilon(x))$ we can have for $\epsilon(x)$ small enough (the blue region being empty). It is then easy to use them to separate different equivalent classes $\mathcal{R}(x) \neq \mathcal{R}(y)$. Details are left to the reader. □

**6.4.2 Definition.** We denote by $\mathbf{p\text{-}PMet}_{\ell\text{-d}}$ (resp. $\mathbf{p\text{-}Met}_{\ell\text{-d}}$) the category of path-pseudometric spaces (resp. path-metric spaces) endowed with their intrinsic distance. The morphisms in these categories are the locally preserving distance maps $f : (X,d_X) \to (Y,d_Y)$.

**6.4.3 Corollary.** *In theorem 6.4.1 assume in addition that* $\mathbf{X}$, $\mathbf{A}$ *and* $\mathbf{B}$ *are path-pseudometric spaces. Denote by* $d_A := d_X|_A$ *and* $d_B := d_X|_B$, *and assume that one of the following conditions hold.*

i) $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are closed, and that, locally,* $\widetilde{d}_X|_A = \widetilde{d}_A$ *and* $\widetilde{d}_X|_B = \widetilde{d}_B$.

ii) $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are open,* $\mathbf{X}$ *is locally compact, and* $\mathrm{Gr}(\mathcal{R}_\varphi)$ *is closed in* $\mathbf{X}\times\mathbf{X}$.

*Then, the pseudodistances* $\nu_*\widetilde{d}_X$ *and* $\widetilde{d}_{X/\mathcal{R}_\varphi}$ *are locally equal, and the quotient space* $(X/\mathcal{R}_\varphi,\widetilde{d}_{X/\mathcal{R}_\varphi})$ *is a path-pseudometric space. The other claims in 6.4.1-(c) are true. Notably, the map*

$$\tilde{\nu}_{\mathrm{met}} : (X,\widetilde{d}_X) \twoheadrightarrow (X/\mathcal{R}_\varphi,\widetilde{d}_{X/\mathcal{R}_\varphi}), \quad x \mapsto \nu(x), \tag{‡‡}$$

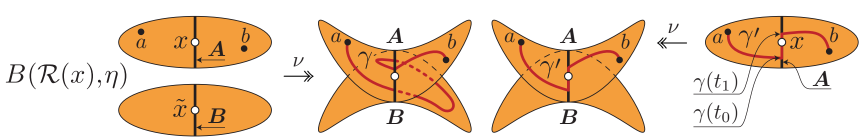
*locally preserves distances and is a quotient in* $\mathbf{PMet}$ *and in* $\mathbf{PMet}_{\ell\text{-d}}$. *Furthermore, if* $\mathbf{X}$ *is a metric space,* (‡‡) *is a quotient in* $\mathbf{Met}$ *and in* $\mathbf{Met}_{\ell\text{-d}}$.

*Proof.* Because $\varphi : (A,d_A) \to (B,d_B)$ is an isometry, the same map induces an isometry $\varphi : (A,\widetilde{d}_A) \to (B,\widetilde{d}_B)$. The conclusions of theorem 6.4.1 are then valid also for the metric space $(X,\widetilde{d}_X)$ in which case, the fact that $\nu_{\mathrm{met}}$ locally preserves distances implies the inequality

$$\widetilde{d}_X(a,b) \geq \widetilde{d}_{X/R}(\nu(a),\nu(b)), \tag{$*$}$$

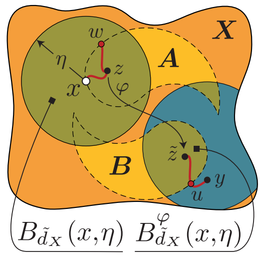
so that, $\tilde{\nu}$ is a 1-Lipschitz map. We now show that $(*)$ is, locally, an equality.

—*When conditions* (i) *are satisfied.* If $x \notin A \sqcup B$, we know from the proof of 6.4.1 that the restriction of $\nu$ to $B_{d_X}(x,\eta)$ is an isometry onto $B_{d_{X/R}}(x,\eta)$, for $\eta$ small enough, so that, again, $(*)$ is an equality on these open balls of $X/\mathcal{R}$. The only possible obstruction to an equality may accur when $x \in A \sqcup B$. In this case, we know that $\nu^{-1}(B_{d_{X/\mathcal{R}}}(\nu(x),\eta)) = B_{d_X}(\mathcal{R}(x),\eta)$, for $\eta$ small enough. This means that, for $a,b \in B_{d_X}(x,\eta)$, a path $\gamma$ in $B_{d_{X/\mathcal{R}}}(\nu(x),\eta)$ joining $\nu(a)$ and $\nu(b)$, may go outside image $\nu(B_{d_X}(x,\eta))$, as shown in the figure below. But then, as $\nu(A)$ is closed, the set $\gamma^{-1}(\nu(A)) \subseteq (0,1) \subseteq I\!\!R$ is compact and has a smallest element $t_0$ and a greater element $t_1$.




Taking $\eta$ smaller if necessary, we can assume that $A \cap B_{d_X}(x,\eta)$ is connected and such that $\widetilde{d}_X(a,b) = \widetilde{d}_A(a,b)$, in which case the portion of $\gamma$ between $t_0$ and $t_1$ can be replaced by a smallest path in $A \cap B_{d_X}(x,\eta)$. The new path $\gamma'$ admits a lift to $B_{d_X}(x,\eta)$ and we then see that $d_X(a,b) \leq \widetilde{d}_{X/\mathcal{R}}(\nu(a),\nu(b))$, and $(*)$ is, locally, an equality.

—*When conditions* (ii) *are satisfied.* The difficulty is concentrated, as in the proof of (c-ii) in the last theorem 6.4.1, in the case where $x \in \overline{A}\backslash A$ and if the virtual ball $B^{\varphi}_{\tilde{d}_X}(x,\eta)$ contains points outside (in blue) the $\mathcal{R}$-saturation of $B_{\tilde{d}_X}(x,\eta)$. But, because $\boldsymbol{X}$ is locally compact, we can take $\eta$ such that the closed ball $\overline{B}^{\varphi}_{\tilde{d}_X}(x,\eta)$ is compact. Il that case, if $\gamma$ is a path going from $\tilde{z}$ to some point outside $B$, there is a first $t$ such as $u := \gamma(t) \notin B$. If we denote by $\gamma'$ the restriction of $\gamma$ to $[0,t]$, then $\tilde{\gamma}' := \varphi^{-1} \circ \gamma'$ is a path in $\overline{B}^{\varphi}_{\tilde{d}_X}(x,\eta)$ and because this is a compact subspace, the limit $w := \tilde{\gamma}'(t)$ exists and belongs to the frontier of $A$. The fact that $\mathcal{R}$ is closed then implies that $(w\mathcal{R}u)$ which contradicts the fact that $u \notin B$. We may thus conclude that

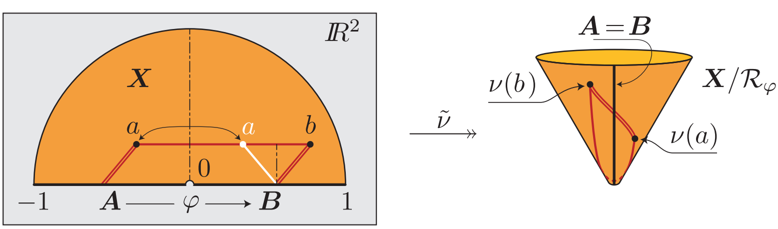



$$\tilde{\nu}_{\mathrm{met}}\big(B_{\tilde{d}_X}(x,\epsilon)\big) = B_{\tilde{d}_{X/R}}(x,\epsilon), \quad \forall \epsilon < \eta,$$

so that the map $\tilde{\nu}_{\mathrm{met}}$ is open.

The remaining claims should now be clear. Details are left to the reader. □

**6.4.4 Example.** A familiar example of glueing subspaces is given by the construction of a *punctured cone*, as shown in the figure below. The glueing rule on the edge $\boldsymbol{A} \sqcup \boldsymbol{B}$ of the domain $\boldsymbol{X} \subseteq \mathbb{R}^2$, is defined by the isometry $\varphi : \boldsymbol{A} \to \boldsymbol{B}$, $(-x,0) \mapsto (x,0)$ (*cf.* 6.2.1). Notice that if the origin 0 were retained, the relation $\mathcal{R}_\varphi$ would not be locally trivial, and the canonical surjection $\nu_{\mathrm{met}}$ would not locally preserve distances around this point.




Once the origin (or a small open ball $B_{\mathbb{R}^2}(0,\epsilon)$) is removed, the conditions of 6.4.3-(i) are satisfied and the map

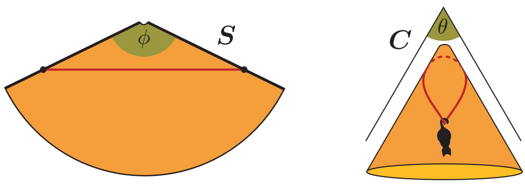
$$\tilde{\nu}_{\mathrm{met}} : (X,\widetilde{d}_X) \twoheadrightarrow (X/\mathcal{R}_\varphi,\widetilde{d}_{X/\mathcal{R}_\varphi}), \quad x \mapsto \nu(x),$$

locally preserves distances and is a quotient in $\mathbf{p}\text{-}\mathbf{Met}_{\ell\text{-d}}$.

As a consequence, the geodesics in $\boldsymbol{X}/\mathcal{R}_\varphi$ are the images through $\nu_{\mathrm{met}}$ of the geodesics in $\boldsymbol{X}$, *i.e.* of line segments. The previous figure gives some visual hints to construct two geodesics in $\boldsymbol{X}/\mathcal{R}_\varphi$ joining the points $\nu(a),\nu(b)$. The length of the shortest one is the distance $\widetilde{d}_{X/\mathcal{R}_\varphi}(\nu(a),\nu(b))$.

**6.4.5 Exercise.** The figure below shows the general construction of a cone $\boldsymbol{C}$ from a circular sector $\boldsymbol{S}$ of *central angle* $\phi$.

1) The *aperture of the cone* is the maximum angle between two of its generatrix lines, $\theta$ in the figure below. Find the relation between $\theta$ and $\phi$.
2) Show that through two points $a,b \in \boldsymbol{C}$ there always passes a positive finite number geodesics, which is smaller or equal to $\lceil 2\pi/\phi \rceil$ (*cf.* 8.2.5).
3) Show that a necklace consisting of a chain without mass and a hanging weight, as in the figure below, will drop out of the cone if $\theta > 60°$.




*Hint: Use the fact that the chain describes a geodesic of the cone.*

**6.4.6 Remark (and exercise).** The hypothesis in theorems 6.4.1 and 6.4.3 can be weakened by replacing the disjoint union $A \sqcup B$ and the isometry $\varphi : A \to B$, by just one subset $Z \subseteq \mathbf{X}$ and an *involutive* isometry $\varphi : Z \to Z$, *i.e.* $\varphi^2 = \mathrm{id}$, with no fixed points. In that case $\mathcal{R}_\varphi(x)$ always has exactly two points when $x \in Z$ and only one otherwise. Check that this is the case. We will need this enhancement, for example, later in this chapter when giving different constructions of the Klein bottle (*cf.* 8.2.6).

**6.5 Locally Homeomorphic Quotients.** Let $\mathbf{X} := (X,d_X)$ be a pseudometric space, and let $\mathcal{R}$ be an equivalence relation on $\mathbf{X}$ such that the canonical surjection $\nu_{\mathrm{top}} : \mathbf{X}_{\mathrm{top}} \to (\mathbf{X}/\mathcal{R})_{\mathrm{top}}$ is a local homeomorphism. An open cover $\mathcal{U} := \{U_i\}_{i\in\mathfrak{A}}$ of $\mathbf{X}$, is called a *trivialization cover of* $\nu_{\mathrm{top}}$ if, for all $i \in \mathfrak{I}$, the set $W_i := \nu(U_i)$ is open in $(\mathbf{X}/\boldsymbol{R})_{\mathrm{top}}$ and the restriction $\nu_i = U_i \to W_i$ is a homeomorphism. In that case, the family $\mathcal{U}$ defines an atlas of $(\mathbf{X}/\mathcal{R})_{\mathrm{top}}$ with transition maps $\varphi_{j,i} := \nu_j^{-1} \circ \nu_i$. The following theorem is almost an immediate consequence.

**6.5.1 Theorem.** *Let $\mathbf{X} := (X,d_X)$ be a pseudometric space, and let $\mathcal{R}$ be an equivalence relation on $\mathbf{X}$ such that $\nu_{\mathrm{top}} : \mathbf{X}_{\mathrm{top}} \to (\mathbf{X}/\mathcal{R})_{\mathrm{top}}$ is a local homeomorphism. A necessary and sufficient condition for the existence of a pseudodistance $d_{X/\mathcal{R}}$ on $(\mathbf{X}/\mathcal{R})_{\mathrm{top}}$ such that*

$$\nu_{\mathrm{met}} : \mathbf{X} \twoheadrightarrow (\mathbf{X}/\mathcal{R})_{\mathrm{met}} := (X/\mathcal{R},d_{X/\mathcal{R}})$$

*is a local isometry and is a quotient in* **PMet**, *in* $\mathbf{PMet}_{\ell\text{-d}}$ *and in* $\mathbf{PMet}_{\ell\text{-iso}}$, *is that there exists a trivialization cover of $\nu_{\mathrm{top}}$ such that the transition maps are isometries. In that case we take $d_{X/\mathcal{R}} := \nu_* d_X$, and $\mathbf{X}$ is path-pseudometric if and only if, $(\mathbf{X}/\mathcal{R})_{\mathrm{met}}$ also is.*

*Proof.* The ideas set out in 1.2.10 show that $\nu_{\text{met}}$ is a local isometry, hence an open map, in which case, the associated topology of $(X/\mathcal{R}, d_{X\mathcal{R}})$ is the quotient topology. That $\nu_{\text{met}}$ is a categorical quotient follows from the fact that this map admits local inverses. Details are left to the reader. □

**6.5.2. Group Actions on Pseudometric Spaces.** An especially important application of the preceding theorem is when the equivalence relation $\mathcal{R}$ is defined by the action of a group. Given a pseudometric space $\boldsymbol{X} := (X, d_X)$, we denote by $\mathrm{Iso}(\boldsymbol{X})$ the set of all the isometries of $\boldsymbol{X}$. If $\phi, \psi \in \mathrm{Iso}(\boldsymbol{X})$, the composition map $\phi \circ \psi$ and the inverse map $\phi^{-1}$ are again isometries. The triplet $\mathrm{Iso}(\boldsymbol{X}) := (\mathrm{Iso}(\boldsymbol{X}), \circ, \mathbf{1}_{\boldsymbol{X}})$ is therefore a group, it is *the group of isometries of* $\boldsymbol{X}$.

**6.5.3 Definitions.** a) An *action by isometries* of a group $\boldsymbol{G}$ on a pseudometric space $\boldsymbol{X}$ is a *group homomorphism* $\rho : \boldsymbol{G} \to \mathrm{Iso}(\boldsymbol{X})$, *i.e.*

$$\rho(\mathbf{1}_{\boldsymbol{G}}) = \mathbf{1}_{\boldsymbol{X}} \quad \textit{and} \quad \rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2) \ (\forall g_1, g_2 \in \boldsymbol{G}).$$

When $\rho$ is understood, we will simply write $g(x)$ for $\rho(g)(x)$.

b) Let $x \in \boldsymbol{X}$. *The orbit of* $\boldsymbol{G}$ *through* $x$ is the set $\boldsymbol{G} \cdot x := \{g(x) \mid g \in \boldsymbol{G}\}$. The *stabilizer of* $x$ is the group $\mathrm{Stab}_{\boldsymbol{G}}(x) := \{g \in \boldsymbol{G} \mid g(x) = x\}$. The map $g \mapsto g(x)$ then induces a bijection between $\boldsymbol{G}/\mathrm{Stab}_{\boldsymbol{G}}(x) \to \boldsymbol{G} \cdot x$, where $\boldsymbol{G}/\mathrm{Stab}_{\boldsymbol{G}}(x)$ denotes the set of left cosets of $\mathrm{Stab}_{\boldsymbol{G}}(x)$ in $\boldsymbol{G}$.

c) The *equivalence relation* $\mathcal{R}_{\boldsymbol{G}}$ *induced on* $\boldsymbol{X}$ *by the action of* $\boldsymbol{G}$, is defined by $(x \, \mathcal{R}_{\boldsymbol{G}} \, y) \Leftrightarrow_{\text{def}} \boldsymbol{G} \cdot x = \boldsymbol{G} \cdot y$. The quotient $\nu_{\mathcal{R}_{\boldsymbol{G}}} : \boldsymbol{X} \twoheadrightarrow \boldsymbol{X}/\mathcal{R}_{\boldsymbol{G}}$ will be denoted by $\nu_{\boldsymbol{G}} : \boldsymbol{X} \twoheadrightarrow \boldsymbol{X}/\boldsymbol{G}$.
We denote by $\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})^{\boldsymbol{G}}$ the subset of $\boldsymbol{G}$*-invariant maps* of $\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$, *i.e.* the maps $\phi : \boldsymbol{X} \to \boldsymbol{Y}$ such that $\phi(g(x)) = \phi(x)$ for all $x \in \boldsymbol{X}$ and $g \in \boldsymbol{G}$, or, still the $\mathcal{R}_{\boldsymbol{G}}$-compatible maps.

d) A point $x \in \boldsymbol{X}$ is a *fixed point of* $\boldsymbol{G}$ if $\boldsymbol{G} \cdot x = \{x\}$. The set of all fixed points of $\boldsymbol{G}$ in $\boldsymbol{X}$ is denoted by $\boldsymbol{X}^{\boldsymbol{G}}$.

e) The action is *free* if $\mathrm{Stab}_{\boldsymbol{G}}(x) = \{\mathbf{1}_{\boldsymbol{G}}\}$, for all $x \in \boldsymbol{X}$, *i.e.* if the map $\boldsymbol{G} \to \boldsymbol{G} \cdot x$, $g \mapsto g(x)$ is bijective, for all $x \in \boldsymbol{X}$.

f) The action is *properly discontinuous without fixed points*, when, for every $x \in \boldsymbol{X}$ there exists $\epsilon(x) > 0$ such that: $g(B_{d_X}(x, \epsilon(x))) \cap B_{d_X}(x, \epsilon(x)) \neq \emptyset$ if and only if $g = \mathbf{1}_{\boldsymbol{G}}$ ([5]).
Note that this is a condition of *local triviality* for $\mathcal{R}_{\boldsymbol{G}}$ (*cf.* 6.2.2).

([5]) The classic definition of a *properly discontinuous action* is somewhat different as it applies to any locally compact topological space $\boldsymbol{X}$. In that case, the definition demands that, for any compact subspace $K \subseteq \boldsymbol{X}$, the set $\{g \in \boldsymbol{G} \mid g \cdot K \cap K \neq \emptyset\}$ is finite. If $\boldsymbol{X}$ is a locally compact *metric* space, this condition implies that for all $\in \boldsymbol{X}$, there exists an open neighborhood $V_x$ such that $\mathrm{Stab}_{\boldsymbol{G}}(x) = \{g \in \boldsymbol{G} \mid g \cdot V_x \cap V_x \neq \emptyset\}$. Hence, if the action is without fixed points, we have our definition (f). It should be noticed that (f) concerns only the metric and that we do not need to assume $\boldsymbol{X}$ to be locally compact.

**6.5.4 Exercise.** Let $\boldsymbol{X}$ be Hausdorff. Show that a group $\boldsymbol{G} \subseteq \mathrm{Iso}(\boldsymbol{X})$ acts properly discontinuously without fixed points on $\boldsymbol{X}$ if and only if, the orbit $\boldsymbol{G} \cdot x$ is discrete in $\boldsymbol{X}$ and $\mathrm{Stab}_{\boldsymbol{G}}(x) = \mathbf{1}_{\boldsymbol{G}}$, for all $x \in \boldsymbol{X}$. If, in addition, $\boldsymbol{G}$ is finite, show that the condition is equivalent to the freeness of the action.

**6.5.5 Theorem.** *Let $\boldsymbol{X} := (X,d_X)$ be a pseudometric space and let $\boldsymbol{G}$ be a group acting by isometries on $\boldsymbol{X}$, freely and properly discontinuously. Then, $\nu_{\mathrm{top}} : \boldsymbol{X} \twoheadrightarrow (\boldsymbol{X}/\boldsymbol{G})_{\mathrm{top}}$ is a covering, and the theorem 6.5.1 applies. The map*

$$\nu_{\mathrm{met}} : \boldsymbol{X} \twoheadrightarrow (\boldsymbol{X}/\boldsymbol{G})_{\mathrm{met}} := (X/\boldsymbol{G},d_{X/\boldsymbol{G}})$$

*is a local isometry and $(\boldsymbol{X}/\boldsymbol{G})_{\mathrm{met}}$ is a quotient in* **PMet**$_*$. *Therefore, the pullback map $\nu_{\mathrm{met}}^* : \mathcal{C}((\boldsymbol{X}/\boldsymbol{G})_{\mathrm{met}}, \boldsymbol{Y}) \to \mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$ is injective with image*

$$\boxed{\nu_{\mathcal{R}}^*(\mathcal{C}(\boldsymbol{X}/\mathcal{R}, \boldsymbol{Y})) = \mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})^{\boldsymbol{G}}}$$

*where $\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})^{\boldsymbol{G}}$ is the set of $\boldsymbol{G}$-invariant continuous maps.*

*– If $\boldsymbol{X}$ is Hausdorff, $(\boldsymbol{X}/\boldsymbol{G})_{\mathrm{met}}$ is also.*

*– The space $\boldsymbol{X}$ is path-pseudometric if and only if $(\boldsymbol{X}/\boldsymbol{G})_{\mathrm{met}}$ also is, in which case $(\boldsymbol{X}/\boldsymbol{G})_{\mathrm{met}}$ is a quotient in* **p-PMet**$_*$.

*Proof.* If $U$ is an open subset of $\boldsymbol{X}$, its $\boldsymbol{G}$-saturation $\boldsymbol{G} \cdot U := \bigcup_{g \in \boldsymbol{G}} g(U)$, is open since each $g(U)$ is so. The map $\nu_{\mathrm{top}}$ is therefore open. The fact that the action is properly discontinuous without fixed points warrants that the restriction of $\nu$ to $B_{d_X}(x,\epsilon(x))$ is a homeomorphism. We then apply **3**♦6.1.1. Let $\boldsymbol{X} := (X,d_X)$ be Hausdorff. If the $\boldsymbol{G}$-orbits $\boldsymbol{G} \cdot x$ and $\boldsymbol{G} \cdot y$ are discrete disjoint, one has $\epsilon := d_X(\boldsymbol{G} \cdot x, \boldsymbol{G} \cdot y) > 0$. But then, the $\boldsymbol{G}$-saturation of the open balls $B_{d_X}(x,\epsilon/2)$ and $B_{d_X}(y,\epsilon/2)$ are disjoint open subsets, and their images under $\nu_{\mathrm{top}}$ are open subsets separating $\nu(x)$ and $\nu(y)$ in $(\boldsymbol{X}/\boldsymbol{G})_{\mathrm{top}}$.
To finish, if $\boldsymbol{X}$ is path-pseudometric, we apply 6.5.1. □

**6.5.6 Exercise.** Show that, under the hypothesis of 6.5.5, and if $\boldsymbol{X}$ is *separable*, *i.e.* if it contains a countable dense subset, the map $\nu_{\mathrm{met}}$ is closed if and only if the group $\boldsymbol{G}$ is finite.

**6.5.7. Factorization of a Group Equivalence Relation.** The action of a group $\boldsymbol{G}$ on a space $\boldsymbol{X}$ induces an action of any of its subgroups $\boldsymbol{H} \subseteq \boldsymbol{G}$ simply by *restricting* the homomorphism $\rho$ in 6.5.3-(a) from $\boldsymbol{G}$ to $\boldsymbol{H}$. It is then clear that $(x \,\mathcal{R}_{\boldsymbol{H}}\, y) \Rightarrow (x \,\mathcal{R}_{\boldsymbol{G}}\, y)$, so that one has a canonical surjection:

$$\nu_{\boldsymbol{H},\boldsymbol{G}} : \boldsymbol{X}/\boldsymbol{H} \twoheadrightarrow \boldsymbol{X}/\boldsymbol{G}, \quad \mu(\boldsymbol{H} \cdot x) = \boldsymbol{G} \cdot x.$$

In turn, this map induces the equivalence $\mathcal{R}_{\nu_{\boldsymbol{H},\boldsymbol{G}}}$ on $\boldsymbol{X}/\boldsymbol{H}$ (*cf.* **2**♦1.5) and we get a bijection $(\boldsymbol{X}/\boldsymbol{H})/\mathcal{R}_{\nu_{\boldsymbol{H},\boldsymbol{G}}} \to \boldsymbol{X}/\boldsymbol{G}$. Note, however, that the relation $\mathcal{R}_{\nu_{\boldsymbol{H},\boldsymbol{G}}}$ is generally *not* related to a group action on $\boldsymbol{X}/\boldsymbol{H}$. This will be the case when $\boldsymbol{H}$ is a *normal* subgroup of $\boldsymbol{G}$, which we denote by $\boldsymbol{H} \triangleleft \boldsymbol{G}$.

**6.5.8 Proposition.** *Let $\boldsymbol{H} \triangleleft \boldsymbol{G}$, and let $\eta : \boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{H}$ be the quotient map.*

a) *The quotient $\boldsymbol{G}/\boldsymbol{H}$ endowed with the product $(\boldsymbol{H}\cdot g)\cdot(\boldsymbol{H}\cdot h) := (\boldsymbol{H}\cdot gh)$ is a group. It naturally acts on $\boldsymbol{X}/\boldsymbol{H}$ by the formula*

$$\eta(\boldsymbol{H}\cdot g)(\boldsymbol{H}\cdot x) := \boldsymbol{H}\cdot g(x), \quad \forall g \in \boldsymbol{G}, \forall x \in \boldsymbol{X}.$$

b) *The canonical surjection $\nu_{\boldsymbol{H},\boldsymbol{G}} : \boldsymbol{X}/\boldsymbol{H} \to \boldsymbol{X}/\boldsymbol{G}$ is compatible with the action of $\boldsymbol{G}/\boldsymbol{H}$,* i.e. *$\nu(\eta(g)(\boldsymbol{H}\cdot x)) = \boldsymbol{G}\cdot x$, for all $g \in \boldsymbol{G}$ and $x \in \boldsymbol{X}$. The following natural map is then bijective:*

$$(\boldsymbol{X}/\boldsymbol{H})/(\boldsymbol{G}/\boldsymbol{H}) \to \boldsymbol{X}/\boldsymbol{G}.$$

**6.5.9 Exercise.** When the subgroup $\boldsymbol{H} \triangleleft \boldsymbol{G}$ admits a supplementary group $\boldsymbol{K} \subseteq \boldsymbol{G}$, *i.e.* a subgroup $\boldsymbol{K}$ such that the map $\boldsymbol{H} \times \boldsymbol{K} \to \boldsymbol{G}$, $(h,k) \mapsto h\cdot k$, is bijective, we write $\boldsymbol{G} := \boldsymbol{H} \rtimes \boldsymbol{K}$. In that case, show that the relation $\mathcal{R}_{\boldsymbol{K}}$ defines an equivalence on $\boldsymbol{X}/\boldsymbol{H}$ by defining $(\boldsymbol{H}\cdot x \;\mathcal{R}_{\boldsymbol{K}}\; \boldsymbol{H}\cdot y) \Leftrightarrow_{\text{def}} (x \;\mathcal{R}_{\boldsymbol{K}}\; y)$. Show, then, that the natural map $(\boldsymbol{X}/\mathcal{R}_{\boldsymbol{H}})/\mathcal{R}_{\boldsymbol{K}} \to \boldsymbol{X}/\mathcal{R}_{\boldsymbol{G}}$ is bijective.

## 7 Assemblages of Path-Metric Spaces

**7.1 The Category of Path-Metric Spaces.** As defined in **2♦**5.2, an atlas in **Met** consists of a family of metric spaces $\mathcal{A} = \{(E_i, d_{E_i})\}_{i\in\mathfrak{I}}$ with transition maps being just homeomorphisms between topological spaces. This kind of assemblage is of no interest to us since we are seeking to glue metric structures rather than just topologies. For this reason, we need the transition maps to be homeomorphisms *and local isometries*. But then, we are confronted with the subtle problem of metrics that would be locally equal but globally different. To circumvent this difficulty and to give us more latitude in glueing metrics, we will consider only metrics that are determined by their local behavior, *i.e.* path-metrics. But, there will still remain one more difficulty: the overlapping subspaces $E_{i,j}$. The corollary 6.4.3 shows that a reasonably good hypothesis to add is that

$$\textit{the distances } \widetilde{d}_{E_i}|_{E_{i,j}} \textit{ and } \widetilde{d}_{E_{i,j}} \textit{ are locally equal,} \qquad (*)$$

hypothesis that we will assume. The goal is now to ensure the existence of assemblages in *the category of path-metric spaces* $\mathbf{p\text{-}Met}_{\ell\text{-d}}$ (6.4.2). As for topological spaces, we restrict to the assemblages of open and closed atlases.

**7.2 Assemblages of Open Atlases of Path-Metric Spaces.** An *open atlas in* $\mathbf{p\text{-}Met}_{\ell\text{-d}}$ is an atlas $\mathcal{A} := \{\boldsymbol{E}_i := (E_i, d_{E_i})\}_{i\in\mathfrak{I}}$, where the $\boldsymbol{E}_i$'s are path-metric spaces, the overlap spaces are *open* subspaces $\boldsymbol{E}_{i,j} \subseteq \boldsymbol{E}_i$, in which case $(*)$ is satisfied by **4♦**6.1.1-(d), and the transition maps $\varphi_{j,i} : \boldsymbol{E}_{i,j} \to \boldsymbol{E}_{j,i}$ are homeomorphic local isometries.

**7.2.1 Theorem.** *Let $\mathcal{A} := \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ be an open atlas in* $\mathbf{p\text{-}Met}_{\ell\text{-d}}$. *Assume that the $\boldsymbol{E}_i$'s are locally compact, and that* $\mathrm{Gr}(\mathcal{R}(\mathcal{A}))$ *is closed in*

$\boldsymbol{E}(\mathcal{A})\times\boldsymbol{E}(\mathcal{A})$. *Then, the quotient* $\mathcal{E}(\mathcal{A}) := \boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}}$ *endowed with the pushforward path-distance is an assemblage of* $\mathcal{A}$ *in* $\mathbf{p\text{-}Met}_{\ell\text{-d}}$ *and in* $\mathbf{p\text{-}Met}_{\ell\text{-iso}}$. *The identification map* $\nu_{\mathcal{A}} : \boldsymbol{E}(\mathcal{A}) \twoheadrightarrow \mathcal{E}(\mathcal{A})$ *is open and induces an isometry between* $(E_i,\widetilde{d}_{E_i})$ *and* $\nu_{\mathcal{A}}(\boldsymbol{E}_i)$.

*Hint.* The ideas for the proof can be found in the justification of propositions 6.4.1 and 6.4.3. We therefore leave the details to the reader.

Notice that the hypothesis of local triviality for the relation $\mathcal{R}(\mathcal{A})$ is now automatic as, if $(a\ \mathcal{R}(\mathcal{A})\ b)$ and $a \neq b$, then $a$ and $b$ belong to different charts, in which case $\widetilde{d}_{\boldsymbol{E}}(a,b) = 1$. Notice also that the hypothesis of local compactness of the charts of $\mathcal{A}$, and the closure of $\mathrm{Gr}(\mathcal{R}(A))$, serve, as in the proof of 6.4.3-(ii), to ensure that if an element $x \in \boldsymbol{E}_i$ belongs to the boundary of some open overlap $\boldsymbol{E}_{i,j}$, then $\nu(B_{\widetilde{d}_{E_i}}(x,\epsilon)) = B_{\widetilde{d}_{\mathcal{E}(\mathcal{A})}}(\nu(x),\epsilon)$, for all $\epsilon < 1$. It then follows that $\nu_{\mathcal{A}}$ is an open map and a local isometry. $\square$

**7.3 Assemblages of Closed Atlases of Path-Metric Spaces.** A *closed atlas in* $\mathbf{p\text{-}Met}_{\ell\text{-d}}$ is an atlas $\mathcal{A} := \{\boldsymbol{E}_i := (E_i,d_{E_i})\}_{i\in\mathfrak{I}}$ where the $\boldsymbol{E}_i$'s are path-metric spaces, the overlap spaces are *closed* subspaces $\boldsymbol{E}_{i,j} \subseteq \boldsymbol{E}_i$ such that $\widetilde{d}_{\boldsymbol{E}_i}|_{\boldsymbol{E}_{i,j}}$ and $\widetilde{d}_{\boldsymbol{E}_{i,j}}$ are locally equal (condition $(*)$), and the transition maps $\varphi_{j,i} : \boldsymbol{E}_{i,j} \to \boldsymbol{E}_{j,i}$ are homeomorphic local isometries.

**7.3.1 Theorem.** *Let* $\mathcal{A} := \{\boldsymbol{E}_i\}_{i\in\mathfrak{I}}$ *be a closed atlas in* $\mathbf{p\text{-}Met}_{\ell\text{-d}}$. *Then, the quotient* $\mathcal{E}(\mathcal{A}) := \boldsymbol{E}(\mathcal{A})/\mathcal{R}_{\mathcal{A}}$ *endowed with the pushforward path distance is an assemblage of* $\mathcal{A}$ *in* $\mathbf{p\text{-}Met}_{\ell\text{-d}}$. *The identification map* $\nu_{\mathcal{A}} : \boldsymbol{E}(\mathcal{A}) \twoheadrightarrow \mathcal{E}(\mathcal{A})$ *is closed and induces an isometry between* $\boldsymbol{E}_i$ *and* $\nu_{\mathcal{A}}(\boldsymbol{E}_i)$.

*Proof.* Left to the reader. $\square$

# 8 Examples of Quotients and Assemblages of Path-Metric Spaces

In this section, we recall the different classic constructions of four well-known flat surfaces: the Cylinder, the Möbius ribbon, the flat 2-dimensional Torus, and the flat Klein bottle. The constructions follow respectively the different procedures described in the last sections:

A) as quotients of $\mathbb{R}^2$ under the action of a groups of isometries; (6.5.5)

B) by glueing sub-boundaries of a domain $\boldsymbol{D} \subseteq \mathbb{R}^2$; (6.4.1-(c-i))

C) by glueing open subsets in an open subspace $\boldsymbol{D}_\epsilon \subseteq \mathbb{R}^2$; (6.4.1-(c-ii))

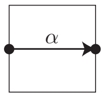
D) as closed assemblage of copies of a domain $\boldsymbol{D} \subseteq \mathbb{R}^2$; (7.3.1)

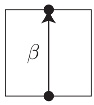
E) as open assemblage of copies of an open subset $\boldsymbol{D}_\epsilon \subseteq \mathbb{R}^2$; (7.2.1)

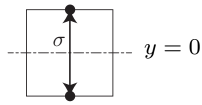
## 8.1 (A) Four Groups of Isometries of $I\!R^2$

In the Euclidean vector space $I\!R^2 := (I\!R^2, d_e)$, consider the maps

$$\alpha : (x,y) \mapsto (x+1,y) \qquad \beta : (x,y) \mapsto (x,y+1) \qquad \sigma : (x,y) \mapsto (x,-y)$$




The first two are translations and the third one is the reflection through the horizontal axis $\{y = 0\}$. They all conserve Euclidean distances and belong to the group $\boldsymbol{G} := \mathrm{Iso}(I\!R^2, d_e)$ of isometries of $I\!R^2$.

Our goal is to explain the following table where, given elements $\alpha_i \in \boldsymbol{G}$, we denote by $\langle \alpha_i \rangle$ the subgroup of $\boldsymbol{G}$ they generate.

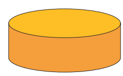
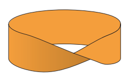
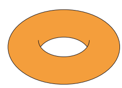
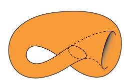
| Name | Subgroup of $\mathrm{Iso}(I\!R^2)$ | Quotient $I\!R^2/\boldsymbol{H}$ |
|---|---|---|
| Cylinder | $\boldsymbol{H}_C := \langle \alpha \rangle$ |  |
| Möbius Ribbon | $\boldsymbol{H}_M := \langle \sigma \circ \alpha \rangle$ |  |
| Torus | $\boldsymbol{H}_T := \langle \alpha, \beta \rangle$ |  |
| Klein Bottle | $\boldsymbol{H}_K := \langle \sigma \circ \alpha, \beta \rangle$ |  |

*Table .1. Four (locally isometric) quotients of the Euclidean plane*

**8.1.1. *H*-Orbits..** The elements $\alpha, \beta, \sigma \in \boldsymbol{G}$ verify the commutation rules

$$\alpha \circ \beta = \beta \circ \alpha, \qquad \alpha \circ \sigma = \sigma \circ \alpha, \qquad \beta \circ \sigma = \sigma \circ \beta^{-1},$$

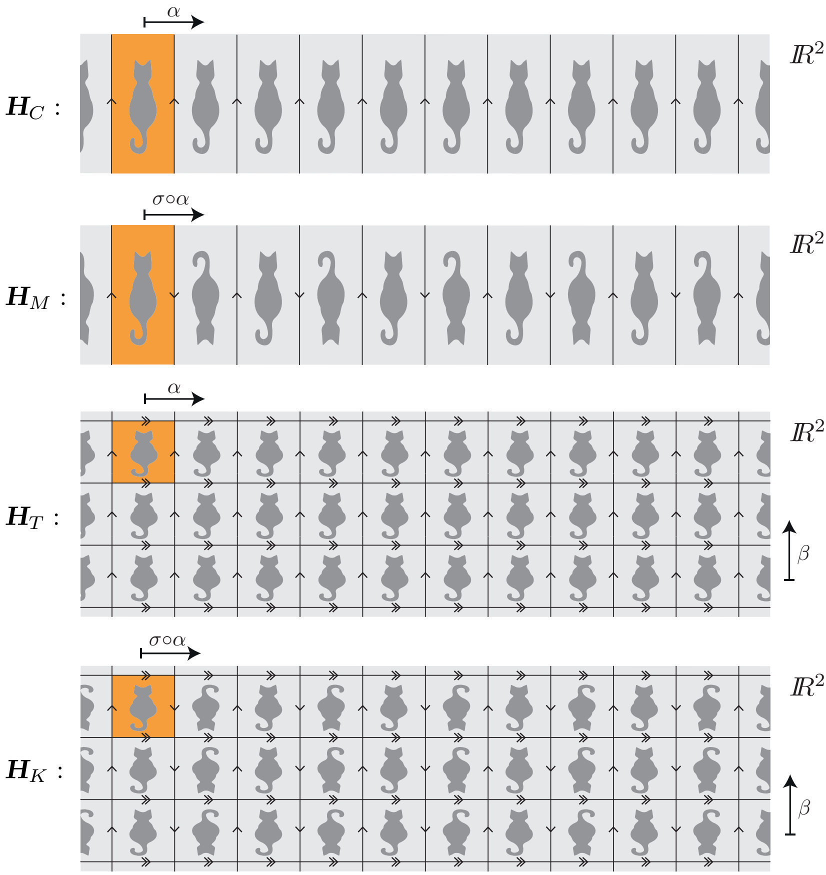
which help to describe the contents of the subgroups $\boldsymbol{H}$. We have:

$$\boldsymbol{H}_C = \{\alpha^n \mid n \in \mathbb{Z}\}, \qquad \boldsymbol{H}_M = \{\sigma^n \circ \alpha^n \mid n \in \mathbb{Z}\},$$
$$\boldsymbol{H}_T = \{\alpha^n \circ \beta^m \mid n,m \in \mathbb{Z}\}, \qquad \boldsymbol{H}_K = \{\sigma^n \circ \alpha^n \circ \beta^m \mid n,m \in \mathbb{Z}\}.$$

The orbits of a point $(x,y) \in I\!R^2$ are then given by the expressions:

$$\begin{aligned}
\boldsymbol{H}_C\cdot(x,y) &= \{(x+n,\ y) \mid n \in \mathbb{Z}\},\\
\boldsymbol{H}_M\cdot(x,y) &= \{(x+n,\ (-1)^n y) \mid n \in \mathbb{Z}\},\\
\boldsymbol{H}_T\cdot(x,y) &= \{(x+n,\ y+m) \mid n,m \in \mathbb{Z}\},\\
\boldsymbol{H}_K\cdot(x,y) &= \{(x+n,\ (-1)^n y+m) \mid n,m \in \mathbb{Z}\},
\end{aligned}$$

which allow us to illustrate the actions of the different groups $\boldsymbol{H}$ on $\mathbb{R}^2$, by the following images.




**8.1.2 Proposition.** *Let $\boldsymbol{H}$ denote one of the four groups $\boldsymbol{H}_C, \boldsymbol{H}_M, \boldsymbol{H}_T, \boldsymbol{H}_K$.*

a) *The action of $\boldsymbol{H}$ on $\mathbb{R}^2$ through the inclusion $\boldsymbol{H} \subseteq \mathrm{Iso}(\mathbb{R}^2, d_{\mathrm{e}})$ is properly discontinuous without fixed points.*
b) *The space $(\mathbb{R}^2/\boldsymbol{H})_{\mathrm{met}} := (\mathbb{R}^2/\boldsymbol{H}, d_{\mathbb{R}^2/\boldsymbol{H}})$ is Hausdorff and path-metric.*
c) *The canonical map $(\mathbb{R}^2/\boldsymbol{H})_{\mathrm{top}} \to (\mathbb{R}^2/\boldsymbol{H})_{\mathrm{met}}$ is a homeomorphism.*
d) *The canonical surjection $\nu_{\boldsymbol{H}} : \mathbb{R}^2 \twoheadrightarrow (\mathbb{R}^2/\boldsymbol{H})_{\mathrm{met}}$ is a covering, a local isometry and a quotient in* **Met**, **p-Met**, **Met**$_{\ell\text{-d}}$ *and* **p-Met**$_{\ell\text{-d}}$.
e) *The geodesics of $(\mathbb{R}^2/\boldsymbol{H})_{\mathrm{met}}$ are the images of the line segments of $\mathbb{R}^2$.*

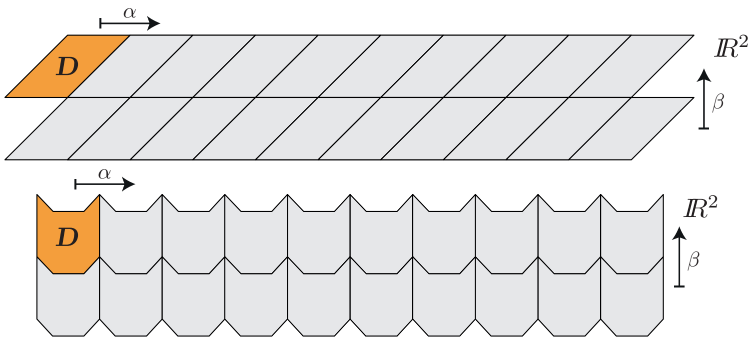
*Proof.* The action of a element $g \in \boldsymbol{H}$ different from the identity map $\mathrm{id}_{\mathbb{R}^2}$, always shifts a point $\vec{v} \in \mathbb{R}^2$ horizontally or vertically by a distance of at least 1. We thus have the inequality $d_{\mathrm{e}}(g(\vec{v}), \vec{v}) \geq 1$, for all $\vec{v} \in \mathbb{R}^2$, and the conditions of properly discontinuous actions without fixed points are obvious. The other claims then result immediately by theorem 6.5.5. $\square$

## 8.2 (B) Glueing Boundaries of Domains

**8.2.1. Fundamental Domains..** A *fundamental domain* for the action of $\boldsymbol{H}$ is a domain $\boldsymbol{D} := (D, d_D) \subseteq \mathbb{R}^2$, *i.e.* the closure of a connected open subset $U \subseteq \mathbb{R}^2$, such that $\nu(\boldsymbol{D}) = \mathbb{R}^2/\boldsymbol{H}$ and such that that the restriction $\nu_{\boldsymbol{H}}|_U : U \to (\mathbb{R}^2/\boldsymbol{H})_{\text{met}}$ is a homeomorphism onto its image.

In the last pictures we represented fundamental domains in orange. Those very simple shapes are obviously not the only ones possible.

For example, in the case of $\boldsymbol{H}_T$, other domains can be the following




as, in both, the domain $\boldsymbol{D}$ has the same surface as the square and, when shifted by $\alpha$ and $\beta$, defines a tessellation of $\mathbb{R}^2$, conditions that characterize fundamental domains (exercise !).

Fundamental domains are important because they are very helpful in understanding the quotient. Indeed, by definition, the restriction

$$\nu_{\boldsymbol{H}}|_D : (\boldsymbol{D}, d_D) \to (\mathbb{R}^2/\boldsymbol{H})_{\text{met}}$$

is a surjective map, and if $\mathcal{R}_{\boldsymbol{H}}$ is the equivalence relation on $\boldsymbol{D}$ induced by $\nu_{\boldsymbol{H}}$, we get a canonical continuous bijection between topological quotients:

$$\boxed{\phi_{\text{top}} : (\boldsymbol{D}/\mathcal{R}_{\boldsymbol{H}})_{\text{top}} \to (\mathbb{R}^2/\boldsymbol{H})_{\text{top}}} \tag{2}$$

This may or may not be a homeomorphism, depending on whether or not the fundamental domain $\boldsymbol{D}$ is compact.

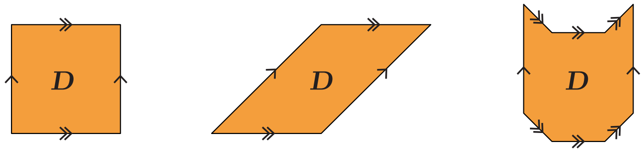
**8.2.2. Compact Fundamental Domains.** When $\boldsymbol{H} \in \{\boldsymbol{H}_T, \boldsymbol{H}_K\}$, the fundamental domain $\boldsymbol{D}$ for $\boldsymbol{H}$ is compact and because $\mathbb{R}^2/\boldsymbol{H}$ is Hausdorff, we immediately conclude that the map (2) is a homeomorphism.

Let us now look more closely at the equivalence relation $\mathcal{R}_{\boldsymbol{H}}$. After the definition of *fundamental domain*, the relation is the identity in the interior $\boldsymbol{D}^\circ$ of $\boldsymbol{D}$. To describe better the relations on the boundary $\partial\boldsymbol{D}$, we shall decompose the group $\boldsymbol{H}$ a *semi-direct product* $\boldsymbol{H} = \langle\alpha'\rangle \rtimes \langle\beta'\rangle$ for certain elements $\alpha', \beta' \in \boldsymbol{H}$, where $\langle\alpha'\rangle \triangleleft \boldsymbol{H}$. We will then have, after exercise 6.5.9, the equality

$$\boldsymbol{D}/\mathcal{R}_{\boldsymbol{H}} = (\boldsymbol{D}/\mathcal{R}_{\alpha'})/\mathcal{R}_{\beta'} . \tag{3}$$

— ***The flat Torus, $\boldsymbol{H} := \boldsymbol{H}_T$.*** The relation $\mathcal{R}_{\boldsymbol{H}}$ identifies the points on the edges having the same number of chevrons

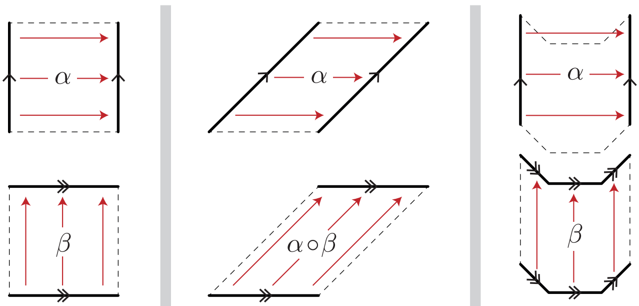
$\boldsymbol{H}_T$ :




(i) (ii) (iii) (4)

following the rules sketched in the images:

$\boldsymbol{H}_T$ :




(i) (ii) (iii)

Note that these rules conform to the procedure for glueing two closed subspaces $\boldsymbol{A}$ and $\boldsymbol{B}$ together, as explained in 6.4.

**8.2.3 Theorem.** *Denote by $(\boldsymbol{D}/\mathcal{R}_{\boldsymbol{H}})_{\mathrm{met}}$ the set $\boldsymbol{D}/\mathcal{R}_{\boldsymbol{H}}$ endowed with the pushforward path distance by the canonical surjection $\nu_{\boldsymbol{H}} : \boldsymbol{D} \twoheadrightarrow \boldsymbol{D}/\mathcal{R}_{\boldsymbol{H}}$. Then, in the natural commutative diagram of quotient spaces*

$$\begin{tikzcd}
\boldsymbol{D} \arrow[r, "\iota_D"] \arrow[d, "\nu_{\boldsymbol{H}}"'] & \mathbb{R}^2 \arrow[d, "\nu_{\boldsymbol{H}}"] \\
(\boldsymbol{D}/\mathcal{R}_{\boldsymbol{H}})_{\mathrm{met}} \arrow[r, "\phi_{\mathrm{met}}"] & (\mathbb{R}^2/\boldsymbol{H})_{\mathrm{met}}
\end{tikzcd}$$

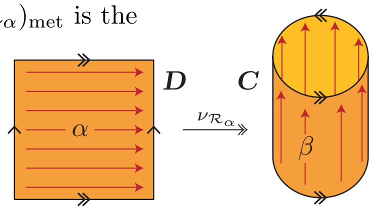
*all the maps are path-length preserving, and $\phi_{\mathrm{met}}$ is a path-distance isometry.*

*Proof.* We explain only the case of the square fundamental domain (i), the other cases follow by the same ideas. We have a direct product decomposition $\boldsymbol{H} = \langle\alpha\rangle\times\langle\beta\rangle$, so that the equivalence $\mathcal{R}_{\boldsymbol{H}}$ is the composition (3) of the equivalences $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$. We then apply theorem 6.4.3-(i) twice, first for $\varphi := \alpha$, second for $\varphi := \beta$.

*First.* The map $\nu_{\boldsymbol{H}}\circ\iota_D$ is compatible with $\mathcal{R}_\alpha$ and factorizes through the canonical surjection $\nu_{\mathcal{R}_\alpha} : D \to (D/\mathcal{R}_\alpha)_{\mathrm{met}}$ in $\phi_{\mathrm{met}} : (\boldsymbol{D}/\mathcal{R}_\alpha)_{\mathrm{met}} \to (\mathbb{R}^2/\boldsymbol{H})_{\mathrm{met}}$:

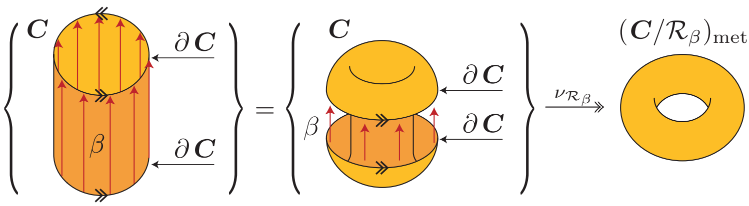
$$\begin{tikzcd}
\boldsymbol{D} \arrow[r, "\iota_D"] \arrow[d, "\nu_{\mathcal{R}_\alpha}"'] \arrow[dr, "\nu_{\boldsymbol{H}}\circ\iota_D"] & \mathbb{R}^2 \arrow[d, "\nu_{\boldsymbol{H}}"] \\
\boldsymbol{C} := (\boldsymbol{D}/\mathcal{R}_\alpha)_{\mathrm{met}} \arrow[r, "\phi_{\mathrm{met}}"] & (\mathbb{R}^2/\boldsymbol{H})_{\mathrm{met}}
\end{tikzcd}$$

Notice that the quotient space $\boldsymbol{C} := (\boldsymbol{D}/\mathcal{R}_\alpha)_{\text{met}}$ is the familiar cylinder endowed with the path-distance. Notice also that the edges of $\boldsymbol{D}$, marked with two chevrons, are sent by $\nu_{\mathcal{R}_\alpha}$ to the boundary $\partial \boldsymbol{C}$ of $\boldsymbol{C}$, where the map $\beta$ defines an isometry between the connected components.




*Second.* The map $\phi_{\text{met}} : \boldsymbol{C} \to (I\!R^2/\boldsymbol{H})_{\text{met}}$ is compatible with $\mathcal{R}_\beta$ and, as the hypotheses of 6.4.3-(i) are satisfied, we infer the existence of the commutative diagram to the right, where one recognizes in the quotient $(\boldsymbol{C}/\mathcal{R}_\beta)_{\text{met}}$, obtained by glueing the components of $\partial \boldsymbol{C}$ following $\beta$, the shape of a *torus* [6], as the images below explain.

$$\begin{tikzcd} \boldsymbol{C} \arrow[r, "\phi_{\text{met}}"] \arrow[d, "\nu_{\mathcal{R}_\beta}"'] & (I\!R^2/\boldsymbol{H})_{\text{met}} \\ (\boldsymbol{C}/\mathcal{R}_\beta)_{\text{met}} \arrow[ur, "\overline{\phi_{\text{met}}}"'] & \end{tikzcd}$$




(5)

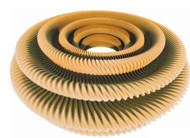
*To Finish.* We check that the composition map $\nu_{\mathcal{R}_\beta} \circ \nu_{\mathcal{R}_\alpha}$ induces on $\boldsymbol{D}$ the relation $\mathcal{R}_{\boldsymbol{H}}$. By the universal property of quotients, there exists a continuous bijection $\phi : \boldsymbol{D}/\mathcal{R}_{\boldsymbol{H}} \to \boldsymbol{C}/\mathcal{R}_\beta$ satisfying $\phi \circ \nu_{\mathcal{R}_\beta} \circ \nu_{\mathcal{R}_\alpha} = \nu_{\boldsymbol{H}}$. This map is bicontinuous since the quotients in question are compact spaces. In this way, we have justified the existence of a commutative diagram

$$\begin{tikzcd} \boldsymbol{D} \arrow[r, equal] \arrow[d, "\nu_{\boldsymbol{H}}"'] & \boldsymbol{D} \arrow[r, "\iota_D"] \arrow[d, two heads, "\nu_{\mathcal{R}_\beta}\circ\nu_{\mathcal{R}_\alpha}"'] & I\!R^2 \arrow[d, two heads, "\nu_{\boldsymbol{H}}"] \\ (\boldsymbol{D}/\mathcal{R}_{\boldsymbol{H}})_{\text{met}} \arrow[r, "\phi", "\sim"'] & (\boldsymbol{C}/\mathcal{R}_\beta)_{\text{met}} \arrow[r, "\bar{\xi}", "\sim"'] & (I\!R^2/\boldsymbol{H})_{\text{met}} \end{tikzcd}$$

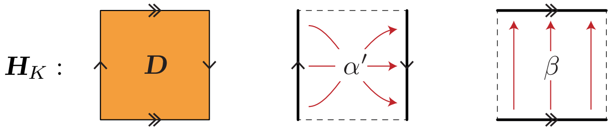
where the maps locally preserve distances and where $\phi$ et $\bar{\xi}$ are homeomorphisms, and therefore establish isometries for path distances. $\square$

---

[6] In fact a *flat* torus, *i.e.* locally isometric to some open disk of the Euclidean plane $I\!R^2$. At this point, the reader should note that, due to flatness, the picture (5) is not faithful, as the (flat) surface will tear when trying to fold its boundary in the middle step. However, the celebrated *Nash embedding theorem* (1954), states that such a torus can actually be isometrically embedded in $I\!R^3$, but the way to do it is so intricate that details are invisible to the naked eye. Indeed, the procedure requires that the surface be thoroughly corrugated at a quite microscopic level as suggested by the image to the right, taken from http://hevea-project.fr/ENPageToreImages.html. See also the video https://youtu.be/RYH_KXhF1SY



—***The Klein Bottle,*** $\boldsymbol{H} := \boldsymbol{H}_K$**.** Limiting ourselves to the case of a square fundamental domain, the relation $\mathcal{R}_{\boldsymbol{H}}$ identifies opposite edges, with glueing rules sketched in the following images, where $\alpha' := \sigma \circ \alpha$.



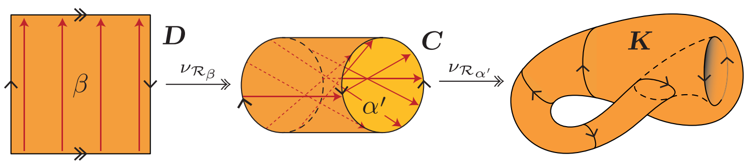
(6)

We have $\langle\beta\rangle \lhd \boldsymbol{H}$, and the factorization (3) becomes:

$$\boldsymbol{D}/\mathcal{R}_{\boldsymbol{H}} = (\boldsymbol{D}/\mathcal{R}_\beta)/\mathcal{R}_{\alpha'}\,. \tag{7}$$

Theorem 8.2.3 holds also for the present data, and follows by the same arguments and ideas, with the difference that we start building the quotient $\boldsymbol{C} := \boldsymbol{D}/\mathcal{R}_\beta$, which appears as a *horizontal* cylinder. The construction of the quotient $\boldsymbol{K} := \boldsymbol{C}/\mathcal{R}_{\alpha'}$ results then in the same way by applying 6.4.3-(i).

Notice that now, one of the components of the boundary $\partial\boldsymbol{C}$ has to be glued by reversing directions ! This can be visualized only by deforming the cylinder in some way, for example, as shown in the figure.



(8)

The path-metric space $\boldsymbol{K} := (\boldsymbol{C}/\mathcal{R}_{\alpha'})_{\mathrm{met}}$ is known as the *Klein Bottle* ([7]). It is a compact surface which is *non-orientable*. From a heuristic point of view, this means that, unlike the torus, or, more commonly, the sphere, the Klein bottle has no interior. As a consequence, such a surface cannot be embedded in the Euclidean space $\mathbb{R}^3$ without tears. This explains why, in picture (8), the surface crosses itself in some places ([8]).

**8.2.4. Noncompact Fundamental Domains.** When $\boldsymbol{H} \in \{\boldsymbol{H}_C, \boldsymbol{H}_M\}$, a fundamental domain $\boldsymbol{D}$ for $\boldsymbol{H}$ can be the product $[0,1]\times\mathbb{R}$. This domain *is not* compact, and the map (2):

$$\phi_{\mathrm{top}} : (\boldsymbol{D}/\mathcal{R}_{\boldsymbol{H}})_{\mathrm{top}} \to (\mathbb{R}^2/\boldsymbol{H})_{\mathrm{top}}\,,$$

*is not* a homeomorphism, as it is not an open map (exercise !). However, theorem 8.2.3 remains true since the action of the generators of $\boldsymbol{H}$, respectively

([7]) After the german mathematician Felix Klein who first described the surface in 1882.

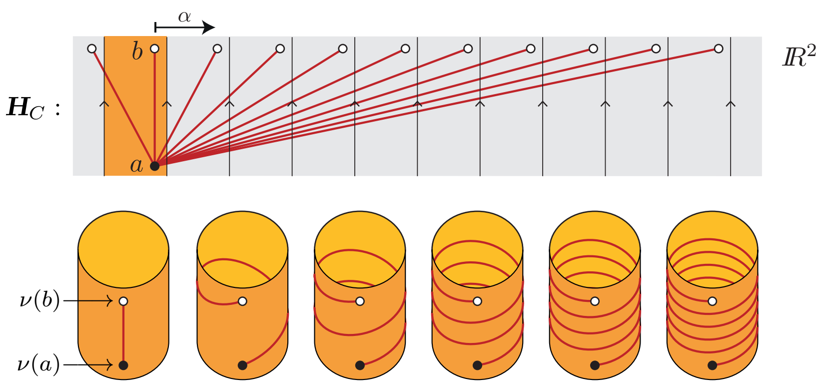
([8]) This problem can be avoided by adding an extra dimension, such as time, in which case the embedding can go through the conflicting area at different times so that there will be no further crossings.

$\alpha$ and $\alpha' := \sigma\circ\alpha$, on $\partial\boldsymbol{D}$ continue to satisfy the hypotheses of 6.4.3-(i) used in the proof, as can immediately be checked in the following images.


(9)

and the glueing rules of the boundaries are as shown here

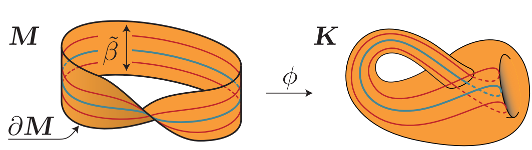

**8.2.5 Exercise.** Show that the set of geodesics between two generic points $\nu(a)$ and $\nu(b)$ in $(\mathbb{R}^2/\boldsymbol{H}_C)_{\mathrm{met}}$ is in bijection with the set of segments $[a,g(b)]$ in $\mathbb{R}^2$, with $g\in\boldsymbol{H}_C$. The set is, hence, countably infinite.




Determine the analog statement for the other three groups $\boldsymbol{H}_M,\boldsymbol{H}_T,\boldsymbol{H}_K$.

**8.2.6 Exercise.** Although this exercise is about compact fundamental domains, we show it here because we need the *compact* Möbius ribbon, obtained by restricting the above action of $\boldsymbol{H}_M$ from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}\times[0,1]$. In that case the compact square $\boldsymbol{D} := [0,1]\times[0,1]$ is a fundamental domain, and the canonical surjection onto the Klein bottle $\nu_{\boldsymbol{H}_K} : \boldsymbol{D}\to\boldsymbol{K} := \mathbb{R}^2/\boldsymbol{H}_K$ is compatible with the action of the group $\langle\alpha'\rangle$ in the sense that $\nu_{\boldsymbol{H}_K}\circ\alpha' = \nu_{\boldsymbol{H}_K}$. As a consequence, $\nu_{\boldsymbol{H}_K}$ factors through the *compact* Möbius ribbon $\boldsymbol{M} := \boldsymbol{D}/\mathcal{R}_{\alpha'}$:

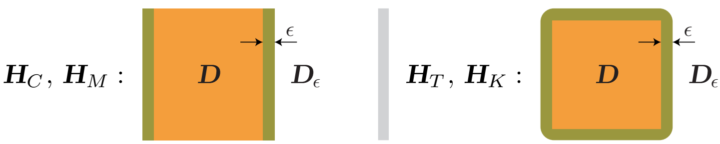
$$\begin{tikzcd} \boldsymbol{D} \arrow[r, two heads, "\nu_{\boldsymbol{H}_K}"] \arrow[d, two heads, "\nu_{\mathcal{R}_{\alpha'}}"'] & \boldsymbol{K} \\ \boldsymbol{M} \arrow[ur, two heads, "\phi"'] & \end{tikzcd}$$



Let $\mathcal{R}_\phi$ be the equivalence relation on $\boldsymbol{M}$, induced by $\phi : \boldsymbol{M} \twoheadrightarrow \boldsymbol{K}$. Show that $\mathcal{R}_\phi$ is the relation $\mathcal{R}_{\tilde{\beta}}$ generated by the isometry $\tilde{\beta} : \partial\boldsymbol{M} \to \partial\boldsymbol{M}$, which associates a point with its symmetric relative to the medial (blue) line in $\boldsymbol{M}$, as shown in the figure above. Check that the conditions to apply theorem 6.4.3-(i), as modified in 6.4.6, are satisfied. Conclude that there exists an isometry of path-metric spaces between $(\boldsymbol{M}/\mathcal{R}_{\tilde{\beta}})_{\mathrm{met}}$ et $\boldsymbol{K}$.

## 8.3 (C) Glueing Overlaps of Tubular Neighborhoods of Domains

**8.3.1. Tubular Neighborhoods of Fundamental Domains.** If $\boldsymbol{D}$ is a fundament domain for $\boldsymbol{H}$, we denote by $\boldsymbol{D}_\epsilon$ the open subspace of points $x \in \mathbb{R}^2$ verifying $d_{\mathrm{e}}(x,\boldsymbol{D}) < \epsilon$. When $\epsilon < 1$, we have




As in the study of compact fundamental domains, the map

$$\nu_{\boldsymbol{H}}|_{D_\epsilon} : (\boldsymbol{D}_\epsilon, d_{D_\epsilon}) \to (\mathbb{R}^2/\boldsymbol{H})_{\mathrm{met}}$$

is surjective, and if we denote by $\mathcal{R}_{\boldsymbol{H}}$ the equivalence relation it induces on $\boldsymbol{D}_\epsilon$, we get a continuous bijection between the underlying topological spaces

$$\phi_{\mathrm{top}} : (\boldsymbol{D}_\epsilon/\mathcal{R}_{\boldsymbol{H}})_{\mathrm{top}} \to (\mathbb{R}^2/\boldsymbol{H})_{\mathrm{top}} \tag{10}$$

**8.3.2 Theorem.** a) *The canonical surjection* $\nu_{\mathcal{R}_{\boldsymbol{H}}} : \boldsymbol{D}_\epsilon \to (\boldsymbol{D}_\epsilon/\mathcal{R}_{\boldsymbol{H}})_{\mathrm{top}}$ *is a local homeomorphism and the map* (10) *is a homeomorphism.*

b) *Denote by* $(\boldsymbol{D}_\epsilon/\mathcal{R}_{\boldsymbol{H}})_{\mathrm{met}}$ *the quotient set* $\boldsymbol{D}_\epsilon/\mathcal{R}_{\boldsymbol{H}}$ *endowed with the push-forward distance defined by* $\nu_{\mathcal{R}_{\boldsymbol{H}}}$*. Then, the natural diagram of quotients*

$$\begin{tikzcd}
\boldsymbol{D}_\epsilon \arrow[r, "\iota_{D_\epsilon}"] \arrow[d, "\nu_{\mathcal{R}_{\boldsymbol{H}}}"'] & \mathbb{R}^2 \arrow[d, "\nu_{\boldsymbol{H}}"] \\
(\boldsymbol{D}_\epsilon/\mathcal{R}_{\boldsymbol{H}})_{\mathrm{met}} \arrow[r, "\phi_{\mathrm{met}}", "\simeq"'] & (\mathbb{R}^2/\boldsymbol{H})_{\mathrm{met}}
\end{tikzcd}$$

*is commutative and the map* $\phi_{\mathrm{met}}$ *is an isometry of path-metric spaces.*

c) *The topologies of* $(\boldsymbol{D}_\epsilon/\mathcal{R}_{\boldsymbol{H}})_{\mathrm{top}}$ *and* $(\boldsymbol{D}_\epsilon/\mathcal{R}_{\boldsymbol{H}})_{\mathrm{met}}$ *coincide.*

*Proof.* (a) Since $\nu_{\boldsymbol{H}} : \mathbb{R}^2 \to \mathbb{R}^2/\boldsymbol{H}$ is a local homeomorphism (6.5.5), its restriction to $\boldsymbol{D}_\epsilon$ is also. The statement then immediately follows.

(b) We can conclude in two different ways. The most direct is to apply theorem 6.5.1. Indeed, as the map $\nu_{\boldsymbol{H}} := \mathbb{R}^2 \to (\mathbb{R}^2/\boldsymbol{H})_{\mathrm{met}}$ is a local isometry, it defines an atlas of $(\mathbb{R}^2/\boldsymbol{H})_{\mathrm{met}}$ whose charts constitute an open cover of $\mathbb{R}^2$ and whose transition maps are isometries. The restriction of those charts to any open subspace $\boldsymbol{U} \subseteq \mathbb{R}^2$ containing $\boldsymbol{D}$, for example $\boldsymbol{D}_\epsilon$, define another atlas of $(\mathbb{R}^2/\boldsymbol{H})_{\text{met}}$ of the same kind. The canonical surjection $\boldsymbol{U} \twoheadrightarrow (\boldsymbol{U}/\mathcal{R}_{\boldsymbol{H}})$ is a local isometry, after the same theorem, and (b) follows.

The other way to conclude is to use theorem 6.4.3-(ii) on the glueing of open subsets as follows.

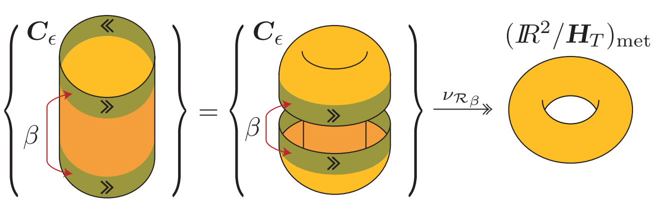
When $\boldsymbol{H} \in \{\boldsymbol{H}_C, \boldsymbol{H}_M\}$, we note, as in 8.2.4, that the action of the generators of $\boldsymbol{H}$, respectively $\alpha$ and $\alpha' := \sigma \circ \alpha$, on the overlaps satisfy the conditions to apply 6.4.3, as it results from the inspection of the images:


the arrows indicate the glueing directions:


When $\boldsymbol{H} \in \{\boldsymbol{H}_T, \boldsymbol{H}_K\}$, we proceed as in the case of compact fundamental domains. For example, if $\boldsymbol{H} := \boldsymbol{H}_T$ we apply the same arguments as for theorem 8.2.3. We begin constructing the cylinder $\boldsymbol{C}_\epsilon := (\boldsymbol{D}_\epsilon/\mathcal{R}_\alpha)_{\text{met}}$. We then note that the induced map $\phi_{\text{met}} : \boldsymbol{C}_\epsilon \to (\mathbb{R}^2/\boldsymbol{H})_{\text{met}}$ is a local isometry compatible with $\mathcal{R}_\beta$, and we get the commutative diagram

$$\begin{tikzcd}
\boldsymbol{C}_\epsilon \arrow[r, "\phi_{\text{met}}"] \arrow[d, "\nu_{\mathcal{R}_\beta}"'] & (\mathbb{R}^2/\boldsymbol{H})_{\text{met}} \\
(\boldsymbol{C}_\epsilon/\mathcal{R}_\beta)_{\text{met}} \arrow[ur, "\overline{\phi_{\text{met}}}"'] &
\end{tikzcd}$$

where all the maps are local isometries, after 6.4.3-(ii). In particular, $\overline{\phi_{\text{met}}}$ is a path-metric space isometry.



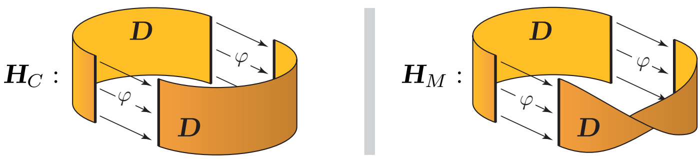

The case where $\boldsymbol{H} := \boldsymbol{H}_K$ follows from the same ideas and is left to the motivated reader. □

## 8.4 (D,E) Assemblages of Copies of a Domain in $\mathbb{R}^2$

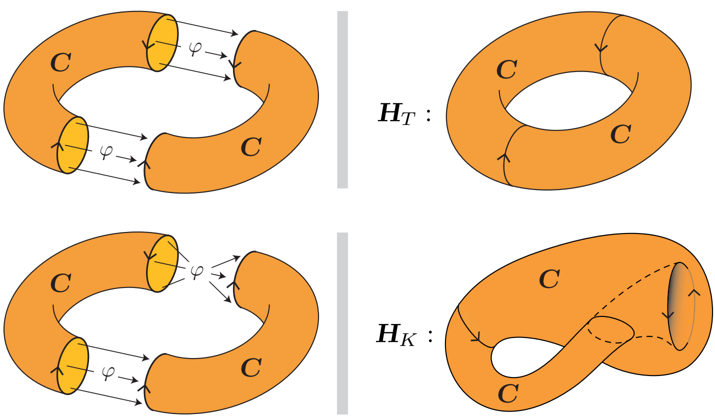
In this section we come back to our initial aims in Chapter **2**, which were to construct spaces by assembling copies of a single space.

**8.4.1. Closed Assemblages.** The charts are copies of the same fundamental domain $\boldsymbol{D} \subseteq \mathbb{R}^2$ with boundary $\partial\boldsymbol{D}$, and the transition maps are isometries between closed subspaces of $\partial\boldsymbol{D}$. When these subspaces are *line segments*, the conditions for the assemblage of closed atlases of path-metric spaces of 7.3 are satisfied and we can apply the theorem 7.3.1 which states the existence of assemblages endowed with a path-metric space structure.

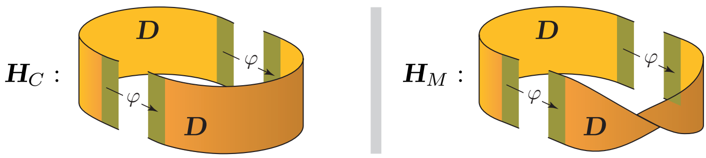
When $\boldsymbol{H} \in \{\boldsymbol{H}_C, \boldsymbol{H}_M\}$, the fundamental domain is $\boldsymbol{D} := [0,1]\times\mathbb{R}$. Its boundary is the subspace $\{0,1\}\times\mathbb{R}$. The atlases consist of two copies of $\boldsymbol{D}$ and the images below show the isometric transition maps. The assemblages give respectively the path-metric spaces $\boldsymbol{C}$ and $\boldsymbol{M}$.




When $\boldsymbol{H} \in \{\boldsymbol{H}_T, \boldsymbol{H}_K\}$, the fundamental domain is $\boldsymbol{D} := [0,1]\times[0,1]$. Its boundary is the subspace $(\{0,1\}\times[0,1]) \cup ([0,1]\times\{0,1\})$. The atlases consist of four copies of $\boldsymbol{D}$, the two pairs of which are used to first assemble two identical cylinders $\boldsymbol{C}$. These are then glued by their boundaries, as shown in the pictures below, to get respectively the torus and the Klein bottle.



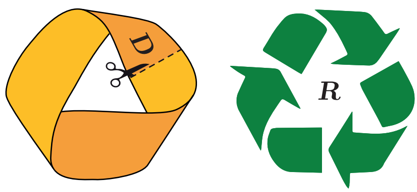

**8.4.2. Open Assemblages..** The only difference is that here the charts are tubular neighborhoods of fundamental domains $\boldsymbol{D}$. The conditions for the assemblage of open atlases of path-metric spaces of 7.2 are simpler and are satisfied. We apply the existence theorem for open assemblages 7.2.1.



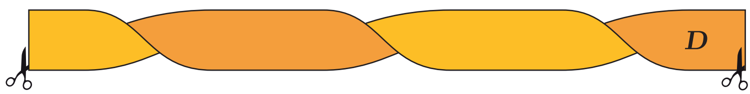

The details are left to the reader.

### 8.5. Example of Isometries on Quotients

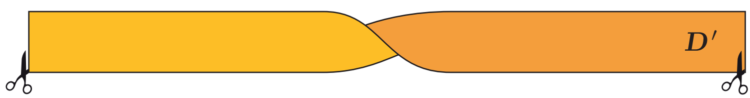
The techniques of assemblages and quotients allow us to construct isometries between spaces which may at first sight seem non-isometric. For example, the *Universal Recycling Logo* $\boldsymbol{R}$, created for the first *Earth Day* (april 22) in 1970, obtained by making *three* half-turns on a rectangular ribbon $\boldsymbol{D}$ before glueing the small edge, is isometric to the Möbius strip $\boldsymbol{M}$, defined in 8.1 by making just *one* half-turn on $\boldsymbol{D}$ before the glueing.



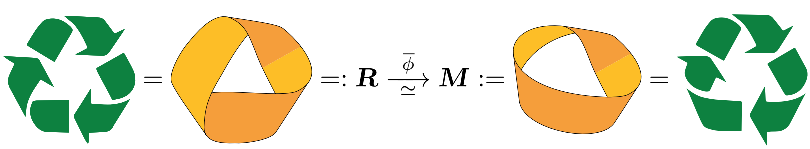

We can see this by cutting $\boldsymbol{R}$ with a pair of scissors as shown in the previous picture. In this way we get the original ribbon $\boldsymbol{D}$:




which is clearly isometric to the same ribbon unfolded twice $\boldsymbol{D}'$, and this, in a manner compatible with the scissors' cutting direction.

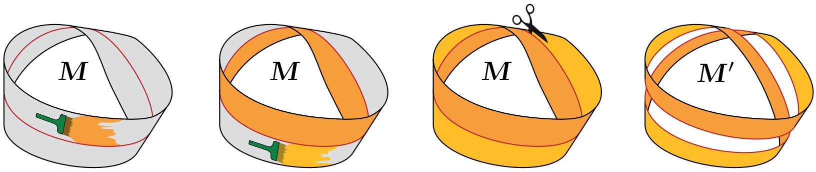



Let $\phi : \boldsymbol{D} \to \boldsymbol{D}'$ denote such isometry. As $\phi$ is compatible with the equivalence relations that identify the edge points to be glued together, it induces, by 6.4.3-(i), an isometry between the respective path-metric quotients, *i.e.* between the recycling logo space $\boldsymbol{R}$ and the Möbius strip $\boldsymbol{M}$:



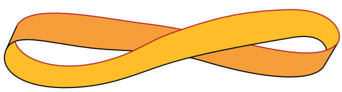
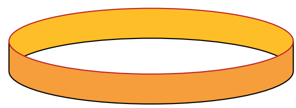

It is worth emphasizing that this isometry *cannot* be extended to a global isometry of $\mathbb{R}^3$. (This is the very starting point to *Knot Theory*).

**8.5.1 Exercise.** We already know that the Möbius Strip $\boldsymbol{M}$ has only one side, but if we draw the medial axis (red), and begin painting a region in some color, we end up painting just one half of the strip, the other half being at its rear. Paint this rear region in another color, and cut the strip following the medial line, as shown in the figure below.

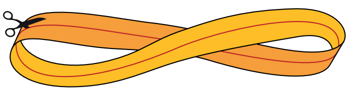
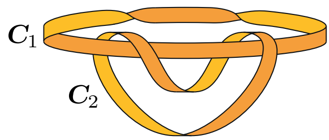



The result will be the surface $\boldsymbol{M}'$: a ribbon with two sides, each colored distinctly, the same as the surface in the left-hand figure below.

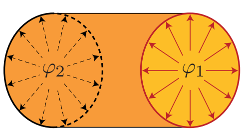
1) Following the procedure in Example 8.5, show that $\boldsymbol{M}'$ is isometric to the cylinder $\boldsymbol{C}$.

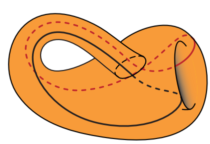
 $=: \boldsymbol{M}' \xrightarrow[\simeq]{} \boldsymbol{C} :=$ 

2) Show that repeating the same exercise by cutting $\boldsymbol{M}'$ by its medial axis, one obtains two isometric cylinders $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ (intertwined in their natural embedding in $\mathbb{R}^3$).

 $=: \boldsymbol{M}' \quad \xrightarrow[\simeq]{}$ 

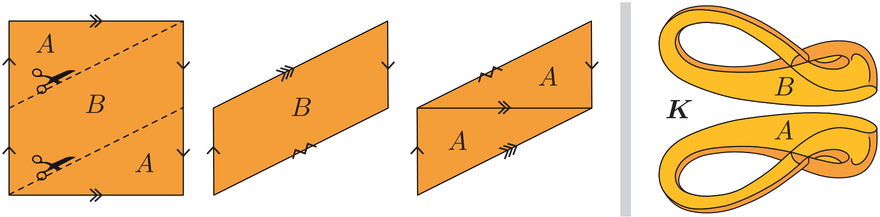

3) In exercise 8.2.6, we saw that the Klein bottle is obtained from the Möbius Strip $\boldsymbol{M}$ by identifying the points on its boundary $\partial\boldsymbol{M}$ which are symmetric relative to the medial axis. Track this identification in the ribbon $\boldsymbol{M}'$ and in the cylinder $\boldsymbol{C}$ of question (1). Conclude that the Klein bottle can be built by identifying opposite points in each connected component of the boundary of a cylinder as shown in the figure below.



$=: \boldsymbol{C} \xrightarrow{\nu} \boldsymbol{K} :=$ 

(*Hint: apply theorem 6.4.3-(i), as modified in 6.4.6. Notice that the intermediate quotient* $\boldsymbol{C}/\mathcal{R}_{\varphi_1}$ *is also isometric to the original Möbius strip* $\boldsymbol{M}$*, but such a construction is, in practice, impossible to do in our three dimensional space.*)

**8.5.2 Exercise.** By cutting a fundamental domain for the Klein bottle $\boldsymbol{K}$ as shown in the figure




prove that $\boldsymbol{K}$ is the assemblage of two Möbius strips.

## Chapter Contents

# Chapter 6

# Flat Riemannian Manifolds with Piecewise Linear Boundary

In this chapter and the next, we will introduce the working category where to study isospectrality questions. Its objects will be more than mere metric spaces $\boldsymbol{M}$, since we need to make them vibrate, which means, according to our preliminary discussions of chapter **1**, that we need to give meaning to the Euler-Lagrange differential equation (**1**♦1.3). We must therefore enrich the notion of metric space in a way we can talk of its differentiable functions and of its Laplace operator. We need to attach to $\boldsymbol{M}$ *the vector space with linear operator* $(\mathcal{C}^\infty(\boldsymbol{M}), \Delta_{\boldsymbol{M}})$. The general answer is given by the *Riemannian manifolds with boundary*, but they are technical and go far beyond our purposes, for which we only need to consider, among these, those which are *flat*, whence the title of this chapter. These are simpler to introduce and still have a very rich structure, we will call them *PL-manifolds*, '*PL*' for *Piecewise Linear*.

The word '*flat*' has often been used in this book, but never gave it a mathematical meaning. It is now the time to fill this gap. This will allow us to introduce the *simple flat spaces*, which are the generalizations, in any dimension, of the familiar planar domains with polygonal boundary. Then, following the preliminaries of chapter **2**, we will build, by assemblage, the most general flat spaces, the PL-manifolds.

Another important question concerns the notion of *morphisms* between *PL*-manifolds. Indeed, these cannot be just continuous maps $\phi : \boldsymbol{M} \to \boldsymbol{M'}$, as they will have to link together differentiable functions and Laplace operators. The next chapter will show that this forces morphisms to be *local isometries*, which is why we will attach particular interest to these in this chapter.

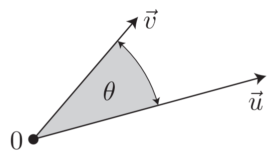
## 1 Affinity of Isometries in Euclidean Vector Spaces

In this section we look closely to the important phenomenon by which any *metric* isometry $\phi : U \to V$ between open subsets of an Euclidean vector space $I\!R^n$, is automatically *affine*. More precisely, $\phi$ is the composition of an *orthogonal linear transformation* and a *translation*. This observation will be very useful later, when we introduce differentiability and Laplacians, as it immediately explains the *isospectrality* of isometric flat Riemannian manifolds.

### 1.1 The Euclidean Structure of $\mathbb{R}^n$

We recall some definitions, notations and known facts of Euclidean Geometry.

- The real vector space $\mathbb{R}^n$ is equipped with the *canonical scalar product*:
$$\langle \_,\_ \rangle : \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}, \quad \langle \vec{u},\vec{v} \rangle := \textstyle\sum_{i=1,\dots,n} u_i v_i,$$
which is a nondegenerate positive definite bilinear form.

- The *Euclidean norm* of a vector $\vec{u} \in \mathbb{R}^n$ is defined by Pythagoras formula
$$\|\vec{u}\| := \sqrt{\langle \vec{u},\vec{u} \rangle} = \sqrt{\textstyle\sum_{i=1,\dots,n} u_i^2}.$$
The *Euclidean distance* between $\vec{u},\vec{v} \in \mathbb{R}^n$ is then defined as:
$$d_{\mathrm{e}}(\vec{u},\vec{v}) := \|\vec{u} - \vec{v}\|.$$
That this is a distance results from the *triangular inequality for the norm*, $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$, itself equivalent to the *Cauchy-Schwarz inequality*, which states that
$$\langle \vec{a},\vec{b} \rangle \leq \|\vec{a}\|\,\|\vec{b}\|.$$


The *angle* $\theta$ between $\vec{u}$ and $\vec{v}$ verifies the equality:
$$\cos(\theta) = \frac{\langle \vec{u},\vec{v} \rangle}{\|\vec{u}\|\,\|\vec{v}\|}.$$
The vectors are said to be *orthogonal* if $\langle \vec{u},\vec{v} \rangle = 0$, we then write $\vec{u} \perp \vec{v}$.

- A family of vectors $\{\vec{v}_i\}_{i \in \mathfrak{I}}$ is an *orthogonal family* if $\langle \vec{v}_i,\vec{v}_j \rangle = 0$ for all $i \neq j$. Is is *orthonormal* if, in addition, $\|\vec{v}_i\| = 1$, for all $i \in \mathfrak{I}$.
  - An orthogonal family of nonzero vectors is linearly independent.
  - The canonical basis $\{e_i := (0,\dots,1_i,\dots,0)\}_{i=1,\dots,n}$ of $\mathbb{R}^n$, is orthonormal.
  - The *Gram-Schmidt process* shows that any linear subspace of $\mathbb{R}^n$ admits orthonormal bases.
  - For a subset $S \subseteq \mathbb{R}^n$, the set of *orthogonal vectors to $S$*, *i.e.*
  $$S^\perp := \{\vec{v} \in \mathbb{R}^n \mid (\vec{v} \perp \vec{s}) \quad \forall \vec{s} \in S\},$$
  is a vector subspace of $\mathbb{R}^n$ (exercise!). If, in addition, $S$ is a vector subspace of $\mathbb{R}^n$, one has $S \cap S^\perp = 0$ and $\mathbb{R}^n = S \oplus S^\perp$, which is why the subspace $S^\perp$ is called the *orthogonal complement* of $S$.

**1.1.1 Exercise.** Let $H \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ be vector subspaces with the induced scalar products and Euclidean distances. Let $\alpha : H \to L$ be a linear map. Show the equivalence of the following properties.

(i) $\alpha$ preserves Euclidean distances; (ii) $\alpha$ preserves scalar products; (iii) $\alpha$ preserves norms; (iv) $\alpha$ preserves angles. (v) $\alpha$ transforms a particular orthonormal basis in an orthonormal family; And, if $n \geq 2$, (vi) $\alpha$ preserves orthogonality; (vii) $\alpha$ preserves othonormality.

**1.2 Euclidean Linear Maps.** Let $H \subseteq I\!R^n$ and $L \subseteq I\!R^m$ be vector subspaces with the induced Euclidean distances. A map $\alpha : H \to L$, is said to be *an Euclidean linear map*, if it is linear and preserves the Euclidean distance:

$$d_{\mathrm{e}}\big(\alpha(\vec{u}),\alpha(\vec{v})\big) = d_{\mathrm{e}}(\vec{u},\vec{v}), \quad \forall \vec{u},\vec{v} \in H.$$

**1.2.1 Exercise (and definition).** A linear map $\alpha : I\!R^n \to I\!R^n$ is called *orthogonal* if it is an *Euclidean linear isomorphism*. The *orthogonal group* $O(n)$ is then the group of orthogonal operators of $I\!R^n$. Show that $\alpha \in O(n)$ if and only if $M(\alpha)M(\alpha)^{\mathrm{tr}} = \mathbf{1}_n$, where $M(\alpha)$ denotes the matrix of $\alpha$ relative to some orthonormal basis $\mathcal{B}$.

**1.3 Linearity of Isometries fixing 0.** We now show the basic fact that a map of $I\!R^n$ into itself, which is only assumed to fix 0 and to preserve the Euclidean distance, is necessary linear.

**1.3.1 Proposition.** a) *Given a finite set* $\mathcal{V} := \{\vec{v}_1,\ldots,\vec{v}_s\}$ *of vectors in* $I\!R^n$, *denote by* $\mathrm{Vec}(\mathcal{V})$ *its linear span,* i.e. *the smallest vector subspace of* $I\!R^n$ *containing the* $\vec{v}_i$*'s. Let* $\langle\!\langle \mathcal{V} \rangle\!\rangle$ *be the symmetric* $s\times s$ *matrix whose* $(i,j)$ *coefficient is* $\langle \vec{v}_i,\vec{v}_j \rangle$. *Then*

$$\mathrm{rank}\langle\!\langle \mathcal{V} \rangle\!\rangle = \dim_{I\!R}\big(\mathrm{Vec}(\mathcal{V})\big).$$

b) *Given a set* $\mathcal{B} := \{\vec{v}_1,\ldots,\vec{v}_d\}$ *of linearly independent vectors in* $I\!R^n$, *and an orthonormal basis* $\mathcal{E} := \{\vec{\varepsilon}_1,\ldots,\vec{\varepsilon}_d\}$ *of* $\mathrm{Vec}(\mathcal{B})$, *let* $P$ *be the change of basis matrix from* $\mathcal{B}$ *to* $\mathcal{E}$, *in short* $\mathcal{E} = P\mathcal{B}$, i.e. *the matrix* $P \in M(d,I\!R)$ *such that*

$$\begin{pmatrix} \vec{\varepsilon}_1 \\ \vdots \\ \vec{\varepsilon}_d \end{pmatrix} = P \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_d \end{pmatrix}.$$

*Then, the map*

$$\big\langle P,\mathcal{B},\_ \big\rangle : \mathrm{Vec}(\mathcal{B}) \to I\!R^d, \quad (\_) \mapsto P \begin{pmatrix} \langle \vec{v}_1 \_ \rangle \\ \vdots \\ \langle \vec{v}_d, \_ \rangle \end{pmatrix}$$

*is an isomorphism of Euclidean vector spaces.*

c) *Let* $S$ *be a subset of* $I\!R^n$ *such that* $0 \in S$ *with the induced Euclidean metric. If* $\phi : S \to I\!R^n$ *verifies* $\phi(0) = 0$ *and preserves distances, then there exists one and only one isometry* $\tilde{\phi} : \mathrm{Vec}(S) \to \mathrm{Vec}(\phi(S))$ *extending* $\phi$. *The map* $\tilde{\phi}$ *is moreover linear and there exists* $\Phi \in O(n)$, *unique when* $\mathrm{Vec}(S) = I\!R^n$, *such that*

$$\Phi(s) = \phi(s), \quad \forall s \in S.$$

*In particular, an isometry of* $(I\!R^n,d_{\mathrm{e}})$ *fixing the origin, belongs to* $O(n)$.

*Proof.* (a) Write the matrix $\langle\!\langle \mathcal{V} \rangle\!\rangle$ in the following more suggestive way.

$$\langle\!\langle \mathcal{V} \rangle\!\rangle = \left\langle \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_s \end{pmatrix}, \big(\vec{v}_1,\ldots,\vec{v}_s\big) \right\rangle. \tag{$\diamond$}$$

Fix any an orthonormal basis $\{w_1,\ldots,w_d\}$ of $\mathrm{Vec}(\mathcal{V})$, and let $P$ be the $s\times d$-matrix such that

$$\begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_s \end{pmatrix} = P \begin{pmatrix} \vec{w}_1 \\ \vdots \\ \vec{w}_d \end{pmatrix}.$$

Replacing this expression in ($\diamond$), one gets $\langle\!\langle \mathcal{V} \rangle\!\rangle = PP^{\mathrm{tr}}$, whence

$$\mathrm{rank}(PP^{\mathrm{tr}}) \le d. \tag{$\ddagger$}$$

For the same reasons, there exists a $d\times s$ matrix $Q$ such that

$$\begin{pmatrix} \vec{w}_1 \\ \vdots \\ \vec{w}_d \end{pmatrix} = Q \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_s \end{pmatrix}.$$

in which case $Q(PP^{\mathrm{tr}})Q^{\mathrm{tr}} = \mathbf{1}_d$, which can only happen if $\mathrm{rank}(PP^{\mathrm{tr}}) \ge d$. Therefore $\mathrm{rank}(PP^{\mathrm{tr}}) = d$, after ($\ddagger$).

(b) As the map $\langle \mathcal{E},\mathcal{B},\_ \rangle$ is clearly linear, the whole statement will result immediately from the verification that this map preserves the scalar product, in which case, injectivity is a consequence of the preservation of norms.

By definition, we have

$$\big\langle \mathcal{E},\mathcal{B},\_ \big\rangle = \begin{pmatrix} \langle \vec{\varepsilon}_1\,\_ \rangle \\ \vdots \\ \langle \vec{\varepsilon}_d,\_ \rangle \end{pmatrix},$$

and if $\vec{a},\vec{b} \in \mathrm{Vec}(\mathcal{B})$, then

$$\big\langle \langle \mathcal{E},\mathcal{B},\vec{a} \rangle, \langle \mathcal{E},\mathcal{B},\vec{b} \rangle \big\rangle = \sum\nolimits_i \langle \vec{\varepsilon}_i,\vec{a} \rangle \langle \vec{\varepsilon}_i,\vec{b} \rangle = \Big\langle \sum\nolimits_i \vec{\varepsilon}_i \langle \vec{\varepsilon}_i,\vec{b} \rangle, \vec{a} \Big\rangle = \langle \vec{a},\vec{b} \rangle.$$

(c) Obvious if $\#S \le 1$. In the general case, if $\vec{v},\vec{w} \in S$, we have

$$\|\vec{v}-\vec{w}\|^2 = \big\langle \vec{v}-\vec{w},\vec{v}-\vec{w} \big\rangle = \|\vec{v}\|^2 - 2\big\langle \vec{v},\vec{w} \big\rangle + \|\vec{w}\|^2,$$

so that

$$\big\langle \vec{v},\vec{w} \big\rangle = \frac{1}{2}\Big( d_{\mathrm{e}}(\vec{v},0)^2 + d_{\mathrm{e}}(\vec{w},0)^2 - d_{\mathrm{e}}(\vec{v},\vec{w})^2 \Big),$$

and we immediately conclude that

$$\big\langle \vec{v},\vec{w} \big\rangle = \big\langle \phi(\vec{v}),\phi(\vec{w}) \big\rangle, \quad \forall \vec{v},\vec{w} \in S. \tag{$*$}$$

**Lemma (A).** *Let $\mathcal{B}$ is a basis of* $\mathrm{Vec}(S)$*, let $\mathcal{E}$ is an orthonormal basis of* $\mathrm{Vec}(S)$*, and let $P$ be the transformation matrix from $\mathcal{B}$ to $\mathcal{E}$. Then, $\phi(\mathcal{B})$ is a basis of* $\mathrm{Vec}(\phi(S))$ *and $P\phi(\mathcal{B})$ is an orthonormal basis of* $\mathrm{Vec}(\phi(S))$.

*Proof of Lemma.* The sets $\phi(\mathcal{B})$ and $\phi(\mathcal{E})$ are bases of $\mathrm{Vec}(\phi(S))$, after (a). On the other hand, thanks to $(*)$, we have

$$\Big\langle P\begin{pmatrix}\phi(\vec{v}_1)\\ \vdots\\ \phi(\vec{v}_d)\end{pmatrix},\big(\phi(\vec{v}_1),\dots,\phi(\vec{v}_d)\big)P^{\mathrm{tr}}\Big\rangle = P\Big\langle\begin{pmatrix}\phi(\vec{v}_1)\\ \vdots\\ \phi(\vec{v}_d)\end{pmatrix},\big(\phi(\vec{v}_1),\dots,\phi(\vec{v}_d)\big)\Big\rangle P^{\mathrm{tr}}$$

$$= P\Big\langle\begin{pmatrix}\vec{v}_1\\ \vdots\\ \vec{v}_d\end{pmatrix},\big(\vec{v}_1,\dots,\vec{v}_d\big)\Big\rangle P^{\mathrm{tr}} = \Big\langle\begin{pmatrix}\vec{\varepsilon}_1\\ \vdots\\ \vec{\varepsilon}_d\end{pmatrix},\big(\vec{\varepsilon}_1,\dots,\vec{\varepsilon}_d\big)\Big\rangle = \mathbf{1}_d.$$

which proves that the basis $P\phi(\mathcal{B})$ is orthonormal. $\boxminus$

Thanks to this lemma, we can apply (b) and consider the diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\subseteq"] \arrow[d, "\phi"] & S \arrow[r, "\subseteq"] \arrow[d, "\phi"] & \mathrm{Vec}(S) \arrow[r, "\langle P{,}\mathcal{B}{,}_\rangle", "\simeq"'] \arrow[d, phantom, "(\mathbf{I})"] & I\!R^d \arrow[d, equal, "\mathrm{id}"] \\
\phi(\mathcal{B}) \arrow[r, "\subseteq"] & \phi(S) \arrow[r, "\subseteq"] & \mathrm{Vec}(\phi(S)) \arrow[r, "\langle P{,}\phi(\mathcal{B}){,}_\rangle", "\simeq"'] & I\!R^d
\end{tikzcd}$$

where the sub-diagram $(\mathbf{I})$ is commutative. Indeed, for all $\vec{s}\in S$,

$$\big\langle P,\mathcal{B},\vec{s}\big\rangle = P\begin{pmatrix}\langle\vec{v}_1,\vec{s}\rangle\\ \vdots\\ \langle\vec{v}_d,\vec{s}\rangle\end{pmatrix} = P\begin{pmatrix}\langle\phi(\vec{v}_1),\phi(\vec{s})\rangle\\ \vdots\\ \langle\phi(\vec{v}_d),\phi(\vec{s})\rangle\end{pmatrix} = \big\langle P,\phi(\mathcal{B}),\phi(\vec{s})\big\rangle,$$

after $(*)$. Consequently, the *linear map*

$$\tilde{\phi} := \big\langle P,\phi(\mathcal{B}),\_\big\rangle^{-1}\circ\big\langle P,\mathcal{B},\_\big\rangle : \mathrm{Vec}(S)\to\mathrm{Vec}(\phi(S))$$

extends the metric space isometry $\phi : S\to\phi(S)$.

Coming back to the initial data, which is just $\phi : S\to I\!R^n$, the fact that on a different basis $\mathcal{B}'$ we still have equality of maps $\phi=\tilde{\phi}$ on $S$, ensures that the extension $\tilde{\phi}$ is independent from the choice of $\mathcal{B}$. In the same lines, for any $S'\supseteq S$ such that $\mathrm{Vec}(S)=\mathrm{Vec}(S')$, and for *any* extension of $\phi$ by a map of metric spaces $\phi' : S'\to I\!R^n$ preserving distances, we can choose the same $\mathcal{B}$ to define $\tilde{\phi}$ and $\tilde{\phi}'$ and hence conclude that these extensions coincide. In the extreme case where $S'=I\!R^n$, this is saying that any *metric* extension of $\phi$ to the whole $I\!R^n$ is necessarily *linear*, as it coincides with $\tilde{\phi}$.

To finish, if $\mathrm{Vec}(S) \subsetneq \mathbb{R}^n$, let $Z := \mathrm{Vec}(S)^\perp$ and $Z' := \mathrm{Vec}(\phi(S))^\perp$. Then the $\Phi \in O(n)$ extending $\phi$ are of the form

$$\Phi : Z \oplus \mathrm{Vec}(S) \to Z' \oplus \mathrm{Vec}(\phi(S)), \quad (\vec{z}, \vec{u}) \mapsto (\psi(\vec{z}), \tilde{\phi}(\vec{u}))$$

where $\psi : Z \to Z'$ is any Euclidean isomorphism. $\square$

**1.3.2 Exercise.** Complete lemma A, showing that $\phi(\mathcal{E}) = P\phi(\mathcal{B})$.

**1.4 Affinity of Euclidean Isometries.** In order to generalize 1.3 to general isometries, we need to recall the notion of affinity.

**1.4.1 Definitions.** Let $\boldsymbol{V}$ be a real vector space.

a) Given finite families $\{x_i\}_{i=1,\ldots,r} \subseteq \mathbb{R}$ and$\{\vec{v}_i\}_{i=1,\ldots,r} \subseteq \boldsymbol{V}$, the vector $\sum_i x_i \vec{v}_i$ is *the affine combination of the $\vec{v}_i$'s with coefficients $x_i$*. When the coefficients verify the condition $\sum_i x_i = 1$, the family $\{x_i\}$ is called *a family of weights* and the affine combination $\sum_i x_i \vec{v}_i$ is then called *the barycenter of the $\vec{v}_i$'s with weights $x_i$*.

b) A map between vector spaces $\alpha : \boldsymbol{V} \to \boldsymbol{V}'$ is *affine* if it preserves *barycenters*, *i.e.* if for any family of weights $\{x_i\}$ and any family of vectors $\{\vec{v}_i\}$, we have
$$\alpha\Big(\sum_i x_i \vec{v}_i\Big) = \sum_i x_i \alpha(\vec{v}_i). \qquad (\diamond)$$

c) A map $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ will be called *Euclidean affine*, if it is affine and if it preserves the Euclidean distance.

d) A subset $A \subseteq \boldsymbol{V}$ is called *an affine subspace of* $\boldsymbol{V}$ if it is stable under the operation of taking barycenters, *i.e.* for any family of weights $\{x_i\}$ and any family of vectors $\{\vec{a}_i\} \subseteq A$, we have $\sum_i x_i \vec{a}_i \in A$.

e) Given a subset $S \subseteq \boldsymbol{V}$, the *affine envelop of $S$*, denoted by $\mathrm{Aff}(S)$ is the set of all possible barycenters of finite families $\{\vec{s}_i\} \subseteq S$. In particular, $\mathrm{Aff}(\emptyset) = \emptyset$ and $\mathrm{Aff}(A) = A$ if $A$ is affine.

We now summarize in a proposition some well-known elementary properties of affine subspaces. Readers who are not aware of some should consider them as exercises.

**1.4.2 Proposition.** *Let $\boldsymbol{V}$ be a real vector space.*

a) *A non empty subset $A \subseteq \boldsymbol{V}$ is an affine subspace if and only if, for every $\vec{a} \in A$, the set $A - \vec{a}$ is a vector subspace. This vector subspace is independent of the choice of $\vec{a}$, it is denoted by $L(A)$ and is called the direction or the tangent space of $A$. In particular, an affine subspace is parallel to a unique vector subspace of $\boldsymbol{V}$, the subspace $L(A)$. The dimension of $A$ is then defined by* $\dim(A) := \dim(L(A))$.

b) *The intersection of a family* $\{A_i\}_{i\in\mathfrak{A}}$ *of affine subspaces of* $\boldsymbol{V}$ *is again an affine subspace of* $\boldsymbol{V}$. *In particular, if* $S \subseteq \boldsymbol{V}$ *is a subset, then*

$$\mathrm{Aff}(S) = \bigcap\bigl\{A \mid A \textit{ is an affine subspace of } \boldsymbol{V} \textit{ and } A \supseteq S\bigr\}.$$

*If, in addition,* $S \neq \emptyset$, *then, for all* $\vec{s} \in S$,

$$\mathrm{Aff}(S) = \mathrm{Vec}(S - \vec{s}) + \vec{s}.$$

c) *An affine map* $\alpha : \boldsymbol{V} \to \boldsymbol{V}'$ *preserves affinity*, i.e.

$$\alpha(\mathrm{Aff}(S)) = \mathrm{Aff}(\alpha(S)).$$

*In particular, if* $A \subseteq \boldsymbol{V}$ *is affine, then* $\alpha(A) \subseteq \boldsymbol{V}'$ *is affine.*

d) *If* $S \subseteq \boldsymbol{V}$ *and* $S' \subseteq \boldsymbol{V}'$, *then*

$$\mathrm{Aff}(S \times S') = \mathrm{Aff}(S) \times \mathrm{Aff}(S') \subseteq \boldsymbol{V} \oplus \boldsymbol{V}'.$$

e) *An affine map* $\alpha : \boldsymbol{V} \to \boldsymbol{V}'$ *decomposes in a unique way as the composition of a linear map* $L(\alpha) : \boldsymbol{V} \to \boldsymbol{V}'$ *and a translation* $T(\alpha) : \boldsymbol{V} \to \boldsymbol{V}'$. *More precisely, if we set*

$$L(\alpha)(\_) := \alpha(\_) - \alpha(0) \quad \textit{and} \quad T_{\alpha(0)}(\_) = (\_) + \alpha(0),$$

*then*

$$\alpha = T_{\alpha(0)} \circ L(\alpha).$$

*Hints.* (d) Assume $(\vec{s},\vec{s}') \in S \times S'$, and apply (b), to justify the equalities:

$$\begin{aligned}\mathrm{Aff}(S \times S') &= \mathrm{Vec}((S - \vec{s}) \times (S' - \vec{s}')) + (\vec{s},\vec{s}')\\ &= \mathrm{Vec}(S - \vec{s}) \oplus \mathrm{Vec}(S' - \vec{s}') + (\vec{s},\vec{s}') = \mathrm{Aff}(S) \times \mathrm{Aff}(S')\end{aligned}$$

(e) Use formula ($\diamond$) in 1.4.1-(b) to show that an affine map $\beta : \boldsymbol{V} \to \boldsymbol{V}'$ is linear if and only if it fixes the origin, *i.e.* $\beta(0) = 0$, for example by taking $\vec{v}_1 = 0$. Use again ($\diamond$) to show that $L(\alpha)$ is affine and fixes the origin. □

We can now state the fundamental fact that isometries on Euclidean vector spaces are always Euclidean affine maps.

**1.4.3 Proposition.** *Let* $S$ *be a subset of* $\mathbb{R}^n$. *A map* $\phi : (S,d_\mathrm{e}) \to (\mathbb{R}^m,d_\mathrm{e})$, *which preserves distances, extends to an Euclidean affine map* $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^m$, *which is unique if and only if* $\mathrm{Aff}(S)$ *is the whole ambient space* $\mathbb{R}^n$.
*– In particular, an isometry of* $(\mathbb{R}^n,d_\mathrm{e})$ *is an Euclidean affine isomorphism.*

*Proof.* We can assume $S \neq \emptyset$, choose $\vec{s}_0 \in S$, and set $S' := S - \vec{s}_0$ and $\phi'(s) := \phi(s) - \phi(\vec{s}_0)$. Clearly $\phi$ preserves distances and the origin, we can then apply 1.3.1-(c) and conclude that $\phi'$ has an Euclidean linear extension $\tilde{\phi}'$. The composition of Euclidean affine maps $\tilde{\phi} := T_{\phi(s_0)} \circ \tilde{\phi}' \circ T_{-s_0}$, then gives the required Euclidean affine extension of $\phi$. The uniqueness statement is left to the reader. □

**1.4.4 Corollary.** *Let $U$ be a connected open subset of an affine subspace $L$ of $(I\!R^n, d_e)$. A map $\phi : U \to I\!R^m$ that locally preserves the Euclidean distance is the restriction of a unique Euclidean affine isometry $\Phi : \mathrm{Aff}(U) \to \mathrm{Aff}(\phi(U))$. In particular, $\phi : U \to I\!R^m$ globally preserves the Euclidean distance.*

*Proof.* If $\Phi : \mathrm{Aff}(F) \to I\!R^m$ is an Euclidean affine map, the set $V_F(\Phi) \subseteq F$ where $\phi|_F$ and $\Phi|_F$ *locally* coincide is tautological open in $F$. On the other hand, as $\phi$ locally preserves the Euclidean distance, we can apply 1.4.3 *locally* and state that, for all $x \in F$, there exists an Euclidean affine map $\Phi_x : \mathrm{Aff}(F) \to I\!R^m$ such that $\Phi_x = \phi$ on a neighborhood of $x$ in $F$. Hence,

$$F = \textstyle\bigcup_{\Phi} V_F(\Phi). \qquad (*)$$

This union is disjoint. Indeed, if the set $V_F(\Phi_1) \cap V_F(\Phi_2)$ is nonempty, it contains a nonempty open ball $B \subseteq \mathrm{Aff}(F)$, where $\Phi_1$ end $\Phi_2$ coincide, but then $\Phi_1 = \Phi_2$, because they are affine and that $B$ generates $\mathrm{Aff}(F)$.

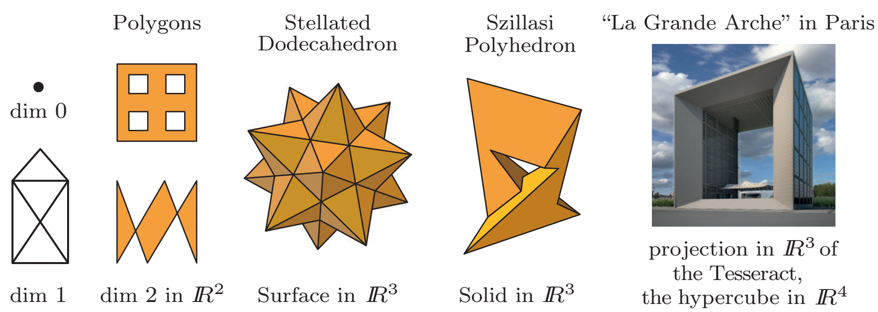
The fact that $(*)$ is disjoint implies that each $V_F(\Phi)$ is open and closed in $F$. Therefore, $F = V(\Phi)$ for a unique $\Phi$, because $F$ is connected by definition.□

**1.4.5. On Terminology..** Caution must be taken that when using the term *Euclidean* for a subset $X \subseteq I\!R^n$, we are referring only to the metric induced by the Euclidean vector space $(I\!R^n, d_e)$. We are thereby *not* presupposing any vector or affine structure on $X$. In the same vein, when we say that a map $\phi : X \to Y$ between subsets of Euclidean vector spaces, preserves Euclidean distances, or that it is a local or global Euclidean isometry, we will refer only to metrics, not presupposing in any way $\phi$ to be affine or linear. This is the sense, for example, of the title of the section 1.4.

# 2 Piecewise Linear Sets

## 2.1 Dots, Polygonal Chains, Polygons, Polyhedra, Polytopes, ...

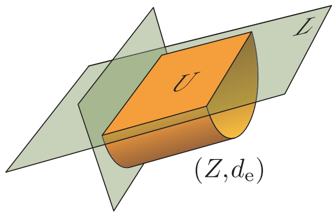
Roughly speaking, *piecewise linear spaces* are geometric objects with *flat* faces and boundaries. In dimensions zero and one, they are points and polygonal chains. In dimension two, they are polygons, regular or not, self-intersecting or not. In dimensions three they are surface or solid polyhedra. In higher dimensions they are *polytopes* whose shapes defy imagination.



Although everybody is familiar with polygons and polyhedra, to give the underlying idea of *flatness* in arbitrary dimension requires some work.

**2.2 Flat open sets.** The first manifestation of *flatness* appears in the context of Euclidean geometry, where the affine subspaces of $\mathbb{R}^n$ are the archetype of flat objets that we seek to generalize.

**2.2.1 Definition.** A *flat open subset* $U$ of a subspace $(Z,d_e) \subseteq (\mathbb{R}^n,d_e)$ is an open subset of $Z$, which is an open subset of some *affine* subspace $L \subseteq \mathbb{R}^n$.




**2.2.2 Proposition and definitions.** a) *If $U \neq \emptyset$ is a subset of $\mathbb{R}^n$, there is at most one affine subspace $L \subseteq \mathbb{R}^n$ where $U$ can be open, it is the subspace $L = \mathrm{Aff}(U)$. In particular, if $\emptyset \neq U_1 \subseteq U_2$ are flat open subsets of some $Z \subseteq \mathbb{R}^n$, then $\mathrm{Aff}(U_1) = \mathrm{Aff}(U_2)$.*

b) *Let $Z \subseteq \mathbb{R}^n$. Given an affine subspace $L \subseteq \mathbb{R}^n$, there is a largest flat open subset of $Z$ which is open in $L$. Denoted by $\mathcal{O}(Z)_L$, if it is nonempty, its dimension is $\dim(\mathcal{O}(Z)_L) := \dim_{\mathbb{R}}(L)$. If $L_1 \neq L_2$, we have*

$$\mathcal{O}(Z)_{L_1} \cap \mathcal{O}(Z)_{L_2} = \emptyset.$$

c) *The flat interior of $Z$ is the set*

$$\mathcal{O}(Z) := \textstyle\bigsqcup_L \mathcal{O}(Z)_L. \qquad (*)$$

i) *Each $\mathcal{O}(Z)_L$ is open and closed in $\mathcal{O}(Z)$.*

ii) *Each connected component $F$ of $\mathcal{O}(Z)$ is locally compact, locally path-connected, path connected and is contained in a unique $\mathcal{O}(Z)_L$; it is called an open $\ell$-face, where $\ell$ is $\dim(F) := \dim_{\mathbb{R}}(L)$.*
*For $\ell \in \mathbb{N}$, the $\ell$-dimensional flat interior of $Z$ is*

$$\mathcal{O}_\ell(Z) := \textstyle\bigcup_L \big\{\mathcal{O}(Z)_L \;\big|\; \dim(L) = \ell\big\}$$

*and the set of open $\ell$-faces of $Z$ is*

$$\mathcal{F}^\circ_\ell(Z) := \big\{F \;\big|\; F \text{ is a connected component of } \mathcal{O}_\ell(Z)\,\big\}.$$

*The set of all open faces of $Z$ is then the set:*

$$\mathcal{F}^\circ(Z) := \textstyle\bigcup_\ell \mathcal{F}^\circ_\ell(Z)$$

d) *Let* $\phi : (Z,d_e) \subseteq \mathbb{R}^n \to (Z',d_e) \subseteq \mathbb{R}^m$ *be a local isometry.*

i) *If* $F'$ *is an open* $\ell$*-face of* $Z'$*, then* $\phi^{-1}(F')$ *is a union of* $\ell$*-faces of* $Z$*. Moreover,*

$$\mathcal{O}_\ell(Z) = \phi^{-1}\big(\mathcal{O}_\ell(Z')\big).$$

ii) *Denote by* $(\overline{\ })$ *the closure in the ambient space* $\mathbb{R}^n$ *(resp.* $\mathbb{R}^m$*).* *If* $S$ *is a subset of an open face* $F \in \mathcal{F}^\circ(Z)$ *such that* $\overline{S} \subseteq Z$*, then*

$$\phi(\overline{S}) = \overline{\phi(S)}.$$

iii) *If* $\phi : (Z,d_e) \to (Z',d_e)$ *is a global isometry, the statement (d-i) can be enhanced. In that case, if* $F'$ *is an open* $\ell$*-face of* $Z'$*, then* $\phi^{-1}(F')$ *is an* $\ell$*-face of* $Z$*. Therefore,*

$$\mathcal{F}^\circ_\ell(Z) = \phi^{-1}\big(\mathcal{F}^\circ_\ell(Z')\big).$$

e) *If* $Z \subseteq \mathbb{R}^n$ *and* $Z' \subseteq \mathbb{R}^m$ *be subsets, then*

$$\mathcal{F}^\circ_\ell(Z \times Z') = \coprod_{\ell=a+b} \mathcal{F}^\circ_a(Z) \times \mathcal{F}^\circ_b(Z').$$

*Proof.* (a) If $U \neq \emptyset$ is an open subset of $L$, it contains a generating family for $L$, in which case $L = \operatorname{Aff}(U)$. Next, if $U_1$ is open in $U_2$ which is open in $\operatorname{Aff}(U_2)$, then $U_1$ is open in $\operatorname{Aff}(U_2)$, and then $\operatorname{Aff}(U_1) = \operatorname{Aff}(U_2)$.

(b) Let $(\mathcal{U}(Z),\subseteq)$ be the set of flat open subsets of $Z$, partially ordered by the inclusion relation. Thanks to (a), every increasing chain in $\mathcal{U}(Z)$ has an upper bound. We can the therefore apply Zorn's lemma and state that $\mathcal{U}(Z)$ has maximal elements. These are the $\mathcal{O}(Z)_L$'s. Now, if $s \in \mathcal{O}(Z)_{L_1} \cap \mathcal{O}(Z)_{L_2}$, there exists an open set $V \ni s$ which is open in both $L_i$'s, and $L_1 = L_2$.

(c) Because $\mathcal{O}(Z)_L$ is open in $L$, it is locally compact and locally path-connected, and because it is open in $\mathcal{O}(Z)$ and the union $(*)$ is disjoint, it is also closed in $\mathcal{O}(Z)$. It follows that, if $F$ is a connected component of $\mathcal{O}(Z)$, first, it is open, locally compact and locally path-connected, and therefore path-connected (**3**♦8.3.2-(c)), and, second, it is contained in a unique $\mathcal{O}(Z)_L$.

(d-i) Let $F'$ be an open $\ell$-face of $Z'$, the set $V := \phi^{-1}(F')$ is open in $Z$ and $\phi|_V : V \to \operatorname{Aff}(F)$ is a local isometry. For $x \in V$, take $\epsilon > 0$ such that $B := B(x,\epsilon) \subseteq Z$ is contained in $V$ and such that $\phi|_B : B \to \phi(B)$ is an isometry. The map $(\phi|_B)^{-1} : \phi(B) \to B \subseteq \mathbb{R}^n$ then verifies the hypothesis of 1.4.4, and, therefore, extends to an Euclidean affine isometry $\Phi : \operatorname{Aff}(F) \to \operatorname{Aff}(B)$, proving that $B$ is open in $\operatorname{Aff}(B)$, and, thereafter, that $B$ is a flat open subset of $Z$ of dimension $\ell$.

Let $F_x$ be the open $\ell$-face of $Z$ containing $B$, then, again by 1.4.4, the set $\phi(F_x)$ is flat open in $Z'$ of dimension $\ell$. We thus have an inclusion of connected flat open subsets $\phi(B) \subseteq \phi(F_x)$, which implies that $\phi(F_x) \subseteq F'$. We conclude that $V$ is a union of $\ell$-faces of $Z$ and (d-i) follows.

(d-ii) For $S \subseteq F \in \mathcal{F}^\circ(Z)$, its closure $\overline{S}$ in $I\!R^n$ is also its closure in $\mathrm{Aff}(F)$, because $\mathrm{Aff}(F)$ is already closed in $I\!R^n$. On the other hand, $\phi : \overline{F} \to I\!R^m$ is the restriction of an isometry $\Phi : \mathrm{Aff}(F) \to \mathrm{Aff}(\phi(F))$, thanks to 1.4.4, in which case $\phi(\overline{S}) = \Phi(\overline{S}) = \overline{\Phi(S)} = \overline{\phi(S)}$.

(d-iii) After (d-i), we know that if $F' \in \mathcal{F}^\circ_\ell(Z')$, the set $\phi^{-1}(F')$ is a union of open $\ell$-faces of $Z$, but because now $\phi$ is a homeomorphism, the set $\phi^{-1}(F')$ is connected and is just one open $\ell$-face.

(e) The inclusion '$\subset$' is obvious. For the converse, let $p : I\!R^n \times I\!R^m \twoheadrightarrow I\!R^n$ be the projection onto the first factor $(\vec{v}, \vec{w}) \mapsto \vec{v}$. If $U$ is flat open and connected in $Z \times Z'$, then $p(U) \subseteq Z$ has, locally, the same properties, hence globally. Same conclusion for the projection $p'(U) \subseteq Z'$ onto the second factor. But $U \subseteq p(U) \times p'(U)$, and the latter is connected flat open. Therefore, if $U$ is a face, *i.e.* maximal connected flat open, then necessarily $U = p(U) \times p'(U)$. □

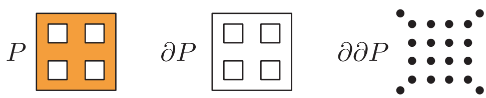
**2.3 *PL*-sets.** We now introduce the concept of *piecewise linear subset*, in short *PL-set*, of the Euclidean subspace $(I\!R^n, d_\mathrm{e})$. The definition, inspired by that of *topological stratified spaces* of Whitney ([1]), is recursive.

P-1) In dimension 0, they are the finite subsets of $(I\!R^n, d_\mathrm{e})$.

P-2) In dimension $d$, they are the *closed* subsets $P \subseteq (I\!R^n, d_\mathrm{e})$, such that

i) $\mathcal{O}(P)$ has finitely many connected components and $\dim(\mathcal{O}(P)) = d$.

ii) The *boundary* $\partial P := P \smallsetminus \mathcal{O}(P)$ is a *PL*-set of dimension $d' < d$.




Up to these two conditions, the set $P$ decomposes as the finite disjoint union of the open faces of the successive boundaries $\partial \dots \partial P$,

$$P = \mathcal{O}(P) \sqcup \mathcal{O}(\partial P) \sqcup \mathcal{O}(\partial^2 P) \sqcup \cdots \sqcup \mathcal{O}(\partial^d P).$$

Denote by $\mathcal{F}_\ell(P)$ the set of open $\ell$-faces of all of the $\partial^i P$'s. The elements of $\mathcal{F}_\ell(P)$ will be simply called the *$\ell$-faces of $P$*. We are thus setting:

$$\mathcal{F}_\ell(P) := \mathcal{F}^\circ_\ell(P) \sqcup \mathcal{F}^\circ_\ell(\partial P) \sqcup \mathcal{F}^\circ_\ell(\partial P) \sqcup \cdots \cup \mathcal{F}^\circ_\ell(\partial^d P).$$

The set $\mathcal{F}(P)$ of *faces of $P$*, is then

$$\mathcal{F}(P) := \textstyle\bigcup_\ell \mathcal{F}_\ell(P).$$

In our definition of *PL*-sets, a third condition, known as the *Whitney frontier property*, will be required.

iii) The boundary of a face of $P$ is a union of faces of $P$. In other words, given two faces $F, F' \in \mathcal{F}(P)$, if $F' \cap \overline{F} \neq \emptyset$, then $F' \subseteq \overline{F}$.

([1]) Whitney, H. *Tangents to an analytic variety.* Ann. of Math. (2) 81 (1965) 496–549.

**2.3.1. Terminology for *PL*-sets.** Let $P$ be a $PL$-set.

- The *dimension of $P$* is $\dim(P) := \max\{\dim(F) \mid F \in \mathcal{F}^\circ(P)\}$;
- $P$ is *$d$-equidimensional* if $\dim(F) = d$, for all $F \in \mathcal{F}^\circ(P)$;
- $P$ is *irreducible*, if there exists only one open face, *i.e.* if $|\mathcal{F}^\circ(P)| = 1$.
- A *sub-PL-set* of $P$ is a closed subset of $P$ which is union of faces of $P$.

**2.3.2 Proposition.** *Let $P$ be a PL-set.*

a) *For $\ell \in \mathbb{N}$, the union of faces of dimension $\leq \ell$,* i.e. *the set*

$$P_{\leq \ell} := \textstyle\bigcup_{i \leq \ell} \big(\bigcup \mathcal{F}_i(P)\big),$$

*is closed in $P$. We get in this way an increasing filtration by PL-sets:*

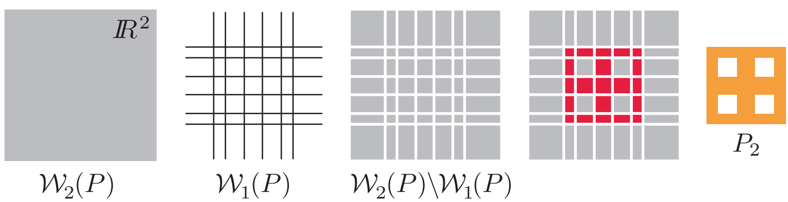
$$\emptyset \subseteq P_0 \subseteq P_{\leq 1} \subseteq P_{\leq 2} \cdots \subseteq P_{\leq d} = P,$$

*where $P_{\leq \ell} \setminus P_{\leq \ell - 1} = \bigcup \mathcal{F}_\ell(P)$.*

b) *Define the set of walls of dimension $\ell$ of $P$, as the set of affine spaces*

$$\mathcal{W}_\ell(P) := \bigcup \big\{L \mid \dim(L) = \ell \text{ and } (\exists i)\big(\mathcal{F}^\circ(\partial^i P)_L \neq \emptyset\big)\big\}.$$

*Then, the closure of an $\ell$-face of $P$ is the closure of a well defined family of connected components of the set $\big(\bigcup \mathcal{W}_\ell(P)\big) \backslash \big(\bigcup \mathcal{W}_{\ell-1}(P)\big)$.*




*Conversely, the knowledge of $\mathcal{W}(P)$ and of those families of connected components allows to reconstruct the PL-set $P$. In particular, any affine map $\phi : \mathbb{R}^n \to \mathbb{R}^m$ transforms PL-sets in PL-sets.*

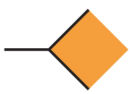
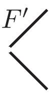
*Proof.* Left to the reader. □

**2.3.3 Exercises.** 1) Show that in (P-2-ii), the condition '$d' < d$' can be replaced by '$\mathcal{O}(P)$ is dense in $P$'.

2) Show that a $PL$-set $P$ is irreducible if and only if it is not the union of two sub-$PL$-sets strictly contained in $P$.

3) Let $P_1$ and $P_2$ be $PL$-sets of $\mathbb{R}^n$ such that $Q := P_1 \cap P_2$ is a sub-$PL$-set of both $P_1$ and $P_2$.
(i) Show that the union $P_1 \cup P_2$ is a $PL$-set. (ii) Give an example where the natural injection $P_1 \hookrightarrow P_1 \cup P_2$ does not locally preserve distances. (iii) Show that if $Q$ is the closure of a face, then $P_i \hookrightarrow P_1 \cup P_2$ does locally preserve distances. *Hint. Review* **5♦***7.1.*

4) Show that if $P_1$ and $P_2$ are $PL$-sets, then $P_1 \times P_2$ is also a $PL$-set.
   *Hint. Use 2.2.2-(e) to show that $\mathcal{O}(P_1 \times P_2) = \mathcal{O}(P_1) \times \mathcal{O}(P_2)$, and, hence, that $\partial(P_1 \times P_2) = (\partial P_1 \times P_2) \cup (P_1 \times \partial P_2)$. Then use induction and (3).*
5) Show that if $P_1$ and $P_2$ are $PL$-sets of $\mathbb{R}^n$, then, for all $x, y \in \mathbb{R}$, the set $\{x\vec{p}_1 + y\vec{p}_2 \mid \vec{p}_i \in P_i\}$ is also a $PL$-set of $\mathbb{R}^n$.
   *Hint.* By (4), $P_1 \times P_2$ is a $PL$-set of $\mathbb{R}^{2n}$, then apply 2.3.2-(b) with the map $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi : (\vec{v}, \vec{w}) \mapsto x\vec{v} + y\vec{w}$.
6) Let $\pi : \mathbb{R}^{a+b} \twoheadrightarrow \mathbb{R}^a$ be the projection over the first $a$ coordinates. Show that if $P \subseteq \mathbb{R}^a$ is a $PL$-set of dimension $d$ then $\pi^{-1}(P)$ is a $PL$-set of dimension $d + b$.
7) Show that if $P \subseteq \mathbb{R}^n$ is a $PL$-set and $H \subseteq \mathbb{R}^n$ is an affine subspace, then $P \cap H$ is a $PL$-set.
8) The following examples of subsets of $\mathbb{R}^2$ verify the conditions (P-2-i) and (P-2-ii). Say which verify the frontier property (P-2-iii).

| | $P$ | $\mathcal{O}(P)$ | $\mathcal{O}(\partial P)$ | $\partial\partial P$ | frontier property |
|---|---|---|---|---|---|
| $A$ |  | $F$  |  $F'$ | $\emptyset$ | $F' \cap \overline{F} \neq \emptyset$<br>$F' \not\subseteq \overline{F}$ |
| $B$ |  | $F$  | $F'$  | $F''$ • | $F' \cap \overline{F} = \emptyset$<br>$F'' \cap \overline{F} \neq \emptyset$<br>$F'' \subseteq \overline{F}$ |

Conclude that unions and intersections of $PL$-sets are not always $PL$-sets.

**2.4 *PL*-sets as Path-Metric Spaces.** An important feature about $PL$-sets, is that they are path-metrizable by the path-distance $\widetilde{d}_{\mathrm{e}}$. As such, they become objects of the category $\mathbf{p\text{-}Met}_{\ell\text{-d}}$ (**5**♦6.4.2), where we can take advantage of the tools introduced in chapter **5**, notably assemblages and quotients.

**2.4.1 Proposition.** *Let $P \subseteq (\mathbb{R}^n, d_{\mathrm{e}})$ be a PL-set.*

a) *$P$ is totally locally star-shaped (4♦6.2.1).*
b) *The map $\mathrm{id}_P : (P, \widetilde{d}_{\mathrm{e}}) \to (P, d_{\mathrm{e}})$ is a homeomorphism.*
c) *If $Z$ is a convex subset of $P$, $\widetilde{d}_{\mathrm{e}}|_Z = d_{\mathrm{e}}|_Z$. In particular, for all $F \in \mathcal{F}(P)$,*

$$\widetilde{d}_{\mathrm{e}}|_F,\ d_{\mathrm{e}}|_F \text{ and } \widetilde{d_{\mathrm{e}}|_F} \text{ are locally equal in } F.$$

d) *If $U$ is open in $P$, the inclusion $(U, \widetilde{d}_{\mathrm{e}}) \hookrightarrow (P, \widetilde{d}_{\mathrm{e}})$ is a local isometry.*

*Proof.* (a) A point $x \in P$ belongs to a unique open face of an unique $\partial^i P$. In particular, there exists an open neighborhood $x \in V_x \subseteq P$ such that for any wall $L \in \mathcal{W}(P)$, we have either $L \cap V_x = \emptyset$ or $V_x \subseteq L$. Then, we can take $\epsilon > 0$ small enough, which is possible because $\mathcal{W}(P)$ is finite, so as to have

$$B_{d_{\mathrm{e}}}(x,\mathbb{R}^n) \cap P = \bigcup_{L \in \mathcal{W}(L)} \{B_{d_{\mathrm{e}}}(x,\mathbb{R}^n) \cap L\},$$

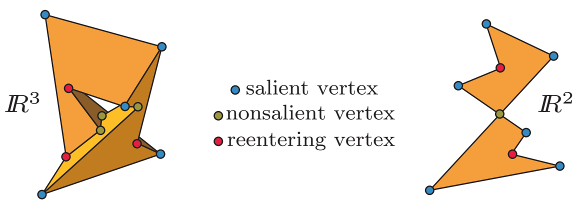
and, thus, conclude that $P$ is locally star-shaped with center in $x$, as this is the case for each $B_{d_{\mathrm{e}}}(x,\mathbb{R}^n) \cap L$. The rest of the proposition results from the path-metrizability theorem **4♦**6.1.1. □

**2.4.2 Exercise.** Give examples of sub-*PL*-sets $P' \subseteq P$ where the inclusion map $P' \hookrightarrow P$ does not locally preserve path-distances.

*Hint. Look at example* $(B)$ *in 2.3.3-(8)* and $P' := \partial P$.

**2.4.3. On Terminology.** As a *PL*-set $P$ is a subset of some Euclidean space $(\mathbb{R}^n, d_{\mathrm{e}})$, the comments on terminology 1.4.5 apply. We thus call *the Euclidean distance of P*, the induced distance $d_{\mathrm{e}}|_P$. This distance will be denoted simply by $d_{\mathrm{e}}$ when no confusion is likely to arise. For example, we will write "$(P', d_{\mathrm{e}}) \subseteq (P, d_{\mathrm{e}})$". On the contrary, the notation for path-distances is more delicate. Indeed, as we want to denote the path-distance in $(P, d_{\mathrm{e}})$ by simply $\widetilde{d}_{\mathrm{e}}$, we could be led to write something like "$(P', \widetilde{d}_{\mathrm{e}}) \subseteq (P, \widetilde{d}_{\mathrm{e}})$", which is incorrect, as $\widetilde{d}_{\mathrm{e}} \neq \widetilde{d}_{\mathrm{e}}|_{P'}$ in general. Although they are always locally equal, the correct notation would rather be "$\mathrm{id} : (P', \widetilde{d}_{\mathrm{e}}) \hookrightarrow (P, \widetilde{d}_{\mathrm{e}})$".

**2.4.4 Comment.** Proposition 2.4.1 is a generalization of theorem **4♦**6.2.2. Observe that the condition that says that $x \in \partial P$ has a convex neighborhood, corresponds to that, in dimension 2, saying that $x$ is a vertex of *salient angle*.




Solid Szillasi Polyhedron Self-intersecting Polygon

**2.5 Stability of *PL*-sets under Local Isometries.** The fact that maps that locally preserve Euclidean distances are continuous and affine, in conjunction with the fact that *PL*-sets are determined by their topological and affine structures, makes it clear that many attributes of *PL*-sets should be preserved through these maps. The following proposition clarifies this remark.

**2.5.1 Proposition.** *Let $P$ be a PL-set of $\mathbb{R}^n$ and let $\phi : (P, d_{\mathrm{e}}) \to \mathbb{R}^m$ be a local isometry onto its image $P' := \phi(P)$. Then, $P'$ is a PL-set of $\mathbb{R}^m$ of dimension* $\dim(P') = \dim(P)$, *and, moreover,*

$$\mathcal{O}_\ell(P) = \phi^{-1}\big(\mathcal{O}_\ell(P')\big), \quad \partial P = \phi^{-1}(\partial P') \quad \text{and} \quad P_{\leq \ell} = \phi^{-1}(P'_{\leq \ell}).$$

*If, in addition,* $\phi : (P,d_{\mathrm{e}}) \to (P',d_{\mathrm{e}})$ *is an isometry, then*

$$\phi(?(P)) = ?(P'), \quad \textit{for } ? \in \big\{\mathcal{O}_\ell, \partial, \mathcal{F}_\ell, (\_)_{\le \ell}\big\}.$$

*Proof.* Proposition 2.2.2-(d-i) already showed that $\mathcal{O}_\ell(P) = \phi^{-1}(\mathcal{O}_\ell(P'))$, and the equality $\partial(P) = \phi^{-1}(\partial(P'))$ follows straightforwardly.

The set $P'$ is closed in $I\!R^m$. Indeed, $P$ is closed in $I\!R^n$ and $P = \bigcup_{F\in\mathcal{F}^\circ(P)} \overline{F}$. If we apply 2.2.2-(d-ii), we get $P' = \bigcup_{F\in\mathcal{F}^\circ(P)} \phi(\overline{F}) = \bigcup_{F\in\mathcal{F}^\circ(P)} \overline{\phi(F)}$, and $P'$ is closed, as it as a finite union of closed subsets in $I\!R^m$.

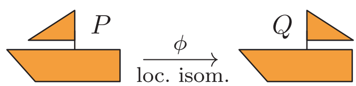
To prove that $P'$ is a *PL*-set, it now only remains to prove that $\partial P$ is a *PL*-set. But since $\phi(\partial P) = \partial P'$ and since the restriction $\phi : \partial P \to \partial P'$ is a local isometry, where $\dim(\partial P) < \dim(P)$, can assume the statement true following a recursive argument.

The case where $\phi$ is an isometry is left to the reader. □

**2.5.2 Exercise.** 1) Let $P \subseteq I\!R^n$ be an $n$-equidimensional *PL*-set. Show that a map $\phi : (P,d_{\mathrm{e}}) \to (I\!R^m,d_{\mathrm{e}})$ locally preserves Euclidean distances, if and only it globally does so.

2) a) Show that the two *PL*-sets $P,Q \subseteq (I\!R^2 d_{\mathrm{e}})$ in the image below are locally isometric but are not globally isometric.

b) Show that there is one and only one isometry between $(P,\widetilde{d}_{\mathrm{e}})$ and $(Q,\widetilde{d})$.




3) Same as (2), for $P_{\le 1}, Q_{\le 1} \subseteq (I\!R^2,d_{\mathrm{e}})$. Compare with (1).

**2.5.3. Locally closed *PL*-sets.** By this term, we refer to open subsets of *PL*-sets. These sets verify the axioms (P-1,P-2) except that they are locally closed, and that they may not satisfy any finiteness condition. As a consequence, the definitions and statements who did not use these properties remain valid for locally closed *PL*-sets.

**2.5.4 Proposition.** *Let* $\phi : (P,\widetilde{d}_{\mathrm{e}}) \to (P',\widetilde{d}_{\mathrm{e}})$ *be a local isometry between locally closed PL-sets. Then,*

$$\mathcal{O}_\ell(P) = \phi^{-1}\big(\mathcal{O}_\ell(P')\big), \quad \partial P = \phi^{-1}(\partial P') \quad \textit{and} \quad P_{\le \ell} = \phi^{-1}(P'_{\le \ell}).$$

*If, in addition,* $\phi : (P,\widetilde{d}_{\mathrm{e}}) \to (P',\widetilde{d}_{\mathrm{e}})$ *is an isometry,* $\dim(P) = \dim(P')$, *and*

$$\phi(?(P)) = ?(P'), \quad \textit{for } ? \in \big\{\mathcal{O}_\ell, \partial, \mathcal{F}_\ell, (\_)_{\le \ell}\big\}.$$

*Proof.* The proof will be the same as for 2.5.1, but because now $\phi$ is only assumed to locally respect path-distances $\widetilde{d}_{\mathrm{e}}$ and not Euclidean distances $d_{\mathrm{e}}$, we have to justify how to fill this gap. Now, the only place where $d_{\mathrm{e}}$ plays an important rôle, is in 2.2.2-(d) to extend $\phi|_F : F \to I\!R^m$ to an Euclidean affine map $\varPhi : \mathrm{Aff}(F) \to I\!R^m$. But, as $F$ is open in $\mathrm{Aff}(F)$, it is locally convex, and then $d_{\mathrm{e}}$ and $\widetilde{d}_{\mathrm{e}}|_F$ locally coincide. The present hypothesis about $\widetilde{d}_{\mathrm{e}}$ therefore allows the use of 2.2.2-(d). Details are left to the reader. □

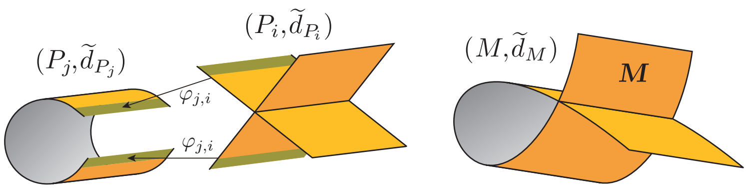
**2.5.5 Corollary.** *Let $\phi : (P,\widetilde{d}_{\mathrm{e}}) \to (P',\widetilde{d}_{\mathrm{e}})$ be a local isometry between locally closed PL-sets. Then, $Q := \phi(P)$ is a locally closed PL-set,* $\dim(Q) = \dim(P)$, *and $?(Q) = Q \cap ?(P')$, for $? \in \{\mathcal{O}_\ell, \partial, \mathcal{F}_\ell, (\_)_{\leq \ell}\}$. Moreover, proposition 2.5.4 remains valid for $Q$ instead of $P'$.*

*Proof.* Because $\phi$ is open, $\phi(P)$ is a locally closed *PL*-space, and we can therefore apply 2.5.4 for the inclusion $\phi(P) \hookrightarrow P'$. ☐

## 3 The Category *PL*-Manifolds

**3.1 *PL*-Manifolds.** Thanks to 2.4.1, locally closed *PL*-sets are objects of the category $\mathbf{p\text{-}Met}_{\ell\text{-iso}}$ (**5**♦6.4.2) where assemblages of open atlases are well defined. It is now that the ideas of the introductory chapter **2** will be used. The locally closed *PL*-sets will be the *simple* objects out of which we will build the more general spaces, classically known under the name of *flat Riemannian manifolds with piecewise linear boundary*, here, briefly called *PL*-manifolds.

**3.1.1 Definitions.** A *PL-manifold* $\boldsymbol{M} := (M,\widetilde{d}_M)$ *of dimension $n$* is the assemblage in the category $\mathbf{p\text{-}Met}_{\ell\text{-d}}$ of an open atlas (*cf.* **5**♦7.2) whose charts $(P_i,\widetilde{d}_{P_i})$ are locally closed *PL*-sets equidimensional of dimension $n$.




*The Categories of PL-Manifolds.* We denote by $PL\text{-}\mathbf{Man}_{\ell\text{-iso}}$ and $PL\text{-}\mathbf{Man}_{\ell\text{-d}}$ the categories whose objects are *PL*-manifolds and whose morphisms are respectively local isometries and locally preserving distance maps ([2]).

**3.1.2. Boundary and Interior of a *PL*-Manifold.** In **3**♦7.2 we introduced the concept of an *open atlas* as an atlas $\mathcal{A} = \{(M_i,\widetilde{d}_{M_i})\}_{i\in\mathfrak{I}}$ in which the overlapping spaces $M_{i,j} \subseteq M_i$ and $M_{j,i} \subseteq M_j$ are open and glued together through transition maps $\varphi_{j,i} : M_{i,j} \to M_{j,i}$ which are isometries. In the present context of *PL*-manifolds, the overlapping spaces are locally closed *PL*-sets, and proposition 2.5.4 applies to state that the transition maps respect dimensions, interiors, boundaries, $\ell$-faces, etc. We have

$$\varphi_{j,i}(?(M_{i,j})) = ?(M_{j,i}),$$

where '$?$' stands for $\mathcal{O}_\ell$, $\partial$, $\mathcal{F}_\ell$ or $(\_)_{\leq \ell}$. Thanks to this, the notions of dimension, interior, boundary, $\ell$-faces, etc., can be defined for *PL*-manifolds.

([2]) Following the standard terminology, one says that $PL\text{-}\mathbf{Man}_{\ell\text{-}*}$ is the *full subcategory* of $\mathbf{p\text{-}Met}_{\ell\text{-}*}$ whose objects are the *PL*-manifolds.

**3.1.3 Definitions.** Let $\boldsymbol{M}$ be a *PL*-manifold built up as the assemblage of the open atlas $\mathcal{A} := \{(M_i, \widetilde{d}_{M_i})\}$. Let $\nu : \sqcup_{i\in\mathfrak{I}} M_i \to \boldsymbol{M}$ be the identification map, and denote by $\tilde{M}_i := \nu(M_i) \subseteq \boldsymbol{M}$ (see **2♦**5.1 for the notation).

a) The *interior $\mathcal{O}(\boldsymbol{M})$ of $\boldsymbol{M}$* is the set of $x \in M$ such that, if $x \in \tilde{M}_i$ then $x \in \nu(\mathcal{O}(M_i))$. In other words, we set

$$\mathcal{O}(\tilde{M}_i) := \nu(\mathcal{O}(M_i)) \quad \text{and} \quad \mathcal{O}(\boldsymbol{M}) := \bigcup\nolimits_{i\in\mathfrak{I}} \mathcal{O}(\tilde{M}_i).$$

The *$\ell$-dimensional interior $\mathcal{O}_\ell(\boldsymbol{M})$* is then defined by replacing $\mathcal{O}$ by $\mathcal{O}_\ell$.

$$\mathcal{O}_\ell(\tilde{M}_i) := \nu_\ell(\mathcal{O}(M_i)) \quad \text{and} \quad \mathcal{O}_\ell(\boldsymbol{M}) := \bigcup\nolimits_{i\in\mathfrak{I}} \mathcal{O}_\ell(\tilde{M}_i).$$

A connected component of $\mathcal{O}(\boldsymbol{M})$ is a connected component of some $\mathcal{O}_\ell(\boldsymbol{M})$ (exercise!), it will be called *an open $\ell$-face*, and also *a smooth, or an Euclidean* open subset of $\boldsymbol{M}$.

As for *PL*-sets, $\mathcal{F}^\circ_\ell(\boldsymbol{M})$ denotes the set of *open $\ell$-faces*, and the set of *all open faces* is then $\mathcal{F}^\circ(\boldsymbol{M}) := \bigcup_\ell \mathcal{F}^\circ_\ell(\boldsymbol{M})$.

b) The *boundary of $\boldsymbol{M}$*, is defined as $\partial(\boldsymbol{M}) := \boldsymbol{M} \backslash \mathcal{O}(\boldsymbol{M})$. In other words,

$$\partial(\tilde{M}_i) := \nu(\partial M_i) \quad \text{and} \quad \partial(\boldsymbol{M}) := \bigcup\nolimits_{i\in\mathfrak{I}} \partial(\tilde{M}_i).$$

When $\partial\boldsymbol{M} = \emptyset$, we will say that the *PL*-manifold $\boldsymbol{M}$ is a *flat* or *Euclidean* manifold. In that case the points of $\boldsymbol{M}$ admit neighborhoods isometric to Euclidean open balls of the form $I\!\!B_{I\!\!R^n}(0,\epsilon)$, for $\epsilon$ small enough.

**3.1.4. Terminology for *PL*-manifolds.** We extend the terminology introduced in 2.3.1 for *PL*-sets to *PL*-manifolds. Let $\boldsymbol{M}$ be a *PL*-manifold.

– The *dimension of $\boldsymbol{M}$* is $\dim(\boldsymbol{M}) := \max\{\dim(F) \mid F \in \mathcal{F}^\circ(\boldsymbol{M})\}$;
– $\boldsymbol{M}$ is *d-equidimensional* if $\mathcal{O}(\boldsymbol{M}) = \mathcal{O}_d(\boldsymbol{M})$;
– $\boldsymbol{M}$ is *irreducible* if $\mathcal{O}(\boldsymbol{M})$ is connected.
– A *sub-PL-manifold* of $\boldsymbol{M}$ is a closed union of faces of $\boldsymbol{M}$.

The next theorem generalizes 2.4.1 and 2.5.4 from *PL*-sets to *PL*-manifolds.

**3.1.5 Theorem.** *Let $\boldsymbol{M}$ be a PL-manifold.*

a) *If $\phi : \boldsymbol{M} \to \boldsymbol{M}'$ is a local isometry between PL-manifolds, then*

$$\mathcal{O}_\ell(\boldsymbol{M}) = \phi^{-1}\big(\mathcal{O}_\ell(\boldsymbol{M}')\big),\ \partial\boldsymbol{M} = \phi^{-1}(\partial\boldsymbol{M}') \textit{ and } \boldsymbol{M}_{\le\ell} = \phi^{-1}(\boldsymbol{M}'_{\le\ell}).$$

*If, in addition, $\phi : \boldsymbol{M} \to \boldsymbol{M}'$ is an isometry,* $\dim(\boldsymbol{M}) = \dim(\boldsymbol{M}')$, *and*

$$\phi(?(\boldsymbol{M})) = ?(\boldsymbol{M}'), \quad \textit{for } ? \in \big\{\mathcal{O}_\ell, \partial, \mathcal{F}_\ell, (\_)_{\le\ell}\big\}.$$

*In particular, the definitions in 3.1.3 are independent from the choice of the atlas defining $\boldsymbol{M}$.*

b) *The subspace $\mathcal{O}_\ell(\boldsymbol{M})$ is empty or is a Euclidean manifold of dimension $\ell$, and the subspace $\partial(\boldsymbol{M})$ is a PL-manifold of dimension* $< \dim(\boldsymbol{M})$.

c) *For $\ell \in \mathbb{N}$, the subspace $\boldsymbol{M}_{\leq \ell} := \bigcup_{i \leq \ell} \big(\bigcup \mathcal{F}_i(\boldsymbol{M})\big)$, is a closed PL-submanifold of $\boldsymbol{M}$ of dimension $\ell$, if nonempty. We get in this way an increasing filtration by sub-PL-manifolds:*

$$\emptyset \subseteq \boldsymbol{M}_0 \subseteq \boldsymbol{M}_{\leq 1} \subseteq \boldsymbol{M}_{\leq 2} \cdots \subseteq \boldsymbol{M}_{\leq n} = \boldsymbol{M},$$

*where $\boldsymbol{M}_{\leq \ell} \smallsetminus \boldsymbol{M}_{\leq \ell-1}$ is an Euclidean manifold of dimension $\ell$, if nonempty.*

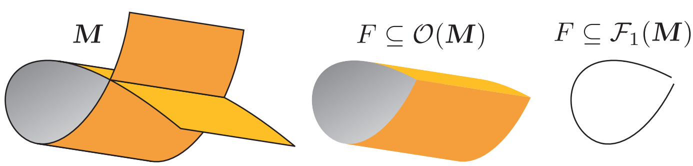
*Proof.* Left to the reader. □

**3.1.6 Exercises.** 1) Verify that Buser surfaces $\boldsymbol{M}_i$ in **1**♦6.5, Buser-Bérard surfaces $\boldsymbol{D}_i$ in **1**♦6.7.1, and all the examples in **5**♦8: cylinders, Möbius ribbons, toruses, Klein bottles, etc., are *PL*-manifolds of dimension 2.

2) A well known theorem in the theory of $\mathcal{C}^\infty$-manifolds states that a Hausdorff manifold can always be embedded as a closed submanifold of some vector space $\mathbb{R}^N$. One may wonder if the analog statement for *PL*-manifolds is true: is a Hausdorff *PL*-manifold always *PL*-subset of some Euclidean space $\mathbb{R}^N$? This is not the case!

a) In 2.4.1-(c) we stated that if $F$ is a face in a *PL*-set $P$, the inclusion $F \hookrightarrow P$ globally preserves the path distance. Show now that if $F$ is a face in a *PL*-manifold $\boldsymbol{M}$, the inclusion map $F \hookrightarrow \boldsymbol{M}$ *locally* preserves path-distances.

b) For the figure below, prove that the map $F \hookrightarrow \boldsymbol{M}$ does not globally preserve path-distances in the two cases, $F \subseteq \mathcal{O}(\boldsymbol{M})$ and $F \subseteq \mathcal{F}_1(\boldsymbol{M})$.




c) Conclude that Hausdorff *PL*-manifolds are not always *PL*-subsets of some Euclidean space $\mathbb{R}^n$.

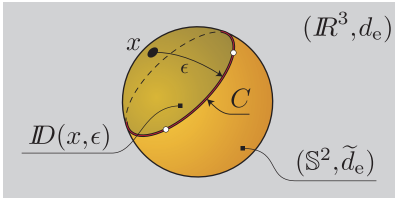
3) An open subset $U$ of a metric space $\boldsymbol{X} := (X, d_{\boldsymbol{X}})$, is called *Euclidean* if it is isometric to an open subset of some Euclidean space $(\mathbb{R}^n, d_{\mathrm{e}})$, in which case $n$ is the *dimension of $U$*. The subset $U$ is called *locally Euclidean*, if every point in $U$ is the center of an Euclidean open ball of $\boldsymbol{X}$.

a) i) Show that Euclidean open subsets of different dimensions are disjoint. Conclude that a connected component of a locally Euclidean open set has a well defined dimension.

ii) For each $n \in \mathbb{N}$, show that there exists a maximal open subset $\mathcal{E}_n(\boldsymbol{X})$ which is an $n$-dimensional Euclidean manifold, if nonempty.

b) Let $\boldsymbol{M}$ be a *PL*-manifold of dimension $n$. Show that $\mathcal{E}_n(\boldsymbol{M})$ is a open dense Euclidean manifold of dimension $n$ in $\boldsymbol{M}$.

c) Show that the sphere $(\mathbb{S}^2, d_{\mathbb{S}}) \subseteq \mathbb{R}^3$, where $d_{\mathbb{S}}$ is the induced metric $d_{\mathrm{e}}$ or the path-distance $\widetilde{d}_{\mathrm{e}}$, is not a *PL*-manifold.




*Hint. Given $x,x' \in \mathbb{S}^2$, there exists an isometry $\alpha : \mathbb{S}^2 \to \mathbb{S}^2$ such that $x' = \alpha(x)$. Therefore, if $\mathbb{S}^2$ were isometric to some PL-manifold $\boldsymbol{M}$, we would have $\partial \boldsymbol{M} = \emptyset$ and $\boldsymbol{M}$ would be an Euclidean manifold.*

*For $x \in \mathbb{S}^2$ and $\epsilon > 0$, let $\mathbb{D}(x,\epsilon) := \{y \in \mathbb{S}^2 \mid d_{\mathbb{S}}(x,y) \leq \epsilon\}$. If $(\mathbb{S}^2, d_{\mathbb{S}})$ were isometric to a PL-manifold $\boldsymbol{M}$ of some dimension $n$, the boundary circle $C$ of $\mathbb{D}(x,\epsilon)$, for $\epsilon$ small, would be isometric to some sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, and, since $C$ becomes disconnected when removing two of its points, this forces $n$ to be 2. But then, we would have $\pi = \ell(C, d_{\mathbb{S}})/2\epsilon$, which is not true for any of the two distances considered on $\mathbb{S}^2$.*

4) Generalize the exercise 2.3.3-(3) to the context of *PL*-manifolds. Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be two *PL*-manifolds of the same dimension. Let $F_i \subseteq \boldsymbol{M}_i$ be faces with isometric closures $\varphi_{2,1} : \overline{F}_1 \to \overline{F}_2$. Then the assemblage of the closed atlas determined by these data is an assemblage in *PL*-**Man**$_*$.

5) Show that the product of two *PL*-manifolds is a *PL*-manifold, but, to be a categorical product, it is necessary that one of them be of dimension 0.
*Hint. Use exercise 2.3.3-(5) to show that the product $P_1 \times P_2$, because it has an open cover of products of locally closed PL-sets, it is also an open assemblage of PL-sets.*

**3.2 Coproducts and *PL*-Submanifolds.** As for the categories of pseudometric spaces, the categories of *PL*-manifolds have categorical coproducts and subspaces, which we will call *PL-Submanifolds*. We state these facts in a single statement, and leave the proof, which is quite easy, to the reader.

**3.2.1 Proposition.** a) *Let $\mathcal{F} := \{\boldsymbol{M}_i := (M_i, d_{M_i})\}_{i \in \mathfrak{I}}$ be a family of PL-manifolds. The space*

$$\coprod \mathcal{F} := \left(\bigsqcup_i \boldsymbol{M}_i, d_{\mathcal{F}}\right),$$

*where $d_{\mathcal{F}}$ is the coproduct distance (**5**♦4.1), and with the inclusion maps $\iota_i : \boldsymbol{M}_i \hookrightarrow \coprod \mathcal{F}$, is a coproduct of $\mathcal{F}$ in the categories PL-***Man**$_*$.

b) *If $\phi : \boldsymbol{M} \to \boldsymbol{M}'$ is a local isometry of PL-manifolds, the image $\phi(\boldsymbol{M})$ is an open subset of $\boldsymbol{M}'$, which endowed with the induced path-distance and the induced decomposition as disjoint union of faces, is a PL-manifold. The inclusion map $\iota : \phi(\boldsymbol{M}) \hookrightarrow \boldsymbol{M}'$ is then a subspace of $\boldsymbol{M}'$ in PL-***Man**$_*$.

## 3.3 Quotients of a *PL*-Manifold by a group of Isometries

**3.3.1. Groups Acting on a *PL*-Manifold.** Recall that in **5**♦6.5.2, we denoted by $\mathrm{Iso}(\boldsymbol{M})$ the group of isometries of *PL*-manifold $\boldsymbol{M} := (M,\widetilde{d}_M)$. In view of the usefulness of the existence theorem for quotients **5**♦6.5.5, we will be mainly interested in subgroups $\boldsymbol{G} \subseteq \mathrm{Iso}(\boldsymbol{M})$ whose natural action on $\boldsymbol{M}$ is free and properly discontinuous. The following theorem summarizes some properties of these actions that will constantly be used in the sequel.

**3.3.2 Theorem.** *Let $\boldsymbol{G}$ be a group acting on a PL-manifold $\boldsymbol{M}$ by isometries.*

a) *The subspaces $\mathcal{O}_\ell(\boldsymbol{M})$, $\partial\boldsymbol{M}$, $\boldsymbol{M}_{\leq \ell}$, etc., are stabilized by $\boldsymbol{G}$.*

b) *Assume that $\boldsymbol{G}$ acts freely on $\boldsymbol{M}$ and either $\boldsymbol{G}$ is finite or it acts properly discontinuously on $\boldsymbol{M}$. Then, the following holds.*

   i) *The quotient*
$$\nu_{\mathrm{met}} : \boldsymbol{M} \to \boldsymbol{M}/\boldsymbol{G}$$
   *exist in PL-*$\mathbf{Man}_{\ell\text{-iso}}$. *We have* $\dim(\boldsymbol{M}/\boldsymbol{G}) = \dim(\boldsymbol{M})$ *and*
$$?(\boldsymbol{M}) = \nu_{\mathrm{met}}^{-1}(?(\boldsymbol{M}/\boldsymbol{G})) \quad \textit{and} \quad \nu_{\mathrm{met}}(?(\boldsymbol{M})) = ?(\boldsymbol{M}/\boldsymbol{G}),$$
   *for* $? \in \big\{\mathcal{O}_\ell,\partial,\mathcal{F}_\ell,(\_)_{\leq \ell}\big\}$.

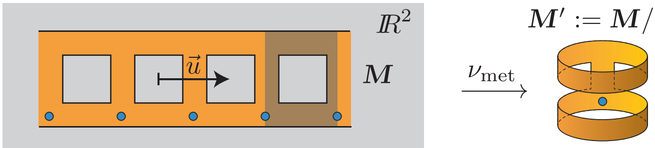
   ii) *The group $\boldsymbol{G}$ acts freely and properly discontinuously on $\mathcal{O}(\boldsymbol{M})$, $\partial\boldsymbol{M}$ and $\boldsymbol{M}_{\leq\ell}$, and there are canonical factorizations in PL-*$\mathbf{Man}_{\ell\text{-iso}}$
$$\begin{tikzcd}
\mathcal{O}(\boldsymbol{M}) \arrow[r, "\nu_{\mathrm{met}}"] \arrow[d, "\nu_{\boldsymbol{G}}"'] & \mathcal{O}(\boldsymbol{M}/\boldsymbol{G}) \\
\mathcal{O}(\boldsymbol{M})/\boldsymbol{G} \arrow[ur, leftrightarrow, "\simeq"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
\partial(\boldsymbol{M}) \arrow[r, "\nu_{\mathrm{met}}"] \arrow[d, "\nu_{\boldsymbol{G}}"'] & \partial(\boldsymbol{M}/\boldsymbol{G}) \\
\partial(\boldsymbol{M})/\boldsymbol{G} \arrow[ur, leftrightarrow, "\simeq"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
\boldsymbol{M}_{\leq\ell} \arrow[r, "\nu_{\mathrm{met}}"] \arrow[d, "\nu_{\boldsymbol{G}}"'] & (\boldsymbol{M}/\boldsymbol{G})_{\leq\ell} \\
\boldsymbol{M}_{\leq\ell}/\boldsymbol{G} \arrow[ur, leftrightarrow, "\simeq"'] &
\end{tikzcd}$$
   *where the diagonal arrows are isometries.*

   iii) *If $\boldsymbol{M}$ is Hausdorff (resp. irreducible), the quotient $\boldsymbol{M}/\boldsymbol{G}$ also is.*

*Proof.* If we stay in the category $\mathbf{p\text{-}Met}_{\ell\text{-d}}$, everything is quite immediate after **5**♦6.5.5, **5**♦6.5.4 and 3.1.5. To see that $\boldsymbol{M}/\boldsymbol{G}$ is a *PL*-manifold in the sense of definition 3.1.1, we use the fact that since $\nu_{\mathrm{met}}$ is a surjective local isometry, it defines trivialization covers $\mathcal{U} = \{U_i\}_{i\in\mathfrak{A}}$ of $\boldsymbol{M}$ (see **5**♦6.5, ??). In these, the open subsets $U_i$ can be chosen as small as desired, for example contained in some open chart of the defining atlas of $\boldsymbol{M}$. In doing so, $\mathcal{U}$ is an open atlas whose charts are locally closed *PL*-sets, and $\boldsymbol{M}/\boldsymbol{G}$, being isometric to the assemblage of $\mathcal{U}$, is a *PL*-manifold. □

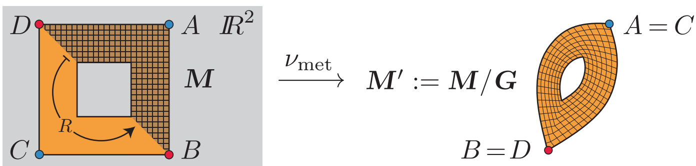
**3.3.3 Examples.** 1) Let $\boldsymbol{M}$ be the ladder shaped *PL*-set of $\mathbb{R}^2$, which extends indefinitely to the right and the left as shown in the figure below. The translation $T_{\vec{u}}$ is an isometry of $\boldsymbol{M}$, both, for the Euclidean distance and the path-distance. The group $\boldsymbol{G} = \langle T_{\vec{u}}\rangle$ acts clearly freely on $\boldsymbol{M}$, the orbits (blue dots) are discrete and the action is properly discontinuous.

The shadowed region is a fundamental domain (**5**♦8.2.1) for the action.




Note that while the boundary $\partial\boldsymbol{M}$ has infinitely many connected components, the boundary $\partial(\boldsymbol{M}')$ has only three connected components.

2) Now $\boldsymbol{M}$ is a square shaped *PL*-set of $I\!R^2$ and the group $\boldsymbol{G}$ is generated by the rotation $R$ of argument $\pi$ as shown in the figure below. The group has only two elements $\boldsymbol{G} = \{R, \mathbf{1}_M\}$ and the action is free. The shadowed region is a fundamental domain for the action.



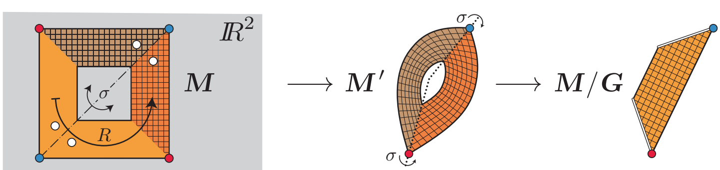

Note that, as in the previous example, the quotient $\boldsymbol{M}' := \boldsymbol{M}/\boldsymbol{G}$ cannot be embedded in $I\!R^2$ without folds or tears, *i.e.* it is not a *planar* domain.

3) For the same $\boldsymbol{M}$ as in (2), let $\boldsymbol{G} := \langle R, \sigma\rangle$ be now the group generated by the same rotation $R$, plus the reflexion $\sigma$ of axis $\overline{AC}$. As the group is abelian, $R\sigma = \sigma R$, we can apply **5**♦6.5.9 and get a canonical identification

$$\boldsymbol{M}/\boldsymbol{G} = \boldsymbol{M}'/\langle\sigma\rangle \tag{$*$}$$

where $\boldsymbol{M}' := \boldsymbol{M}/\langle R\rangle$. The orbits of $\boldsymbol{G}$ in $\boldsymbol{M}$ have either 2 (red and blue) or 4 (white) elements. The action is without fixed points but it is not free, and theorem 3.3.2 cannot be applied.

A way to understand $\boldsymbol{M}/\boldsymbol{G}$ is to use $(*)$ and remark that the action of $\sigma$ on $\boldsymbol{M}'$ is the folding through the dotted lines. In this action, the involution $\sigma$ has fixed points, those belonging to the folding segments, elsewhere the action is free.




The quotient $\boldsymbol{M}/\boldsymbol{G}$ is however a nice *PL*-manifold, although it is neither true that $\nu_{\mathrm{met}} : \boldsymbol{M} \to \boldsymbol{M}/\boldsymbol{G}$ is a local isomorphism, nor that the equality $\nu^{-1}(\partial(\boldsymbol{M}/\boldsymbol{G})) = \partial\boldsymbol{M}$ holds. Note that $\boldsymbol{M}/\boldsymbol{G}$ is now a *planar* domain. This kind of example will be crucial for the construction of planar isospectral domains. They will be thoroughly studied in a forthcoming chapter.

**3.4 Assemblages of open atlases of *PL*-manifolds.** The same questions for assemblages of open atlases of *PL*-manifolds have also positive answer. The following proposition states these facts, the proof, left to the reader, follows by the same arguments as for quotients.

**3.4.1 Proposition.** *Let* $\mathcal{A} := \{\boldsymbol{E}_i\}_{i\in\mathfrak{A}}$ *be open atlas of PL-manifolds with transition isometries* $\{\varphi_{j,i} : \boldsymbol{E}_{i,j} \to \boldsymbol{E}_{j,i}\}$ *and identification map*

$$\nu_{\mathcal{A}} : \boldsymbol{E}(\mathcal{A}) := \sqcup_i \boldsymbol{E}_i \to \mathcal{E}(\mathcal{A}).$$

a) *The transition maps respect the operators* $? \in \{\mathcal{O}_\ell, \partial, \mathcal{F}_\ell, (\_)_{\leq\ell}\}$, *in the sense that* $\varphi_{j,i}(?\boldsymbol{E}_{i,j}) = ?\boldsymbol{E}_{j,i}$, *and define open atlases* $?\mathcal{A} := \{?\boldsymbol{M}_i\}_{i\in\mathfrak{A}}$ *whose assemblages*

$$\nu_{?\mathcal{A}} : \boldsymbol{E}(?\mathcal{A}) := \sqcup_i ?\boldsymbol{E}_i \to \mathcal{E}(?\mathcal{A}),$$

*canonically identify*

$$\mathcal{E}(?\mathcal{A}) \cong ?\mathcal{E}(\mathcal{A}).$$

b) *The assemblages* $\mathcal{E}(\mathcal{A})$ *and* $\mathcal{E}(?\mathcal{A})$ *are all PL-manifolds.*

**3.5 Functorialities in the category $\boldsymbol{PL}$-Man$_{\ell\text{-iso}}$**

In 3.3.2-(b-i), we established that the quotient of a *PL*-manifold $\boldsymbol{M}$ by a group $\boldsymbol{G}$ acting freely and properly discontinuously is again a *PL*-manifold. We also showed that $\boldsymbol{G}$ stabilizes the all the spaces $?\boldsymbol{M}$ inducing quotients compatible with $?(\boldsymbol{M}/\boldsymbol{G})$, in the sense that we have categorical identifications $(?\boldsymbol{M})/\boldsymbol{G} \cong ?(\boldsymbol{M}/\boldsymbol{Q})$ as *PL*-manifolds. The same phenomenon was established for open assemblages in 3.4.1-(a). We can say then that in the category *PL*-**Man**$_{\ell\text{-iso}}$ is *stable* all these operations are compatible each other.

At this point, it is worth noting that the right way to interpret these compatibilities is through the fact that the operators $? \in \{\mathcal{O}_\ell, \partial, (\_)_{\leq\ell}\}$ are actually functors:

$$? : PL\text{-}\mathbf{Man}_{\ell\text{-iso}} \rightsquigarrow PL\text{-}\mathbf{Man}_{\ell\text{-iso}}\,. \tag{$\diamond$}$$

Indeed, thanks to the fact that for local isometries $\phi : \boldsymbol{M} \to \boldsymbol{M}'$, one has $\phi^{-1}(?\boldsymbol{M}') = ?\boldsymbol{M}$, the restriction of $\phi$ to $?\boldsymbol{M}$, denoted by $?\phi$, has values in $?\boldsymbol{M}'$, and the map

$$?\phi : ?\boldsymbol{M} \to ?\boldsymbol{M}', \tag{$\diamond\diamond$}$$

is a well defined local isometry.

**3.5.1 Proposition.** *For each operator* $? \in \{\mathcal{O}_\ell, \partial, (\_)_{\leq\ell}\}$, *the correspondences* ($\diamond$) $\boldsymbol{M} \rightsquigarrow ?\boldsymbol{M}$ *and* ($\diamond\diamond$) $\phi \rightsquigarrow ?\phi$, *define a covariant functor compatible with group quotients and open and closed assemblages, in the sense of* **5**♦*7.1.*

*Hint.* The equalities $?\mathrm{id}_{(\_)} = \mathrm{id}_{?(\_)}$ and $?(\phi_1 \circ \phi_2) = ?(\phi_1) \circ ?(\phi_2)$ are obvious. The remaining has already been proved in 3.1.5, 3.3.2 and 3.4.1. □

## Chapter Contents

# Chapter 7

# Differentiable *PL*-Manifolds The Euler-Lagrange Equation on *PL*-Manifolds

We now introduce the two last ingredients necessary to *make vibrate* a path-metric space $\boldsymbol{M}$, namely *the vector space of differentiable functions* $\mathcal{C}^\infty(\boldsymbol{M})$ and the *Laplacian* $\Delta_{\boldsymbol{M}}$. Indeed, as we recalled in **1**♦1.3, by analogy with membranes, the height $f(p,t)$ of a point $p \in \boldsymbol{M}$ at time $t$ when $\boldsymbol{M}$ *vibrates*, should be a solution of the *Euler-Lagrange partial differential equation*

$$\partial_t^2 f(p,t) = \Delta_{\boldsymbol{M}} f(p,t), \qquad (*_{\boldsymbol{M}})$$

where $\Delta_{\boldsymbol{M}}$ is the *Laplace-Beltrami* operator on $\mathcal{C}^\infty(\boldsymbol{M})$, so that we need to give a meaning to the couple $(\mathcal{C}^\infty(\boldsymbol{M}),\Delta_{\boldsymbol{M}})$. Once, this is achieved, the modes of vibration of $\boldsymbol{M}$, under Dirichlet ($\mathcal{D}$) or Neumann ($\mathcal{N}$) boundary conditions, are determined by the extended spectra $\mathrm{Spec}_\beta(\boldsymbol{M})$ of $\boldsymbol{M}$, *i.e.* by the families of dimensions of the subspaces $\mathcal{H}_{\lambda,\beta}(\boldsymbol{M})$ of functions $\phi \in \mathcal{C}^\infty(\boldsymbol{M})$ verifying $\lambda\phi = -\Delta_{\boldsymbol{M}}(\phi)$ and the boundary condition $\beta \in \{\mathcal{D},\mathcal{N}\}$ (*cf.* **1**♦1.5, **1**♦5.2).

Once this has been done on each individual *PL*-manifold $\boldsymbol{M}$, we will *enlarge* the concept of a *PL*-manifold $\boldsymbol{M}$ to the triple $(\boldsymbol{M},\mathcal{C}^\infty(\boldsymbol{M}),\Delta_{\boldsymbol{M}})$. This, in turn, will entail to *enlarge* the category $PL\text{-}\mathbf{Man}_{\ell\text{-iso}}$, extending the definition of morphism from that of local isometry $\phi : \boldsymbol{M} \to \boldsymbol{M}'$, by incorporating to it the natural linear map

$$\phi^* : (\mathcal{C}^\infty(\boldsymbol{M}'),\Delta_{\boldsymbol{M}'}) \to (\mathcal{C}^\infty(\boldsymbol{M}),\Delta_{\boldsymbol{M}}), \quad f \mapsto f \circ \phi, \qquad (\diamond)$$

which also happens to respects the boundary conditions.

This enlarged category is well suited to state and study the fundamental fact by which *isometries preserve sounds*, *i.e. if* $\phi : \boldsymbol{N} \to \boldsymbol{M}$ *is a local isometry and if* $f(p,t)$ *is a solution of* $(*_{\boldsymbol{M}})$, *then* $f(\phi(p),t)$ *is a solution of* $(*_{\boldsymbol{N}})$. The spot will then focus on the key example of this phenomenon where $\boldsymbol{M}'$ is the quotient *PL*-manifold $\boldsymbol{M}/\boldsymbol{G}$ and where $(\diamond)$ is the canonical map. In that case $\phi^*$ induces an *identification* of vector spaces

$$\phi^* : (\mathcal{C}^\infty(\boldsymbol{M}'),\Delta_{\boldsymbol{M}'}) \xrightarrow[\simeq]{} (\mathcal{C}^\infty(\boldsymbol{M})^{\boldsymbol{G}},\Delta_{\boldsymbol{M}}),$$

opening to Sunada's method, subject to be covered in the next chapter.

# 1 The Differential Structure of $I\!R^n$

## 1.1 The Algebra of Real Functions

A *(real) function* on a subset $\boldsymbol{S} \subseteq I\!R^n$ is simply a map $f : \boldsymbol{S} \to I\!R$. We denote by $\mathrm{Maps}(\boldsymbol{S},I\!R)$ the set of functions ([1]). Thanks to the algebraic structure of $I\!R$, the set $\mathrm{Maps}(\boldsymbol{S},I\!R)$ can be equipped with the structure of an *$I\!R$-algebra*. For all $f,f' \in \mathrm{Maps}(\boldsymbol{S},I\!R)$ and $\lambda \in I\!R$, set $\lambda \cdot f$, $f + f'$ and $f{\times}f'$ as the functions that associate with $s \in \boldsymbol{S}$ the following real numbers

$$\begin{cases} (f+f')(s) := f(s) + f'(s), \\ (\lambda \cdot f)(s) := \lambda \cdot f(s), \\ (f{\times}f')(s) := f(s) \cdot f'(s). \end{cases} \tag{‡}$$

Denote by $\mathbf{0}_{\boldsymbol{S}},\mathbf{1}_{\boldsymbol{S}} : \boldsymbol{S} \to I\!R$ the constant maps onto 0 and 1 respectively. Then,

$$\big(\mathrm{Maps}(\boldsymbol{S},I\!R),+,\mathbf{0}_{\boldsymbol{S}},\times,\mathbf{1}_{\boldsymbol{S}},\cdot,1\big)$$

is a commutative $I\!R$-algebra ([2]).

**1.2 The Algebra of Continuous Functions.** In the last paragraph, we induced an algebraic structure on the set $\mathrm{Maps}(\boldsymbol{S},I\!R)$ from that of $I\!R$. We can go much further taking advantage of the Euclidean structure of the vector space $(I\!R^n,\|\_\|)$. The following proposition summarizes well known continuity properties. The reason for stating it is that we want to pinpoint where exactly one has to prove something new, after what the other results come out almost automatically from universal properties. For example, in next 1.2.1, once (a) is established, the implications (a)⇒(b) and (a)⇒(c) are immediate. It is worth noting that this same proposition will still be true when replacing *continuous* by *$r$-differentiable* (1.3.2).

### 1.2.1 Proposition

a) *Any multilinear map $\alpha : I\!R^{n_1} \times \cdots \times I\!R^{n_r} \to I\!R$ is continuous.*
b) *Any polynomial map* ([3]) *$\alpha : I\!R^n \to I\!R$ is continuous.*
c) *For any subset $\boldsymbol{S} \subseteq I\!R^n$, the set of continuous functions $\mathcal{C}(\boldsymbol{S},I\!R)$ is a subalgebra of the $I\!R$-algebra* $\mathrm{Maps}(\boldsymbol{S},I\!R)$*,* i.e. *the functions in* (‡) *are continuous if $f,f'$ are so.*

---

([1]) We adhere here to the usage by which the term *function*, in speaking of a map, designates its codomain to be the reference coefficients field: *the ground field*. In here, it is the field of real numbers $(I\!R,+,0,\cdot,1)$, but in other contexts, it can differ. For example, for complex manifolds, it is the field of complex number field $\mathbb{C}$, and for algebraic varieties in positive characteristic, it is the algebraic closure of a finite field $\overline{\mathbb{F}}_p$.

([2]) $(\boldsymbol{A},+,\mathbf{0}_{\boldsymbol{A}},\times,\mathbf{1}_{\boldsymbol{A}},\cdot,1)$ is an *$I\!R$-algebra* if $(\boldsymbol{A},+,\mathbf{0},\cdot,1)$ is a vector space, $(\boldsymbol{A},+,\mathbf{0}_{\boldsymbol{A}},\times,\mathbf{1}_{\boldsymbol{A}})$ is a ring, and the associativity law $\lambda \cdot (f{\times}f') = (\lambda \cdot f){\times}f' = f{\times}(\lambda \cdot f')$ is satisfied.

([3]) A function $\alpha$ on a vector space $\boldsymbol{V}$ is *polynomial* if the description of $\alpha(\vec{v})$ in terms of the coefficients $(x,\ldots,x_n)$ of $\vec{v}$ relative to a basis $\mathcal{B}$ of $\boldsymbol{V}$, is given by a polynomial in those coefficients. This *polynomiality* is independent of the choice of the basis $\mathcal{B}$, which implies that the *degree*, the *homogeneity*, etc., of a polynomial function, are all intrinsic notions.

*Hint.* (a) If $\{\vec{e}_i\}$ is the canonical basis of $\mathbb{R}^{n_j}$, we can write vectors in terms of their coordinates as $\vec{a} = x^i\vec{e}_i$, following Einstein notation, in which case if

$$|\alpha(\vec{a}_1,\ldots,\vec{a}_r)| = |x_1^{i_1}x_2^{i_2}\ldots x_r^{i_r}\alpha(\vec{e}_{i_1},\ldots,\vec{e}_{i_r})| \le C\|\vec{a}_1\|\ldots\|\vec{a}_r\|$$

where $C = \sup\{|\alpha(\vec{e}_{i_1},\ldots,\vec{e}_{i_r})|\}$ and $\|(x_1,\ldots,x_r)\| := \sum_j |x_j|$ (*cf.* **4**♦1.1.2-(2)). It is then easy to see that

$$\Big|\alpha(\vec{a}_1,\ldots,\vec{a}_r) - \alpha(\vec{b}_1,\ldots,\vec{b}_r)\Big| = \Big|\sum\nolimits_j \alpha(\vec{a}_1,\vec{a}_2,\ldots,(\vec{a}_j - \vec{b}_j),\vec{b}_{j+1},\ldots,\vec{b}_r)\Big|$$
$$\le \sum\nolimits_j C\|\vec{a}_j - \vec{b}_j\|\|\vec{a}_1\|\ldots\widehat{\|\vec{a}_j\|}\ \|\vec{b}_{j+1}\|\ldots\|\vec{b}_j\|$$

and the continuity of $\alpha$ results from that of $\|\_\| := d_1(\_,0)$ (*cf.* **4**♦2.2.2).

(b) A polynomial map $\alpha : \mathbb{R}^n \to \mathbb{R}$ is a linear combination of monomial maps of the form

$$X_1^{m_1}\cdots X_n^{m_n} : (x_1,\ldots,x_n) \mapsto x_1^{m_1}\cdots x_n^{m_n}\,, \tag{$*$}$$

so that $\alpha$ is continuous if each monomial $(*)$ is so, after (a). Now, $(*)$ clearly decomposes as the inclusion

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}\,, \quad (x_1,\ldots,x_n) \mapsto \big(\delta_{m_1}(x_1)\cdots\delta_{m_n}(x_n)\big)\,,$$

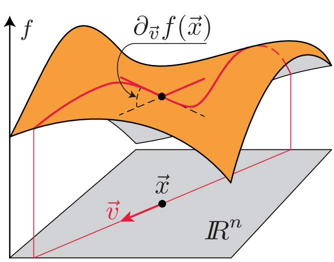
with $\delta_k(x) = (x,x,\ldots,x) \in \mathbb{R}^k$, which is obviously continuous because linear, followed by the multilinear map $\cdot : \mathbb{R}^{m_1+\cdots+m_r} \to \mathbb{R}$, $(x_i)_i \mapsto \Pi_i x_i$, continuous after (a).

(c) Given $f,f' \in \mathrm{Maps}(\boldsymbol{S},\mathbb{R})$, the map $(f,f') : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}$ is continuous, after the universal property of the product (**3**♦3.1.1-(2)). The composition of $(f,f')$ with the maps $+,\cdot : \mathbb{R}^2 \to$, continuous after (a), shows that $f + f'$ and $f \times f'$ are continuous. □

**1.2.2 Notation.** For reasons that will become soon clear, we denote by $\mathcal{C}^0(\boldsymbol{S})$ the $\mathbb{R}$-algebra of continuous functions $\mathcal{C}(\mathbb{R}^n,\mathbb{R})$.

## 1.3 The Algebra of Differentiable Functions of $\mathbb{R}^n$

**1.3.1. Directional derivatives on $\mathbb{R}^n$.** For $\vec{v} \in \mathbb{R}^n$, we denote by $\partial_{\vec{v}}$ the *derivation along the vector $\vec{v}$*, which applied to a function $f : \mathbb{R}^n \to \mathbb{R}$, is defined, on a point $\vec{x} \in \mathbb{R}^n$, by the well known expression




$$\partial_{\vec{v}} f(\vec{x}) := \lim_{\epsilon \mapsto 0} \frac{f(\vec{x} + \epsilon\vec{v}) - f(\vec{x})}{\epsilon}\,, \tag{$\diamond$}$$

where the limit may or may not exist depending on the function.

If $\{\vec{e}_1,\ldots,\vec{e}_n\}$ is the canonical basis of $\mathbb{R}^n$, the notations $\partial_i$ and $\frac{d}{dx_i}$ will be preferred to $\partial_{\vec{e}_i}$.

**1.3.1 Definitions.** a) A function $f : I\!R^n \to I\!R$ is said to be *derivable* if, for all $\vec{v}$ and $\vec{x}$ in $I\!R^n$, the limit ($\diamond$) is a well defined real number. We then denote by $\partial_{\vec{v}} f : I\!R^n \to I\!R$ the resulting function.

b) A derivable function $f : I\!R^n \to I\!R$ such that $\partial_{\vec{v}} f$ is continuous for all $\vec{v}$, is said to be *continuously derivable* or *of class $\mathcal{C}^1$*. The set of these functions, denoted by $\mathcal{C}^1(I\!R^n)$. We have $\mathcal{C}^0(I\!R^n) \supseteq \mathcal{C}^1(I\!R^n)$.

c) More generally, $f$ is *$r$-continuously derivable* or *of class $\mathcal{C}^r$*, if all of its directional derivatives $\partial_{\vec{v}} f$ are $(r-1)$-continuously derivable. We denote by $\mathcal{C}^r(I\!R^n)$ the set of such functions. We have $\mathcal{C}^{r-1}(I\!R^n) \supseteq \mathcal{C}^r(I\!R^n)$.

d) So far, we have defined the infinite decreasing sequence of sets

$$\mathcal{C}^0(I\!R^n) \supseteq \mathcal{C}^1(I\!R^n) \supseteq \mathcal{C}^2(I\!R^n) \supseteq \cdots$$

We say that $f$ is *infinitely derivable*, or *of classe $\mathcal{C}^\infty$*, if it belongs to all the $\mathcal{C}^r(I\!R^n)$, *i.e.* if it belongs to the set

$$\boxed{\mathcal{C}^\infty(I\!R^n) := \bigcap\nolimits_{r\in\mathbb{N}} \mathcal{C}^r(I\!R^n)}$$

The *top class of $f$* is the highest $r \in \mathbb{N} \cup \{\infty\}$ such that $f \in \mathcal{C}^r(I\!R^n)$.

**Convention.** In the sequel, the superscript $r$ in '$\mathcal{C}^r$' will always be consider as belonging to $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.

The following proposition states basic properties of directional derivatives.

**1.3.2 Proposition**

a) *Any multilinear map $\alpha : I\!R^{n_1} \times \cdots \times I\!R^{n_r} \to I\!R$ is differentiable.*

b) *Any polynomial map $\alpha : I\!R^n \to I\!R$ is differentiable.*

c) *The set $\big(\mathcal{C}^r(I\!R^n), +, \mathbf{0_S}, \times, \mathbf{1_S}, \cdot, 1\big)$ is a commutative $I\!R$-algebra. In particular, the sequence*

$$\mathcal{C}^0(I\!R^n) \supseteq \mathcal{C}^1(I\!R^n) \supseteq \mathcal{C}^2(I\!R^n) \supseteq \cdots \supseteq \mathcal{C}^\infty(I\!R^n)$$

*is a decreasing sequence of $I\!R$-algebras.*

d) *For all $\vec{v} \in I\!R^n$, the map $\partial_{\vec{v}} : \mathcal{C}^\infty(I\!R^n) \to \mathcal{C}^\infty(I\!R^n)$ is an $I\!R$-derivation,* i.e. *it is an $I\!R$-linear map verifying the Leibntiz rule*

$$\partial_{\vec{v}}(f_1 f_2) = \partial_{\vec{v}}(f_1) f_2 + f_1 \partial_{\vec{v}}(f_2).$$

*Proof.* (a) A vector $\vec{v} \in I\!R^{n_1} \times \cdots \times I\!R^{n_r}$ is an $r$-tuple of vectors $\vec{v} = (\vec{v}_1, \ldots, \vec{v}_r)$. Proceeding as in the proof of 1.2.1-(a), now, thanks to the multilinearity of $\alpha$, we have

$$\frac{\alpha(\vec{a}_1, \ldots, \vec{a}_r) - \alpha(\vec{b}_1, \ldots, \vec{b}_r)}{\epsilon} = \sum\nolimits_j \alpha\Big(\vec{a}_1, \vec{a}_2, \ldots, \frac{\vec{a}_j - \vec{b}_j}{\epsilon}, \vec{b}_{j+1}, \ldots, \vec{b}_r\Big),$$

and if $\vec{a}_i := \vec{b}_i + \epsilon \vec{v}_i$, we immediately get

$$\partial_{\vec{v}} \alpha(\vec{b}_1, \ldots, \vec{b}_r) = \sum\nolimits_j \alpha(\vec{b}_1, \ldots, \vec{b}_{j-1}, \vec{v}_j, \vec{b}_{j+1}, \ldots, \vec{b}_r)$$

which shows that $\partial_{\vec{v}}\alpha : I\!R^{n_1} \times \cdots \times I\!R^{n_r} \to I\!R$ is again multilinear, and as we already showed that such maps are continuous, an inductive argument shows that $\alpha$ is $r$-continuously derivable for all $r > 0$, and is therefore differentiable. (b,c) are consequences of (a), details are left to the reader. For (d), one shows that $\partial_{\vec{v}} : \mathcal{C}^{r+1}(I\!R^n) \to \mathcal{C}^r(I\!R^n)$ is an $I\!R$-derivation for all $r \in \mathbb{N}$, which also comes out easily by (a), the conclusion for $\mathcal{C}^\infty(I\!R^n)$ is then clear. □

### 1.3.3. The Algebra of Differentiable Functions of $U \subseteq I\!R^n$

The notions of continuity and derivability being *local*, we can restrict the considerations of the previous paragraphs to functions $f : \boldsymbol{U} := (U, d_e) \to I\!R$, where $\boldsymbol{U}$ is an open subset of $(I\!R^n, \|\_\|)$. We then obtain, in the same way, the $I\!R$-algebras $(\mathcal{C}^r(\boldsymbol{U}), +, \mathbf{0}_{\boldsymbol{S}}, \times, \mathbf{1}_{\boldsymbol{S}}, \cdot, 1)$, hence generalizing 1.3.2-(c).

**1.3.4 Exercise.** Check that, if $\boldsymbol{U}_1 \supseteq \boldsymbol{U}_2$ are open subsets of $I\!R^n$, the restriction of $f \in \mathcal{C}^r(\boldsymbol{U}_1)$ to $\boldsymbol{U}_2$ belongs to $\mathcal{C}^r(\boldsymbol{U}_2)$, and that the map

$$(\_)|_{U_2} : \mathcal{C}^r(\boldsymbol{U}_1) \to \mathcal{C}^r(\boldsymbol{U}_2),$$

is a morphism of $I\!R$-algebras verifying $(\partial_{\vec{v}} f)|_{U_2} = \partial_{\vec{v}}(f|_{U_2})$, for all $\vec{v} \in I\!R^n$.

## 1.4 Differentiable Maps between Open Euclidean Sets

### 1.4.1. Coordinate Functions

- If $\mathcal{B} := \{\vec{w}_i\}$ is a basis of a vector space $\mathbf{V}$, one has the decomposition

  $$\vec{a} := \sum_i x_{\mathcal{B},i}(\vec{a})\, \vec{w}_i,$$

  where $x_{\mathcal{B},i}(\vec{a})$ is the *i-th coordinate of $\vec{a}$ relative to $\mathcal{B}$*. The maps

  $$x_{\mathcal{B},i} : \mathbf{V} \to I\!R,$$

  are the *coordinate functions on $\mathbf{V}$ relative to $\mathcal{B}$*. They are linear functions. If $\mathcal{B}$ is the canonical basis of $I\!R^n$, we will write '$x_i$' instead of '$x_{\mathcal{B},i}$'.

- Given $U \subseteq I\!R^n$ and $U' \subseteq I\!R^m$, a map $\phi : U \to U'$ is totally described by its coordinates $\phi_i := x_i \circ \phi$, as we clearly have

  $$\phi = (\phi_1, \ldots, \phi_n).$$

  Moreover, the definition of the directional derivative $\partial_{\vec{v}}$, given by the formula 1.3.1-($\diamond$), applies as is to $\phi$, and, since the topology of $(I\!R^m, \|\_\|)$ is the product topology of $(I\!R, |\_|)^m$, we clearly have

  $$\partial_{\vec{v}}\phi = (\partial_{\vec{v}}\phi_1, \ldots, \partial_{\vec{v}}\phi_n).$$

  This says that the fact that $f$ is continuous, derivable, differentiable, etc., is equivalent to the same fact for its coordinate functions. We are thus lead to the definition of *differentiable map between open Euclidean sets*.

**1.4.2 Definition.** Given open subspaces $\boldsymbol{U} \subseteq I\!R^n$ and $\boldsymbol{U}' \subseteq I\!R^m$, a map $\phi : \boldsymbol{U} \to \boldsymbol{U}'$ is *continuous*, *r-continuously derivable*, *differentiable* (resp. *of class $\mathcal{C}^0$, $\mathcal{C}^r$, $\mathcal{C}^\infty$*), if its coordinates functions $\phi_i : \boldsymbol{U} \to I\!R$ are so. The sets of these functions will be denoted by $\mathcal{C}^r(\boldsymbol{U},\boldsymbol{U}')$. As in the case of functions, we have the decreasing sequence of sets

$$\mathcal{C}^0(\boldsymbol{U},\boldsymbol{U}') \supseteq \mathcal{C}^1(\boldsymbol{U},\boldsymbol{U}') \supseteq \cdots \supseteq \mathcal{C}^\infty(\boldsymbol{U},\boldsymbol{U}') := \textstyle\bigcap_{r\in\mathbb{N}} \mathcal{C}^r(\boldsymbol{U},\boldsymbol{U}').$$

**1.4.3. The Pullback Map.** Given $\phi : U \to U'$ and $f : U' \to Z$, the composition $\phi^*(f) := f \circ \phi \in \mathrm{Maps}(U',Z)$ is the *pullback of $f$ by $\phi$.*

Given $U \xrightarrow{\phi} U' \xrightarrow{\psi} U''$, we clearly have:

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*.$$

$$\begin{tikzcd}
U \arrow[r, "\phi"] \arrow[d, dotted] & U' \arrow[r, "\psi"] \arrow[d, dotted] \arrow[l, phantom, "f\circ\psi\circ\phi \longleftarrow\!\shortmid" description, shift right=6ex] & U'' \arrow[d, "f"] \arrow[l, phantom, "f\circ\psi \longleftarrow\!\shortmid" description, shift right=6ex] \\
Z \arrow[r, equal] & Z \arrow[r, equal] & Z
\end{tikzcd} \qquad (\diamond)$$

And when $Z := I\!R$, we get a *morphism* of $I\!R$-algebras

$$\phi^* : \mathrm{Maps}(U',I\!R) \to \mathrm{Maps}(U,I\!R).$$

**1.4.4 Proposition.** *Given open subspaces $\boldsymbol{U} \subseteq I\!R^n$ and $\boldsymbol{U}' \subseteq I\!R^m$, and a map $\phi : \boldsymbol{U} \to \boldsymbol{U}'$, we have:*

$$\big(\phi \in \mathcal{C}^r(\boldsymbol{U},\boldsymbol{U}')\big) \Leftrightarrow \big(\phi^*(\mathcal{C}^r(\boldsymbol{U}')) \subseteq \mathcal{C}^r(\boldsymbol{U})\big)$$

*In which case, the pullback*

$$\phi^* : \mathcal{C}^r(\boldsymbol{U}') \to \mathcal{C}^r(\boldsymbol{U})$$

*is well defined and is an $I\!R$-algebra morphism.*

$$\begin{tikzcd}
U \arrow[r, "\phi"] \arrow[dr, dotted, "\phi^* f\in\mathcal{C}^r"'] & U' \arrow[d, "f\in\mathcal{C}^r"] \arrow[dl, phantom, "\longleftarrow\!\shortmid" description] \\
& I\!R
\end{tikzcd}$$

*Proof.* Since the coordinate functions $x_i : \boldsymbol{U}' \to I\!R$ belong to $\mathcal{C}^r(\boldsymbol{U}')$, the implication '$\Leftarrow$' is immediate. For the converse, we have to prove that if $\phi_i$ and $f$ are $\mathcal{C}^r$, then $\phi^*(f)$ is $\mathcal{C}^r$, or, what amounts the same, that $\partial_{\vec{v}}(f \circ \phi)$ is $\mathcal{C}^{r-1}$. For that, we will use the well know formula for the *derivative of a composition of functions*, which says that for all $\vec{v} \in I\!R^n$, we have the equality

$$\partial_{\vec{v}}(f \circ \phi) = \sum\nolimits_{i=i}^{m} \partial_i f(\phi(\vec{a}))\, \partial_{\vec{v}}\phi_i(\vec{a}). \qquad (*)$$

As we already know that the functions $\partial_{\vec{v}}\phi_i$ are $\mathcal{C}^{r-1}$, and, since $\mathcal{C}^{r-1}(\boldsymbol{U})$ is an algebra (1.3.2(c)), we will be able to conclude that $\partial_{\vec{v}}(f \circ \phi)$ is $\mathcal{C}^{r-1}$ with the help of $(*)$, if we can justify that $\partial_i f \circ \phi$ is $\mathcal{C}^{r-1}$. But here, $\partial_i f$ and $\phi$ are both $\mathcal{C}^{r-1}$, and we can assume that $\partial_i f \circ \phi$ is $\mathcal{C}^{r-1}$, by induction. ☐

**1.4.5 Comment.** It is worth noting the underlying idea in this proposition. It says that, for a map $\phi : \boldsymbol{U} \to \boldsymbol{U}'$, to be of class $\mathcal{C}^r$ is equivalent that it preserves, by pullback, the class of functions of class $\mathcal{C}^{r'}$, for all $r' \leq r$ ([4]).

---

([4]) Care should be taken with this equivalence, as it is not generally true. Indeed, already for topological spaces, if it were verified by all maps $\phi : (\_) \to \boldsymbol{Y}$, and some space $\boldsymbol{Z}$

Because of this equivalence, an alternative definition for *the class* of a map $\phi : \boldsymbol{U} \to \boldsymbol{U}'$ would be

$$\text{class of } f = \max\big\{r \in \mathbb{N} \;\big|\; \phi^*(\mathcal{C}^r(\boldsymbol{U}')) \subseteq \mathcal{C}^r(\boldsymbol{U})\big\}.$$

We will soon see that this property naturally extends to more elaborated spaces, notably the *differentiable* assemblages of open subspaces of $\mathbb{R}^n$, more commonly called *differentiable manifolds* (*cf.* prop. 2.4.2-(b)).

**1.4.6 Proposition.** *If $\boldsymbol{U} \xrightarrow{\phi} \boldsymbol{U}' \xrightarrow{\psi} \boldsymbol{U}''$ are differentiable maps of class $\mathcal{C}^r$ between open subspaces of Euclidean vector spaces, the composed map $\psi \circ \phi : \boldsymbol{U} \to \boldsymbol{U}''$ is also differentiable of class $\mathcal{C}^r$.*

*Proof.* The class of $\psi\circ\phi$ is then given by 1.4.4 and the obvious inclusions $(\psi\circ\phi)^*(\mathcal{C}^r(\boldsymbol{U}''))=\phi^*(\psi^*(\mathcal{C}^r(\boldsymbol{U}''))\subseteq\phi^*(\mathcal{C}^r(\boldsymbol{U}'))\subseteq\mathcal{C}^r(\boldsymbol{U})$, thanks to 1.4.3-($\diamond$). □

**1.4.7. The Tangent map to a Differentiable Map**

Let $\boldsymbol{U} \subseteq \mathbb{R}^n$ and $\boldsymbol{U}' \subseteq \mathbb{R}^m$. If $\phi : \boldsymbol{U} \to \boldsymbol{U}'$ is a derivable map, the *tangent map to $\phi$ at $\vec{a} \in \boldsymbol{U}$* is the map

$$T_{\vec{a}}(\phi) : \mathbb{R}^n \to \mathbb{R}^m, \quad \vec{v} \mapsto \partial_{\vec{v}}\phi(\vec{a}).$$

**1.4.8 Proposition.** *Let $\phi : \boldsymbol{U} \to \boldsymbol{U}'$ be of class $\mathcal{C}^1$.*

a) *The tangent maps $T_{\vec{a}}(\phi) : \mathbb{R}^n \to \mathbb{R}^m$ are linear maps.*

b) *The tangent map $T_{\vec{a}}(\phi) : \mathbb{R}^n \to \mathbb{R}^m$ is the only linear map such that*

$$\phi(\vec{a}+\vec{v}) = \phi(\vec{a}) + T_{\vec{a}}(\phi)(\vec{v}) + o(\|\vec{v}\|), \quad (\forall \vec{v} \in \mathbb{R}^n).$$

c) *Given, in addition, $\psi : \boldsymbol{U}' \to \boldsymbol{U}''$ also of class $\mathcal{C}^1$, we have*

$$T_{\vec{a}}(\psi\circ\phi) = T_{\phi(\vec{a})}(\psi) \circ T_{\vec{a}}(\phi).$$

d) *If $\phi$ is invertible,* i.e. *bijective and such that $\phi^{-1}$ is of class $\mathcal{C}^1$, then the tangent maps $T_{\vec{a}}(\phi)$ are isomorphisms, for all $\vec{a} \in \boldsymbol{U}$. In particular,* $\dim \boldsymbol{U} = \dim \boldsymbol{U}'$, i.e. *they are open subspaces of the same $\mathbb{R}^n$.*

*Proof.* (a) We have to show that

$$\partial_{\lambda\vec{v}+\vec{w}}\phi(\vec{a}) = \lambda\partial_{\vec{v}}\phi(\vec{a}) + \partial_{\vec{w}}\phi(\vec{a}). \qquad (\diamond)$$

For that, we consider the elementary decomposition

$$\frac{\phi(\vec{a}+\epsilon(\lambda\vec{v}+\vec{w}))-\phi(\vec{a})}{\epsilon} = \frac{\phi(\vec{a}+\epsilon\lambda\vec{v}+\epsilon\vec{w})-\phi(\vec{a}+\epsilon\lambda\vec{v})}{\epsilon} + \frac{\phi(\vec{a}+\epsilon\lambda\vec{v})-\phi(\vec{a})}{\epsilon}.$$

instead of $\mathbb{R}$, we would immediately conclude, after the universal property of products, that $\boldsymbol{Y}$ is *locally* a subspace of $\boldsymbol{Z}^{\mathcal{C}(\boldsymbol{Y},\boldsymbol{Z})}$, which is not always the case, and which is, in fact, the condition for the equivalence to be true.

In it, the last term can be easily settled, since for $\lambda \neq 0$, we have

$$\lim_{\epsilon \mapsto 0} \frac{\phi(\vec{a} + \epsilon\lambda\vec{v}) - \phi(\vec{a})}{\epsilon} = \lambda \lim_{\epsilon \mapsto 0} \frac{\phi(\vec{a} + \lambda\epsilon\vec{v}) - \phi(\vec{a})}{\lambda\epsilon} = \lambda\partial_{\vec{v}}\phi(\vec{a}).$$

For the other term, we write $\phi = (\phi_1, \ldots, \phi_m)$, and look at the real functions

$$h_i(t) := \phi_i(\vec{a} + \epsilon\lambda\vec{v} + t\vec{w}) \in I\!R,$$

where $\epsilon$ is fixed and small, and $t \in (-\eta, \eta)$ for $\eta$ small. Here, we can apply the *mean value theorem* and conclude that there exists $e(\epsilon) \in [0, \epsilon]$ such that

$$\frac{h_i(\epsilon) - h_i(0)}{\epsilon} = h_i'(e(\epsilon)).$$

Therefore,

$$\lim_{\epsilon \mapsto 0} \frac{\phi_i(\vec{a} + \epsilon\lambda\vec{v} + \epsilon\vec{w}) - \phi_i(\vec{a} + \epsilon\lambda\vec{v})}{\epsilon} = \lim_{\epsilon \mapsto 0} \partial_{\vec{w}}\phi\big(\vec{a} + \epsilon\vec{v} + e(\epsilon)\vec{w}\big) = \partial_{\vec{w}}\phi_i\big(\vec{a}\big),$$

because each $\phi_i$ is $\mathcal{C}^1$. The equality ($\diamond$) then follows easily.

(b) For the existence, it suffices to consider the case where $\phi$ is a function ($m = 1$), in which case the equality results from *Taylor's development with remainder*. For the unicity, if $\alpha : I\!R^n \to I\!R^m$ is linear and such that $\|T_{\vec{a}}(\phi)(\vec{v}) - \alpha(\vec{v})\| = o(\|\vec{v}\|)$, then

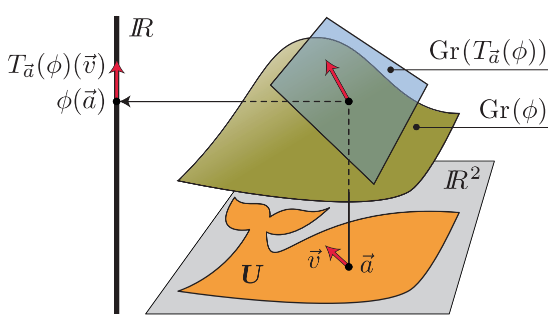
$$(T_{\vec{a}}(\phi) - \alpha)(\vec{v}) = \lim_{\epsilon \mapsto 0} \frac{(T_{\vec{a}}(\phi) - \alpha)(\epsilon\vec{v})}{\epsilon} = 0.$$

(c) Results from (b), because one has

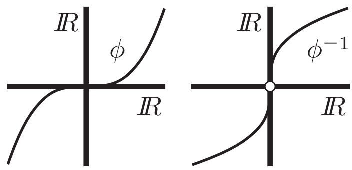
$$\begin{aligned} \psi(\phi(\vec{a} + \vec{v})) &= \psi\big\{\phi(\vec{a}) + T_{\vec{a}}(\phi)(\vec{v}) + o(\|\vec{v}\|)\big\} \\ &= \psi(\phi(\vec{a}) + T_{\phi(\vec{a})}(\psi)\big\{T_{\vec{a}}(\phi)(\vec{v}) + o(\|\vec{v}\|)\big\} \\ &= \psi(\phi(\vec{a}) + \big(T_{\phi(\vec{a})}(\psi) \circ T_{\vec{a}}(\phi)\big)(\vec{v}) + o(\|\vec{v}\|) \end{aligned}$$

and we conclude by the unicity of the tangent maps.
(c) is obvious. $\square$

**1.4.9 Comments.** a) If a function $\phi : \boldsymbol{U} \to I\!R$ is represented by its graph $\mathrm{Gr}(\phi)$, which is some surface in $\boldsymbol{U} \times I\!R$ (in green), the tangent function appears as the tangent space (in blue) to $\mathrm{Gr}(\phi)$ at the point $(\vec{a}, \phi(\vec{a}))$.



b) Care must be taken that if a map $\phi : \boldsymbol{U} \to \boldsymbol{U}'$ is a homeomorphism of class $\mathcal{C}^r$, the inverse map can fail to be of the same class. For example, $\phi : I\!R \to I\!R$, $t \mapsto t^3$, is a homeomorphism of class $\mathcal{C}^\infty$, but $\phi^{-1}$ is not derivable at 0. When $\phi$ is invertible and $\phi$ and $\phi^{-1}$ are both $\mathcal{C}^r$, the map $\phi$ is called a *diffeomorphism of class* $\mathcal{C}^r$. The *Inverse function theorem* ([L], I.§5, p.13) deals with this question. It says that: *if $\phi$ is a bijection of class $\mathcal{C}^r$, the necessary and sufficient condition for $\phi^{-1}$ to be of class $\mathcal{C}^r$, is that $T_{\vec{a}}\phi$ be a linear isomorphism for all $\vec{a} \in \boldsymbol{U}$* (*cf.* 1.4.8-(d)). We will not use this result.




# 2 The Category of Manifolds

The aim of this section is to define the category of *differentiable spaces*, *i.e.* spaces in which we have a mean to decide if a function is derivable at a point, if it is differentiable and even determine its class of differentiability. These properties are already present in the open subsets $\boldsymbol{U}$ of $I\!R^n$ for which we have defined the algebras $\mathcal{C}^r(\boldsymbol{U})$ of functions of class $\mathcal{C}^r$. In this respect, we can say that the couples $(\boldsymbol{U}, \mathcal{C}^r(\boldsymbol{U}))$ are the most basic differentiable spaces. To go further, we will patch together these, following the glueing procedure described in chapter **2**, to build the more elaborated differential spaces $\boldsymbol{M}$, called *manifolds*. For these, we will need:

(i) an underlying topological space $\boldsymbol{M}$, (ii) the algebras of functions $\mathcal{C}^r(\boldsymbol{M})$, (iii) a the sets $\mathcal{C}^r(\boldsymbol{M}, \boldsymbol{M}')$ of differentiable maps between such spaces.

At the end we will have constructed the categories $\mathcal{C}^r$-**Man**, that include the spaces $(\boldsymbol{U}, \mathcal{C}^r(\boldsymbol{U}))$, and are stable by open assemblages (see theorem 2.7.3).

## 2.1 The Category of Open Subspaces of $I\!R^n$

In the previous sections we defined the notion of map of class $\mathcal{C}^r$ between open subspaces of Euclidean spaces $I\!R^n$. These classes contain the identity maps, and the composition of two maps of class $\mathcal{C}^r$ is still of the same class. This is all we need to define categories (*cf.* **2**♦2.1).

**2.1.1 Definition.** Let $\mathcal{C}^r$**-Opn** be the category whose objects are the couples $(\boldsymbol{U}, \mathcal{C}^r(\boldsymbol{U}))$, where $\boldsymbol{U}$ is an open subspace of some $I\!R^n$, and whose *morphisms* from $(\boldsymbol{U}, \mathcal{C}^r(\boldsymbol{U}))$ to $(\boldsymbol{U}', \mathcal{C}^r(\boldsymbol{U}'))$ are the couples $\phi^\natural := (\phi, \phi^*)$ with $\phi \in \mathcal{C}^r(\boldsymbol{U}, \boldsymbol{U}')$ and $\phi^* : \mathcal{C}^r(\boldsymbol{U}) \to \mathcal{C}^r(\boldsymbol{U})$ its associated *pullback* (*cf.* 1.4.4).

Notice that, as $\phi^\natural$ is totally determined by $\phi$, a morphism in $\mathcal{C}^r$**-Opn** can (and will) be denoted simply its space component: $\phi : \boldsymbol{U} \to \boldsymbol{U}'$.

The underlying set $\sigma((\boldsymbol{U}, \mathcal{C}^r(\boldsymbol{U})))$ is obviously the set $U := \sigma(\boldsymbol{U})$ itself, so that $\mathcal{C}^r$**-Opn** is a category of spaces in the sense of **2**♦2.4.

**2.1.2 Exercises.** 1) Show that for $(\boldsymbol{U},\mathcal{C}^r(\boldsymbol{U})) \in \mathcal{C}^r$**-Opn**, the dimension $n$, such that $\boldsymbol{U} \subseteq I\!R^n$, is uniquely determined, *i.e.* if $(\boldsymbol{U},\mathcal{C}^r(\boldsymbol{U}))$ and $(\boldsymbol{U}',\mathcal{C}^r(\boldsymbol{U}'))$ are isomorphic in $\mathcal{C}^r$**-Opn**, then $\dim(\boldsymbol{U}) = \dim(\boldsymbol{U}')$. This number $n$ will be called *the dimension of* $\boldsymbol{U}$, and will be denoted $\dim(\boldsymbol{U})$. *Hint. Use 1.4.8-(d).*

2) a) Show that the correspondence $(\_)_{\mathrm{top}} : \mathcal{C}^r$**-Opn** $\rightsquigarrow$ **Top** which associates $(\boldsymbol{U},\mathcal{C}^r(\boldsymbol{U})) \rightsquigarrow \boldsymbol{U}$ and $\phi^\natural \rightsquigarrow \phi$, is functorial and faithful. ([5])

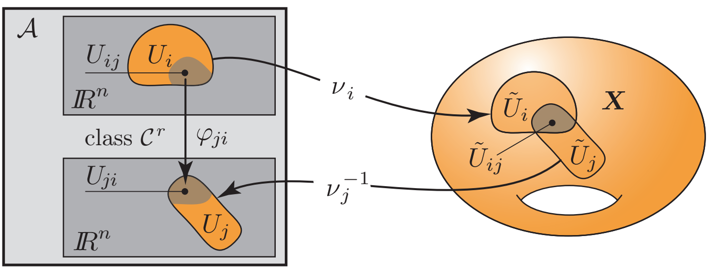
b) Show that the category $\mathcal{C}^r$**-Opn** has finite products, and open subspaces, but that it is not stable for coproducts nor for open assemblages, although they exist in **Top**.

c) (**) Same as (2-a) for the correspondence $\mathcal{O} : \mathcal{C}^r$**-Opn** $\rightsquigarrow I\!R$-Alg which associates $(\boldsymbol{U},\mathcal{C}^r(\boldsymbol{U})) \rightsquigarrow \mathcal{C}^r(\boldsymbol{U})$ and $\phi^\natural \rightsquigarrow \phi^*$.

**Terminology.** After 2.1.2-(1), we denote by $\mathcal{C}^r$**-Opn**$(n)$ the *full subcategory* of $\mathcal{C}^r$**-Opn** whose objects are of dimension $n$ (*cf.* **5♦**2.1).

## 2.2 Differentiable Manifolds

**2.2.1 Definition.** We say that a topological space $\boldsymbol{X}$ admits a *differentiable structure of class* $\mathcal{C}^r$ *and dimension* $n$, if it can be built by glueing together open subsets of $I\!R^n$ by a *smooth* glueing of class $\mathcal{C}^r$.




After the discussion of **3♦**7, this can be reformulated in mathematical terms by saying that $\boldsymbol{X}$ is the assemblage in **Top** of an open atlas $\mathcal{A}$ in $\mathcal{C}^r$**-Opn**$(n)$, or, equivalently, after **3♦**7.3.3, that there exists a continuous *open* map

$$\nu(\mathcal{A}) : \boldsymbol{E}(\mathcal{A}) \to \boldsymbol{X}, \tag{$\diamond$}$$

inducing on $\boldsymbol{E}(\mathcal{A})$ the relation $\mathcal{R}(\mathcal{A})$ generated by the transition maps of $\mathcal{A}$.

**Terminology and Notations.** – A topological space endowed with differentiable structure of class $\mathcal{C}^r$ is called *a differentiable manifold of class* $\mathcal{C}^r$, in short *a* $\mathcal{C}^r$*-manifold*. A $\mathcal{C}^r$-manifold is therefore a couple $\boldsymbol{M} := (\boldsymbol{M}_{\mathrm{top}},\nu(\mathcal{A}))$, where $\boldsymbol{M}_{\mathrm{top}}$ denotes the underlying topological space, so far denoted $\boldsymbol{X}$.

– The map $\nu(\mathcal{A})$ is *the definition atlas of the differentiable structure* of $\boldsymbol{M}$.

**Convention.** The word *manifold* without the specification of the differentiability class will stand for *manifold of a certain class* $\mathcal{C}^r$.

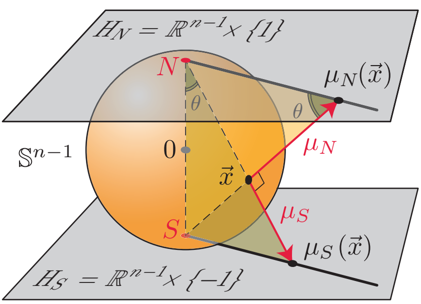
([5]) A functor is *faithful* if its action on morphisms is injective.

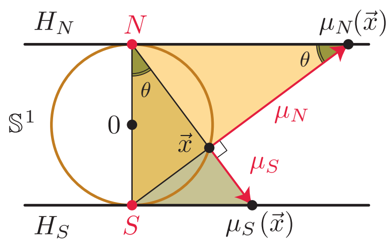
• The *induced differentiable structure* by $\boldsymbol{M} := (\boldsymbol{M}_{\text{top}}, \nu(\mathcal{A}))$ on an open subspace $\boldsymbol{V}_{\text{top}} \subseteq \boldsymbol{M}_{\text{top}}$ is the one given by the map $\nu(\mathcal{A} \Cap V) : \boldsymbol{E}(\mathcal{A} \Cap V) \to \boldsymbol{V}_{\text{top}}$, where $\mathcal{A} \Cap V$ is the open restriction of $\mathcal{A}$ introduced in **3**♦7.5.1. The terminology is justified by **3**♦7.5.2, which shows that $\nu(\mathcal{A} \Cap V)$ is an open assemblage of $\boldsymbol{V}_{\text{top}}$, and by the fact that the restrictions of the transition maps are still of class $\mathcal{C}^r$. We will denote $\boldsymbol{V} := (\boldsymbol{V}_{\text{top}}, \nu(\mathcal{A} \Cap V))$, and then write $\boldsymbol{V} \subseteq \boldsymbol{M}$ to mean that $\boldsymbol{V}$ is an *open submanifold* of $\boldsymbol{M}$.

• The *canonical structure $\mathcal{C}^r$-manifold* of $\mathbb{R}^n$, is the one given by the definition atlas consisting in one only chart $\mathcal{A} := \{(\mathbb{R}^n, \mathcal{C}^r(\mathbb{R}^n))\}$ and the trivial identification map $\nu(\mathcal{A}) := \mathrm{id}_{\mathbb{R}^n}$. Subsequently, the canonical structure of manifold of an open subspace $\boldsymbol{V}_{\text{top}} \subseteq \mathbb{R}^n$ is that of open submanifold of $\mathbb{R}^n$.

**2.2.2 Exercise.** Show that if in the definition 2.2.1, we drop the condition on the dimension $n$ and define a manifold simply as an assemblage $\boldsymbol{M}$ of an open atlas in $\mathcal{C}^r$**-Opn**, then each connected component of $\boldsymbol{M}$ will nevertheless be of a well defined dimension. *Hint. 1.4.8-(d), 2.1.2-(1).*

**2.2.3. Example: The Sphere.** Let $\mathbb{S}^{n-1}$ be the unit *sphere* of $(\mathbb{R}^n, \|\_\|)$, *i.e.* the set of vectors of norm 1, endowed with the induced metric. We will recall how $\mathbb{S}^{n-1}$ is traditionally endowed with a differentiable structure of class $\mathcal{C}^\infty$ of dimension $n-1$, thanks to the *stereographic projection*, by two charts isomorphic to $(\mathbb{R}^2, \mathcal{C}^\infty(\mathbb{R}^2))$ and overlapping set $\mathbb{R}^2 \backslash \{0\}$.




Denote by $N := (0,\dots,0,1)$ and $S := (0,\dots,0,-1)$ the poles of $\mathbb{S}^{n-1}$, and consider the following subsets of $\mathbb{R}^n$

$$\begin{cases} \mathbb{S}_N := \mathbb{S}^{n-1} \backslash \{S\} = \{(x_1,\dots,x_n) \in \mathbb{S}^{n-1} \mid x_n > -1\} \\ \mathbb{S}_S := \mathbb{S}^{n-1} \backslash \{N\} = \{(x_1,\dots,x_n) \in \mathbb{S}^{n-1} \mid x_n < 1\} \\ H_N := \{x_n = 1\} \text{ and } H_S := \{x_n = -1\} \end{cases}$$

The *stereographic projections*, respectively *of center* $N$ and $S$, are the maps

$$\mu_S : \mathbb{S}_S \to H_S = \mathbb{R}^{n-1} \times \{-1\}, \qquad \mu_N : \mathbb{S}_N \to H_N = \mathbb{R}^{n-1} \times \{1\}.$$

defined by the *radial projection* relative to the corresponding center as shown by the red arrows the previous pictures.

Elementary arguments of Euclidean geometry show that the colored triangles are similar. Therefore, $\tan(\theta) = \|\mu_N(\vec{x})\| = 1/\|\mu_S(\vec{x})\|$ and

$$\mu_N(\vec{x}) = \frac{\mu_S(\vec{x})}{\|\mu_S(\vec{x})\|^2}, \quad \forall \vec{x} \in \mathbb{S}_N \cap \mathbb{S}_S. \tag{$*$}$$

Identify $H_S \equiv (I\!R^{n-1}, \|\_\|)$ and $H_S \equiv (I\!R^{n-1}, \|\_\|)$. The inverse maps

$$\mu_N^{-1} : H_N \to \mathbb{S}_N, \quad \mu_S^{-1} : H_S \to \mathbb{S}_S. \tag{$\diamond$}$$

then define an open atlas $\mathcal{A} := \{H_S, H_N\}$ in the category **Top** whose assemblage is $\mathbb{S}^{n-1}$. The overlapping set in $\mathbb{S}^{n-1}$ is the intersection $\mathbb{S}_N \cap \mathbb{S}_S$, which corresponds through the maps $(\diamond)$ to the open sets $H'_? \equiv I\!R^{n-1} \backslash \{0\}$.
The transition map, given by $(*)$, is then the homeomorphism

$$\mu_S \circ \mu_N^{-1} : H'_N \to H'_S, \quad \vec{z} \mapsto \frac{\vec{z}}{\|\vec{z}\|^2},$$

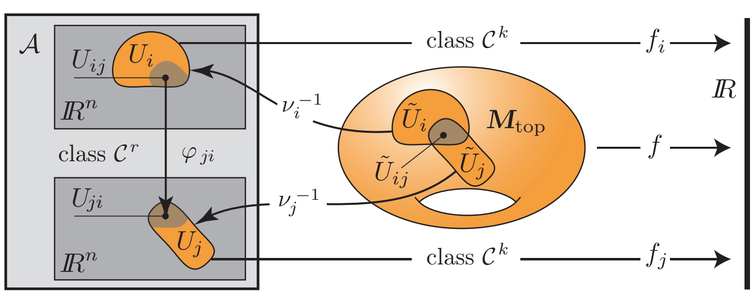
which can be shown to be of class $\mathcal{C}^\infty$ using elementary rules of derivation. The atlas $\mathcal{A}$ can therefore be seen as an open atlas in $\mathcal{C}^\infty$-**Opn**$(n-1)$. This is the canonical structure of $\mathcal{C}^\infty$-manifold of $\mathbb{S}^{n-1}$ announced.

### 2.3 Differentiable Functions of a Manifold

Inspired by the description of morphisms between assemblages (**2**♦6.4, **3**♦7.4), we will now define the notion of *differentiable function* of a manifold $\boldsymbol{M}$, and, in the next section, de notion of *differentiable map* between manifolds.

**2.3.1 Definition.** Let $\mathcal{A} := \{\boldsymbol{U}_i\}_{i\in\mathfrak{I}}$ be an open atlas in $\mathcal{C}^r$-**Opn**$(n)$ and let $\nu(\mathcal{A}) : \boldsymbol{E}(\mathcal{A}) \to \boldsymbol{M}_{\text{top}}$ be the definition atlas of a manifold $\boldsymbol{M}$. For $0 \le k \le r$, a *function of* $\boldsymbol{M}$ *of class* $\mathcal{C}^k$, is a family of functions $\mathfrak{f} := \{f_i \in \mathcal{C}^k(\boldsymbol{U}_i)\}_{i\in\mathfrak{I}}$ compatible with the transition maps $\varphi_{ji}$ of $\mathcal{A}$, *i.e.* such that

$$f_i|_{U_{ij}} = f_j \circ \varphi_{ji}, \qquad (\forall i,j \in \mathfrak{I}).$$




We denote by $\mathcal{C}^k(\boldsymbol{M})$ the set of these functions.
Notice that by **2**♦6.4.1, these data define already a continuous map $f : \boldsymbol{X} \to I\!R$, and that we have natural inclusions

$$\mathcal{C}^r(\boldsymbol{M}) \subseteq \mathcal{C}^k(\boldsymbol{M}) \subseteq \mathcal{C}^0(\boldsymbol{M}) = \mathcal{C}^0(\boldsymbol{M}_{\text{top}}).$$

**2.3.2. The Algebra $\mathcal{C}^k(\boldsymbol{M})$.** The following proposition extends the definition of the $\mathbb{R}$-algebra structure of $\mathcal{C}^k(\boldsymbol{U})$ to $\mathcal{C}^k(\boldsymbol{M})$, generalizing thereby the statement 1.3.2-(c) to differentiable structures.

**2.3.3 Proposition.** *Let $\boldsymbol{M} := (\boldsymbol{M}_{\text{top}}, \nu(\mathcal{A}))$ be a $\mathcal{C}^r$-manifold as in 2.3.1.*

a) *Given families $\mathfrak{f} := \{f_i\}_{i\in\mathfrak{I}}$ and $\mathfrak{f}' := \{f_i'\}_{i\in\mathfrak{I}}$ defining $f, f' \in \mathcal{C}^k(\boldsymbol{M})$, the families* (*cf.* 1.3)*:*

$$\mathfrak{f}+\mathfrak{f}':=\{f_i+f_i'\},\ \lambda\cdot\mathfrak{f}:=\{\lambda\cdot f_i\},\ \mathfrak{f}\times\mathfrak{f}':=\{f_i\times f_i'\},\ \mathbf{0}_{\boldsymbol{M}}:=\{\mathbf{0}_{\boldsymbol{U}_i}\},\ \mathbf{1}_{\boldsymbol{M}}:=\{\mathbf{1}_{\boldsymbol{U}_i}\},$$

*are compatible with the transition maps in $\mathcal{A}$, and therefore define functions in $\mathcal{C}^k(\boldsymbol{M})$. The resulting tuple $\big(\mathcal{C}^r(\boldsymbol{M}), +, \mathbf{0}_{\boldsymbol{M}}, \times, \mathbf{1}_{\boldsymbol{M}}, \cdot, 1\big)$ has then the structure an $\mathbb{R}$-algebra.*

b) *Let $\boldsymbol{U}$ be an open submanifold of $\boldsymbol{M}$. The restriction of $f \in \mathcal{C}^k(\boldsymbol{M})$ to $\boldsymbol{U}_{\text{top}}$ belongs to $\mathcal{C}^k(\boldsymbol{U})$, and the restriction map*

$$(\_)|_{\boldsymbol{U}} : \mathcal{C}^k(\boldsymbol{M}) \to \mathcal{C}^k(\boldsymbol{U})$$

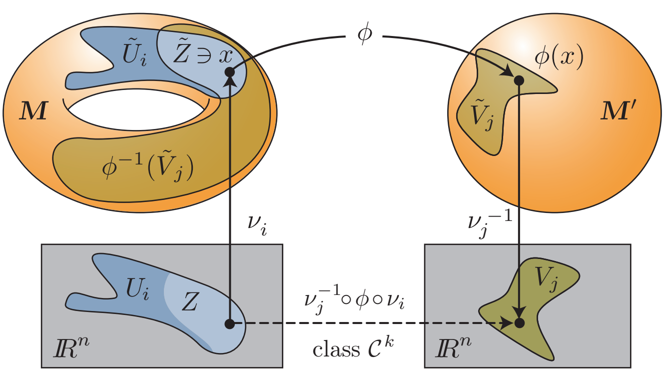
*is a morphism of $\mathbb{R}$-algebras.*

*Proof.* (a) Clear since by 1.3.4 and 1.4.4, the restrictions $\mathcal{C}^k(\boldsymbol{U}_i) \to \mathcal{C}^k(\boldsymbol{U}_{ij})$ and pullbacks $\varphi_{ji}^* : \mathcal{C}^k(\boldsymbol{U}_{ji}) \to \mathcal{C}^k(\boldsymbol{U}_{ij})$ are morphisms of $\mathbb{R}$-algebras.

(b) Left to the reader. □

## 2.4 Differentiable Maps between Manifolds

**2.4.1 Definition.** Let $\boldsymbol{M} := (\boldsymbol{M}_{\text{top}}, \nu(\mathcal{A}))$ and $\boldsymbol{M}' := (\boldsymbol{M}'_{\text{top}}, \nu(\mathcal{A}'))$ be $\mathcal{C}^r$-manifolds. For $0 \le k \le r$, a *continuous* map $\phi : \boldsymbol{M}_{\text{top}} \to \boldsymbol{M}'_{\text{top}}$ is said to be *$\mathcal{C}^k$-differentiable in the neighborhood of $x \in \boldsymbol{M}_{\text{top}}$*, if for every pair of charts $\nu_i : U_i \to \tilde{U}_i \subseteq \boldsymbol{M}$ and $\nu_j : V_i \to \tilde{V}_j \subseteq \boldsymbol{M}'$, such that $(x, \phi(x)) \in \tilde{U}_i \times \tilde{V}_j$, the map $\nu_j^{-1} \circ \phi \circ \nu_i^{-1} : Z \to V_j$, where $Z := \nu^{-1}(\tilde{U}_i \cap \phi^{-1}(\tilde{V}_j))$, is of class $\mathcal{C}^k$.




• The map $\phi : \boldsymbol{M} \to \boldsymbol{M}'$ is *$\mathcal{C}^k$-differentiable*, if it is so in the neighborhood of every $x \in \boldsymbol{M}$. We denote by $\mathcal{C}^k(\boldsymbol{M}, \boldsymbol{M}')$ the set of these maps.

• A homeomorphism $\phi : \boldsymbol{M} \to \boldsymbol{M}'$ is said to be *a $\mathcal{C}^k$-diffeomorphism*, if both, $\phi$ and $\phi^{-1}$, are differentiable maps of class $\mathcal{C}^k$ (*cf.* 1.4.9-(b)).

**Convention.** In order to simplify terminology, we will simply say that the map is *differentiable* when its class $k$ is the highest possible, *i.e.* when $k = r$.

**2.4.2 Proposition.** *Let $\boldsymbol{M}$, $\boldsymbol{M}'$, $\boldsymbol{M}''$ be $\mathcal{C}^r$-manifolds.*

a) *In the definition 2.4.1, it suffices that the defining condition be satisfied by one given pair of charts $(\tilde{U}_i,\tilde{V}_j)$ containing $(x,\phi(x))$, for it to be satisfied by any other such pair.*

b) *Let $0 \le k \le r$. A map $\phi : \boldsymbol{M} \to \boldsymbol{M}'$ is $\mathcal{C}^k$-differentiable if and only if the pullback morphism $\phi^* : \mathcal{C}^0(\boldsymbol{M}') \to \mathcal{C}^0(\boldsymbol{M})$ preserves differentiability,* i.e.

$$\phi^*(\mathcal{C}^k(\boldsymbol{M}') \subseteq \mathcal{C}^k(\boldsymbol{M}).$$

c) *If $\boldsymbol{M} \xrightarrow{\phi} \boldsymbol{M}' \xrightarrow{\psi} \boldsymbol{M}''$ are differentiable maps of class $\mathcal{C}^k$, the composed map $\psi \circ \phi : \boldsymbol{M} \to \boldsymbol{M}''$ is also differentiable of class $\mathcal{C}^k$.*

*Proof.* (a) If we change charts, the map $\nu_j^{-1} \circ \phi \circ \nu_i$ of class $\mathcal{C}^k$, becomes the map $\varphi_{j',j} \circ (\nu_j^{-1} \circ \phi \circ \nu_i) \circ \varphi_{i,i'}$, which is also of class $\mathcal{C}^k$, since the transition maps $\varphi$'s, are of higher class.

(b) For any open submanifold $\tilde{\boldsymbol{Z}}' \subseteq \boldsymbol{M}'$ such that $\tilde{Z}' \subseteq \tilde{V}$, for some chart $\nu' : V \to \tilde{V}$ of $\boldsymbol{M}'$, one has the tautological equality $\nu'^*(\mathcal{C}^r(\tilde{\boldsymbol{Z}}')) = \mathcal{C}^r(Z')$, where $Z' = \nu'^{-1}(\tilde{Z}')$.

We proceed in the same way in $\boldsymbol{M}$ but we take care that the open submanifold $\tilde{\boldsymbol{Z}}$ verifies in addition the condition $\tilde{\boldsymbol{Z}} \subseteq \phi^{-1}(\tilde{\boldsymbol{Z}}')$.

If we now define $\tilde{\varPhi} : \tilde{\boldsymbol{Z}} \to \tilde{\boldsymbol{Z}}'$ to be the restriction of $\phi$, and then $\varPhi : Z \to Z'$ to be the composition $\varPhi := \nu'^{-1} \circ \phi \circ \nu$, we get the commutative diagram

$$\begin{array}{ccccc} \tilde{\boldsymbol{Z}} & \xrightarrow{\tilde{\varPhi}} & \tilde{\boldsymbol{Z}}' & \qquad \tilde{\varPhi}^*(\mathcal{C}^r(\tilde{\boldsymbol{Z}}')) & \qquad \mathcal{C}^r(\tilde{\boldsymbol{Z}}) \\ {\scriptstyle\nu}\uparrow & & \uparrow{\scriptstyle\nu'} & {\scriptstyle\nu'^*}\| & \|{\scriptstyle\nu^*} \\ Z & \xrightarrow{\varPhi} & Z' & \qquad \varPhi^*(\mathcal{C}^r(Z')) & \subseteq \ \mathcal{C}^r(Z) \end{array}$$

and the equivalence

$$\tilde{\varPhi} \in \mathcal{C}^r(\tilde{\boldsymbol{Z}},\tilde{\boldsymbol{Z}}') \ \Leftrightarrow \ \tilde{\varPhi}^*(\mathcal{C}^r(\tilde{\boldsymbol{Z}}')) \subseteq \mathcal{C}^r(\tilde{\boldsymbol{Z}}). \tag{$*$}$$

But then, since $\mathcal{C}^r(\tilde{\boldsymbol{Z}}') = \iota^*(\mathcal{C}^r(\boldsymbol{M}'))$, where $\iota : \tilde{\boldsymbol{Z}}' \hookrightarrow \boldsymbol{M}'$ is the inclusion map, and that $\big(\tilde{\varPhi} \in \mathcal{C}^r(\tilde{\boldsymbol{Z}},\tilde{\boldsymbol{Z}}')\big) \Leftrightarrow \big(\phi|_{\tilde{\boldsymbol{Z}}} \in \mathcal{C}^r(\tilde{\boldsymbol{Z}},\boldsymbol{M}')\big)$, we can rewrite $(*)$ as

$$\phi|_{\tilde{Z}} \in \mathcal{C}^r(\tilde{Z},\boldsymbol{M}') \ \Leftrightarrow \ (\phi|_{\tilde{Z}})^*(\mathcal{C}^r(\boldsymbol{M}')) \subseteq \mathcal{C}^r(\tilde{Z}), \tag{$\diamond$}$$

and (b) results, since the open submanifolds $\tilde{\boldsymbol{Z}}$ cover $\boldsymbol{M}$, implying that

$$\phi \in \mathcal{C}^r(\boldsymbol{M},\boldsymbol{M}') \ \Leftrightarrow \ \phi|_{\tilde{\boldsymbol{Z}}} \in \mathcal{C}^r(\tilde{\boldsymbol{Z}},\boldsymbol{M}') \qquad (\forall \tilde{\boldsymbol{Z}})$$

$$\phi^*(\mathcal{C}^r(\boldsymbol{M}')) \subseteq \mathcal{C}^r(\boldsymbol{M}) \ \Leftrightarrow \ (\phi|_{\tilde{\boldsymbol{Z}}})^*(\mathcal{C}^r(\boldsymbol{M}')) \subseteq \mathcal{C}^r(\tilde{\boldsymbol{Z}}) \quad (\forall \tilde{\boldsymbol{Z}})$$

(c) Immediate after (b), as in 1.4.6. □

## 2.5 Complete Atlases of a Manifold

As the reader has certainly already understood, the usefulness of atlases comes from their ability to translate local questions on manifolds into equivalent questions but simply between open subspaces of the affine spaces $\mathbb{R}^n$, where one disposes a priori of more tools to find answers. In this sense, an atlas is particularly efficient for obtaining global results by local analysis. However, in practice, while setting up such approaches, it is often the case that the atlas of definition of a manifold, or its complete atlas, are not well adapted to the problem one seeks to solve. The purpose of this section is to show that there is in fact a lot of freedom in the choice of atlases, something that generally helps in making proofs clearer and shorter.

**2.5.1. Equivalent Definition Atlases.** Given two differentiable structures $\boldsymbol{M}_1 := (\boldsymbol{X},\nu(\mathcal{A}_1))$ and $\boldsymbol{M}_2 := (\boldsymbol{X},\nu(\mathcal{A}_2))$ on the same topological space $\boldsymbol{X}$, we say that the definition atlases $\nu(\mathcal{A})$, $\nu(\mathcal{A}')$ are *$\mathcal{C}^r$-equivalent*, and we write $\nu(\mathcal{A}) \sim_r \nu(\mathcal{A}')$, if they define the *same differentiable structure of class $\mathcal{C}^r$*, *i.e.* if the set $\mathcal{C}^r(\boldsymbol{X})$ is the same whether it is defined by $\nu(\mathcal{A})$ or by $\nu(\mathcal{A}')$, or, equivalently (2.4.2-(b)), if the map $\mathrm{id}_{\boldsymbol{X}} : \boldsymbol{M}_1 \to \boldsymbol{M}_2$ is a $\mathcal{C}^r$-diffeomorphism.

A complementary approach to this question is also given by the concept of open covers, as defined in **3♦**7.3. Indeed, the definition atlases $\nu(\mathcal{A}_i) : \boldsymbol{E}(\mathcal{A}_i) \to \boldsymbol{X}$ are open covers of $\boldsymbol{X}$ and the transition maps in the sum $\mathcal{A}_1 \Cup_X \mathcal{A}_2$ (**3♦**7.5.3) are $\mathcal{C}^r$-diffeomorphisms since $\mathrm{id}_{\boldsymbol{X}} : \boldsymbol{M}_1 \to \boldsymbol{M}_2$ is so. Therefore, $\mathcal{A}_1 \Cup_X \mathcal{A}_2$ is an open atlas in $\mathcal{C}^r$**-Opn** and the coproduct of $\nu(\mathcal{A}_i)$, *i.e.* the map

$$\nu(\mathcal{A}_1 \Cup_X \mathcal{A}_2) : \boldsymbol{E}(\mathcal{A}_1 \Cup_X \mathcal{A}_2) \to \boldsymbol{X},$$

defines a differentiable structure of class $\mathcal{C}^r$ on $\boldsymbol{X}$.

**2.5.2 Proposition.** *Let $\boldsymbol{M}_i := (\boldsymbol{X},\nu(\mathcal{A}_i))$, $i = 1,2$, be $\mathcal{C}^r$-manifolds with the same underlying topological space $\boldsymbol{X}$. The following statements are equivalent.*

a) $\mathrm{id}_{\boldsymbol{X}} : \boldsymbol{M}_1 \to \boldsymbol{M}_2$ *is a $\mathcal{C}^r$-diffeomorphism.*

b) $\mathcal{C}^k(\mathcal{M}_1) = \mathcal{C}^k(\boldsymbol{M}_2)$, *for all* $0 \le k \le r$.

c) $\mathcal{C}^r(\boldsymbol{M}_1) = \mathcal{C}^r(\boldsymbol{M}_2)$.

d) $\boldsymbol{M}_{12} := (\boldsymbol{X},\nu(\mathcal{A}_1 \Cup_X \mathcal{A}_2))$ *is a $\mathcal{C}^r$-manifold.*

*Proof.* (a)⇒(b)⇒(c)⇒(a) Obvious. (d)⇒(a) The maps $\mathrm{id}_{\boldsymbol{X}} : \boldsymbol{M}_i \to \boldsymbol{M}_{12}$ are $\mathcal{C}^r$-diffeomorphism by 2.4.2-(a). (a)⇒ (d) By proposition **3♦**7.5.4, and since the transition maps in $\mathcal{A}_1 \Cup_X \mathcal{A}_2$ are $\mathcal{C}^r$-diffeomorphisms. □

**2.5.3. The Complete Definition Atlas.** Once the differential structure has been defined, there exists a canonical definition atlas, known as *the complete definition atlas of class $\mathcal{C}^r$ of $\boldsymbol{M}$*, 'complete' because it contains all the others atlases $\mathcal{C}^r$-equivalent. The definition is quite straightforward: the complete definition atlas consists on *all* the possible $\mathcal{C}^r$-diffeomorphisms $\boldsymbol{U} \to \boldsymbol{V}$ from any open subspace $\boldsymbol{U} \subseteq \mathbb{R}^n$ onto any open submanifold of $\boldsymbol{V} \subseteq \boldsymbol{M}$. The transition maps are then automatically $\mathcal{C}^r$-diffeomorphisms.

**2.5.4 Proposition.** *Given a $\mathcal{C}^r$-manifold $(\mathbf{X}, \nu(\mathcal{A}))$, denote by $\nu(\hat{\mathcal{A}})$ the complete atlas of class $\mathcal{C}^r$ it defines. Then for any sub-atlas $\mathcal{B} \subseteq \hat{\mathcal{A}}$ (3♦7.3.5) the following definition atlases are $\mathcal{C}^r$-equivalent:*

$$\nu(\mathcal{A}) \sim_r \nu(\hat{\mathcal{A}}) \sim_r \nu(\mathcal{B}).$$

*Proof.* This is a straightforward application of 2.4.2-(a) to the map $\mathrm{id}_{\mathbf{X}}$ between the three manifolds $(\mathbf{X}, \nu(\mathcal{A}))$, $(\mathbf{X}, \nu(\hat{\mathcal{A}}))$ and $(\mathbf{X}, \nu(\mathcal{B}))$. □

**2.5.5 Comment.** Thanks to these statement, we can replace the definition atlas of a manifold by any sub-atlas of it complete definition atlas without altering the differential structure. We will son see the usefulness of this idea in the proof of proposition 2.6.2.

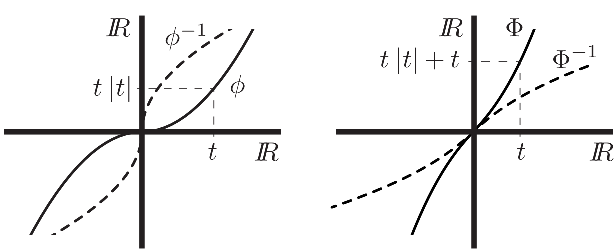
**2.5.6 Exercise.** Define the *top class* of an open atlas $\mathcal{A}$ in $\mathcal{C}^0$-$\mathbf{Opn}(n)$ as the upper bound of the top classes of the transition maps of $\mathcal{A}$ (1.3.1-(d)).

1) Give examples of an open atlas $\mathcal{A}$ of top class strictly lower than that of a sub-atlas $\mathcal{B} \subseteq \mathcal{A}$.
2) Define a manifold $\boldsymbol{M} := (\mathbf{X}, \nu(\mathcal{A}))$ such that for some open subspace $\emptyset \neq \boldsymbol{U} \subseteq \boldsymbol{M}$, the top class of $\mathcal{A} \Cap U$ is strictly higher than that of $\mathcal{A}$.
3) However, show that if $\mathcal{A}$ is the complete atlas of class $\mathcal{C}^r$ of a manifold $\boldsymbol{M}$, then the restricted atlases $\mathcal{A} \Cap U$ are of top class $r$, for all $U \neq \emptyset$. *Hint. In dimension $n > 0$, you have to use the fact that for any open subset $U \subseteq \mathbb{R}^n$, the inclusion $\mathcal{C}^r(U) \supseteq \mathcal{C}^{r+1}(U)$ is strict.*

**2.5.7 Discussion.** As the last exercise suggests, the definition of manifold has more subtleties than appears at first sight. Let us look more closely the example the real line $\mathbb{R}$.

– The function $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(t) = t|t|$, is strictly increasing and its derivative is the continuous function $t \mapsto 2|t|$, so that $\phi$ is a homeomorphism of top class $\mathcal{C}^1$, as it is not two times derivable. On the other hand, the function $\phi^{-1}$ is not derivable at the origin so that it is of top class $\mathcal{C}^0$.

– The function $\varPhi : \mathbb{R} \to \mathbb{R}$, $\varPhi(t) = \phi(t) + t$ is also a homeomorphism of top class $\mathcal{C}^1$, and, since $\partial_t \varPhi > 0$, the inverse function $\varPhi^{-1}$ is also of top class $\mathcal{C}^1$.




We now define a structure of manifold $\boldsymbol{M}$ on $\mathbb{R}$ in five different ways.

*Only one chart*: • $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \boldsymbol{M}_1$; • $\phi : \mathbb{R} \to \boldsymbol{M}_2$; • $\Phi : \mathbb{R} \to \boldsymbol{M}_3$.

*Two charts*: • $\nu_0 = \mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \boldsymbol{M}_4$, $\nu_1 = \phi : \mathbb{R} \to \boldsymbol{M}_4$, $\varphi_{10} = \phi^{-1}$,

$$\begin{tikzcd} \mathbb{R} \arrow[d, "\varphi_{10}"', "\phi^{-1}"] \arrow[dr, "\nu_0=\mathrm{id}_{\mathbb{R}}"] & \\ \mathbb{R} \arrow[r, "\nu_1=\phi"'] & \boldsymbol{M}_4 \end{tikzcd}$$

*Two charts*: • Same, replacing $\phi$ by $\Phi$ and $\boldsymbol{M}_4$ by $\boldsymbol{M}_5$.

**Exercise.** Prove the following statements.

- $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{M}_3$, are $\mathcal{C}^\infty$-diffeomorphic $\mathcal{C}^\infty$-manifolds, two by two different, *i.e.* the identity map between them is not a $\mathcal{C}^\infty$-diffeomorphism.
- $\boldsymbol{M}_4$ is of top class $\mathcal{C}^0$. • $\boldsymbol{M}_5$ is of top class $\mathcal{C}^1$.

### 2.6 Local Diffeomorphisms, Coverings and Quotients

Following the same approach as in chapters **3** and **5**, where we saw the interest of the concepts of local homeomorphisms and of topological coverings in relation with group quotients (see **5**♦1.2, **5**♦6.5), we now extend these notions to the context of manifolds. The definitions are *mutatis mutandis* the same, it suffices to replace the words: 'toplogical or pseudomeric space' by 'manifold', 'continuous map' by 'differentiable map', 'open subspace' by 'open submanifold', 'homeomorphism' by 'diffeomorphism', etc.

**2.6.1 Definitions.** • A differentiable map between manifolds $\phi : \boldsymbol{M} \to \boldsymbol{M}'$, is said to be *a local diffeomorphism*, if for every $x \in \boldsymbol{M}$ there exists an open submanifold $\boldsymbol{V}_x \ni x$ such that $\boldsymbol{V}_{\phi(x)} := \phi(\boldsymbol{V}_x)$ is open in $\boldsymbol{Y}$, and $\phi|_{\boldsymbol{V}_x} : \boldsymbol{V}_x \to \boldsymbol{V}_{\phi(x)}$ is a diffeomorphism (see **3**♦7.1).

- The map $\phi : \boldsymbol{M} \to \boldsymbol{M}'$ is *a differential covering* if, for every $b \in \boldsymbol{M}'$ there exists an open submanifold $\boldsymbol{V}_b \ni b$ such that the inverse image $\phi^{-1}(\boldsymbol{V}_b)$ is a family of disjoint open submanifolds $\{\boldsymbol{V}_x \ni x\}$, indexed by $x \in \phi^{-1}(b)$, such that the restrictions $\phi|_{V_x} : \boldsymbol{V}_x \to \boldsymbol{V}_b$ are *diffeomorphisms*, for all $x \in F_b$. We call such $\boldsymbol{V}_b$ *a trivializing open submanifold for* $\phi$ (*cf.* **5**♦1.2).
- The set $\mathcal{C}^k$-Diffeom$(\boldsymbol{M})$ of $\mathcal{C}^k$-diffeomorphisms of a manifold $\boldsymbol{M}$ is a subgroup of the group of homeomorphisms of $\boldsymbol{M}$, after 2.4.2-(c). *An action of a group $\boldsymbol{G}$ on $\boldsymbol{M}$ by $\mathcal{C}^k$-diffeomorphisms*, is then a *group homomorphism*

$$\rho : \boldsymbol{G} \to \mathcal{C}^k\text{-Diffeom}(\boldsymbol{X})$$

(see **5**♦6.5.3). When $\rho$ is understood, we will simply write $g(x)$ for $\rho(g)(x)$.

– The action is said to be *properly discontinuous without fixed points*, if, for every $x \in \boldsymbol{M}$ there exists an open submanifold $\boldsymbol{V}_x \ni x$ such that: $g(\boldsymbol{V}_x) \cap \boldsymbol{V}_x \neq \emptyset$ if and only if $g = \mathbf{1}_{\boldsymbol{G}}$ (see fn. page 121). We call such $\boldsymbol{V}_x$ a *$\boldsymbol{G}$-distinguished open submanifold of $\boldsymbol{M}$.*

Notice that if $\boldsymbol{M}$ is a coproduct of manifolds $\boldsymbol{M}_i$ of different dimensions, a differential action stabilizes each $\boldsymbol{M}_i$ (*cf.* 1.4.8-(d), 2.2.2).

The next theorem summarizes results already proved in the context of topological and pseudometric spaces. The content of sections **5♦**1.2.6–10 cover the subject and we recommend to reread them keeping in mind the new context of differential objets.

**2.6.2 Theorem** a) *Let $\phi : \boldsymbol{X} \to \boldsymbol{M}'$ be a local homeomorphism onto a $\mathcal{C}^r$-manifold $\boldsymbol{M}'$. There exists one and only one structure of $\mathcal{C}^r$-manifold on $\boldsymbol{X}$ such that $\phi : \boldsymbol{M} \to \boldsymbol{M}'$ is a local $\mathcal{C}^r$-diffeomorphism.*

b) *Let $\boldsymbol{G}$ be a group acting by $\mathcal{C}^k$-diffeomorphisms and properly discontinuously without fixed points on a $\mathcal{C}^r$-manifold $\boldsymbol{M}$.*

i) *There exists one and only one structure of $\mathcal{C}^k$-manifold on the topological quotient $(\boldsymbol{M}/\boldsymbol{G})_{\mathrm{top}}$ such that the quotient map $\nu_{\boldsymbol{G}} : \boldsymbol{M} \to \boldsymbol{M}/\boldsymbol{G}$ is a $\mathcal{C}^k$-differential covering.*

ii) *Let $\mathcal{C}^k(\boldsymbol{M})^{\boldsymbol{G}}$ denote the subalgebra of $\boldsymbol{G}$-invariant functions of $\boldsymbol{M}$,* i.e. *functions $f : \boldsymbol{M} \to \mathbb{R}$ such that $f(g(x)) = f(x)$, for all $x \in \boldsymbol{M}$ and $g \in \boldsymbol{G}$. The pullback map $\nu_{\boldsymbol{G}}^* : \mathcal{C}^k(\boldsymbol{M}/\boldsymbol{G}) \to \mathcal{C}^k(\boldsymbol{M})$ is injective and induces an algebra isomorphism*

$$\nu_{\boldsymbol{G}}^* : \mathcal{C}^k(\boldsymbol{M}/\boldsymbol{G}) \xrightarrow[\simeq]{} \mathcal{C}^k(\boldsymbol{M})^{\boldsymbol{G}}$$

*Proof.* All the statements are simple applications of proposition 2.5.4.

(a) Take as definition atlas $\mathcal{A}'$ of $\boldsymbol{M}'$ to be the atlas of open submanifolds $\boldsymbol{V} \subseteq \boldsymbol{M}'$ which are both: trivializing for $\phi$ and diffeormorphic to some open subspace of $\mathbb{R}^n$. The pullback atlas $\mathcal{A} := \phi^*(\mathcal{A}')$ (*cf.* **5♦**1.2.6) then defines on $\boldsymbol{X}$ a differential structure of $\mathcal{C}^r$-manifold $\boldsymbol{M}$, satisfying the required properties. Any other such differential structure on $\boldsymbol{X}$ would be locally diffeormorphic to $\boldsymbol{M}'$, hence to $\boldsymbol{M}$, therefore, the same as $\boldsymbol{M}$.

(b-i) Because of the hypothesis on the action of $\boldsymbol{G}$, the map $\nu_{\boldsymbol{G}}$ is a covering. Let $\nu(\mathcal{A})$ be the defining atlas of $\boldsymbol{M}$ that consists of all $\boldsymbol{G}$-distinguished submanifolds $\boldsymbol{V} \subseteq \boldsymbol{M}$ which are diffeormorphic to some open subspace of $\mathbb{R}^n$. The composition $\nu_{\boldsymbol{G}} \circ \nu(\mathcal{A}) : \boldsymbol{E}(\mathcal{A}) \to (\boldsymbol{M}/\boldsymbol{G})$top is an open cover and it defines a structure of manifold of class $\mathcal{C}^k$ on $(\boldsymbol{M}/\boldsymbol{G})$top if and only if the transition maps are of class $\mathcal{C}^k$. Let us prove this.

For $\boldsymbol{G}$-distinguished open charts $\boldsymbol{V}_i \subseteq \boldsymbol{M}$, denote by $\tilde{\boldsymbol{V}}_i$ the (open) image $\nu_{\boldsymbol{G}}(\boldsymbol{V}_i)$. The overlapping subspace $\tilde{\boldsymbol{V}}_i \cap \tilde{\boldsymbol{V}}_j$ is homeomorphic, through the canonical surjection $\nu_{\boldsymbol{G}}$, to the open subspace $\boldsymbol{V}_j^i := \boldsymbol{V}_j \cap (\boldsymbol{G}\boldsymbol{V}_i)$ and since $\boldsymbol{V}_i$ is distinguished, it has a disjoint open decomposition indexed by $g \in \boldsymbol{G}$:

$$\boldsymbol{V}_j^i = \coprod_{g \in \boldsymbol{G}} \boldsymbol{V}_j \cap g\boldsymbol{V}_i.$$

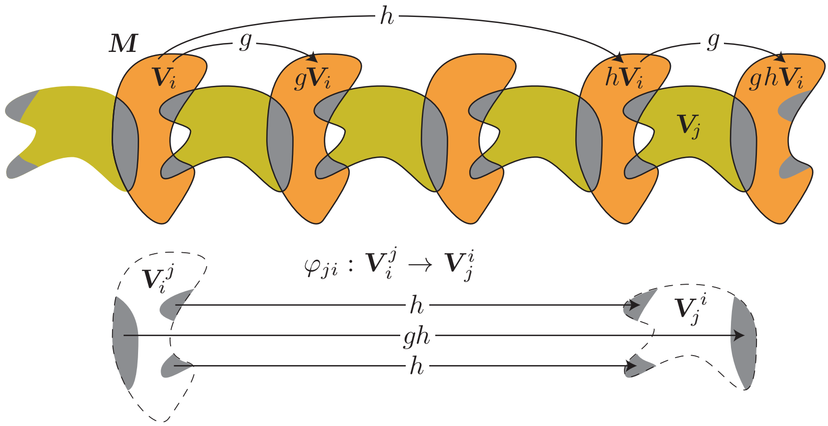
Exchanging $j \leftrightarrow i$ and $g \leftrightarrow g^{-1}$, we get in the same way the decomposition

$$\boldsymbol{V}_i^j := \boldsymbol{V}_i \cap (\boldsymbol{G}\boldsymbol{V}_j) = \coprod_{g \in \boldsymbol{G}} g^{-1}\boldsymbol{V}_j \cap \boldsymbol{V}_i.$$

Hence, the overlapping condition between $\tilde{\boldsymbol{V}}_i$ and $\tilde{\boldsymbol{V}}_j$ is read on $\boldsymbol{V}_i$ and $\boldsymbol{V}_j$ as a transition map $\varphi_{ji} : \boldsymbol{V}_i^j \to \boldsymbol{V}_j^i$ given by the coproduct of the maps

$$g : \left(g^{-1}\boldsymbol{V}_j \cap \boldsymbol{V}_i\right) \to \left(\boldsymbol{V}_j \cap g\boldsymbol{V}_i\right),$$

each being of class $\mathcal{C}^k$ by hypothesis. The definition atlas $(\nu_{\boldsymbol{G}} \circ \nu)(\mathcal{A})$ is therefore of this class $\mathcal{C}^k$.




The uniqueness of the differential structure in $\boldsymbol{M}/\boldsymbol{G}$ results, as in (a), from the fact that the structure is locally, therefore globally, determined by that of $\boldsymbol{M}$, through the topological covering $\nu_{\boldsymbol{G}} : \boldsymbol{M}$
*onto*$\boldsymbol{M}/\boldsymbol{G}$.

(b-ii) Left to the reader. □

### 2.7 The category of $\mathcal{C}^r$-manifolds

We are now at the same point where we were in 2.1, but now, instead of open sets $\boldsymbol{U}$, we have $\mathcal{C}^r$-manifolds $\boldsymbol{M}$, algebras of differentiable functions $\mathcal{C}^r(\boldsymbol{M})$, and sets of differentiable maps $\mathcal{C}^r(\boldsymbol{M},\boldsymbol{M}')$. And, again, the identity maps are of class $\mathcal{C}^r$, and the composition of maps of class $\mathcal{C}^r$ are still of the same class.

**2.7.1 Definition.** The *category of $\mathcal{C}^r$-manifolds, denoted by $\mathcal{C}^r$-***Man**, is the category whose objects are the $\mathcal{C}^r$-manifolds $\boldsymbol{M}$ and where the morphisms are the maps $\phi \in \mathcal{C}^r(\boldsymbol{M},\boldsymbol{M}')$. The underlying topological space $\sigma\boldsymbol{M}$ is $\boldsymbol{M}_{\mathrm{top}}$, and $\mathcal{C}^r$-**Man** is a category of spaces in the sense of **2**♦2.4.

A manifold of dimension 1 is called *a (differentiable) curve*, and *a (differentiable) surface* if it is of dimension 2.

**Conventions.** As in the definitions of manifolds and of differentiable maps (see 2.2.1, 2.4.1), the expression *category of manifolds* and the notation **Man**, without the specification of the differentiability class will will have a default signification, that of $\mathcal{C}^\infty$-manifolds.

**2.7.2 Comments.** a) When defining differentiable structures, it is frequent to add more topological constraints, for example: to be Hausdorff, countably infinite, etc. We will come back on these later, but for the moment they are not necessary and we will not dwell on them.

b) In dimension 1, the open subsets of $\mathbb{R}$ are unions of open intervals, and, each being obviously diffeomorphic to $\mathbb{R}$, the domains of the charts can naturally taken to be all equal to $\mathbb{R}$. The situation is so simple that it is not difficult to prove that *any* connected differentiable curve is isomorphic either to the real line $\mathbb{R}$ or to the circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$.

The classification of manifolds up to isomorphism is a very difficult problem, and even, at higher dimensions, hopeless. Above dimension 1, there are quite broad answers in dimensions 2 and 3 related to Perelman's (6) results on the *Thurston's geometrization conjecture* and the *Poincaré conjecture*, but in higher dimensions the question is *undecidable*, which means that there cannot exist any effective procedure to decide wether any two manifolds are isomorphic or not. (7)

**2.7.3 Theorem.** a) *The category* **Man** *has coproducts, finite products, open subspaces, open assemblages, and quotients by groups acting by properly discontinuous free actions.*

b) *The category* $\mathcal{C}^r$**-Opn** *is a full subcategory of* $\mathcal{C}^r$**-Man**.

*Proof.* Corollary of 2.6.2. Details are left to the reader. □

**2.7.4 Comment.** One last remark before leaving this section. It is worth noting that if the concept of manifold catches the idea of differentiability, it is still not sufficient to catch the concept of differential operator, and therefore to study solutions of partial differential equations.

## 3 Differential Operators of Finite Order

## 4 The Laplacian on $\mathbb{R}^n$

**4.1 Le groupe des isométries.** On considère $\mathbb{R}^n$ muni du produit scalaire canonique $\langle \_,\_ \rangle$ et de la structure euclidienne correspondante (??). On note

- $\mathbb{I}\mathrm{so}(n,\mathbb{R})$, le "*groupe des isométries de $\mathbb{R}^n$*".

(6) Fields medal 2006.

(7) Excerpt from Wikipedia article "Differentiable manifold": *The classification of $n$-manifolds for $n$ greater than three is known to be impossible, even up to homotopy equivalence. Given any finitely presented group, one can construct a closed 4-manifold having that group as fundamental group. Since there is no algorithm to decide the isomorphism problem for finitely presented groups, there is no algorithm to decide if two 4-manifolds have the same fundamental group. [...] In addition, since even recognizing the trivial group is undecidable, it is not even possible in general to decide if a manifold has trivial fundamental group, i.e. if it is simply connected.*

- $O(n) := \mathrm{Stab}_{\mathbb{I}\mathrm{so}(n)}(\vec{0})$, le "*groupe orthogonal de $I\!R^n$*", ses éléments sont les applications *linéaires* orthogonales (voir **9**♦1.0.1).
- $\mathbb{T}(n,I\!R)$, le "*groupe des translations de $I\!R^n$*". Pour $\vec{v} \in I\!R^n$, la "*translation (de vecteur $\vec{v}$)*" est l'isométrie $\tau_{\vec{v}} : \vec{x} \mapsto \vec{x} + \vec{v}$. On a

$$\tau_{\vec{0}} = \mathrm{id}_{I\!R_n}\,, \qquad \tau_{\vec{v}} \circ \tau_{\vec{w}} = \tau_{\vec{v}+\vec{w}}\,, \qquad \theta\tau_{\vec{v}}\theta^{-1} = \tau_{\theta(\vec{v})}\,,\ \forall\theta \in O(n),$$

**4.1.1 Proposition.** a) *Le groupe des translations $\mathbb{T}(n,I\!R)$ est distingué dans $\mathbb{I}\mathrm{so}(n,I\!R)$.*

b) *L'application*

$$\Psi : O(n) \times \mathbb{T}(n,I\!R) \to \mathbb{I}\mathrm{so}(n,I\!R), \quad (\theta,\tau) \mapsto \theta \circ \tau,$$

*est bijective et c'est un isomorphisme de groupes si l'on munit $O(n) \times \mathbb{T}(n,I\!R)$ de la loi de composition interne:*

$$(\theta',\tau_{\vec{x}})(\theta,\tau_{\vec{y}}) := (\theta'\theta,\tau_{\theta^{-1}(\vec{x})+\vec{y}})$$

**4.2 L'algèbre $\mathbf{End}_{I\!R}(\mathcal{C}^\infty(I\!R^n))$.** Étant donné un $I\!R$-espace vectoriel $V$, l'ensemble $\mathrm{End}_{I\!R}(V)$ des ses "*endomorphismes $I\!R$-linéaires*"([8]) est muni de trois opérations: le "*produit par des scalaires* '$\cdot$'", l'"*addition* '$+$'" et la "*composition* '$\circ$'". Ces opérations sont telles que

- $(\mathrm{End}_{I\!R}(V),+,0,\cdot,1)$ est un espace vectoriel sur $I\!R$,
- $(\mathrm{End}_{I\!R}(V),+,0,\circ,\mathrm{id}_V)$ est un anneau,
- pour tous $\lambda \in I\!R$, $\alpha,\beta \in \mathrm{End}_{I\!R}(V)$, on a $\lambda\cdot(\alpha\circ\beta) = (\lambda\cdot\alpha)\circ\beta = \alpha\circ(\lambda\cdot\beta)$.
- la composition '$\circ$' est "*associative*", *i.e.* $\alpha\circ(\beta\circ\gamma) = (\alpha\circ\beta)\circ\gamma$

On dit alors que $(\mathrm{End}_{I\!R}(V),+,0,\circ,\mathrm{id}_V,\cdot,1)$ est une "*$I\!R$-algèbre associative*".

Nous allons nous intéresser maintenant à l'algèbre $(\mathrm{End}_{I\!R}(\mathcal{C}^\infty(I\!R^n)),+,\circ,\cdot)$ des endomorphismes de l'espace vectoriel $\mathcal{C}^r(I\!R^n)$ (**??**). Voici quelques fanilles importantes de ses éléments.

a) Pour tout $h \in \mathcal{C}^\infty(I\!R^n)$, la "*multiplication par $h$*" (*cf.* **??**)

$$\mu_h : \mathcal{C}^\infty(I\!R^n) \to \mathcal{C}^\infty(I\!R^n), \quad f \mapsto h\cdot f,\ \forall f \in \mathcal{C}^\infty(I\!R^n).$$

b) Pour tout $\alpha \in \mathcal{C}^\infty(I\!R^n,I\!R^n)$, la "*composition par $\alpha$*"

$$\alpha^\natural : \mathcal{C}^\infty(I\!R^n) \to \mathcal{C}^\infty(I\!R^n), \quad \alpha^\natural(f) = f\circ\alpha,\ \forall f \in \mathcal{C}^\infty(I\!R^n).$$

Si $\alpha,\beta \in \mathcal{C}^\infty(I\!R^n,I\!R^n)$, on a (*cf.* **??**)

$$(\beta\circ\alpha)^\natural = \alpha^\natural\circ\beta^\natural. \tag{1}$$

([8]) C'est-à-dire, des application $I\!R$-linéaires de $V$ dans lui-même.

c) Pour tout $\vec{v} \in I\!R^n$, la "*dérivée directionnelle de direction $\vec{v}$*"

$$\partial_{\vec{v}} : \mathcal{C}^\infty(I\!R^n) \to \mathcal{C}^\infty(I\!R^n).$$

Si $f \in \mathcal{C}^\infty(I\!R^n)$, la fonction $\partial_{\vec{v}}(f)$ associe à $\vec{x} \in I\!R^n$ la valeur

$$\partial_{\vec{v}} f(\vec{x}) := \frac{d}{dt} f(\vec{x} + t\vec{v})|_{t=0}.$$

La fonction $\partial_{\vec{v}}(f) : I\!R^n \to I\!R$ est encore $\mathcal{C}^\infty$ et $\partial_{\vec{v}}$ est bien $I\!R$-linéaire (6.1.2).

d) Les "*dérivées partielles (canoniques)*". Si $\{\vec{\mathbf{e}}_i\}_{i=1,\dots,n}$ désigne la base canonique de $I\!R^n$, la "*est la i-ième dérivée partielle*" est

$$\partial_i := \partial_{\vec{\mathbf{e}}_i} : \mathcal{C}^\infty(I\!R^n) \to \mathcal{C}^\infty(I\!R^n).$$

e) Le "*Laplacien*"

$$\boxed{\Delta := \partial_1^2 + \cdots + \partial_n^2 : \mathcal{C}^\infty(I\!R^n) \to \mathcal{C}^\infty(I\!R^n)} \tag{2}$$

**4.2.1 Proposition.** *L'application*

$$\partial : I\!R^n \to \mathrm{End}_{I\!R}(\mathcal{C}^\infty(I\!R^n)), \quad \vec{v} \mapsto \partial_{\vec{v}}$$

*est une injection* $I\!R$-linéaire*. En particulier, si $\vec{v} = x_1\vec{\mathbf{e}}_1 + \cdots + x_n\vec{\mathbf{e}}_n$, on a:*

$$\boxed{\partial_{\vec{v}} = x_1\partial_1 + \cdots + x_n\partial_n} \tag{3}$$

*Hint* Pour la linéarité voir 6.1.2. Pour l'injectivité, montrer l'égalité $\partial_{\vec{v}}(\gamma_j) = x_j$, où $\gamma_j$ est la fonction $j$-ème coordonnée $\gamma_j : I\!R^n \to I\!R$, $\vec{x} \mapsto x_j$. □

**4.3 L'algèbre Diff$_{I\!R}(I\!R^n)$.** Dans ces notes, on entendra par "*opérateur différentiel sur $I\!R^n$*" tout élément de la sous-algèbre 'Diff$_{\boldsymbol{R}}(I\!R^n)$' de $\mathrm{End}_{I\!R}(\mathcal{C}^\infty(I\!R^n))$ engendrée par les multiplications (a) et par les dérivées partielles (d).

**Conventions notionnelles.** Dans le but d'alléger les notations, il est d'usage de ne pas indiquer les opérations de multiplication par scalaires '$\cdot$' et de composition '$\circ$' dans l'algèbre Diff$_{\boldsymbol{R}}(I\!R^n)$ et de simplifier le parenthésage autant que les règles de l'algèbre le permettent. L'écriture $\lambda \cdot (\alpha \circ \beta)$ devient donc $\lambda\alpha\beta$.

De même, pour $h \in \mathcal{C}^\infty(I\!R^n)$, la multiplication $\mu_h \in \mathrm{End}_{I\!R}(\mathcal{C}^\infty(I\!R^n))$ est simplement appelé "fonction" et noté "$h$", le contexte étant suffisamment clair pour éliminer toute ambiguïté

Le lemme suivant tient compte de ces conventions et rappelle les principales caractéristiques de cette sous-algèbre.

**4.3.1 Lemma.** a) *Règle de dérivation de Leibniz. Pour tout* $\vec{v} \in I\!R^n$ *et toute* $f \in \mathcal{C}^\infty(I\!R^n)$, *on a*

$$\partial_{\vec{v}} f - f\partial_{\vec{v}} = \partial_{\vec{v}}(f). \tag{4}$$

b) *Pour tous* $\vec{v},\vec{v}' \in I\!R^n$, *on a*

$$\partial_{\vec{v}}\partial_{\vec{v}'} - \partial_{\vec{v}'}\partial_{\vec{v}} = 0.$$

c) *Un élément* $P \in \mathrm{Diff}_{I\!R}(I\!R^n)$ *s'écrit d'une et d'une seule manière comme polynôme en* $\{\partial_1,\dots,\partial_n\}$ *à coefficients dans* $\mathcal{C}^\infty(I\!R^n)$, *soit*

$$P = \sum\nolimits_{(i_1,\dots,i_n)\in\mathbb{N}^n} f_{i_1,\dots,i_n}\partial_1^{i_1}\cdots\partial_n^{i_n}.$$

*où* $f_{i_1,\dots,i_n} \in \mathcal{C}^\infty(I\!R^n)$ *et* $f_{i_1,\dots,i_n} = 0$ *pour presque tout* $(i_1,\dots,i_n) \in \mathbb{N}^n$. *On résume ce résultat par la notation*

$$\mathrm{Diff}_{I\!R}(I\!R^n) = \mathcal{C}^\infty(I\!R^n)[\partial_1,\dots,\partial_n]. \tag{5}$$

*où l'on rappelle que* $\mathcal{C}^\infty(I\!R^n)$ *possède la structure d'un anneau (??).*

*Proof* Exercice. *Indication pour* (c): Raisonner par induction sur le nombre des dérivées partielles intervenant dans un monôme $T_1T_2\cdots T_r$, où $T_j$ désigne soit une fonction soit l'un des opérateurs $\partial_i$. La règle de Leibniz permet la descente inductive. □

**4.3.2 Remark.** Il faut bien retenir le fait que contrairement au produit des fonctions dans $\mathcal{C}^\infty(I\!R^n)$ et à la composition des dérivées partielles qui sont des opérations commutatives, le produit d'une fonction et d'une dérivation ne l'est pas: il obéit à la règle de Leibniz (4). La notation (5) ne doit donc pas induire cette erreur par une analogie hâtive avec l'algèbre des polynômes.

**4.3.3 Exercise.** Soient $(i_1,\dots,i_n)$ et $(j_1,\dots,j_n)$ des $n$-uplets d'entiers naturels On note $X_1,\dots,X_n$ les "*fonctions coordonnées*" sur $I\!R^n$, *i.e.* $X_i(\vec{x}) = x_i$. Montrer que si $i_k \le j_k$ pour $k = 1,\dots,n$, on a

$$\frac{\partial_1^{i_1}}{i_1!}\cdots\frac{\partial_n^{i_n}}{i_n!}(X_1^{j_1}\cdots X_n^{j_n}) = \binom{j_1}{i_1}\cdots\binom{j_n}{i_n}X_1^{j_1-i_1}\cdots X_n^{j_n-i_n}.$$

Que devient l'égalité si $i_k > j_k$ pour l'un des $k$ ?

Les fractions $\partial^i/i!$ sont appelées "*puissances divisées*", elles jouent un rôle central dans diverses disciplines mathématiques.

**4.4 L'algèbre $\mathrm{Diff}_{I\!R}(U)$.** Soit $U$ un ouvert de $I\!R^n$. Dans la liste d'opérations (D1-D5), la multiplication par une fonction, les dérivées partielles, et donc le Laplacien, opèrent également sur les fonctions de $\mathcal{C}^\infty(U)$, quel que soit l'ouvert $U \subseteq I\!R^n$. De ce fait, on a une application naturelle de restriction

$$\mathrm{Diff}_{I\!R}(I\!R^n) \to \mathrm{Diff}_{I\!R}(U).$$

Le lemme 4.3.1 s'applique également à cette situation et donne l'égalité

$$\mathrm{Diff}_{I\!R}(U) = \mathcal{C}^\infty(U)[\partial_1,\ldots,\partial_n].$$

**4.4.1 Exercise.** Montrer que pour que dans $\mathrm{Diff}_{I\!R}(U)$ une fonction commute aux dérivées partielles, il faut et il suffit qu'elle soit localement constante.

**4.5 Action de $\mathbf{Aff}(n,I\!R)$ sur $\mathbf{Diff}_{I\!R}(I\!R^n)$.** Un cas particulièrement important d'opérateurs du type (b) est fourni par les automorphismes *affines* de $I\!R^n$, *i.e.* par les composées des éléments $\theta \in \mathrm{Gl}(n,I\!R)$: le "*groupe général linéaire de $I\!R^n$*", et des translations $\tau_{\vec{v}} \in \mathbb{T}(n,I\!R)$: le "*groupe des translations de $I\!R^n$*". Une telle bijection $\phi = \theta\circ\tau_{\vec{v}}$ est évidemment de classe $\mathcal{C}^\infty$ (exercice) de même que son inverse $\phi^{-1}$. L'élément $\phi^\natural \in \mathrm{End}_{I\!R}(I\!R^n)$ est donc inversible dans cette algèbre.

**Lemma.** *Soit* $\phi \in \mathrm{Aff}(n,I\!R)$.

a) *Pour tout* $h \in \mathcal{C}^\infty(I\!R^n)$, $(\phi^{-1})^\natural \circ \mu_h \circ \phi^\natural = \mu_{h\circ\phi}$.

b) *Pour tout* $\vec{v} \in I\!R^n$, $\begin{cases}(\phi^{-1})^\natural \circ \partial_{\vec{v}} \circ \phi^\natural = \partial_{\phi(\vec{v})} & \textit{si } \phi \in \mathrm{Gl}(n,I\!R),\\ (\phi^{-1})^\natural \circ \partial_{\vec{v}} \circ \phi^\natural = \partial_{\vec{v}} & \textit{si } \phi \in \mathbb{T}(n,I\!R).\end{cases}$

c) *L'application*

$$(\phi^{-1})^\natural\circ(\_)\circ\phi^\natural : \mathrm{End}_{I\!R}(\mathcal{C}^\infty(I\!R^n)) \to \mathrm{End}_{I\!R}(\mathcal{C}^\infty(I\!R^n)), \quad \varphi \mapsto (\phi^{-1})^\natural\circ\varphi\circ\phi^\natural$$

*est un automorphisme d'algèbre qui stabilise* $\mathrm{Diff}_{I\!R}(I\!R^n)$, i.e.

$$(\phi^{-1})^\natural \circ \mathrm{Diff}_{I\!R}(I\!R^n) \circ \phi^\natural = \mathrm{Diff}_{I\!R}(I\!R^n)$$

*Proof.* (b). On applique $(\theta^{-1})^\natural \circ \partial_{\vec{v}} \circ \theta^\natural$ à une fonction $f \in \mathcal{C}^\infty(I\!R^n)$:

$$\begin{aligned}\big((\theta^{-1})^\natural \circ \partial_{\vec{v}} \circ \theta^\natural\big)(f)(\vec{x}) &= \big(\partial_{\vec{v}} \circ \theta^\natural\big)(f)(\theta^{-1}(\vec{x}))\\ &= \frac{d}{dt}\theta^\natural(f)(\theta^{-1}(\vec{x}) + t\vec{v})\Big|_{t=0} = \frac{d}{dt}(f)(\vec{x} + t\theta(\vec{v}))\Big|_{t=0}\\ &= \partial_{\theta(\vec{v})}(f)(\vec{x})\end{aligned}$$

Le reste du lemme est laissé en exercice. □

**4.6 Action de $\mathbf{Gl}(n,I\!R)$ sur le Laplacien.** Comme nous l'avons remarqué dans l'exercice 3.1, l'ensemble $\{\partial_1,\ldots,\partial_n\}$ est linéairement indépendant dans $\mathrm{Diff}_{I\!R}(I\!R^n)$ et engendre un sous-espace vectoriel de dimension $n$ noté $\partial_{I\!R^n}$. On exprime les éléments de $\partial_{I\!R^n}$ par l'écriture matricielle habituelle

$$\partial_{\vec{v}} = x_1\partial_1 + \cdots + x_n\partial_n = (x_1,\ldots,x_n)\begin{pmatrix}\partial_1\\ \vdots\\ \partial_n\end{pmatrix}.$$

L'assertion (b) du lemme 4.5, exprime alors le fait que

$$(\theta^{-1})^\natural \circ \partial_{\vec{v}} \circ \theta^\natural = \partial_{\theta(\vec{v})} = \theta(x_1,\dots,x_n)\begin{pmatrix}\partial_1\\ \vdots\\ \partial_n\end{pmatrix},$$

et en prenant $\vec{v} = \vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_n$, on obtient l'égalité

$$\begin{pmatrix}(\theta^{-1})^\natural \circ \partial_1 \circ \theta^\natural\\ \vdots\\ (\theta^{-1})^\natural \circ \partial_n \circ \theta^\natural\end{pmatrix} = {}^t((\theta_{ij}))\begin{pmatrix}\partial_1\\ \vdots\\ \partial_n\end{pmatrix}, \tag{6}$$

utile pour démontrer la proposition suivante.

**4.6.1 Theorem.** a) *Si $\theta \in \mathrm{Gl}(n,I\!R)$, on a*

$$(\theta^{-1})^\natural \circ \Delta \circ \theta^\natural = (\partial_1,\dots,\partial_n)\ ((\theta_{ij}))\,{}^t((\theta_{ij}))\begin{pmatrix}\partial_1\\ \vdots\\ \partial_n\end{pmatrix}$$

b) *Si $\vec{v} \in I\!R^n$, on a*

$$\tau_{-\vec{v}}^\natural \circ \Delta \circ \tau_{\vec{v}}^\natural = \Delta$$

c) *Le Laplacien commute aux isométries de $I\!R^n$*

$$\boxed{\phi^\natural \circ \Delta = \Delta \circ \phi^\natural, \quad \forall \phi \in \mathbb{I}\mathrm{so}(n,I\!R)}$$

d) *Une isométrie $\phi : U \to V$ entre deux ouverts de l'espace euclidien $I\!R^n$ induit l'application $I\!R$-linéaire $\phi^\natural : \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(U)$. On a alors*

$$\boxed{\phi^\natural \circ \Delta_V = \Delta_U \circ \phi^\natural, \quad \forall \phi \in \mathbb{I}\mathrm{so}(U,V)}$$

*où l'on a noté $\Delta_U$ et $\Delta_V$ les restrictions respectives du Laplacien (4.4).*

*Hint* On écrit le Laplacien sous forme de produit matriciel

$$\Delta = (\partial_1,\dots,\partial_n)\begin{pmatrix}\partial_1\\ \vdots\\ \partial_n\end{pmatrix}$$

on utilise alors le fait que $\phi^\natural \circ (\_) \circ (\phi^{-1})^\natural$ est un automorphisme d'algèbre, on applique (6) et le fait que l'on a $\theta\,{}^t\theta = \mathrm{id}$ pour tout $\theta \in O(n)$. □

**4.6.2 Corollary.** *Pour toute base orthonormée $\{\vec{\mathbf{w}}_1,\dots,\vec{\mathbf{w}}_n\}$ de $I\!R^n$, on a*

$$\Delta = \partial^2_{\vec{\mathbf{w}}_1} + \cdots + \partial^2_{\vec{\mathbf{w}}_n}$$

# 5 Le Laplacien sur les variétés plates

Nous avons introduit dans la section section ?? l'espace $\mathcal{C}^\infty(\boldsymbol{M})$ des fonctions de classe $\mathcal{C}^\infty$ sur une variété $\boldsymbol{M}$ de même classe ce qui comprend le cas des variétés plates.

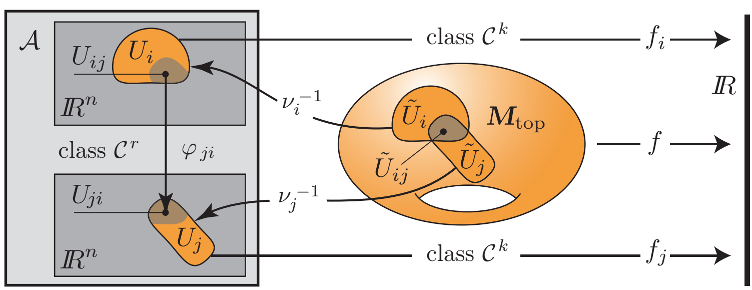
Le but de cette section est de montrer que la définition du Laplacien $\Delta \in \mathrm{End}_{\mathbb{R}}(\mathcal{C}^\infty(\mathbb{R}^n))$ s'étend de manière canonique à l'algèbre $\mathrm{End}_{\mathbb{R}}(\mathcal{C}^\infty(\boldsymbol{M}))$ où $\boldsymbol{M}$ est une variété plate de dimension $n$.

**5.1 Fonctions sur une variété.** Soit $\boldsymbol{M}$ un espace topologique muni d'une structure de variété de dimension $n$ et de classe $\mathcal{C}^\infty$ à l'aide d'un atlas (de définition) $\mathcal{A} = \{\nu_i : U_i \to \tilde{U}_i \subseteq \boldsymbol{M}\}_{i\in\mathfrak{I}}$.

**5.1.1 Proposition.** *Il y a équivalence entre la donnée de $f \in \mathcal{C}^\infty(\boldsymbol{M})$ (d'après la définition ??), et la donnée d'une famille de fonctions $\{\phi_i \in \mathcal{C}^\infty(U_i)\}_{i\in\mathfrak{I}}$ telle que pour tout couple $(i,j) \in \mathfrak{I}^2$ on ait*

$$\phi_i = \phi_j \circ \varphi_{ji}, \ \text{sur l'ouvert } U_{ij}. \tag{$*$}$$

*où $\varphi_{ji} : U_{ij} \to U_{ji}$ désigne la fonction de transition.*




*Proof.* Si $f \in \mathcal{C}^\infty(\boldsymbol{M})$, on pose $\phi_i := f\circ\nu_i$ et $(*)$ est claire. Réciproquement, si $x \in \boldsymbol{M}$, on pose $f(x) := \phi_i(\nu^{-1}(x))$ si $x \in \tilde{U}_i$. La condition $(*)$ sert à montrer que $f : \boldsymbol{M} \to \mathbb{R}$ est ainsi bien définie. La condition de différentiabilité de ?? est alors automatique. □

**5.1.2 Théorème et définition.** Soit $\boldsymbol{M}$ est une variété plate d'atlas plat complet $\mathcal{A} = \{\nu_i : U_i \to \tilde{U}_i \subseteq \boldsymbol{M}\}$ (**9**♦2.0.3) . Pour toute fonction $f \in \mathcal{C}^\infty(\boldsymbol{M})$, la famille de fonctions

$$\{\Delta_{U_i}(f\circ\nu_i) : U_i \to \mathbb{R}\}_{i\in\mathfrak{I}}$$

définit une fonction $\Delta(f) \in \mathcal{C}^\infty(\boldsymbol{M})$. L'opérateur $\Delta : \mathcal{C}^\infty(\boldsymbol{M}) \to \mathcal{C}^\infty(\boldsymbol{M})$ ainsi défini est $\mathbb{R}$-linéaire, on l'appelle "***le Laplacien de M***".

*Proof.* Il suffit de vérifier que la famille en question satisfait la condition de recollement $(*)$ de 5.1.1. Notons $\phi_j := \Delta(f\circ\nu_j)$, on a sur l'ouvert $U_{ij}$

$$\phi_j\circ\alpha_{ji} = (\alpha_{ji}^\natural\circ\Delta)(f\circ\nu_j) \underset{1}{=} (\Delta\circ\alpha_{ji}^\natural)(f\circ\nu_j) = \Delta(f\circ\nu_j\circ\varphi_{ji}) = \phi_i,$$

où $(\underset{1}{=})$ est la commutation du Laplacien et les isométries (4.6.1-(d)). □

## 6 Le Laplacien sur les ouverts à coin

Nous suivons la même démarche que dans le cas des variétés plates, mais nous devons auparavant préciser ce que l'on va appeler "*fonction de classe* $\mathcal{C}^\infty$" sur un ouvert à coin $V$ de $I\!R^2$ (**12**♦4.3, p. 240).

**6.1 Fonctions de classe $\mathcal{C}^r$ sur les ouverts à coin de $I\!R^2$.** Il est évident que ce sera une fonction $f : V \to I\!R$ continue et de classe $\mathcal{C}^r$ sur $V\setminus\partial V$, mais la condition de dérivabilité sur les points de $\partial V$ mérite quelques commentaires.

Dans un ouvert à coin $V$, pour tout $\vec{y}\in\partial V$ et pour tout $\vec{v}\in I\!R^2$ tel que

$$\vec{y}\pm\epsilon\vec{v}\in V,\ \text{pour } \epsilon>0 \text{ assez petit,} \tag{7}$$

on peut étudier l'existence de la dérivée directionnelle "latérale" (183)

$$\partial_{\vec{v}}(f)(\vec{y}) := \lim_{\epsilon\to 0\pm}\frac{f(\vec{y}+\epsilon\vec{v})-f(\vec{y})}{\epsilon}, \tag{8}$$

où $\lim_{\epsilon\to 0\pm}$ indique que l'approche de 0 se fera par des valeurs tantôt négatives, tantôt positives, de manière à garantir que $\vec{y}+\epsilon\vec{v}\in V$.

**6.1.1 Lemma.** *Soit $f : V\to I\!R$ continue et telle que les dérivées partielles $\partial_i(f)$ existent sur $V\setminus\partial V$ et admettent un prolongement continu à $V$ tout entier. Alors:*

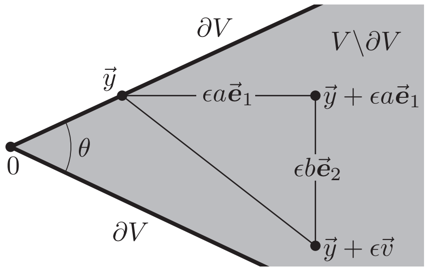
a) *Pour tout $\vec{y}\in\partial V$ et tout $\vec{v}\in I\!R^2$ vérifiant la condition (7), la dérivée directionnelle latérale $\partial_{\vec{v}}(f)(\vec{y})$ existe et l'on a*

$$\partial_{\vec{v}}(f)(\vec{y}) = \lim\nolimits_{\vec{x}\in V\setminus\partial V,\,\vec{x}\to\vec{y}}\ a\partial_1(f)(\vec{x}) + b\partial_2(f)(\vec{x}).$$

b) *Pour tout $y\in V$ qui n'est pas un "coin" dans $\partial V$,* i.e. $y\neq 0$, *les dérivées directionnelles $\partial_{\vec{v}}(f)$ existent pour tout vecteur $\vec{v}=(a,b)\in I\!R^2$ et l'on a*

$$\partial_{\vec{v}}(f)(\vec{y}) = a\partial_{\vec{e}_1}(f)(\vec{y}) + b\partial_{\vec{e}_2}(f)(\vec{y}).$$

*Proof.* Cela résulte d'appliquer le théorème des accroissements finis à la situation décrite dans la figure.




On a grâce à ce théorème

$$\frac{f(\vec{y}+\epsilon\vec{v})-f(\vec{y})}{\epsilon} = \frac{f(\vec{y}+\epsilon\vec{v})-f(\vec{y}+\epsilon a\vec{\mathbf{e}}_1)+f(\vec{y}+\epsilon a\vec{\mathbf{e}}_1)-f(\vec{y})}{\epsilon}$$
$$= b\partial_{\vec{\mathbf{e}}_2}(f)(\vec{y}+\epsilon a\vec{\mathbf{e}}_1+\epsilon' b\vec{\mathbf{e}}_2)+a\partial_{\vec{\mathbf{e}}_1}f(\vec{y}+\epsilon'' a\vec{\mathbf{e}}_1)$$

où $\epsilon', \epsilon'' \in ]0, \epsilon[$, et l'hypothèse de prolongement continu des dérivées partielles de $f$ depuis l'intérieur de $V$ permet de conclure. □

**6.1.2 Remark.** L'énoncé du lemme et sa démonstration ne sont autres que l'adaptation au contexte des ouverts à coin du théorème, déjà cité dans ces notes (??), qui affirme l'équivalente pour une fonction $f : I\!R^n \to I\!R$ entre être de classe $\mathcal{C}^1$ et avoir des dérivées partielles continues. *Ce lemme est très important puisqu'il nous permettra de faire du calcul différentiel sur des ouverts à coin et, ultérieurement, sur les surfaces plates à coins.*

Forts de ces observations, on définit une fonction de classe $\mathcal{C}^r$ sur un ouvert à coin de la manière suivante

**6.1.3 Definition.** Soit $V$ un ouvert à coin de $I\!R^2$ et soit $r$ un entier positif. Une fonction $f : V \to I\!R$ est dite de classe $\mathcal{C}^r$ si

- $f : V \to I\!R$ est continue;
- $f$ est de classe $C^1$ sur $V \backslash \partial V$;
- chaque dérivée partielle $\partial_i(f)$, définie sur $V \backslash \partial V$, se prolonge continûment à $V$ tout entier en une fonction que nous notons pareillement $\partial_i(f) : V \to I\!R$;
- les applications $\partial_i(f) : V \to I\!R$ sont de classe $\mathcal{C}^{r-1}$.

On pose alors

$$\mathcal{C}^\infty(V) := \bigcap\nolimits_{r\in\mathbb{N}} \mathcal{C}^r(V).$$

**6.2 Le Laplacien sur $\mathcal{C}^\infty(V)$.** La définition de fonction de classe $\mathcal{C}^\infty$ sur un ouvert à coin $V$ introduit les dérivées partielles, et donc directionnelles, en faisant agir ces opérateurs à l'intérieur des ouverts à coin concernés, puis en passant à la limite sur $\partial V$. Il s'ensuit que les propriétés des opérateurs différentiels de 4.3 continuent d'être vérifiées pour les ouverts à coin. Les applications

$$\partial_{\vec{v}} : \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(V)$$

sont bien définies et $\mathbb{R}$-linéaires, le lemme 4.3.1 est vérifié et l'on a

$$\mathrm{Diff}_{\mathbb{R}}(V) := \mathcal{C}^\infty(V)[\partial_1,\partial_2].$$

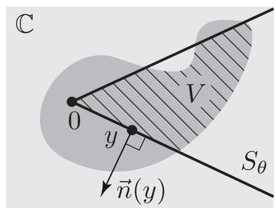
Le lapacien sur $\mathcal{C}^\infty(V)$ est défini par la même formule

$$\boxed{\Delta = \partial_1^2 + \partial_2^2}, \tag{9}$$

et pour toute base orthonormée $\{\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2\}$ de $\mathbb{R}^2$, on a (4.6.2)

$$\Delta = \partial^2_{\vec{\mathbf{w}}_1} + \partial^2_{\vec{\mathbf{w}}_2}. \tag{10}$$

**6.3. Dérivée normale au bord..** Un ouvert à coin $V$ à été défini dans **12♦**4.3 comme un ouvert d'un secteur angulaire $S_\theta$ de $\mathbb{C}$. Il est évident que pour tout $y \in \partial V$ différent de $0 \in \mathbb{C}$, il existe un unique vecteur normal au bord de norme 1 et sortant de l'ouvert, on le notera $\vec{n}(y)$. L'application $\vec{n} : \partial V \setminus \{0\} \to \mathbb{C}_u$, $y \mapsto \vec{n}(y)$ est localement constante.




On appèlera "*dérivée normale (en $y \in \partial V$)*" l'opérateur

$$\partial_{\vec{n}} : \mathcal{C}^\infty(\partial V \setminus \{0\}) \to \mathcal{C}^\infty(\partial V \setminus \{0\})$$

défini par (8), c'est-à-dire

$$\partial_{\vec{n}}(f)(y) := \partial_{\vec{n}(y)}(f)(y) = \lim_{\epsilon \to 0^+} \frac{f(\vec{y}) - f(\vec{y} - \epsilon \vec{n}(y))}{\epsilon}.$$

**6.3.1 Exercise.** Monter que la fonction $\partial_{\vec{n}}(f)$ est de classe $\mathcal{C}^\infty$ sur $\partial V \setminus \{0\}$, quelle que soit $f \in \mathcal{C}^\infty(V)$.

## 7 Le Laplacien sur les ouverts plats à coins

**7.1 Fonctions sur une surface plate à coins.** Soit $\boldsymbol{M}$ une variété plate à coins. L'espace $\mathcal{C}^\infty(\boldsymbol{M})$ est défini, comme pour les variétés plates (5), à l'aide de la proposition de recollement 5.1.1. Il suffit juste de comprendre que l'isométrie (affine) de transition $\varphi_{ji} : V_{ij} \to V_{ji}$ échange intérieurs, bords, coins et dérivées directionnelles qu'elles soient latérales ou non. L'application $\varphi_{ji}^\natural : \mathcal{C}^\infty(V_{ji}) \to \mathcal{C}^\infty(V_{ij})$, $\phi_j \mapsto \phi_j \circ \varphi_{ji}$, est donc bien définie. Á partir de là, la définition de $\mathcal{C}^\infty(\boldsymbol{M})$ est calquée de la proposition 5.1.1.

**7.2 Le Laplacien sur une surface plate à coins.** Tout comme nous avons procédé dans les paragraphes précédents, la vérification de la commutation des Laplaciens et de l'isométrie $\varphi_{ji}$, *i.e.* l'égalité

$$\varphi_{ji}^{\natural}\big(\Delta_{V_{ji}}(f)\big) = \Delta_{V_{ij}}\big(\varphi_{ji}^{\natural}(f)\big), \quad \forall f \in \mathcal{C}^\infty(V_{ji}),$$

est automatique dans la mesure ou $\varphi_{ji}$ respecte les bords et ramène la question de la commutation aux intérieurs des ouverts en question sur les quels le résultat est déjà connu, un passage à la limite achève alors la vérification. La proposition 4.6.1, notamment son assertion (d), se voit ainsi établie pour les ouverts à coin ce qui est l'ingrédient qui valide le théorème-définition 5.1.2. Ce théorème donne maintenant la définition du Laplacien d'une surface plate à coins.

**7.3 Applications localement isométriques.** Une application entre variétés plates (à bords, à coins, ...) $\alpha : \boldsymbol{M}_1 \to \boldsymbol{M}_2$, est dite "*localement isométrique*" lorsque tout $x \in \boldsymbol{M}_1$ admet un voisinage ouvert $V_x$ tel que $\alpha(V_x)$ est ouvert dans $\boldsymbol{M}_2$ et $\alpha : V_x \to \alpha(V_x)$ est une isométrie inversible.

On dira aussi que $\alpha$ "*respecte bords et coins*" lorsque

$$\alpha^{-1}(\text{bord}) = \text{bord et } \alpha^{-1}(\text{coin}) = \text{coin}.$$

**7.3.1 Proposition.** *Désignons par $\boldsymbol{M}$ une variété plate (à bords, à coins, ...).*

a) *Une application $\alpha : \boldsymbol{M}_1 \to \boldsymbol{M}_2$, localement isométrique, respecte bords et coins et "commute aux Laplaciens",* i.e.

$$\alpha^{\natural} \circ \Delta_{\boldsymbol{M}_2} = \Delta_{\boldsymbol{M}_1} \circ \alpha^{\natural}.$$

b) *Soit $\boldsymbol{G}$ un groupe agissant par des isométries sur $\boldsymbol{M}$. Alors,*
   i) *L'action de $\boldsymbol{G}$ sur $\mathcal{C}^\infty(\boldsymbol{M})$ commute au Laplacien $\Delta_{\boldsymbol{M}}$. En particulier,*
      - *$\boldsymbol{G}$ agit sur les sous-espaces propres de $\Delta_{\boldsymbol{M}}$ dans $\mathcal{C}^\infty(\boldsymbol{M})$.*
      - *$\Delta_{\boldsymbol{M}}$ agit sur l'espace $\mathcal{C}^\infty(\boldsymbol{M})^{\boldsymbol{G}}$ des fonctions $f : \boldsymbol{M} \to \mathbb{R}$ de classe $\mathcal{C}^\infty$ qui sont $\boldsymbol{G}$-invariantes,* i.e. *telles que $f \circ g = f$ pour tout $g \in \boldsymbol{G}$.*

   ii) *Lorsque $\boldsymbol{G}$ est fini et agit **librement** sur $\boldsymbol{M}$, La surjection canonique*

$$\nu : \boldsymbol{M} \to \boldsymbol{M}/\boldsymbol{G}$$

*respecte bords et coins et c'est une isométrie locale. L'application induite*

$$\nu^{\natural} : \mathcal{C}^r(\boldsymbol{M}/\boldsymbol{G}) \to \mathcal{C}^r(\boldsymbol{M})^{\boldsymbol{G}}, \quad \phi \mapsto \phi \circ \nu,$$

*est un isomorphisme pour tout $r$, et commute aux Laplaciens si $r \geq 2$.*

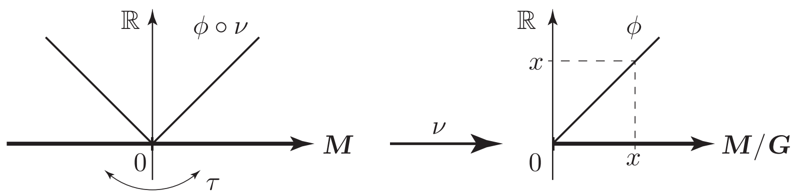
*Hint.* Exercice (utiliser **10**♦2.1.2-(c)). □

**7.3.2 Remark.** Soit $\boldsymbol{M} = I\!R$. L'application $\tau : I\!R \to I\!R$, $x \mapsto -x$, est une isométrie involutive, *i.e.* $\tau^2 = \mathrm{id}$. Le groupe $\boldsymbol{G} = \{\boldsymbol{e},\tau\}$ engendré par $\tau$ n'opère pas librement car $\tau(0) = 0$. La demi-droite fermée $I\!R_+ := [0,+\infty[$ est un domaine fondamental fermé (**10**♦2.4.1) et s'identifie à la variété plate à bord $\boldsymbol{M}/\boldsymbol{G}$.

On a bien un isomorphisme

$$\nu^\natural : \mathcal{C}^0(\boldsymbol{M}/\boldsymbol{G}) \to \mathcal{C}^0(\boldsymbol{M})^{\boldsymbol{G}}, \quad \phi \mapsto \phi \circ \nu,$$

mais sa restriction à $\mathcal{C}^\infty(\boldsymbol{M}/\boldsymbol{G})$ n'est pas à valeurs dans $\mathcal{C}^\infty(\boldsymbol{M})^{\boldsymbol{G}}$, ni d'ailleurs même pas dans $\mathcal{C}^1(\boldsymbol{M})^{\boldsymbol{G}}$. En effet, la fonction $\phi : I\!R_+ \to I\!R$, $x \mapsto x$, est de classe $\mathcal{C}^\infty$ mais $\nu^\natural(\phi)$, $x \mapsto |x|$, n'est pas dérivable en 0.




Par cet exemple, nous montrons que la liberté de l'action de $\boldsymbol{G}$ sur $\boldsymbol{M}$ est indispensable dans l'assertion (b)-(ii) de la proposition précédente.

Ce le genre de difficulté dont il faut s'attendre lorsque l'action de $\boldsymbol{G}$ n'est pas libre. Le preuve du théorème 3.3.1 est un exemple où ce phénomène exige une attention particulière.

## 8 Vibrations des surfaces plates à coin

Lors de la vibration d'une membrane de tambour $\boldsymbol{M}$, c'est-à-dire d'un domaine à bord $\partial\boldsymbol{M}$ du plan $I\!R^2$, chaque point de $\vec{x} \in \boldsymbol{M}$ a une coordonnée verticale "$h$" qui varie avec le temps "t". Ce phénomène est repéré par une fonction à deux variables $h(t,\vec{x})$. On démontre que si $h$ est supposée de classe $\mathcal{C}^2$, elle vérifie une certaine équation au dérivées partielles ("*une EDP*"): l'"*équation des membranes vibrantes*", a savoir

$$\partial_t^2(h) = c^2(\partial_1^2 + \partial_2^2)(h), \quad \text{et} \quad h(t,\vec{x}) = 0,\ \forall\vec{x} \in \partial\boldsymbol{M},\ \forall t \in I\!R \tag{11}$$

où $c \in I\!R$ est une certaine constante positive qui relève des caractéristiques physiques de la membrane, p.e. densité, elasticité, tension.

La transposition au contexte des surfaces plates à coins est immédiate.

**8.0.1 Definition.** Une "*vibration*" d'une surface plate à coin $\boldsymbol{M}$ est la donnée d'une fonction $h : \boldsymbol{M} \times I\!R \to I\!R$ de classe $\mathcal{C}^2$ astreinte à vérifier l'équation différentielle

$$\boxed{\partial_t^2(h) = c^2 \Delta_{\boldsymbol{M}}(h)} \tag{12}$$

et satisfaisant la "*condition au bord*"

$$h(t,x) = 0 \text{ pour tout } x \in \partial \boldsymbol{M} \text{ et tout } t \in I\!R. \tag{13}$$

**8.1 Vibrations et fonctions propres du Laplacien.** Si dans (12) on pose

$$h(t,x) = \big(A\sin(\omega t) + B\cos(\omega t)\big) f(x), \tag{14}$$

on voit que $h$ est une vibration si et seulement si

$$-\omega^2 f = c^2 \Delta(f) \tag{15}$$

soit donc

$$\boxed{-\Delta(f) = \lambda f} \tag{16}$$

pour $\lambda = (\omega/c)^2 \in I\!R$.

Une telle fonction fonction, si elle est non nulle, est appellée "*fonction propre*" de $-\Delta_{\boldsymbol{M}}$ (parfois appelé "*le moins-Laplacien*"), et le nombre réel $\lambda$ est la "*valeur propre*"([9]) associée.

**8.2 Conditions aux limites.** Deux conditions au bord pour les fonctions propres vont jouer un rôle important dans la suite

- (D) *Conditions aux limites de Dirichlet.* Elle traduit la condition (13):

$$\boxed{f|_{\partial \boldsymbol{M}} = 0}$$

- (N) *Conditions aux limites de Neumann.*

$$\boxed{\partial_\nu(f)(x) = 0, \quad \forall x \in \partial \boldsymbol{M}}$$

où $\partial_{\vec{n}}$ est l'opérateur de dérivée directionnelle normale au bord (6.1-(8)).

**Notation.** On désignera par '$\beta$' l'une des deux conditions, soit $\beta \in \{\mathcal{D},\mathcal{N}\}$.

**8.2.1 Definition.** Le "*spectre*" $\sigma(\boldsymbol{M})$ (du moins-Laplacien) est l'ensemble des valeurs propres de $-\Delta_{\boldsymbol{M}}$. Le "*spectre de Dirichlet*" $\sigma_D(\boldsymbol{M})$ est de même, mais ne concerne que les fonctions propres vérifiant la condition de Dirichlet, et pareillement pour "*spectre de Neumann*" $\sigma_N(\boldsymbol{M})$.

**8.2.2 Remark.** On observera que compte tenu des égalités (15) et (16) le spectre de Dirichlet $\sigma_D(\boldsymbol{M})$ est, à un facteur près, l'ensemble des fréquences des vibrations de $\boldsymbol{M}$.

([9]) Respectivement "*Eigenfunction*" et "*eigenvalue*" en anglais.

**8.3 La question de Marc Kac.** ([10]) En 1966, il synthétise un problème de géométrie spectrale sous la forme d'une question devenue célèbre : "*Peut-on entendre la forme d'un tambour ?*" [K] ([11])

Compte tenue de la remarque 8.2.2, cette question se transcrit naturellement dans le contexte des surfaces plates à coins, par la question suivante.

**8.3.1 Question.** Soient $\boldsymbol{M}_1$ et $\boldsymbol{M}_2$ deux surfaces plates à coins telles que $\sigma_D(\boldsymbol{M}_1) = \sigma_D(\boldsymbol{M}_2)$. Est-ce que $\boldsymbol{M}_1$ est $\boldsymbol{M}_2$ sont isométriques, *i.e.* existe-t-il une bijection isométrique $\phi : \boldsymbol{M}_1 \to \boldsymbol{M}_2$ ?

Mais avant d'aborder cette question (*cf.* §**15**), nous rappelons quelques résultats sur les fonctions propres du Laplacien qui nous seront indispensables.

## 9 Fonctions propres du Laplacien

L'étude quantitative et qualitative des fonctions propres du Laplacien est un vaste sujet dans la littérature mathématique. La plupart du temps, on cherche de telles fonctions satisfaisant des "*conditions limites*", et dans un petit nombre de cas on sait expliciter toutes les fonctions possibles.

**9.1 L'exemple de la membrane carrée.** ([12]) Soit $\boldsymbol{M} = [0,1] \times [0,1] \subseteq \mathbb{R}^2$. Les fonctions

$$f_{m,n}(x,y) = \sin(m\pi x)\sin(n\pi y),$$

vérifient

$$-\Delta(f_{m,n}) = (n^2 + m^2)\pi^2 f_{m,n}\,, \quad f_{m,n}|_{\partial \boldsymbol{M}} = 0\,.$$

Il s'agit donc des fonctions propres du Laplacien satisfaisant la condition de Dirichlet sur $\partial \boldsymbol{M}$ et de valeurs propres associées

$$\lambda = (n^2 + m^2)\pi^2\,.$$

On remarquera le fait qu'il peut fort bien exister des fonctions propres indépendantes (*i.e.* non colinéaires) associées à la même valeur propre. Ici par

([10]) Extrait de Wikipedia: *Mark Kac (ou Marek Kac) (1914-1984) (prononcer : katz) était un mathématicien américain d'origine polonaise, spécialiste de la théorie des probabilités.*

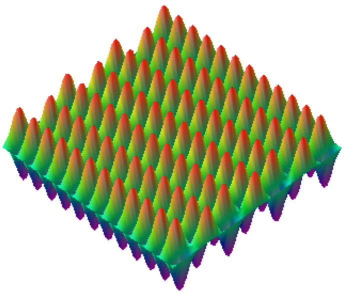
([11]) Voici une citation antérieure de la même nature due à Sir Arthur Schuster (1851-1934), physicien germano-britannique : *To find out the different tunes sent out by a vibrating system is a problem which may or may not be solvable in certain special cases, but it would baffle the most skillful mathematician to solve the inverse problem and to find out the shape of a bell by means of the sounds which it is capable of giving out. And this is the problem which ultimately spectroscopy hopes to solve in the case of light. In the meantime we must welcome with delight even the smallest step in the right direction (Schuster, 1882.).*

*Note du rédacteur. En spectroscopie, on cherche à identifier un corps en étudiant la lumière qu'il émet.*

([12]) La référence pour cette partie est le paragraphe §V.5 *The Vibrating Membrane*, p. 297, et §V.5.4 *Rectangular Membrane*, p. 300 du livre de Courant & Hilbert [CH].

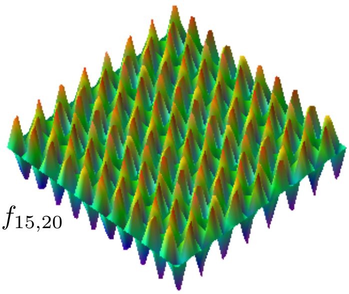
exemple, la question se ramène à la recherche des triangles pythagoriciens de même hypoténuse. Voici le plus petit exemple de tels triangles non semblables.([13])

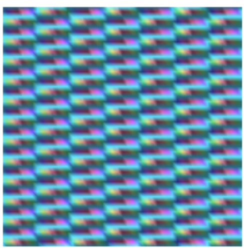
$$25^2 = 7^2 + 24^2 = 15^2 + 20^2$$



$f_{7,24}$

$f_{15,20}$





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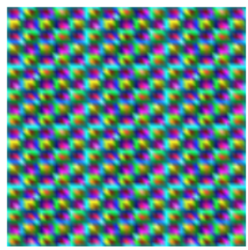

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Et bien évidemment, toute combinaison linéaire non triviale

$$f = a f_{7,24} + b f_{15,20},$$

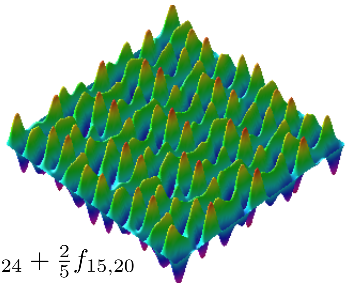
([13]) Voici la liste des premiers triangles pythagoriciens vérifiant cette propriété, trouvée sur le site http://villemin.gerard.free.fr/Wwwgvmm/Addition/TripTrg.htm

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $7^2$ | + | $24^2$ | = | $15^2$ | + | $20^2$ | = | 625 | = | $25^2$ |
| $14^2$ | + | $48^2$ | = | $30^2$ | + | $40^2$ | = | 2 500 | = | $50^2$ |
| $16^2$ | + | $63^2$ | = | $25^2$ | + | $60^2$ | = | 4 225 | = | $65^2$ |
| | | | = | $33^2$ | + | $56^2$ | = | 4 225 | = | $65^2$ |
| | | | = | $39^2$ | + | $52^2$ | = | 4 225 | = | $65^2$ |
| $21^2$ | + | $72^2$ | = | $45^2$ | + | $60^2$ | = | 5 625 | = | $75^2$ |
| $13^2$ | + | $84^2$ | = | $36^2$ | + | $77^2$ | = | 7 225 | = | $85^2$ |
| | | | = | $40^2$ | + | $75^2$ | = | 7 225 | = | $85^2$ |
| | | | = | $51^2$ | + | $68^2$ | = | 7 225 | = | $85^2$ |
| $28^2$ | + | $96^2$ | = | $60^2$ | + | $80^2$ | = | 10 000 | = | $100^2$ |
| $35^2$ | + | $120^2$ | = | $44^2$ | + | $117^2$ | = | 15 625 | = | $125^2$ |
| | | | = | $75^2$ | + | $100^2$ | = | 15 625 | = | $125^2$ |
| $32^2$ | + | $126^2$ | = | $50^2$ | + | $120^2$ | = | 16 900 | = | $130^2$ |
| | | | = | $66^2$ | + | $112^2$ | = | 16 900 | = | $130^2$ |
| | | | = | $78^2$ | + | $104^2$ | = | 16 900 | = | $130^2$ |
| $17^2$ | + | $144^2$ | = | $24^2$ | + | $143^2$ | = | 21 025 | = | $145^2$ |
| | | | = | $87^2$ | + | $116^2$ | = | 21 025 | = | $145^2$ |
| | | | = | $100^2$ | + | $105^2$ | = | 21 025 | = | $145^2$ |
| $42^2$ | + | $144^2$ | = | $90^2$ | + | $120^2$ | = | 22 500 | = | $150^2$ |
| $80^2$ | + | $150^2$ | = | $102^2$ | + | $136^2$ | = | 28 900 | = | $170^2$ |

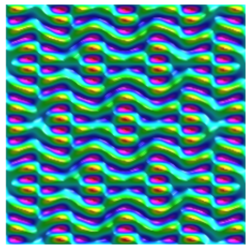
est aussi une fonction propre de $\Delta_{\boldsymbol{M}}$ pour la même valeur propre, car

$$\begin{aligned}\Delta(f) &= \Delta(af_{7,24}+bf_{15,20}) = \Delta(af_{7,24})+\Delta(bf_{15,20})\\ &= a\Delta(f_{7,24})+b\Delta(f_{15,20}) = a\lambda f_{7,24}+b\lambda f_{15,20} = \lambda f\,.\end{aligned}$$

La fonction $\frac{3}{5}f_{7,24}+\frac{2}{5}f_{15,20}$ appartient donc aussi à $\mathcal{H}_{(25\pi)^2}(\boldsymbol{M})$.



$\frac{3}{5}f_{7,24}+\frac{2}{5}f_{15,20}$



Vue d'en haut

**9.1.1 Remark.** L'inspection de cet exemple et d'autres où l'on arrive à expliciter les solutions cherchées, emmène à se poser un certain nombre de questions. En voici deux très importantes et dont l'étude demande des techniques très sophistiqués, bien au delà de la portée de ces notes.

- Combien de fonctions propres linéairement indépendantes peuvent exister pour une valeur propre donnée ?
- Est-ce que deux fonctions propres peuvent coïncider au voisinage d'un point tout en étant différentes?

Les sections suivantes apportent quelques éléments de réponse.

**9.2 Sous-espaces de fonctions propres.** Soit $\boldsymbol{M}$ une surface plate à coin.

On note $\mathcal{H}_\lambda(\boldsymbol{M}) \subseteq \mathcal{C}^\infty(\boldsymbol{M})$ (resp. $\mathcal{H}_{\lambda,D}(\boldsymbol{M})$ et $\mathcal{H}_{\lambda,N}(\boldsymbol{M})$) l'ensemble de toutes les fonctions $f \in \mathcal{C}^\infty(\boldsymbol{M})$ telles que $-\Delta_{\boldsymbol{M}}(f) = \lambda f$ (et qui satisfont respectivement la condition de Dirichlet ou Neumann).

**9.2.1 Theorem.** *On rappelle que $\beta$ l'une des lettres $D$,$N$.*

a) $\mathcal{H}_{\lambda,\beta}(\boldsymbol{M})$ *est un sous-espace vectoriel de* $\mathcal{C}^\infty(\boldsymbol{M})$.

b) $\mathcal{H}_{\lambda,\beta}(\boldsymbol{M}) \cap \mathcal{H}_{\lambda',\beta}(\boldsymbol{M}) \neq 0$, *si et seulement si* $\lambda = \lambda'$.

c) $\mathcal{H}_{\lambda,\beta}(\boldsymbol{M}) = 0$, *pour tout* $\lambda < 0$. $\mathcal{H}_{0,D}(\boldsymbol{M}) = 0$ *et* $f \in \mathcal{H}_{0,N}(\boldsymbol{M})$, *si et seulement si $f$ est localement constante.*

d) *Si $\boldsymbol{M}$ est compacte la dimension de $\mathcal{H}_{\lambda,\beta}(\boldsymbol{M})$, appelée "multiplicité de $\lambda$ (pour la condition $\beta$)", est finie.*

e) *Si* $\alpha : \boldsymbol{M}_1 \to \boldsymbol{M}_2$ *est une bijection isométrique, l'application*

$$\alpha^\natural : \mathcal{C}^\infty(\boldsymbol{M}_2) \to \mathcal{C}^\infty(\boldsymbol{M}_1), \quad f \mapsto f \circ \alpha,$$

*induit un isomorphisme de* $\mathcal{H}_{\lambda,\beta}(\boldsymbol{M}_2)$ *sur* $\mathcal{H}_{\lambda,\beta}(\boldsymbol{M}_1)$ *pour tous* $\lambda$ *et* $\beta$*. On dit alors que les surfaces* $\boldsymbol{M}_1$ *et* $\boldsymbol{M}_2$ *son "isospectrales". En particulier,*

$$\sigma_\beta(\boldsymbol{M}_1) = \sigma_\beta(\boldsymbol{M}_2).$$

*Hints.* (a) car $\Delta$ est un opérateur $I\!R$-linéaire. (b) car $-\Delta(f) = \lambda f$ et $-\Delta(f) = \lambda' f$, impliquent $(\lambda - \lambda')f = 0$, et alors $\lambda = \lambda'$, puisque $f \neq 0$.

(c) se démontre à l'aide des formules de Green ([14]). Elles montrent, pour une fonction propre $f$ du moins-lapacien telle que $\partial_{\vec{n}}(f)f|_{\partial \boldsymbol{M}} = 0$, l'égalité remarquable

$$\lambda = \|f\|^{-2} \int_{\boldsymbol{M}} |\mathrm{Grad} f|^2 \, dV,$$

on en déduit, d'une part, que $\lambda \geq 0$, et d'autre part, que $\lambda = 0$, si et seulement si $\mathrm{Grad} f = 0$, autrement dit, si et seulement si $f$ est localement constante.

(d) par contre, est bien plus profonde. Le résultat sous cette forme peut être consulté dans [Bé$_1$] Th. 18, p. 53. Les conditions de régularité du bord $\partial \boldsymbol{M}$ sont assez cruciales et il est difficile de trouver une référence les donnant précisément. On peut dire grosso modo que tout se passe bien si $\boldsymbol{M}$ est compacte et $\partial \boldsymbol{M}$ est de classe $\mathcal{C}^\infty$ par morceaux, ce qui est une condition largement satisfaite dans notre contexte où $\partial \boldsymbol{M}$ est une réunion de segments de la droite euclidienne $I\!R$.

(e) est claire puisque $\alpha^\natural$ commute aux Laplaciens (7.3.1). □

**9.3 Unicité locale des fonctions propres du Laplacien.** Les résultats du paragraphe précédent concernent les valeurs propres du Laplacien mais ne disent rien sur la nature des fonctions propres. À quoi ressemblent-elles ? Dans l'exemple de la membrane carrée (**1**♦4.3), on a des produits de fonctions trigonométriques, mais qu'en est-il dans le cas général. C'est encore un énorme sujet dans la littérature sur les EDP et plus particulièrement sur ce que l'on appelle "*les solutions des opérateurs différentiels elliptiques*" (le Laplacien en est un!). L'un des acteurs le plus importants dans cette discipline fut Lipman Bers ([15]), il a prouvé ([Ber], 1955) un résultat qui, replacé dans notre contexte, dit:

([14]) Chapitre I du livre d'Isaac Chavel [Cha], formule (46), page 8. La validité des formules de Green sur $\boldsymbol{M} \backslash \partial \boldsymbol{M}$ implique leur validité partout par passage à la limite

([15]) Extrait de Wikipedia: Lipman "Lipa" Bers (1914?1993) was an American mathematician born in Riga who created the theory of pseudoanalytic functions and worked on Riemann surfaces and Kleinian groups. He was also known for his work in human rights activism.

**9.3.1 Theorem (L. Bers).** *Soit $\Delta$ le Laplacien de $\mathbb{R}^2$. Soit $B(\vec{0},r)$ une boule ouverte de $\mathbb{R}^2$ et soit $f$ une fonction de $\mathcal{C}^\infty(B(\vec{0},r))$ vérifiant*

$$f(\vec{0}) = 0, \quad \textit{et} \quad -\Delta(f) = \lambda f, \textit{ pour un certain } \lambda \in \mathbb{R},$$

*Il n'y a alors que deux cas possibles*

- *il existe un polynôme **non nul** $P(x,y)$ homogène en $\{x,y\}$ et tel que*

$$\begin{cases} f(x,y) = P(x,y) + O(\|(x,y)\|^{\deg P+1}), \\ -\Delta(P) = \lambda P. \end{cases} \tag{$*$}$$

- *ou bien $f = 0$ sur toute la boule $B(\vec{0},r)$,*

**9.3.2 Remark.** Dans cet énoncé, il doit être clair que le rôle joué par l'origine de $\mathbb{R}^2$ est purement cosmétique. En effet, on aurait pu le remplacer par n'importe quel $\vec{x} \in \mathbb{R}^2$, à condition bien évidemment de remplacer en même temps le développement limité autour de $\vec{0}$ dans $(*)$ par un développement limité autour de $\vec{x}$. La justification mathématique de cette remarque est l'invariance du Laplacien par isométries de $\mathbb{R}^2$ (4.6.1).

Grâce au théorème de Bers, le corollaire suivant, version édulcorée d'un théorème de Cheng ([Che], 1976), est de démonstration élémentaire.

**9.3.3 Corollary (Cheng).** *Soit $\boldsymbol{M}$ une surface plate à coins **connexe** et soit $f$ une fonction propre du Laplacien $\Delta_{\boldsymbol{M}}$.*

a) *L'ensemble $f^{-1}(0)$, appelé "ensemble nodal de $f$", est d'intérieur vide.*
b) *Soient $f$ et $g$ deux fonctions propres de $\Delta_{\boldsymbol{M}}$ (pas forcément pour la même valeur propre), alors $f = g$ sur $\boldsymbol{M}$ tout entier, si et seulement si $f = g$ au voisinage d'un point de $\boldsymbol{M}$.*

*Hint.* (a) Soit $f \in \mathcal{C}^\infty(\boldsymbol{M})$ une fonction propre du Laplacien. Notons $\mathcal{Q}_f$ l'ensemble des points $y \in \boldsymbol{M}$ tels que $f$ s'annule sur un voisinage ouvert $W_y$ de $y$. L'ensemble $\mathcal{Q}_f$ est ouvert dans $\boldsymbol{M}$ puisqu'il est la réunion des ouverts $W_y$. Soit maintenant $x$ un élément dans l'adhérence de $\mathcal{Q}_f$. Il existe une suite $(y_n)_{n\in\mathbb{N}}$ avec $y_n \in \mathcal{Q}_f$ telle que $x = \lim_n y_n$. Mais alors pour tous $m_1,m_2 \in \mathbb{N}$, on a par continuité

$$\partial_1^{m_1}\partial_2^{m_2}(f)(x) = \lim_{n\in\mathbb{N}} \partial_1^{m_1}\partial_2^{m_2}(f)(y_n) = 0$$

car autour de chaque $y_n$ la fonction $f$ s'annule identiquement. Le théorème de Bers nous dit alors que $f$ est, soit identiquement nulle autour de $x$, soit elle admet un développement limité. Or ce deuxième cas est incompatible avec l'annulation de toutes les dérivées partielles, c'est donc que $f = 0$ sur un voisinage autour de $x$. Par conséquent $x \in \mathcal{Q}_f$, et $\mathcal{Q}_f$ est fermé dans $\boldsymbol{M}$.

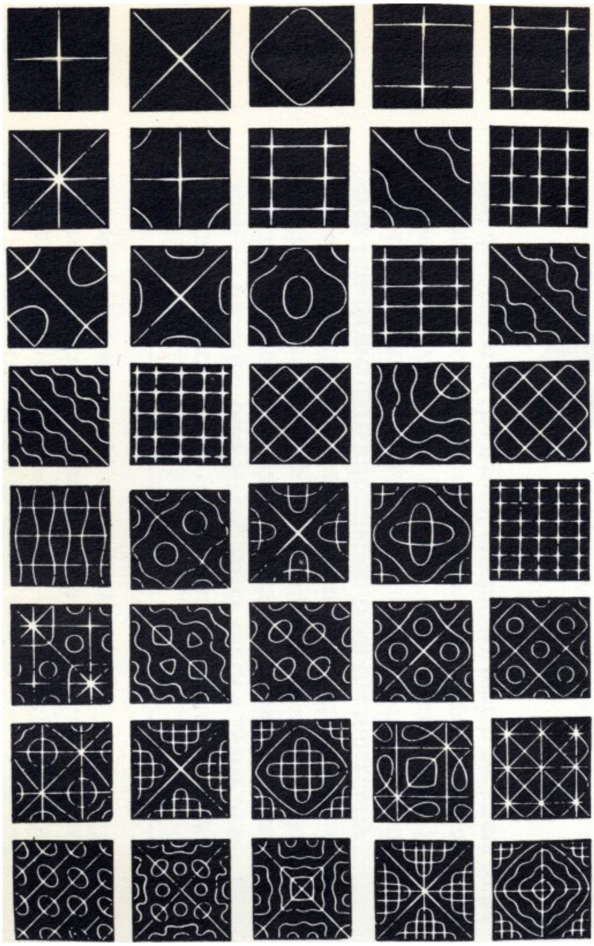
Le théorème de Bers dit par conséquent que l'ensemble $Q_f$ est à la fois ouvert et fermé dans $\boldsymbol{M}$. Comme $\boldsymbol{M}$ est supposée connexe, c'est que $\mathcal{Q}_f$ est soit vide soit $\boldsymbol{M}$ tout entier. L'assertion (a) en résulte aussitôt, puisque $f \neq 0$, c'est que $\mathcal{Q}_f = \emptyset$. Or, $Q_f$ est l'intérieur de $f^{-1}(0)$.

(b) en découle. En effet, supposons que $f = g$ sur un ouvert $U$. Alors, ou bien $f|_U$ et $g|_U$ ne sont pas nulles, et alors nécessairement $f|_U, g|_U \in \mathcal{H}_\lambda(U)$ pour un même $\lambda$ (**1**⧫5.3.1-(b)), auquel cas $f, g \in \mathcal{H}_\lambda(\boldsymbol{M})$ et $f - g \in \mathcal{H}_\lambda(\boldsymbol{M})$ s'annule sur $U$ et donc $f - g = 0$ sur $\boldsymbol{M}$ d'après (a). Ou bien $f|_U = g|_U = 0$, mais alors $f = g = 0$ sur $\boldsymbol{M}$ toujours d'après (a). □

**9.4 Ensembles nodaux.** Lorsque l'on fait vibrer une membrane plane $\boldsymbol{M}$ saupoudrée de farine, sable ou sucre, les grains s'organisent suivant des figures géométrique présentant des symétries parfois très étonnantes


Ces figures ont fasciné le physicien allemand Ernst Chladni ([16]) qui les étudia pendant de nombreuses années. Elles sont appelées “*les figures acoustiques de Chladni*”. En voici des images datant de 1869 dues au physicien irlandais John Tyndall (1820-1893).

([16]) Extrait de Wikipedia : Ernest Chadli. *Ernst Florens Friedrich Chladni (1756-1827), physicien allemand, voyagea toute sa vie. C'est le fondateur de l'acoustique moderne. Il étudiait expérimentalement les vibrations des plaques, en les saupoudrant de sable fin, obtenant ainsi les figures acoustiques qui portent son nom. Il fit plusieurs découvertes intéressantes, inventa un nouvel instrument de musique, l'euphone ou clavicylindre, un instrument dérivé de l'harmonica de verre de Benjamin Franklin. Il publia en 1802 un Traité d'acoustique, en allemand, traduit en français, 1809. On lui doit aussi des Dissertations sur les météores et les aérolithes (Vienne, 1819).*

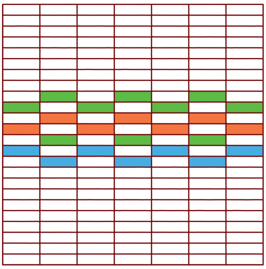


L'explication du phénomène est fort simple. Lors de la vibration de la membrane par une fréquence bien précise, une certaine fonction propre du Laplacien de $\boldsymbol{M}$ se manifeste. La vibration d'un point $x \in \boldsymbol{M}$ est alors de la forme (14)

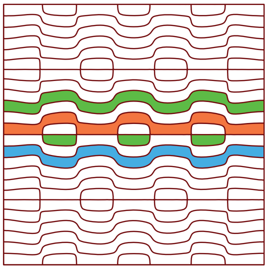
$$h(t,x) = \big(A\sin(\omega t) + B\cos(\omega t)\big) f(x).$$

Lorsque $f(x) \neq 0$, le point $x \in \boldsymbol{M}$ est soumis à des vibrations qui vont secouer toute particule de sable se trouvant à cette place. Au bout de quelques instants, les particules quittent ces endroits et se positionnent naturellement autour des points $x \in \boldsymbol{M}$ qui ne bougent pas, *i.e.* qui vérifient $f(x) = 0$. Les figures de Chladni révèlent donc l'“*ensemble nodal*” $f^{-1}(0) \subseteq \boldsymbol{M}$.

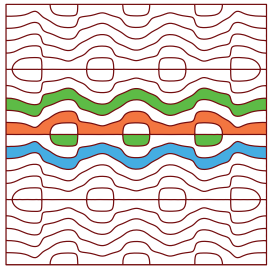
Voici l'ensemble nodal calculé par Maple pour certaines fonctions propres de l'intervalle $[f_{7,24}, f_{15,20}]$ de l'exemple **1**♦4.3 de la membrane carrée. Nous y avons colorié certaines régions pour faciliter l'observation du processus qui fait passer d'une grille de $7 \times 25$ à une autre de $15 \times 20$ (17) (Comparer aux vues d'en haut des images de la page 196 et suivante.)



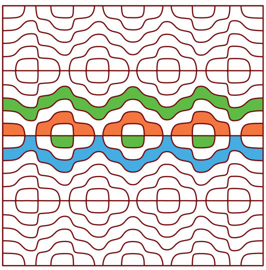
$f_{7,24}=0$



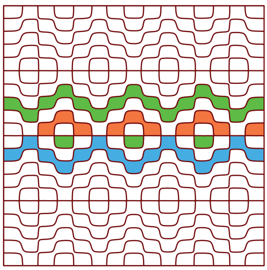
$\frac{4}{5}f_{7,24}+\frac{1}{5}f_{15,20}=0$



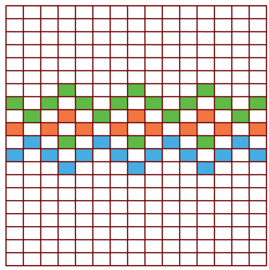
$\frac{3}{5}f_{7,24}+\frac{2}{5}f_{15,20}=0$



$\frac{2}{5}f_{7,24}+\frac{3}{5}f_{15,20}=0$



$\frac{5}{5}f_{7,24}+\frac{1}{4}f_{15,20}=0$



$f_{15,20}=0$

Les théorèmes de Bers et de Cheng nous indiquent que les parties nodales sont d'intérieur vide. Dans ces images, on trouve en fait des réunions de courbes. Ce n'est pas un hasard, c'est un théorème ([Che]).

**9.5 Domaines nodaux.** Si $f$ est une fonction propre de $\Delta_M$, on appelle "*domaine nodal (de $f$)*" chaque "*composante connexe*"(18) du complémentaire de l'ensemble nodal de $f$. Si $\Omega$ est une telle région, la fonction $f$ y garde un signe constant (**3**♦8.3.3-(5)). Le théorème suivant est de grande importance, c'est le "*Théorème de Courant des domaines nodaux*" (*cf.* [Cha], p. 19).

(17) Routine Maple utilisée:

```
>with(plots):
>f1:=sin(07*Pi*x)*sin(24*Pi*y):
>f2:=sin(15*Pi*x)*sin(20*Pi*y):
>for i from 0 to 5 do
   implicitplot(((5-i)/5)*f1+(i/5)*f2=0,x=0..1,y=0..1,numpoints=1000
 end do;
```

(18) "*composante connexe*": c'est une partie connexe maximale d'un espace topologique. Si l'espace est une variété, une composante connexe est constituée d'un point et tous ceux que l'on peut relier à celui-ci par un chemin continu.

**9.5.1 Theorem (R. Courant).** *Soit $\boldsymbol{M}$ une surface plate à coins et compacte. Suite à la proposition $\mathbf{1}$♦5.3.1, notons*

$$\lambda_1(\boldsymbol{M},\beta) \leq \lambda_2(\boldsymbol{M},\beta) \leq \lambda_3(\boldsymbol{M},\beta) \leq \cdots \leq \lambda_k(\boldsymbol{M},\beta) \leq \ldots$$

*la liste complète des valeurs propres du spectre $\sigma_\beta(\boldsymbol{M})$, $\beta \in \{\mathcal{D},\mathcal{N}\}$, chacune répétée autant des fois que sa multiplicité. Soit maintenant*

$$\{f_1,f_2,f_3,\ldots,f_k,\ldots\}$$

*une suite de fonctions linéairement indépendantes et telles que $f_k \in \mathcal{H}_{\lambda_k,\beta}(\boldsymbol{M})$. Alors, le nombre des domaines nodaux de $f_k$ est majoré par $k$, pour $k = 1,2,\ldots$*

**9.5.2 Corollary.** a) *La fonction $f_1$ est de signe constant.*
b) *La valeur propre $\lambda_1(\beta,\boldsymbol{M})$ et de multiplicité 1. Donc $\lambda_1(\boldsymbol{M},\beta) < \lambda_2(\boldsymbol{M},\beta)$.*
c) *La fonction $f_2$ admet exactement deux domaines nodaux.*
d) *La valeur propre $\lambda_1(\beta,\boldsymbol{M})$ est caractérisée pour être la seule valeur propre à posséder une (toute) fonction propre de signe constant.*

*Hint* (a) C'est le théorème de Courant. (b) On remarque que si $f$ et $f'$ appartiennent à $\mathcal{H}_{\lambda_1,\beta}(\boldsymbol{M})$, toute combinaison linéaire $af+bf'$ est de signe constant sur $\boldsymbol{M}$ puisque c'est encore une fonction de $\mathcal{H}_{\lambda_1,\beta}(\boldsymbol{M})$, mais ce n'est possible que si $f_1$ et $f_2$ sont colinéaires (exercice). (c) Dans des variétés compactes à bord, on dispose de la formule de Green([19])

$$\int_{\boldsymbol{M}} \big\{f_1\Delta(f_2) - \Delta(f_1)f_2\big\}dV = \int_{\partial\boldsymbol{M}} \big\{f_1\partial_{\vec{n}}(f_2) - \partial_{\vec{n}}(f_1)f_2\big\}dA. \qquad (*)$$

où $\partial_{\vec{n}}$ désigne la dérivée normale sortant du bord (6.3). Lorsque les $f_i$ vérifient la même condition au bord $\beta$, le second membre de $(*)$ s'annule, et si $f_i \in \mathcal{H}_{\lambda_i,\beta}(\boldsymbol{M})$, on a la relation d'"*orthogonalité*"

$$\int_{\boldsymbol{M}} f_1(V)f_2(V)dV = 0,$$

puisque $\lambda_1 \neq \lambda_2$. Or, ceci est impossible si les deux fonctions gardent leur signe sur $\boldsymbol{M}$ et comme c'est le cas pour $f_1$, c'est que $f_2$ change de signe et possède par conséquent au moins deux domaines nodaux; la majoration de Courant permet de conclure. Enfin, pour (d), on raisonne comme pour (c): l'orthogonalité de $\mathcal{H}_{\lambda_1}(\boldsymbol{M})$ vis-à-vis de chacun des $\mathcal{H}_{\lambda_k}(\boldsymbol{M})$, $k > 1$, implique que les fonctions dans $\mathcal{H}_{\lambda_1}(\boldsymbol{M})$ sont les seules à ne pas changer de signe. □

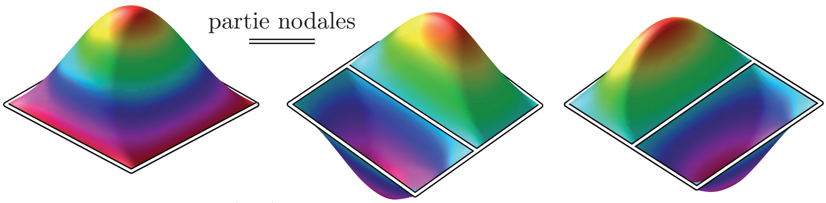
**9.6 Exemple de deux premières valeurs propres.** Dans le cas de la membrane rectangulaire $\mathbf{1}$♦4.3, pour la condition de Dirichlet

$$\lambda_1 = (1^2+1^2)\pi^2 = 2\pi^2 \quad \text{et} \quad \lambda_2 = (1^2+2^2)\pi^2 = 5\pi^2$$

On a les fonctions propres

([19]) Chapitre I du livre d'Isaac Chavel [Cha], formule (40), page 7.

- $f_1 := \sin(\pi x)\sin(\pi y) \in \mathcal{H}_{\lambda_1,D}(\boldsymbol{M})$,
- $\left\{ \begin{array}{l} f_{1,2} := \sin(\pi x)\sin(2\pi y) \\ f_{2,1} := \sin(2\pi x)\sin(\pi y) \end{array} \right\} \in \mathcal{H}_{\lambda_2,D}(\boldsymbol{M})$.




$f_{1,1}\,\mathbb{R} = \mathcal{H}_{\lambda_1,D}(\boldsymbol{M})$

$f_{1,2}\,\mathbb{R} \oplus f_{2,1}\,\mathbb{R} = \mathcal{H}_{\lambda_2,D}(\boldsymbol{M})$

## References

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