# Equivariant Poincaré Duality, Equivariant Gysin Morphisms and Euler Classes

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The is an introduction to equivariant Poincaré duality of oriented $G$-manifolds, for a compact Lie group $G$, and to equivariant Gysin morphisms both for proper and for non-proper maps. As applications, we derive: the equivariant Gysin exact sequence, the equivariant Lefschetz fixed point theorem, the equivariant Thom isomorphism and the definition of the equivariant Thom and Euler classes.

While, for educational purposes, we have chosen to confine ourselves to the very basic setup of smooth manifolds and de Rham equivariant cohomology, and hence the non-trivial restriction that the coefficients field must be at least that of real numbers $\mathbb{R}$, we present in the last section a general approach, based on Grothendieck-Verdier duality formalism, whereby the same results can be proved for any field of arbitrary characteristic.

## Introduction

Given a compact Lie group $G$, we prove the $G$-equivariant Poincaré duality for an oriented $G$-manifold $M$. In doing this, we follow the approach of J.-L. Brylinski when he introduced the equivariant intersection cohomology for $G$-pseudomanifolds ([Br], 1992), restricting it to the category of smooth varieties and cohomology with coefficients in the field of real numbers $\mathbb{R}$ for pedagogical reasons only.

We work in the derived category of $\Omega_G$-differential graded modules $\mathrm{DGM}(\Omega_G)$, that we denote $\mathcal{D}^+(\mathrm{DGM}(\Omega_G))$. In this notation, $\Omega_G$ is the graded ring $S(\mathfrak{g}^\vee)^G$ of $G$-invariant real polynomial functions on $\mathfrak{g} := \mathrm{Lie}(G)$, where $\mathfrak{g}^\vee$ is homogeneous of degree 2. The differential in $\Omega_G$ is zero so that $\Omega_G$ coincides with its cohomology, denoted by $H_G$, which is the $G$-equivariant cohomology of a point. The basic duality in $\mathcal{D}^+(\mathrm{DGM}(\Omega_G))$ is the contravariant functor

$$\mathbb{R}Hom^\bullet_{\Omega_G}(\_,\Omega_G) : \mathcal{D}^+(\mathrm{DGM}(\Omega_G)) \rightsquigarrow \mathcal{D}(\mathrm{DGM}(\Omega_G)). \qquad (\diamond)$$

The complexes of $G$-equivariant differential forms $\Omega_G(M)$ and $\Omega_{G,\mathrm{c}}(M)$ (with compact support), which compute the (de Rham) equivariant cohomologies $H_G(M)$ and $H_{G,\mathrm{c}}(M)$ respectively, belong to $\mathrm{DGM}(\Omega_G)$ and, if $M$ is an oriented $G$-manifold of dimension $d_M$, we can consider the following equivariant analog of the familiar Poincaré pairing,

$$\mathbb{P}_{M,G} : \Omega_G(M) \otimes \Omega_{G,\mathrm{c}}(M) \to \Omega_G\,, \quad \alpha \otimes \beta \mapsto \int_M \alpha \wedge \beta,$$

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which leads to the equivariant Poincaré left and right adjunctions, i.e. the morphisms of $\Omega_G$-differential graded modules:

$$\begin{aligned} \mathbb{D}_{G,M} :\ \Omega_G(M)[d_M] &\to \mathit{Hom}^{\bullet}_{\Omega_G}\big(\Omega_{G,\mathrm{c}}(M),\Omega_G\big) \\ \mathbb{D}'_{G,M} : \Omega_{G,\mathrm{c}}(M)[d_M] &\to \mathit{Hom}^{\bullet}_{\Omega_G}\big(\Omega_G(M),\Omega_G\big) \end{aligned}$$

for which we prove the equivariant analog of the Poincaré duality theorem.

*Theorem (4.5.1). Let $G$ be a compact connected Lie group, and $M$ an oriented $G$-manifold of dimension $d_M$.*

a) *The morphism $\mathbb{D}_{G,M} : \Omega_G(M)[d_M] \to \mathbb{R}\mathit{Hom}^{\bullet}_{\Omega_G}\big(\Omega_{G,\mathrm{c}}(M),\Omega_G\big)$ is an* isomorphism *in $\mathcal{D}^+(\mathrm{DGM}(\Omega_G))$.*

b) *There are natural spectral sequences converging to $H_G(M)[d_M]$*

$$\begin{cases} \mathbb{E}_2^{p,q}(M) = \big(\mathit{Ext}^p_{H_G}\big(H_{G,\mathrm{c}}(M),H_G\big)\big)^{[q]} \Rightarrow H_G^{p+q}(M)[d_M] \\ \mathbb{F}_2^{p,q}(M) = \ H_G^p \otimes_{\mathbb{R}} \mathit{Hom}^{\bullet}_{\mathbb{R}}(H^q_{\mathrm{c}}(M),\mathbb{R}) \ \Rightarrow H_G^{p+q}(M)[d_M] \end{cases}$$

*where $[q]$ denotes the degree of homogeneity in the graded vector space.*

c) *If $\dim(H_\mathrm{c}(M)) < \infty$, the morphism*

$$\mathbb{D}'_{G,M} : \Omega_{G,\mathrm{c}}(M)[d_M] \to \mathbb{R}\mathit{Hom}^{\bullet}_{\Omega_G}\big(\Omega_G(M),\Omega_G\big)$$

*is an isomorphism in $\mathcal{D}^+(\mathrm{DGM}(\Omega_G))$, and the analogues of* (b) *is verified.*

Let G-Man (resp. G-Man$_{\mathrm{pr}}$) denote the category of $G$-manifolds and $G$-equivariant differential (resp. proper) maps.

If $f : M \to M'$ is a proper map between oriented manifolds of dimensions $d_M$ and $d_{M'}$ respectively, the pullback morphism $f^* : \Omega_{G,\mathrm{c}}(M') \to \Omega_{G,\mathrm{c}}(M)$ induces in $\mathcal{D}^+(\mathrm{DGM}(\Omega_G))$, by equivariant Poincaré duality, *the Gysin morphism for proper maps*:

$$f_! := (\mathbb{D}_{G,M})^{-1} \circ f^* \circ \mathbb{D}_{G,M'} : \Omega_G(M)[d_M] \to \Omega_G(M')[d_{M'}],$$

whence the Gysin covariant functor

$$(\_)_! : \text{G-Man}_{\mathrm{pr}} \rightsquigarrow \mathcal{D}^+(\mathrm{DGM}(\Omega_G)), \quad M \rightsquigarrow \Omega_G(M)[d_M], \quad f \rightsquigarrow f_!\,.$$

When, in addition, $\dim(H_\mathrm{c}(M)) < \infty$, we can proceed in the same way with non-proper maps $f : M \to M'$ and thus define in $\mathcal{D}^+(\mathrm{DGM}(\Omega_G))$

$$f_* := (\mathbb{D}'_{G,M})^{-1} \circ f^* \circ \mathbb{D}'_{G,M'} : \Omega_{G,\mathrm{c}}(M)[d_M] \to \Omega_{G,\mathrm{c}}(M')[d_{M'}],$$

and the Gysin covariant functor

$$(\_)_* : \text{G-Man} \rightsquigarrow \mathcal{D}^+(\mathrm{DGM}(\Omega_G)), \quad M \rightsquigarrow \Omega_{G,\mathrm{c}}(M)[d_M], \quad f \rightsquigarrow f_*\,.$$

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One important feature of this approach is that, whereas Gysin morphisms are generally explicitly defined only for projections $p : M \times N \to N$ and closed embeddings $i : M \hookrightarrow M \times N$, and then extended to any $f = M \to N$ by setting $f_* := p_* \circ \mathrm{Gr}(f)_*$, where $\mathrm{Gr}(f) : M \to M \times N$ is the graph map, the formal approach is indifferent to the nature of $f$. This flexibility has many theoretical advantages:

- It automatically justifies functoriality of Gysin morphisms.

- Given a closed inclusion of oriented manifolds $N \subseteq M$, we dualize the exact triangle $\Omega_{G,\mathrm{c}}(M\backslash N) \to \Omega_{G,\mathrm{c}}(M) \to \Omega_{G,\mathrm{c}}(N) \to$ by ($\diamond$) and immediately get, via equivariant Poincaré duality, the Gysin exact triangle:

$$\Omega_G(N)[d_N] \to \Omega_G(M)[d_M] \to \Omega_G(M\backslash N)[d_M] \to$$

  leading to *the Gysin exact sequence of equivariant cohomology*:

$$\to H_G^i(N) \to H_G^{i+d_M-d_N}(M) \to H_G^{i+d_M-d_N}(M\backslash N) \to$$

- In the same easy way, if $E \to B$ is a vector bundle with fibers of dimension $n$ and oriented base space, the zero section map $\iota : B \to E$, being proper, automatically induces the Gysin morphism $\iota_! : \Omega_G(B) \to \Omega_G(E)[n]$ in $\mathcal{D}^+(\mathrm{DGM}(\Omega_G))$, which is an isomorphism. We deduce the *equivariant Thom isomorphism* of $H_G$-graded modules $\iota_! : H_G^i(B) \simeq H_G^{i+n}(E)$, the *Thom equivariant class* then being $\iota_!(1) \in H_G^n(E)$.
- The equivariant Euler classes are also easily introduced for any closed embedding $i : N \subseteq M$ of oriented manifolds, by setting $\mathrm{Eu}_G\big(N,M\big) := i^* i_!(1)$ where $i^* i_! : H_G(N) \to H_G(N)$ is the push-pull operator.

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The book is organized in four parts.

– The first part consists only of section **1**, which summarizes familiar results and constructions in ordinary (nonequivariant) cohomology related to Poincaré duality. It is here that we set the formalism to introduce Gysin morphisms as Poincaré duals of the usual pullback morphisms, formalism that we will extend to the equivariant framework.

– The second part consists of sections **2** to **5**. Following a brief review of the origins of equivariant cohomology in section **2**, we recall in section **3** standard definitions and constructions in equivariant cohomology of manifolds, and in section **4** we integrate them to the Grothendieck-Verdier's framework, which allows us to state in a formal way the equivariant analog of Poincaré duality and, thereby, to introduce Gysin morphisms and related constructions in section **5**.

– The third part, that consists of sections **6** and **7**, we confine ourselves to the case where $G$ is a torus $T$ and address localization questions related to Gysin morphisms. For example, we prove the following localization theorem.

*Proposition (7.3.1). Let $M$ be an oriented $T$-manifold of finite orbit type. For any closed subgroup $H$ of $T$, denote by $\iota_H : M^H \hookrightarrow M$ the inclusion. The following morphisms of $H_T$-graded modules are isomorphisms modulo torsion.*

$$\begin{aligned} \iota_{H!} &: H_T(M^H)[d_{M^H}] \to H_T(M)[d_M] \\ \iota_{H*} &: H_{T,\mathrm{c}}(M^H)[d_{M^H}] \to H_{T,\mathrm{c}}(M)[d_M] \end{aligned}$$

– In the last part, section 8, we briefly explain changes to be done in order to work with coefficients in any field $\Bbbk$ of arbitrary characteristic. Among other changes, the replacement of the category $\mathcal{D}(\mathrm{DGM}(\Omega_G))$ by the derived category $\mathcal{D}(\mathbb{B}G;\Bbbk)$ of the category of sheaves of $\Bbbk$-vector spaces over the classifying space $\mathbb{B}G$, replacement that entails to substitute the complexes of equivariant differential forms $\Omega_{G,\mathrm{c}}(M)$ and $\Omega_G(M)$ respectively by the complexes $\mathbb{R}p_!(\underline{\Bbbk}_{M_G})$ and $\mathbb{R}p_*(\underline{\Bbbk}_{M_G})$.

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These notes started off as an appendix to the forthcoming book of Loring Tu: *Introductory Lectures on Equivariant Cohomology* ([1]), but as it soon became clear that they went beyond the level originally intended, the project was abandoned. We were, in the end, encouraged to publish by the flexibility of our approach, which enables parallel treatment of the equivariant and nonequivariant cases, as well as change of coefficient field from the field of real numbers to fields of arbitrary characteristic.

We assume that the reader has a good level in algebraic topology and is familiar with sheaf cohomology and with the language of derived categories as they appear in the framework of Grothendieck-Verdier duality, for example, in the reference books by Kashiwara-Schapira [$\mathrm{KS}_1$, $\mathrm{KS}_2$]. We also assume that the reader knows the basics of equivariant cohomology, both: as the ordinary cohomology (with arbitrary coefficients) of the Borel construction of $G$-spaces, and as the cohomology of the equivariant de Rham complex in the case of $G$-manifolds. The usual references for these topics are the books of W. Hsiang [H] and Allday-Puppe [AP], for ordinary cohomology, and the book of Guillemin-Sternberg [GS] for de Rham cohomology. Among these books, only [AP] addresses the questions of equivariant Poincaré duality and equivariant Gysin morphisms, and while our approaches are close, at the time they wrote the book, they gave only partial results, which is a further reason for the publication of the present text.

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[1] In Annals of Mathematics Studies, Princeton University Press.

## Table of Contents

# 1. Nonequivariant Background

## 1.1. Category of Cochain Complexes

**1.1.1. Fields in Use.** We denote by $\Bbbk$ an arbitrary field of coefficients, but, as soon as we will consider differential forms on manifolds and de Rham cohomology, it will be specialized to the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. In section 8 we explain changes to be done in order to overcome this restriction.

**1.1.2. Vector Spaces Pairings.** Whenever $\Bbbk$ is understood the expression "*vector space*" means vector space over $\Bbbk$. If $V$ is a vector space, we denote by $V^\vee := \mathrm{Hom}_\Bbbk(V,\Bbbk)$ its *dual*.

Given a bilinear map $\beta : V \times W \to \Bbbk$, also called *a pairing*, consider the two linear maps $\gamma_\beta : V \to W^\vee$ and $\rho_\beta : W \to V^\vee$, defined by $\gamma_\beta(v)(w) = \rho_\beta(w)(v) := \beta(v,w)$ and respectively called the *left and right adjoint maps associated with* $\beta$. One says that $\beta$ *is a nondegenerate pairing* whenever the adjoint maps are injective, and one says that $\beta$ *is a perfect pairing* whenever they are bijective. For example, the canonical pairing $V^\vee \times V \to \Bbbk$, $(\lambda,v) \mapsto \lambda(v)$, is always nondegenerate and it is perfect if and only if $V$ is finite dimensional.

**1.1.3. Graded Vector Spaces.** A *graded space* is a family $V := \{V^m\}_{m\in\mathbb{Z}}$ of vector spaces. A *graded homomorphism* $\boldsymbol{\alpha} : V \to W$ *of degree* $d = \deg(\boldsymbol{\alpha})$ is a family of linear maps $\{\alpha_m : V^m \to W^{m+d}\}_{m\in\mathbb{Z}}$, composition of such is defined degree by degree, i.e. $\boldsymbol{\beta}\circ\boldsymbol{\alpha} = \{\beta_{m+d}\circ\alpha_m\}_{m\in\mathbb{Z}}$. One has $\deg(\boldsymbol{\alpha}\circ\boldsymbol{\beta}) = \deg\boldsymbol{\alpha} + \deg\boldsymbol{\beta}$.

We denote by $\mathit{Homgr}^d_\Bbbk(V,W)$ the space of graded homomorphisms of degree $d$ and by $\mathit{Hom}^\bullet_\Bbbk(V,W)$ the graded space of all graded homomorphisms, i.e. the family

$$\mathit{Hom}^\bullet_\Bbbk(V,W) = \left\{\mathit{Homgr}^d_\Bbbk(V,W)\right\}_{d\in\mathbb{Z}}.$$

When $d = 0$, we may write $\mathit{Homgr}_\Bbbk(V,W)$ for $\mathit{Homgr}^0_\Bbbk(V,W)$.

**1.1.4** The *category* $\mathrm{GV}(\Bbbk)$ *of graded vector spaces* is the category whose objects are graded spaces and whose *morphisms* are the graded homomorphisms of degree 0. We denote equivalently $\mathit{Mor}_{\mathrm{GV}(\Bbbk)}(V,W) := \mathit{Homgr}_\Bbbk(V,W)$ the set of morphisms from $V$ to $W$.

**1.1.5. Differential Graded Vector Space.** A *differential graded vector space* $(V,d)$, *a complex* in short, is a graded vector space $V$ together a *differential* or *coboundary* $d : \mathit{Endgr}^1(V)$ such that $d^2 = 0$. A *morphism of complexes* $\boldsymbol{\alpha} : (V,d) \to (V',d')$ is a morphism $\boldsymbol{\alpha} \in \mathit{Homgr}_\Bbbk(V,V')$ commuting with differentials, i.e. $\boldsymbol{\alpha}\circ d = d'\circ\boldsymbol{\alpha}$. The complexes and their morphisms constitute the *category* $\mathrm{DGM}(\Bbbk)$ *of differential graded vector spaces*.

**1.1.6** A morphism of complexes $\boldsymbol{\alpha} : (V,d) \to (V',d')$ induces a morphism between the graded spaces of cohomologies $H(\boldsymbol{\alpha}) : H(V,d) \to H(V',d')$. The morphism $\boldsymbol{\alpha}$ is a *quasi-isomorphism, quasi-injection, quasi-surjection*, whenever $H(\boldsymbol{\alpha})$ is respectively, an isomorphism, injection, surjection.

**1.1.7** Let $m \in \mathbb{Z}$. If $L$ is a vector space, we denote by $L[m]$ the graded space defined by $L[m]^{-m} = L$ and $L[m]^n = 0$ if $n + m \neq 0$. If $\alpha : V \to W$ is a linear map, we denote by $\alpha[m] : V[m] \to W[m]$ the morphism of graded spaces equal to $\alpha$ in degree $-m$ and 0 otherwise. The correspondence $L \rightsquigarrow (L[m],\mathbf{0})$ and $\alpha \rightsquigarrow \alpha[m]$ is a functor

$$[m] : \mathrm{Vec}(\Bbbk) \to \mathrm{DGM}(\Bbbk).$$

More generally, If $V$ is a graded space we denote by $V[m]$ the graded space $(V[m])^i = V^{m+i}$, and if $\boldsymbol{\alpha} : V \to W$ is a graded homomorphism we denote by $\boldsymbol{\alpha}[m] : V[m] \to W[m]$ the graded homomorphism $\boldsymbol{\alpha}[m]_i = \alpha_{m+i}$. Next, if $(V,d)$ is a complex, $(V,d)[m]$ is the complex $\big(V[m],(-1)^m d[m]\big)$. The correspondence $(V,d) \rightsquigarrow (V,d)[m]$, $\boldsymbol{\alpha} \rightsquigarrow \boldsymbol{\alpha}[m]$ is the *$m$-th shift functor*

$$[m] : \mathrm{DGM}(\Bbbk) \rightsquigarrow \mathrm{DGM}(\Bbbk).$$

**1.1.8** Given two complexes $(V,d)$ and $(V',d')$, we recall the definition of the complexes

$$\big(\mathit{Hom}^\bullet_\Bbbk(V,V'),D\big) \quad \text{and} \quad \big((V \otimes_\Bbbk V')^\bullet,\boldsymbol{\Delta}\big).$$

As graded vector spaces they are

$$m \in \mathbb{Z} \mapsto \begin{cases} \mathit{Hom}^m_\Bbbk(V,V') = \mathit{Homgr}^m_\Bbbk(V,V') \\ (V \otimes_\Bbbk V')^m = \prod_{b+a=m} V^a \otimes_\Bbbk V'^b \end{cases}$$

and their differentials are

$$\begin{cases} D_m(f) = d' \circ f - (-1)^m f \circ d \\ \Delta_m(v \otimes v') = d(v) \otimes v' + (-1)^{|v|} v \otimes d'(v') \end{cases}$$

where $v \otimes v' \in V^{|v|} \otimes V'^{|v'|}$. ([2])

**1.1.9. Exercise.** Verify that the following complexes coincide as graded vector spaces but not as complexes even though they are naturally isomorphic.

$$\mathit{Hom}^\bullet_\Bbbk(V[n],W[m]) \simeq \mathit{Hom}^\bullet_\Bbbk(V,W)[m-n]$$
$$V[n] \otimes W[m] \simeq (V \otimes W)[m+n]$$

**1.1.10** Given a morphism of complexes $\boldsymbol{\varphi} : (V,d) \to (W,d)$ the map

$$\mathit{Hom}^m_\Bbbk(W,V') \to \mathit{Hom}^m_\Bbbk(V,V'), \quad \boldsymbol{\alpha} \mapsto \boldsymbol{\alpha} \circ \boldsymbol{\varphi},$$

is well defined for all $m \in \mathbb{Z}$ and commutes with differentials so that one has a morphism of complexes

$$\mathit{Hom}^\bullet_\Bbbk(\boldsymbol{\alpha},V') : \big(\mathit{Hom}^\bullet_\Bbbk(W,V'),D\big) \to \big(\mathit{Hom}^\bullet_\Bbbk(V,V'),D\big).$$

The correspondence $(V,d) \rightsquigarrow \mathit{Hom}^\bullet_\Bbbk(V,V')$, $\boldsymbol{\alpha} \rightsquigarrow \mathit{Hom}^\bullet_\Bbbk(\boldsymbol{\alpha},V')$ is then a *contravariant* functor

$$\mathit{Hom}^\bullet_\Bbbk(\_,V') : \mathrm{DGM}(\Bbbk) \rightsquigarrow \mathrm{DGM}(\Bbbk).$$

[2] It is worth noting that these formulas are inspired by the super Lie bracket equalities

$$[\![d,f]\!] = df - (-1)^{|d||f|} fd \quad \text{and} \quad [\![d,ab]\!] = [\![d,a]\!] + (-1)^{|a||d|} a[\![d,b]\!]$$

where $[\![d,[\![d,\_]\!]]\!] = 0$ is an immediate consequence of $|d| = 1$ and $d^2 = 0$.

**1.1.11. The Dual Complex.** The functor $Hom^\bullet_{\Bbbk}(\_,\Bbbk[0])$ is the *duality functor*, simply denoted by $(\_)^\vee := Hom^\bullet_{\Bbbk}(\_,\Bbbk[0])$

$$(\_)^\vee : \mathrm{DGM}(\Bbbk) \rightsquigarrow \mathrm{DGM}(\Bbbk).$$

The complex $(V,d)^\vee$ is called *the dual complex associated with* $(V,d)$. One has

$$(V^\vee)^m = \mathrm{Hom}_{\Bbbk}(V^{-m},\Bbbk), \quad D_m = (-1)^{m+1\,t}d_{-(m+1)}$$

**1.1.12. Remark.** One must take care that the natural embedding of vector spaces $V \subseteq V^{\vee\vee}$ gives only an inclusion of complexes $(V,-d) \subseteq (V,d)^{\vee\vee}$ where the sign of the differential has changed ! The canonical isomorphism

$$\boldsymbol{\epsilon} : (V,d) \to (V,-d), \quad \epsilon_m = (-1)^m \mathrm{id}_{V^m} \tag{$\boldsymbol{\epsilon}$}$$

is then necessary to get a canonical embedding $(V,d) \hookrightarrow (V,d)^{\vee\vee}$.

The next statement will be used without mention, it is left as an exercise.

**1.1.13. Proposition**

a) *A morphism of complexes* $\boldsymbol{\alpha} : (V,d) \to (V',d')$ *is a quasi-isomorphism if and only if* $\boldsymbol{\alpha}^\vee$ *is so.*

b) *There exists a canonical isomorphism between the cohomology of the dual and the dual of the cohomology, i.e.*

$$h\big((V,d)^\vee\big) \xrightarrow[\simeq]{} \big(h(V,d)\big)^\vee.$$

*where* $h$ *denotes the graded vector space of the cohomologies of a complex.*

## 1.2. Some Categories of Manifolds

**1.2.1. Manifolds.** The names *manifold* and *map of manifolds* will be shortcuts for *real differentiable manifold* and *differentiable map of class* $\mathcal{C}^\infty$, in short *smooth map*. A manifold is equidimensional if all its connected components are of the same dimension, in that case "$d_M$" denotes this common dimension.

**1.2.2** Man (resp. $\mathrm{Man}^{\mathrm{or}}$) denotes the *category of manifolds (resp. oriented) and smooth maps*. Over Man one has the *(real) de Rham complex* contravariant functor

$$\Omega(\_) : \mathrm{Man} \rightsquigarrow \mathrm{DGM}(\mathbb{R})$$

and the *de Rham cohomology* contravariant functor

$$H(\_) : \mathrm{Man} \rightsquigarrow \mathrm{Mod}^{\mathbb{N}}(\mathbb{R}).$$

**1.2.3** $\mathrm{Man}_{\mathrm{pr}}$ (resp. $\mathrm{Man}^{\mathrm{or}}_{\mathrm{pr}}$) denotes the subcategory of Man (resp. $\mathrm{Man}^{\mathrm{or}}$) with the same objects but with only **proper** maps. Over $\mathrm{Man}_{\mathrm{pr}}$ one has, in addition to the previous functors, the *(real) de Rham complex with* ***compact*** *supports* contravariant functor:

$$\Omega_{\mathrm{c}}(\_) : \mathrm{Man}_{\mathrm{pr}} \rightsquigarrow \mathrm{DGM}(\mathbb{R})$$

and the *de Rham cohomology with compact support* contravariant functor

$$H_{\mathrm{c}}(\_) : \mathrm{Man}_{\mathrm{pr}} \rightsquigarrow \mathrm{DGM}(\mathbb{R}).$$

The inclusion $\Omega_{\mathrm{c}}(\_) \subseteq \Omega(\_)$ induces a morphism of functors $H_{\mathrm{c}}(\_) \to H(\_)$.

**1.2.4. *G*-Manifolds.** Let $G$ denote a Lie group. A manifold endowed with a smooth action of $G$ is called *a $G$-manifold*. A map $f : M \to N$ between $G$-manifolds is called *a $G$-equivariant* if it commutes with the action of $G$. The class of $G$-manifolds and $G$-equivariant maps constitutes the category $G$-Man. The categories $G\text{-Man}^{\mathrm{or}}$, $G\text{-Man}_{\mathrm{pr}}$, $G\text{-Man}_{\mathrm{pr}}^{\mathrm{or}}$ are the analogues of those already introduced in this section.

## 1.3. Poincaré Pairing

The reference for this section is [BT] (**I** §5 p. 44). Let $M$ be an oriented manifold. The composition of the bilinear map $\Omega^{d_M-i}(M)\otimes\Omega_{\mathrm{c}}^{i}(M)\to\Omega_{\mathrm{c}}^{d_M}(M)$, $\alpha\otimes\beta\mapsto\alpha\wedge\beta$, with integration $\int_M : \Omega_{\mathrm{c}}^{d_M}(M)\to\mathbb{R}$, gives rise to a nondegenerate pairing

$$\mathbb{P}_M : \Omega^{d_M-i}(M)\otimes\Omega_{\mathrm{c}}^{i}(M)\to\mathbb{R},\quad \alpha\otimes\beta\mapsto\int_M\alpha\wedge\beta, \tag{$\mathbb{P}$}$$

inducing the *Poincaré pairing in cohomology*

$$\mathscr{P}_M : H^{d_M-i}(M)\otimes H_{\mathrm{c}}^{i}(M)\to\mathbb{R},\quad [\alpha]\otimes[\beta]\mapsto\int_M[\alpha]\cup[\beta]. \tag{$\mathscr{P}$}$$

The left adjoint map associated with $\mathbb{P}$ is

$$\begin{aligned} \mathbb{D}_M : \Omega^{d_M-i}(M) &\longrightarrow \Omega_{\mathrm{c}}^{i}(M)^\vee \\ \alpha \quad &\longmapsto \mathbb{D}(\alpha) := \Big(\beta\mapsto\int_M\alpha\wedge\beta\Big) \end{aligned} \tag{$\mathbb{D}$}$$

and one has

$$\begin{aligned} \mathbb{D}_M\big((-1)^{d_M}d\alpha\big)(\beta) &= \int_M(-1)^{d_M}d\alpha\wedge\beta \\ &= \int_M(-1)^{d_M+|\alpha|+1}\alpha\wedge d\beta = (-1)^{|\beta|}\mathbb{D}_M(\alpha)(d\beta), \end{aligned}$$

Hence, following the conventions introduced in 1.1.7 and 1.1.8, $\mathbb{D}_M$ is a morphism of complexes from $\Omega(M)[d_M]$ to $\Omega_{\mathrm{c}}(M)^\vee$.

**1.3.1. Exercise.** Show that ($\mathbb{P}$) is a nondegenerate pairing.

**1.3.2. Theorem (Poincaré duality).** *Let $M$ be an oriented manifold.*

a) *The morphism of complexes, called* the Poincaré morphism,

$$\mathbb{D}_M : \Omega(M)[d_M]\hookrightarrow\Omega_{\mathrm{c}}(M)^\vee \tag{$*$}$$

*is a quasi-isomorphism, i.e. the morphism it induces in cohomology*

$$\mathscr{D}_M : H(M)[d_M]\xrightarrow[\simeq]{}H_{\mathrm{c}}(M)^\vee, \tag{$**$}$$

*is an isomorphism.*

b) *The Poincaré pairing in cohomology*

$$\mathscr{P}_M : H(M)\otimes H_c(M)\to\mathbb{R} \tag{$***$}$$

*is* nondegenerate, *and it is* perfect *(1.1.2) if and only if* $\dim(H(M))<+\infty$.

*Proof.* The (a) part (cf. [BT] p. 44–, for details) states the bijectivity of the left adjoint map associated with $\mathscr{P}$. Then, for each fixed $i$, one obtains by duality the bijectivity of $\mathscr{D}^\vee_{M,i} : (H_c^{d_M-i}(M))^{\vee\vee} \to H^i(M)^\vee$ and the composition of this map with the canonical embedding $\epsilon_i : H_c^{d_M-i} \hookrightarrow (H_c^{d_M-i})^{\vee\vee}$ is the right adjoint map $\rho_{\mathscr{P}} : H_c(M)[d_M] \to H(M)^\vee$ (see 1.1.2). The "finite dimensional" condition then ensures the bijectivity of $\epsilon_i$, hence of $\rho_{\mathscr{P}}$. □

**1.3.3. Exercise.** Let $M$ and $N$ be oriented manifolds. We denote by

$$I\!D(\_) : \Omega(\_)[d_\_] \to \Omega(\_)^\vee_c\,, \quad I\!D(\alpha)(\beta) = \int \alpha \wedge \beta\,,$$

the left adjoint map of the Poincaré pairing, and by

$$I\!D'(\_) : \Omega_c(\_)[d_\_] \to \Omega(\_)^\vee\,, \quad I\!D'(\beta)(\alpha) = \int \alpha \wedge \beta\,,$$

the right adjoint map.

A pair $(L,R)$ of morphisms of complexes

$$L : \Omega(N) \to \Omega(M) \quad \text{and} \quad R : \Omega_c(M)[d_M] \to \Omega_c(N)[d_N]$$

is a *(Poincaré) adjoint pair* whenever

$$\int_M L(\alpha) \wedge \beta = \int_N \alpha \wedge R(\beta)$$

for all $\alpha \in \Omega(N)$ and $\beta \in \Omega_c(M)$. Show that

a) If $(L,R_1)$ and $(L,R_2)$ are adjoint pairs, then $R_1 = R_2$. One says that $R := R_1$ is the *(Poincaré) right adjoint of* $L$.

b) If $(L_1,R)$ and $(L_2,R)$ are adjoint pairs, then $L_1 = L_2$. One says that $L := L_1$ is the *(Poincaré) left adjoint of* $R$.

c) If $(L,R)$ is an adjoint pair, then

$$I\!D \circ L = R^\vee \circ I\!D\,, \qquad I\!D' \circ R = L^\vee \circ I\!D'\,,$$

i.e. the following diagrams are commutative

$$\begin{tikzcd} \Omega(M) \arrow[r, hook, "I\!D_M", "(\simeq)"'] & \Omega_c(M)^\vee \\ \Omega(N) \arrow[u, "L"] \arrow[r, hook, "I\!D_N", "(\simeq)"'] & \Omega_c(N)^\vee \arrow[u, "R^\vee"'] \end{tikzcd} \qquad \begin{tikzcd} \Omega_c(M) \arrow[r, hook, "I\!D'_M"] \arrow[d, "R"'] & \Omega(M)^\vee \arrow[d, "L^\vee"] \\ \Omega_c(N) \arrow[r, hook, "I\!D'_N"] & \Omega(N)^\vee \end{tikzcd}$$

d) Do the exercise in the cohomological framework, i.e. replace Poincaré pairing $(I\!P)$ by $(\mathscr{P})$, $I\!D$ by $\mathscr{D} : H[d] \to H_c^\vee$, $I\!D'$ by $\mathscr{D}' : H_c[d] \to H^\vee$, and define the notion of *(Poincaré) adjoint pair in cohomology*.

Show that if $(L,R)$ is an adjoint pair of morphisms of complexes, then $(H(L),H_c(R))$ is an adjoint pair in cohomology so that one has

$$\mathscr{D} \circ H(L) = H_c(R)^\vee \circ \mathscr{D}\,, \qquad \mathscr{D}' \circ H_c(R) = H(L)^\vee \circ \mathscr{D}'\,.$$

In particular, $H(L)$ *identifies with the dual of* $H_c(R)$ *via Poincaré duality.*

**1.3.4. Remark.** We shall see that, given $f : M \to N$, the pullback morphism $f^* : \Omega(N) \to \Omega(M)$ may or may not admit a right Poincaré adjoint at the complexes level, but that it will always do so at the cohomology level, this right adjoint is the Gysin morphism $f_* : H_c(M) \to H_c(N)$, so that $(H(f^*),f_*)$ is a

Poincaré adjoint pair in cohomology. On the other hand, when $f$ is a proper map, the pullback $f^* : \Omega_c(N) \to \Omega_c(M)$ is also well defined and one may look for a left Poincaré adjoint to $f^*$, i.e. some morphism $L : \Omega(M)[d_M] \to \Omega(N)[d_N]$

$$\int_N L(\alpha) \wedge \beta = \int_M \alpha \wedge f^*(\beta).$$

Again, this will sometimes be possible at the complex level and will always be possible at the cohomology level leading to the notion of *the Gysin morphism for proper maps* $f_! : H(M) \to H(N)$, so that $(f_!, H_c(f^*))$ is a Poincaré adjoint pair in cohomology.

### 1.4. Manifolds and maps of Finite de Rham Type

**1.4.1. Definitions.** A manifold $M$ is said to be *of finite (de Rham) type* when its de Rham cohomology ring $H(M)$ is finite dimensional. A map between manifolds $f : M \to N$ si said to be *of finite (de Rham) type* if $N$ is the union of a countable ascending chain $\mathscr{U} := \{U_0 \subseteq U_1 \subseteq \cdots\}$ of open subspaces of finite type such that each subspace $f^{-1}(U_m) \subseteq M$ is of finite type.

**1.4.2. Remarks**

a) By Poincaré duality (1.3.2), $M$ is of finite type if and only if its compact support cohomology $H_c(M)$ is finite dimensional.

b) A compact manifold is of finite type ([BT] **5.3** pp. 42-43). An oriented manifold is of finite type if and only if its Poincaré pairing in cohomology is perfect (1.3.2-(b)). This will be soon used in our discussion of the Gysin morphism.

c) Since any manifold is the union of a countable ascending chain $\{\uparrow U_m\}$ of open submanifolds of finite type (*cf.* 1.4.4), any locally trivial fibration of manifolds $f : M \to N$ is of finite type (exercise).

**1.4.3. Ascending Chain Property.** Although general manifolds need not be of finite type, they are always the inductive limit of such. More precisely, any manifold $M$ is the union of an ascending chain $\{U_0 \subseteq U_1 \subseteq \cdots\}$ of open subsets of finite type of $M$. This weaker finiteness property, sufficient for our needs, is generally proved by a riemannian geometry argument ([3]). When a manifold is endowed with the action of a Lie group $G$, we will require in addition that each $U_n$ be $G$-stable.

**1.4.4. Proposition.** *Let $G$ be a compact Lie Group. A $G$-manifold $M$ is the union of a countable ascending chain $\mathscr{U} := \{U_0 \subseteq U_1 \subseteq \cdots\}$ of $G$-stable open subsets of $M$ of finite type.*

The next sections recall some facts needed in the proof of this proposition.

**1.4.5. Existence of Proper Functions.** Recall that a map between manifolds $f : M \to N$ is said to be *proper* whenever $f^{-1}(F)$ is compact for any

[3] In these notes, a *good cover of $M$* (also known as *Leray cover*) is a finite open cover $\mathscr{U} = \{U_i \mid i = 1,\ldots,r\}$ of $M$ such that all intersections $U_{i_1} \cap \ldots \cap U_{i_k}$ are either vacuous or acyclic ([BT], p. 5). The existence of good covers is established in *loc.cit.* §5, p. 42.

compact subset $F$ of $N$. The aim of this section is to show that on a $G$-manifold there are always proper $G$-invariant functions.

Since the existence of proper functions over compact manifolds is clear, let $M$ be a noncompact manifold. Fix a countable, locally finite cover $\mathscr{U} := \{U_n\}_{n\in\mathbb{N}}$ of $M$, where each $U_n$ is a *relatively compact* open subset of $M$, and note that the noncompactness of $M$ implies that the family is necessarily infinite. Next, fix a smooth partition of unity $\{\phi_n\}_{n\in\mathbb{N}}$ subordinate to $\mathscr{U}$. This means in particular that for each $n \in \mathbb{N}$, the equality $\phi_n(x) = 0$ holds whenever $x \notin U_n$. Then one has for every $N \in \mathbb{N}$

$$1 = \sum\nolimits_{n>N} \phi_n(x), \quad \forall x \notin U_0 \cup \cdots \cup U_N \,. \tag{$\diamond$}$$

Now, for every $x \in M$, the infinite sum $\Phi(x) := \sum_{n\in\mathbb{N}} n\cdot\phi_n(x)$ is finite and smooth with respect to $x \in M$, as it is a locally finite sum of smooth functions.

**1.4.6. Lemma.** *The positive function $\Phi : M \to \mathbb{R}$ is unbounded and proper.*

*Proof.* By property ($\diamond$) one has

$$\Phi(x) \geqslant \sum\nolimits_{n>N} n\cdot\phi_n(x) > N\Big(\sum\nolimits_{n>N} \phi_n(x)\Big) = N, \quad \forall x \notin U_0 \cup\cdots\cup U_N \,, \tag{$\diamond\diamond$}$$

which clearly implies that $\Phi$ is an unbounded function on $M$. Now, to see that $\Phi$ is proper, remark that if $F \subseteq \mathbb{R}$ is compact, then $F \subseteq [-N,N]$ for some $N \in \mathbb{N}$ and $\Phi^{-1}(F) \subseteq U_0 \cup \cdots \cup U_N$ by ($\diamond\diamond$). But the closure $\overline{U_0 \cup \cdots \cup U_N}$ is a compact subset of $M$ because each $\overline{U}_i$ is assumed compact. As a closed subset of a compact set, $\Phi^{-1}(F)$ is compact. □

As a corollary of the preceding lemma one has:

**1.4.7. Proposition.** *Manifolds $M$ endowed with a smooth action of a compact Lie group $G$ admit proper $G$-invariant positive functions $\Phi : M \to \mathbb{R}$.*

*Proof.* If $M$ is compact, any positive *constant* map $\Phi$ will do. If $M$ is not compact, let $\phi : M \to \mathbb{R}$ denote any unbounded proper positive function (see 1.4.6), and set:

$$\Phi(x) := \int_G \phi(g\cdot x)\; dg,$$

where $dg$ is a $G$-invariant form of top degree on $G$, such that $1 = \int_G dg$. The correspondence $x \mapsto \Phi(x)$ is clearly a well-defined nonnegative unbounded $G$-invariant function of $M$ into $\mathbb{R}$. Now, for each $N \in \mathbb{N}$, the set

$$M_N := G\cdot\phi^{-1}([-N,N])$$

is compact and $G$-stable, and if $y \notin M_N$, one has $\phi(g\cdot y) > N$ for all $g \in G$, so that

$$\Phi(y) = \int_G \phi(g\cdot y)\; dg > N\,, \tag{$\diamond\diamond\diamond$}$$

and properness of $\Phi$ follows by the same argument as in lemma 1.4.6: If $F$ is a compact subset of $\mathbb{R}$, then $F \subseteq [-N,N]$ for some $N \in \mathbb{N}$, and $\Phi^{-1}(F) \subseteq M_N$ by ($\diamond\diamond\diamond$). Then $\Phi^{-1}(F)$ is compact since it is closed in the compact set $M_N$. □

## 1.5. Manifolds With Boundary

**1.5.1. Proposition.** *The interior of a compact manifold with boundary is of finite type.*

*Proof.* Let $B$ be a compact manifold with boundary and let $M$ be its interior. Gluing $B$ with itself along its boundary $\partial B$, one gets the "double" $B \sqcup_{\partial B} B$, which is a compact manifold without boundary. Then, from the long exact sequence of de Rham cohomology with compact support (see 1.11.1-(a))

$$\cdots \longrightarrow H^i_c(M) \times H^i_c(M) \longrightarrow H^i(B \sqcup_{\partial B} B) \longrightarrow H^0 i(\partial B) \longrightarrow \cdots,$$

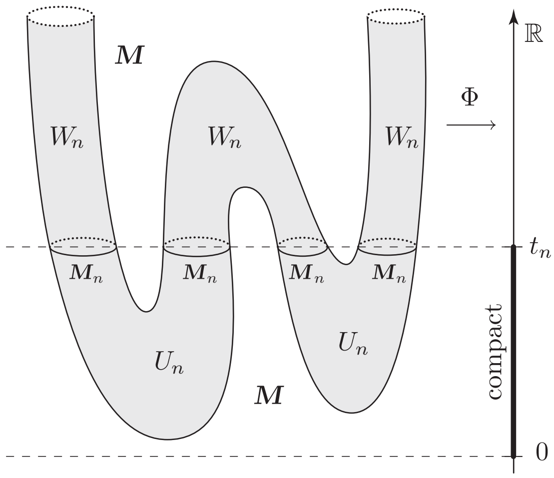
where $H^*(B \sqcup_{\partial B} B)$ and $H^*(\partial B)$ are finite-dimensional, the finiteness of $H^*_c(M)$ follows immediately. The finiteness of $H^*(M)$ results from Poincaré duality $H^*(M) \cong H^*_c(M)^\vee$. □

## 1.6. Proof of Proposition 1.4.4

Recall that the connected components of a manifold $M$ are always open and closed submanifolds of $M$. In particular, if $M = \coprod_{\mathfrak{i}\in\mathfrak{I}} C_\mathfrak{i}$ denotes the decomposition of $M$ in connected components, the indexing set $\mathfrak{I}$ is finite or countable, and the restriction of a proper function $\Phi : M \to \mathbb{R}$ to each $C_i$ remains proper.

If all the connected components of $M$ are compact, we may index them by natural numbers $C_0, C_1, \ldots$ and define $U_n := C_0 \cup C_1 \cup \cdots \cup C_n$. Each $U_n$ is then open in $M$ and is also a compact manifold, hence it is of finite type. The ascending chain $\{U_0 \subseteq U_1 \subseteq \cdots\}$ satisfies the conditions of the proposition.

If $M$ contains a noncompact connected component $C$, fix any proper positive $G$-invariant function $\Phi : M \to \mathbb{R}$, which is possible due to 1.4.7, and note that $\Phi(C)$ is necessarily unbounded, since otherwise $C \subseteq \Phi^{-1}([0,T])$ for some $T \in \mathbb{R}$, and $C$ would be compact as $\Phi$ is proper over $C$. Moreover, there exists $N \in \mathbb{N}$ such that $\Phi(M) \supseteq \Phi(C) \supseteq (]N,+\infty[)$, since $\Phi(C)$ is unbounded *and* connected. Now, by Sard's theorem, the interior of the set of *critical* values of $\Phi : M \to \mathbb{R}$ is empty so that there exists an unbounded increasing sequence of positive real numbers $\{N < t_0 < \cdots < t_n < \cdots\}_{n\in\mathbb{N}}$ which are *regular* values of $\Phi$.



Each subset $M_n := \Phi^{-1}(t_n)$ is then a submanifold of codimension 1 in $M$ and, moreover, it is compact and $G$-stable since $\Phi$ is proper and $G$-invariant. Similarly, the sets $U_n := \Phi^{-1}(]-\infty,t_n[)$ and $W_n := \Phi^{-1}(]t_n,+\infty[)$, clearly nonempty, are $G$-stable open subsets of $M$. One then easily checks that $\overline{U}_n = U_n \sqcup M_n$ and $\overline{W}_n = M_n \sqcup W_n$ are in fact $G$-manifolds with boundary $M_n$ embedded in $M$. Furthermore, $\overline{U}_n$ is compact as one has $\overline{U}_n := \Phi^{-1}(]-\infty,t_n]) = \Phi^{-1}([0,t_n])$ since $\Phi$ is positive.

We can then apply proposition 1.5.1 and state that $U_n$ is a $G$-stable open subset of finite type of $M$. The ascending chain $\{U_0 \subseteq U_1 \subseteq \cdots\}$ satisfies the conditions of the proposition. □

### 1.7. The Gysin Functor

In this section we dualize $\mathbb{D}_M : \Omega(M)[d_M] \to \Omega_c(M)^\vee$, getting an injection $\mathscr{D}'_M : H_c(M)[d_M] \hookrightarrow H(M)^\vee$ whose image will be shown to be functorial on the category $\mathrm{Man}^{\mathrm{or}}$ of oriented manifolds.

**1.7.1. The Right Adjoint Map.** In 1.3 we introduced the left adjoint map associated with Poincaré pairing, i.e. the quasi-isomorphism

$$\mathbb{D}_M : \Omega(M)[d_M] \xrightarrow[(\simeq)]{} \Omega_c(M)^\vee .$$

By duality, this map yields $\mathbb{D}_M^\vee : \Omega_c(M)^{\vee\vee} \to \Omega(M)[d_M]^\vee$ which is also a quasi-isomorphism and, composed with the embedding $\Omega_c(M) \subseteq \Omega_c(M)^{\vee\vee}$, gives rise to the injection and quasi-injection (1.3.1, 1.1.6)

$$\begin{tikzcd} \big(\Omega_c(M)[d_M],d\big) \arrow[r, hook, "\subseteq"] \arrow[rr, bend right, "\mathbb{D}'_M"] & \big(\Omega_c(M)^{\vee\vee}[d_M],-d\big) \arrow[r, "\mathbb{D}^\vee", "(\simeq)"'] & \big(\Omega(M)^\vee,-d\big) \end{tikzcd}$$

(See 1.1.11 for the sign of differentials.) One has (cf. 1.3.3)

$$\mathbb{D}'_M(\beta) = \Big(\alpha \mapsto \int_M \alpha \wedge \beta\Big),$$

which clearly it is the *right adjoint map associated with the Poincaré pairing* $\mathbb{P}$.

The following proposition paraphrases the statement 1.3.2-(b).

**1.7.2. Proposition.** *Let $M$ be an oriented manifold.*

a) *The morphism of complexes*

$$\mathbb{D}'_M : \big(\Omega_c(M)[d_M],d\big) \hookrightarrow \big(\Omega(M)^\vee,-d\big)$$

*is always an injection and a quasi-injection. We denote by*

$$\mathscr{D}'_M : H_c(M)[d_M] \hookrightarrow H(M)^\vee$$

*the induced injection in cohomology.*

b) *The morphism $\mathscr{D}'_M$ is an isomorphism if and only if $M$ is of finite type.*

**1.7.3. The Gysin Morphism.** The last statement shows that for oriented manifolds of finite type, compact support cohomology canonically coincides with the dual of arbitrary support cohomology so that if $M$ and $N$ are as such manifolds, the diagram

$$\begin{tikzcd} H_{\mathrm{c}}(M)[d_M] \arrow[r, hook, "\mathscr{D}'_M", "\simeq"'] \arrow[d, dashed, "f_*"'] \arrow[dr, phantom, "\oplus"] & H(M)^{\vee} \arrow[d, "H(f^*)^{\vee}"] \\ H_{\mathrm{c}}(N)[d_N] \arrow[r, hook, "\mathscr{D}'_N", "\simeq"'] & H(N)^{\vee} \end{tikzcd} \qquad (\mathscr{D}')$$

can be closed as a commutative diagram in a unique way by a morphism of graded spaces $f_* : H_{\mathrm{c}}(M)[d_M] \to H_{\mathrm{c}}(N)[d_N]$. It is then clear that the correspondence defined over the category of oriented finite type manifolds, that assigns $M \rightsquigarrow M_* := H_{\mathrm{c}}(M)[d_M]$ and $f \rightsquigarrow f_*$, is a covariant functor.

When the manifold $N$ in $(\mathscr{D}')$ is not of finite type, $\mathscr{D}'_N$ is still an injection but it is no longer surjective so that it is not obvious that the diagram can be closed. Statement (b) in the next theorem states that this is in fact the case. It is therefore always possible to define the morphism $f_* : M_* \to N_*$, which we call *the Gysin morphism associated with* $f$. The resulting correspondence

$$M \rightsquigarrow M_* := H_{\mathrm{c}}(M)[d_M] \quad \text{and} \quad f \rightsquigarrow f_*$$

will thus appear to be a covariant functor defined over the ***whole*** category $\mathrm{Man}^{\mathrm{or}}$, which will be called *the Gysin functor*.

### 1.7.4. Theorem and definitions

a) *Let $M$ be oriented and endow its open subsets with induced orientations. For any inclusion of open subsets $i : V \subseteq W$, denote by $i_* : \Omega_{\mathrm{c}}(V) \to \Omega_{\mathrm{c}}(W)$ the map that assigns to $\beta \in \Omega_{\mathrm{c}}(V)$ its extension by zero to $W$, called* the pushforward *of $\beta$. Then, the following diagrams*

$$\begin{tikzcd} \Omega_{\mathrm{c}}(V)[d_M] \arrow[r, hook, "\mathbb{D}'_V"] \arrow[d, "i_*"'] & \Omega(V)^{\vee} \arrow[d, "(i^*)^{\vee}"] \\ \Omega_{\mathrm{c}}(W)[d_M] \arrow[r, hook, "\mathbb{D}'_W"] & \Omega(W)^{\vee} \end{tikzcd} \qquad \begin{tikzcd} H_{\mathrm{c}}(V)[d_M] \arrow[r, hook, "\mathscr{D}'_V"] \arrow[d, "H_{\mathrm{c}}(i_*)"'] & H(V)^{\vee} \arrow[d, "H(i^*)^{\vee}"] \\ H_{\mathrm{c}}(W)[d_M] \arrow[r, hook, "\mathscr{D}'_W"] & H(W)^{\vee} \end{tikzcd}$$

*are commutative, i.e. $(i^*, i_*)$ is a Poincaré adjoint pair (1.3.3).*

b) *For any map $f : M \to N$ between oriented manifolds, one has the diagram*

$$\begin{tikzcd} H_{\mathrm{c}}(M)[d_M] \arrow[r, hook, "\mathscr{D}'_M"] \arrow[d, dashed, "f_*"'] & H(M)^{\vee} \arrow[d, "H(f^*)^{\vee}"] \\ H_{\mathrm{c}}(N)[d_N] \arrow[r, hook, "\mathscr{D}'_N"] & H(N)^{\vee} \end{tikzcd} \qquad (\mathscr{D}')$$

*where $H(f^*)^{\vee}\big(\mathrm{Im}(\mathscr{D}'_M)\big) \subseteq \mathrm{Im}(\mathscr{D}'_N)$, so that there exists one and only one morphism of graded spaces*

$$f_* : H_{\mathrm{c}}(M)[d_M] \longrightarrow H_{\mathrm{c}}(N)[d_N] \qquad (\diamond)$$

*called* the Gysin morphism associated with *$f$, making $(\mathscr{D}')$ commutative, i.e. $(H(f^*), f_*)$ is a Poincaré adjoint pair in cohomology, which means that, for any $[\alpha] \in H(N)$ and $[\beta] \in H_{\mathrm{c}}(M)$, the equation in $X$,*

$$\int_M f^*[\alpha] \cup [\beta] = \int_N [\alpha] \cup X\,, \qquad (\diamond\diamond)$$

*admits one and only one solution in $H_{\mathrm{c}}(N)$, namely $X = f_*[\beta]$.*

*Furthermore, $f_*$ in ($\diamond$) is a morphism of $H(N)$-modules, i.e. the equality, called the* projection formula,

$$f_*\big(f^*[\alpha] \cup [\beta]\big) = [\alpha] \cup f_*([\beta]) \tag{$\diamond\diamond\diamond$}$$

*holds for all $[\alpha] \in H(N)$ and $[\beta] \in H_{\mathrm{c}}(M)$.*

c) *The correspondence*

$$(\_)_* : \mathrm{Man}^{\mathrm{or}} \rightsquigarrow \mathrm{GV}(\mathbb{R}) \qquad \text{with} \qquad \begin{cases} M \rightsquigarrow M_* := H_{\mathrm{c}}(M)[d_M] \\ f \rightsquigarrow f_* \end{cases}$$

*is a covariant functor. It will be called the* Gysin functor.

d) *If $M$ and $N$ are oriented of finite type, then $f^* : H(N) \to H(M)$ is an isomorphism if and only if the Gysin morphism $f_* : H_{\mathrm{c}}(M)[d_M] \to H_{\mathrm{c}}(N)[d_N]$ is also an isomorphism.*

*Proof.* (a) The commutativity results from the equality

$$\int_V \alpha|_V \wedge \beta = \int_W \alpha \wedge i_*\beta$$

for $\alpha \in \Omega(W)$ and $\beta \in \Omega_{\mathrm{c}}(V)$, which is evident since the support of $\alpha \wedge i_*\beta$ is contained in $V$.

(b) One must verify that, given $[\beta] \in H_{\mathrm{c}}(M)$, the linear form

$$[\alpha] \in H(N) \mapsto \int_M f^*[\alpha] \cup [\beta]$$

is of the form

$$[\alpha] \in H(N) \mapsto \int_N [\alpha] \cup [\beta']$$

for some $[\beta'] \in H_{\mathrm{c}}(N)$. Now, thanks to proposition 1.4.4, there exists an open subset $W \in N$ of finite type such that $f^{-1}W$ contains the support of $\beta$, denoted $\overline{\beta} := \beta|_{f^{-1}W}$. One then has the following commutative diagram:

$$\begin{tikzcd}
{[\overline{\beta}] \in H_{\mathrm{c}}(f^{-1}W)[d_M]} \arrow[rr, "\mathscr{D}'_{f^{-1}W}"] \arrow[dr, "H_{\mathrm{c}}(i_*)"'] \arrow[drr, phantom, "\mathbf{(I)}"] & & H(f^{-1}W)^\vee \arrow[dr, "(i^*)^\vee"] \arrow[dd, "(f^*)^\vee"'] \arrow[ddr, phantom, "\mathbf{(II)}"] & \\
& {[\beta] \in H_{\mathrm{c}}(M)[d_M]} \arrow[rr, "\mathscr{D}'_M"] & & H(M)^\vee \arrow[dd, "(f^*)^\vee"] \\
{[\beta'] \in H_{\mathrm{c}}(W)[d_N]} \arrow[rr, "\mathscr{D}'_W", "\simeq"'] \arrow[dr, "H_{\mathrm{c}}(i_*)"'] \arrow[drr, phantom, "\mathbf{(I)}"] & & H(W)^\vee \arrow[dr, "(i^*)^\vee"] & \\
& H_{\mathrm{c}}(N)[d_M] \arrow[rr, "\mathscr{D}'_N"] & & H(N)^\vee
\end{tikzcd}$$

where subdiagrams (**I**) are commutative after (c) and the commutativity of (**II**) is just functoriality of pullbacks.

Following the arrows, we see that

$$\begin{aligned}
(f^*)^\vee \circ \mathscr{D}'_M([\beta]) &= (i^*)^\vee \circ (f^*)^\vee \circ \mathscr{D}'_{f^{-1}W}([\overline{\beta}]) \\
&= (i^*)^\vee \circ \mathscr{D}'_W([\beta']) \\
&= \mathscr{D}'_N \circ H_{\mathrm{c}}(i_*)([\beta'])
\end{aligned}$$

where $[\beta'] \in H_c(W)[d_N]$ verifies

$$\mathscr{D}'_W([\beta']) = (f^*)^\vee \circ \mathscr{D}'_{f^{-1}W}([\overline{\beta}])$$

which is possible since $\mathscr{D}'_W$ is **surjective** as $W$ is of finite type !

The statement about the equation $(\diamond\diamond)$ is clear and implies formally the projection formula since

$$\begin{aligned}\int_N [\omega] \cup f_*\big(f^*[\alpha] \cup [\beta]\big) &= \int_M f^*[\omega] \cup f^*[\alpha] \cup [\beta] \\ &= \int_M f^*([\omega] \cup [\alpha]) \cup [\beta] = \int_N [\omega] \cup [\alpha] \cup f_*[\beta].\end{aligned}$$

Finally, (c) is trivial since $\mathscr{D}'$ is bijective over its image, and (d) is clear. □

**1.7.5. Remark.** It is important to note that the main ingredients in the proof are (i) the Poincaré pairings, (ii) Poincaré duality, (iii) the ascending chain property (1.4.3). In later sections of these notes we will show that all these ingredients exist also in the equivariant setting so that the last theorem and its proof will extend *verbatim* to $G$-manifolds and $G$-equivariant cohomology.

**1.7.6. Exercise.** Let $f : M \to N$ be a map between oriented manifolds. Show that the dual of the Gysin morphism $f_* : H_c(M)[d_M] \to H_c(N)[d_N]$ coincides, via Poincaré duality, with the *pullback morphism* $f^* : H(N) \to H(M)$.

**1.7.7. The Image of $\mathscr{D}'_M$.** The next proposition will be used when extending the Gysin functor to the equivariant context. It gives a description of the image of $\mathscr{D}'_M$ in terms of ascending chains of open finite type subsets of $M$, which was the main reason why we proved that such covers always exist (see 1.4.4).

**1.7.8. Proposition.** *Let $\mathscr{U}$ be a filtrant open cover* ([4]) *of a manifold $M$.*

a) *Let $i : V \subseteq W$ denote an inclusion of open subsets of $M$.*

*The map $i_* : \Omega_c(V) \subseteq \Omega_c(W)$, that assigns to $\beta \in \Omega_c(V)$ the differential form $i_*(\beta) \in \Omega_c(W)$ equal to $\beta$ over $V$ and $0$ otherwise, is a well-defined morphism of complexes inducing in cohomology the morphism of graded spaces $H_c(i_*) : H_c(V) \to H_c(W)$. One has also the morphism of complexes $i^* : \Omega(W) \to \Omega(V)$ that restricts a differential form of $W$ to $V$, and the corresponding morphism of graded spaces $H(i^*) : H(W) \to H(V)$.*

*These constructions, applied to the elements of $\mathscr{U}$, give rise to the inductive systems $\{\Omega_c(U)\}_{U\in\mathscr{U}}$ and $\{H_c(U)\}_{U\in\mathscr{U}}$, and to the projective systems $\{\Omega(U)\}_{U\in\mathscr{U}}$ and $\{H(U)\}_{U\in\mathscr{U}}$, whence the canonical maps*

$$\nu : \varinjlim_{U\in\mathscr{U}} \Omega_c(U) \to \Omega_c(M) \quad \text{and} \quad H(\nu) : \varinjlim_{U\in\mathscr{U}} H_c(U) \to H_c(M),$$

$$\mu : \Omega(M) \to \varprojlim_{U\in\mathscr{U}} \Omega(U) \quad \text{and} \quad H(\mu) : H(M) \to \varprojlim_{U\in\mathscr{U}} H(U).$$

*All these maps are bijective.*

[4] We recall that $\mathscr{U} = \{U_i\}_{i\in\mathfrak{I}}$ is said *filtrant* whenever for all $U_1, U_2 \in \mathscr{U}$ there exists $U_3 \in \mathscr{U}$ such that $(U_1 \cup U_2) \subseteq U_3$.

b) *Suppose $M$ is oriented, then the map*

$$\boxed{\begin{array}{rcl} I\!D'_{\mathscr{U}} : \big(\Omega_{\mathrm{c}}(M)[d_M], d\big) & \longrightarrow & \varinjlim_{U\in\mathscr{U}} \big(\Omega(U)^\vee, -d\big) \\ \beta & \mapsto & \Big(\alpha \longmapsto \displaystyle\int_M \alpha\wedge\beta\Big) \end{array}}$$

*is a well-defined morphism of complexes inducing in cohomology the map*

$$\mathscr{D}'_{\mathscr{U}} : H_{\mathrm{c}}(M)[d_M] \to \varinjlim_{U\in\mathscr{U}} H(U)^\vee$$

c) *Suppose further that each $U \in \mathscr{U}$ is of finite type. Then $I\!D'_{\mathscr{U}}$ is a quasi-isomorphism, and one has*

$$\mathrm{Im}(\mathscr{D}'_M) = \varinjlim_{U\in\mathscr{U}} H(U)^\vee \subseteq H(M)^\vee \tag{$\diamond$}$$

*Moreover, the adjoint $\mathscr{D}'^{\vee}_{\mathscr{U}}$ canonically identifies with $\mathscr{D}_M$; more precisely, the following diagram is commutative:*

$$\begin{tikzcd} \varprojlim_{U\in\mathscr{U}} H(U)[d_M] = \big(\varinjlim_{U\in\mathscr{U}} H(U)^\vee\big)^\vee[d_M] \arrow[r, "\mathscr{D}'^{\vee}_{\mathscr{U}}"] & H_{\mathrm{c}}(M)^\vee \arrow[d, equal] \\ H(M)[d_M] \arrow[u, "\simeq"] \arrow[r, "\mathscr{D}_M"] & H_{\mathrm{c}}(M)^\vee \end{tikzcd}$$

*Proof.* (a) The map $\nu : \varinjlim_{U\in\mathscr{U}} \Omega^*_{\mathrm{c}}(U) \to \Omega^*_{\mathrm{c}}(M)$ is injective since it's the limit of a filtrant inductive system of injective maps. The image of $\nu$ is the union of $\Omega^*_{\mathrm{c}}(U)$ for the same reason. Now, if $\omega \in \Omega^*_{\mathrm{c}}(M)$, its support, being compact, is contained in some $U \in \mathscr{U}$ so that $\omega$ is the pushforward of $\omega|_U \in \Omega^*_{\mathrm{c}}(U)$. This justifies the equality $\Omega^*_{\mathrm{c}}(M) = \bigcup_{U\in\mathscr{U}} \Omega^*_{\mathrm{c}}(U)$ and proves that $\nu$ is surjective. Standard arguments on the homology of filtrant inductive systems of complexes prove that $H(\nu)$ is bijective.

The map $\mu : \Omega(M) \to \varprojlim_{U\in\mathscr{U}} \Omega(U)$ is injective, since a differential form is null if and only if it is locally null. To see it is also surjective, let $\{\alpha_U \in \Omega(U)\}_{U\in\mathscr{U}}$ be a given projective system of differential forms, and note that for any $x \in M$, the element $\tilde\alpha(x) := \alpha_U(x)$ is well defined since if $x \in U_1 \in \mathscr{U}$ and $x \in U_2 \in \mathscr{U}$, one chooses $U_3 \in \mathscr{U}$ s.t. $U_1 \cup U_2 \subseteq U_3$, in which case $\alpha_{U_1}(x) = \alpha_{U_3}(x) = \alpha_{U_2}(x)$. Likewise, one verifies the differentiability of $\tilde\alpha$. It is clear that $\tilde\alpha|_U = \alpha_U$, which ends the proof that $\mu$ is surjective.

It remains only to justify why $H(\mu)$ is bijective. This is immediate when $M$ is orientable, since $H(\mu)$ is then just the Poincaré dual of $H_{\mathrm{c}}(\nu)$ which has already been shown to be bijective. Otherwise, when $M$ is not orientable, we lift $\mathscr{U}$ to the orientation manifold $\tilde M$ associated with $M$ through the canonical $\mathbb{Z}/2\mathbb{Z}$-covering $p : \tilde M \twoheadrightarrow M$, setting therefore $\tilde{\mathscr{U}} := \{\tilde U := p^{-1}(U) \,|\, U \in \mathscr{U}\}$. As $\tilde M$ is orientable, the map $H(\tilde M) \to \varprojlim_{U\in\mathscr{U}} H(\tilde U)$ is now bijective, and because this map is also compatible with the reversing-orientation action of $\mathbb{Z}/2\mathbb{Z}$, it induces a bijection between invariants subspaces $H(\tilde M)^{\mathbb{Z}/2\mathbb{Z}} \xrightarrow[\simeq]{} \varprojlim_{U\in\mathscr{U}} H(\tilde U)^{\mathbb{Z}/2\mathbb{Z}}$, and one concludes since $H(U) = H(\tilde U)^{\mathbb{Z}/2\mathbb{Z}}$.

(b) Endow each $U \in \mathscr{U}$ with the induced orientation. Taking the inductive limit of the maps $I\!D'_U : H_{\mathrm{c}}(U)[d_M] \to H(U)^\vee$ and applying (a) one sees immediately that $I\!D_{\mathscr{U}} = \varinjlim_{U\in\mathscr{U}} I\!D'_U$.

(c) By 1.7.2 the maps $I\!D'_U : H_c(U)[d_M] \to H(U)^\vee$ are quasi-isomorphisms for each $U \in \mathscr{U}$, hence $I\!D_{\mathscr{U}} = \varinjlim_{U\in\mathscr{U}} I\!D'_U$ is also a quasi-isomorphism since $\mathscr{U}$ is filtrant. The rest of the statement is then clear by duality. □

## 1.8. The Gysin Functor for Proper Maps

In this section, the Gysin morphism for compact supports

$$f_* : H_c(M)[d_M] \to H_c(N)[d_N]$$

will be extended to arbitrary supports

$$f_! : H(M)[d_M] \to H(N)[d_N]$$

when $f : M \to N$ is a **proper** map. As we will see this case is much simpler than the general one as it results immediately from Poincaré duality.

When $f : M \to N$ is proper, the pullback $f^* : \Omega(N) \to \Omega(M)$ respects compact supports and induces a morphism of complexes $f^* : \Omega_c(N) \to \Omega_c(M)$, giving rise to the *covariant* functor from $\mathrm{Man}_{\mathrm{pr}}$ to $\mathrm{Vec}(\mathbb{R})$

$$M \rightsquigarrow H_c(M)^\vee, \qquad f \rightsquigarrow H_c(f^*)^\vee.$$

When $M$ is oriented, $I\!D'_M$ may be extended from $\Omega_c(M)$ to $\Omega(M)$ by setting (see 1.7.1)

$$I\!D'_M(\alpha) = \Big(\beta \mapsto \int_M \beta\wedge\alpha\Big), \quad \forall\alpha\in\Omega(M), \quad \forall\beta\in\Omega_c(M),$$

so that the diagram

$$\begin{tikzcd} \Omega(M) \arrow[r, "I\!D'_M", "(\simeq)"'] & \Omega_c(M)^\vee \\ \Omega_c(M) \arrow[u, "\subseteq"] \arrow[r, "I\!D'_M"] & \Omega(M)^\vee \arrow[u, two heads] \end{tikzcd}$$

is commutative, and, moreover, with its first line a *quasi-isomorphism* as it is simply the Poincaré duality map $I\!D_M$ up to $\pm 1$.

**1.8.1. Definition.** If $f : M \to N$ is a proper map between oriented manifolds, the *Gysin morphism associated with* $f$ is the map $f_! : H(M)[d_M] \to H(N[d_N])$ making commutative the diagram

$$\begin{tikzcd} H(M)[d_M] \arrow[r, "\mathscr{D}'_M", "\simeq"'] \arrow[d, dashed, "f_!"'] & H_c(M)^\vee \arrow[d, "H_c(f^*)^\vee"] \\ H(N)[d_N] \arrow[r, "\mathscr{D}'_N", "\simeq"'] & H_c(N)^\vee \end{tikzcd}$$

The next theorem, analog to 1.7.4 and almost immediate, is left as an exercise.

**1.8.2. Theorem and definitions**

a) *For $\beta\in H_c(N)$ and $\alpha\in H(M)$, the equation in $X$,*

$$\int_M f^*[\beta]\cup[\alpha] = \int_N [\beta]\cup X, \tag{$\star\star$}$$

*admits one and only one solution in $H(N)$, namely $X = f_![\alpha]$.*

*Furthermore, $f_!$ is a morphism of $H_c(N)$-modules, i.e. the equality, called the* projection formula for proper maps,

$$f_!\big(f^*[\beta] \cup [\alpha]\big) = [\beta] \cup f_![\alpha] \tag{$\star\star\star$}$$

*holds for all $[\beta] \in H_c(N)$ and $[\alpha] \in H(M)$.*

b) *The following correspondence is a covariant functor:*

$$f_! : \mathrm{Man}^{\mathrm{or}}_{\mathrm{pr}} \rightsquigarrow \mathrm{GV}(\mathbb{R}) \qquad \textit{with} \qquad \begin{cases} M \rightsquigarrow M_! := H(M)[d_M] \\ f \rightsquigarrow f_! \end{cases}$$

*We will refer to it as the* Gysin functor for proper maps.

c) *The pullback $f^* : H_c(N) \to H_c(M)$ is an isomorphism if and only if the Gysin morphism $f_! : H(M)[d_M] \to H(N)[d_N]$ is also an isomorphism.*

d) *The natural map $\phi(\_) : H_c(\_)[d\_] \to H(\_)[d\_]$ (1.2.3) is a homomorphism of Gysin functors $(\_)_* \to (\_)_!$ over the category $\mathrm{Man}^{\mathrm{or}}_{\mathrm{pr}}$, i.e. the diagrams*

$$\begin{tikzcd} H_c(M)[d_M] \arrow[r, "\phi(M)"] \arrow[d, "f_*"'] & H(M)[d_M] \arrow[d, "f_!"] \\ H_c(N)[d_N] \arrow[r, "\phi(N)"] & H(N)[d_N] \end{tikzcd}$$

*are natural and commutative.*

## 1.9. Principal Examples of Gysin Morphisms

**1.9.1. Universal Property of the Gysin Morphism.** This property is the statement (b) in theorem 1.7.4, which says that if $f : M \to N$ is a map between oriented manifolds, then for each $[\beta] \in H_c(M)$, the element $f_*([\beta]) \in H_c(N)$ is determined by the equality, for all $[\alpha] \in H(N)$,

$$\boxed{\int_M f^*[\alpha] \cup [\beta] = \int_N [\alpha] \cup f_*[\beta]} \tag{$\diamond\diamond$}$$

The pair $(f_*, f^*)$ is a Poincaré *adjoint pair* in cohomology (1.3.3).

**1.9.2. Constant Map.** Let $M$ be oriented and denote by $c_M : M \to \{\bullet\}$ the constant map. One applies ($\diamond\diamond$) taking $\alpha = 1$:

$$c_{M\,*}([\beta]) = \int_{\{\bullet\}} 1 \cup c_{M\,*}[\beta] = \int_M \beta.$$

so that the Gysin morphism $c_{M\,*} : H_c(M)[d_M] \to H_c(\{\bullet\}) = \mathbb{R}$ is the integration map, Poincaré dual of the graded algebra homomorphism $c_M^* : \mathbb{R} \to H(M)$.

**1.9.3. Exercise.** Show that $c_M^* : \Omega(\{\bullet\}) \to \Omega(M)$ admits a right Poincaré adjoint at the complex level, i.e. $c_{M\,*} : \Omega_c(M)[d_M] \to \Omega(\{\bullet\})$.

**1.9.4. Open Embedding.** Let $M$ be oriented. Given an open embedding $i : U \subseteq M$, endow $U$ with the induced orientation. For any $\beta \in \Omega_c(U)$ one has the tautological equality:

$$\int_U \alpha|_U \wedge \beta = \int_M \alpha \wedge i_*\beta \tag{$*$}$$

where $i_*\beta \in \Omega_c(M)$ denotes the extension by zero of $\beta$. The Gysin morphism $i_* : H_c(U)[d_U] \to H_c(M)[d_M]$ is therefore the pushforward $i_* = H_c(i_*)[d_M]$ (see 1.7.8-(a)). Note also that the equality $(*)$ shows that the pair $(i^*, i_*)$ is a Poincaré adjoint pair (1.3.3).

**1.9.5. Locally Trivial Fibration.** Let $\pi : E \to B$ be a locally trivial fibration with base space $B$ (connected for simplicity) and total space $E$ both assumed oriented, with fiber $F$ of dimension $d_F$ endowed with the induced orientation. Under these assumptions one has the *operation of integration along* $F$ (see [BT] **I**§6 pp. 61-63) which is a morphism of complexes

$$\int_F : \Omega_c(E)[d_F] \to \Omega_c(B)$$

satisfying

$$\int_E \pi^*\alpha \wedge \beta = \int_B \Big(\alpha \wedge \int_F \beta\Big), \tag{$*$}$$

so that after the adjunction property $(\diamond\diamond)$, one has $\pi_* = \int_F [d_B]$ and the Gysin morphism is the shift of integration along fibers. Note again that $(*)$ shows that the pair $(\pi^*, \int_F [d_B])$ is a Poincaré adjoint pair.

**1.9.6. Proposition.** *Let* $(\pi, V, B)$ *and* $(\pi, V', B')$ *be two oriented locally trivial fibrations. Let* $g : B' \to B$ *be a* ***proper*** *map and assume the following diagram cartesian:*

$$\begin{array}{ccc} V' & \xrightarrow{g} & V \\ {\scriptstyle\pi}\big\downarrow & \square & \big\downarrow{\scriptstyle\pi} \\ B' & \xrightarrow{g} & B \end{array}$$

*i.e.* $V' = \big\{(b', v) \in B' \times V \;\big|\; g(b') = \pi(v)\big\}$. *Then*

$$\begin{cases} g^* \circ \pi_* = \pi_* \circ g^* : H_c(V) \to H_c(B') \\ \pi^* \circ g_! = g_! \circ \pi^* : H(B') \to H(V) \end{cases}$$

*Hint.* By adjointness, the first equality is equivalent to the second. The first equality follows from the equality for differential forms $g^*\big(\int_F \omega\big) = \int_F g^*(\omega)$ for all $\omega \in \Omega_c(V)$, that may be verified locally in $B'$ (*loc.cit.*). ⊡

**1.9.7. Zero Section of a Vector Bundle.** Let $(\pi, V, B)$ be a vector bundle and assume $B$ and $V$ oriented. The *zero section map* $\sigma : B \to V$ is a closed embedding, hence proper, so that we have the Gysin morphism for proper maps $\sigma_! : H(B) \to H(V)$. By the adjunction property $(\star\star)$ (see 1.8.2-(a)), one has for all $\beta \in H_c(V)$ and $\alpha \in H(B)$

$$\begin{aligned} \int_V [\beta] \cup \sigma_!([\alpha]) &= \int_B \sigma^*[\beta] \cup [\alpha] = \int_B \sigma^*[\beta] \cup \sigma^*(\pi^*[\alpha]) \\ &= \int_B \sigma^*([\beta] \cup \pi^*[\alpha]) \cup 1 = \int_V [\beta] \cup \pi^*[\alpha] \cup \sigma_!(1) \end{aligned} \tag{$\diamond$}$$

where $\Phi := \sigma_!(1)$ is *the Thom class of the pair* $(B, V)$. The Gysin morphism associated with the zero section of a fiber bundle

$$\sigma_! : H(B)[d_B] \to H(V)[d_V] \tag{!}$$

is then the multiplication by the Thom class

$$\sigma_!([\alpha]) = \pi^*[\alpha] \cup \Phi. \tag{!!}$$

Finally, note that $\sigma_!$ is generally not an isomorphism, since it identifies, via Poincaré duality, with the dual of the proper pullback $\sigma^* : H_c(V) \to H_c(B)$ (see 1.8.1) which is generally not an isomorphism ([5]).

It can be seen ([BT] §I.6 p. 64) that if $\alpha \in H_c(B)$, then $\pi^*[\alpha] \cup \Phi$ naturally belongs to $H_c(V)$ so that the Gysin morphism

$$\sigma_* : H_c(B)[d_B] \to H_c(V)[d_V] \tag{$\star$}$$

is given by the same equality (!!),

$$\sigma_*([\beta]) = \pi^*[\beta] \wedge \Phi. \tag{$\star\star$}$$

On the other hand, the Poincaré lemma for vector bundles asserts that the pullback $\pi^* : H(B) \to H(V)$ is an isomorphism and this implies, via Poincaré duality (see 1.7.6), that $\pi_* : H_c(V)[d_V] \to H_c(B)[d_B]$ is also an isomorphism. Now, by functoriality, one has $\pi_* \circ \sigma_* = \mathrm{id}$, so that $\sigma_*$ is also an isomorphism. This isomorphism is *the Thom isomorphism*.

**1.9.8. Proposition.** *Let $(\pi, V, B)$ and $(\pi, V', B')$ be two oriented vector bundles and assume the cartesian diagram in 1.9.6 with $g : B' \to B$* **proper**. *Denote by $\sigma : B \to V$ and $\sigma : B' \to V'$ the zero section maps. The diagram*

$$\begin{tikzcd} B' \arrow[r, "g"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\sigma"] \\ V' \arrow[r, "g"] & V \end{tikzcd}$$

*is cartesian and the equalities* $\begin{cases} g^* \circ \sigma_* = \sigma_* \circ g^* : H_c(B) \to H_c(V') \\ \sigma^* \circ g_! = g_! \circ \sigma^* : H(V') \to H(B) \end{cases}$ *hold.*

*Hint.* It is a corollary of 1.9.6 since $\sigma_*$ is the inverse of $\pi_*$. ⊡

## 1.10. Constructions of Gysin Morphisms

In this last preliminary section we summarize the steps in the construction of the Gysin morphisms.

**1.10.1. The Proper Case.** Let $f : M \to N$ be a **proper** map of oriented manifolds. To $\alpha \in \Omega(M)$ we assign the linear form on $\Omega_c(N)$ defined by $\mathbb{D}'_f(\alpha) : \beta \mapsto \int_M f^*\beta \wedge \alpha$. In this way we obtain diagram

$$\begin{tikzcd} \Omega(M)[d_M] \arrow[r, dotted, "f_!"] \arrow[dr, "\mathbb{D}'_f"'] & \Omega(N)[d_N] \arrow[d, "\mathbb{D}'_N\ \text{(quasi-iso)}"] \arrow[dl, phantom, "\oplus"] \\ & \Omega_c(N)^\vee \end{tikzcd}$$

---

[5]For example, if $B$ is compact, $H_c(B) = H(B) = H(V)$ and $\sigma^*$ would give a graded isomorphism $H_c(V) \simeq H(V)$, and by Poincaré duality $H^0(V) \simeq H^{d_V}(V)$, which impossible if $V$ is a vector bundle of positive dimension over $B$.

which may be closed in cohomology, since $\mathbb{D}'_N$ is a quasi-isomorphism. Note that the closing arrow $f_!$, the Gysin morphism for proper maps, in general exists *only* at the cohomology level.

**1.10.2. The General Case.** Let $f : M \to N$ be a map of oriented manifolds. To $\beta \in \Omega_c(M)$ we assign the linear form on $\Omega(N)$ defined by $\mathbb{D}'_f(\beta) : \alpha \mapsto \int_M f^*\alpha \wedge \beta$. In this way we obtain the diagram

$$\begin{tikzcd}
\Omega_c(M)[d_M] \arrow[r, dotted, "f_*"] \arrow[dr, "\mathbb{D}'_f"'] \arrow[r, phantom, shift right=4ex, "\bigoplus"] & \Omega_c(N)[d_N] \arrow[d, "\mathbb{D}'_N \left(\substack{\text{quasi-iso if } N \text{ is of} \\ \text{finite type}}\right)"] \\
& \Omega(N)^\vee
\end{tikzcd}$$

which may be closed in cohomology (as in the proper case), when $N$ **is of finite type**, since then $\mathbb{D}'_N$ is a quasi-isomorphism (1.7.2-(b)).

When $N$ is not of finite type, one fixes any filtrant cover $\mathscr{U}$ of $N$ made up of open finite type subsets of $N$ (see 1.4.4), and replaces $\mathbb{D}'_N$ with $\mathbb{D}'_{\mathscr{U}}$. In this way, we get (see 1.7.8-(b,c)), the following diagram:

$$\begin{tikzcd}
\Omega_c(M)[d_M] \arrow[r, dotted, "f_*"] \arrow[dr, "\mathbb{D}'_{f,\mathscr{U}}"'] \arrow[r, phantom, shift right=4ex, "\bigoplus"] & \Omega_c(N)[d_N] \arrow[r, dotted, "="'] \arrow[d, "\mathbb{D}'_{\mathscr{U}}\ \text{(quasi-iso)}"] & \Omega_c(N)[d_N] \arrow[d, dotted, "\mathbb{D}'_N"] \\
& \varinjlim_{U \in \mathscr{U}} \Omega(U)^\vee \arrow[r, dotted, "\subseteq"'] & \Omega(N)^\vee
\end{tikzcd}$$

where $\mathbb{D}'_{f,\mathscr{U}}$ is defined as follows. For $\beta \in \Omega_c(M)$ denote by $|\beta|$ its support and by $\mathscr{U}_\beta \subseteq \mathscr{U}$ the system consisting of $U \in \mathscr{U}$ s.t. $|\beta| \subseteq f^{-1}U$. One has a natural map $\varinjlim_{\mathscr{U}_\beta} \Omega(U)^\vee \to \varinjlim_{\mathscr{U}} \Omega(U)^\vee$ (which is in fact is bijective). Now, for every $U \in \mathscr{U}_\beta$ the linear map $\left(\int_M f^*(\_) \wedge \beta\right) : \Omega(U) \to \mathbb{R}$, is well defined and is compatible with restriction, so that it defines an element of $\varinjlim_{\mathscr{U}_\beta} \Omega(U)^\vee$, and then of $\varinjlim_{\mathscr{U}} \Omega(U)^\vee$. This element is $\mathbb{D}'_{f,\mathscr{U}}(\beta)$ by definition.

The closing arrow $f_*$, *the Gysin morphism associated with a general map* $f$, is then defined in cohomology as the composition $\mathscr{D}'_{\mathscr{U}}{}^{-1} \circ H(\mathbb{D}_{f,\mathscr{U}})$.

**1.10.3. Remark.** In all cases, the Gysin morphism appears as the composition of a morphism of complexes with the "inverse" of a quasi-isomorphism, which obviously is possible in cohomology but also in the *derived category of complexes* since this is its main property, i.e. a morphism in derived category is an isomorphism if and only if it induces an isomorphism in cohomology. Gysin morphisms are therefore well defined in the derived category.

## 1.11. Exercises

**1.11.1. Gysin Long Exact Sequence.** Let $i : F \subseteq M$ be a closed embedding of oriented manifolds. Assume $F$ compact, for simplicity. Put $U := M \backslash F$ and $j : U \subseteq M$ the canonical injection.

a) i) Let $\mathscr{F}$ denote the set of open neighborhood of $F$. Restriction morphisms $R^W_V : \Omega(W) \to \Omega(V)$ for all $W \supseteq V \supseteq F$, give rise to a filtrant inductive system $\{R_V^W \mid W \supseteq V \text{ in } \mathscr{F}\}$ and a canonical morphism of complexes $R^M_{\mathscr{F}} : \Omega(M) \to \varinjlim_{V\in\mathscr{F}} \Omega(V)$. Show that the short sequence

$$\mathbf{0} \to \Omega_{\mathrm{c}}(U) \xrightarrow{j_*} \Omega_{\mathrm{c}}(M) \xrightarrow{R^M_{\mathscr{F}}} \varinjlim_{\mathscr{F}} \Omega(V) \to \mathbf{0}$$

where $j_*$ is the pushforward morphism, is exact.

ii) Restrictions $R_F^V : \Omega(V) \to \Omega(F)$ for $V \supseteq F$, define a morphism of the inductive system $\{R_V^W \mid W \supseteq V \text{ in } \mathscr{F}\}$ into $\Omega(F)$. Denote by $R_F^{\mathscr{F}} := \varinjlim_{\mathscr{F}} R_F^V$. Show that

$$R_F^{\mathscr{F}} : \varinjlim_{\mathscr{F}} \Omega(V) \to \Omega(F)$$

is a quasi-isomorphism.

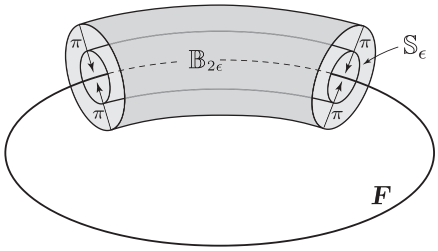
iii) Derive the existence of *the long exact sequence of compact support cohomology*

$$\cdots \to H^k_{\mathrm{c}}(U) \xrightarrow{i_*} H^k_{\mathrm{c}}(M) \xrightarrow{i^*} H^k(F) \xrightarrow{c_k} H^{k+1}_{\mathrm{c}}(U) \to \cdots \qquad (\diamond)$$

iv) Endow $M$ with a Riemannian metric $d : M \times M \to \mathbb{R}$. For each $\epsilon \in \mathbb{R}$, denote

$$\begin{cases} \mathbb{B}_\epsilon(F) := \{m \in M \mid d(m,F) < \epsilon\} \\ \mathbb{S}_\epsilon(F) := \{m \in M \mid d(m,F) = \epsilon\} \end{cases}$$

If $\epsilon$ is small enough, $\mathbb{B}_{2\epsilon}(F)$ is a fiber bundle with fiber $\mathbb{R}^{d_M - d_F}$ over $F$ via the geodesic projection $\pi : \mathbb{B}_{2\epsilon} \to F$. By restriction, $\pi : \mathbb{S}_\epsilon \to F$ is a fiber bundle with compact fiber $\mathbb{S}^{d_M - d_F - 1}$. Denote by $\ell : \mathbb{S}_\epsilon \hookrightarrow \mathbb{B}_{2\epsilon}\backslash F$ the canonical injection. We have the following maps




$$\begin{tikzcd} \mathbb{S}_\epsilon \arrow[r, "\ell"] \arrow[d, two heads, "\pi"'] & \mathbb{B}_{2\epsilon}\backslash F \arrow[r, "j_\epsilon", "\subseteq"'] & U \arrow[d, "j"', "\subseteq"] \\ F \arrow[rr, hook, "i", "\subseteq"'] & & M \end{tikzcd}$$

Show that the connecting morphism $c : H(F) \to H_{\mathrm{c}}(U)[1]$ is given by the composition of the following morphisms

$$\begin{tikzcd} H(F) \arrow[r, "\pi^*"] \arrow[rrr, bend right, "c"'] & H_{\mathrm{c}}(\mathbb{S}_\epsilon) \arrow[r, "\ell_*[-d_{\mathbb{S}_\epsilon}]"] & H_{\mathrm{c}}(\mathbb{B}_{2\epsilon}\backslash F)[1] \arrow[r, "j_{\epsilon,*}"] & H_{\mathrm{c}}(U)[1] \end{tikzcd}$$

where $\ell_* : H_{\mathrm{c}}(\mathbb{S}_\epsilon)[d_{\mathbb{S}_\epsilon}] \to H_{\mathrm{c}}(\mathbb{B}_{2\epsilon}\backslash F)[d_{\mathbb{B}_{2\epsilon}}]$ denotes the Gysin morphism associated with $\ell$.

b) i) Dualizing and shifting the long exact sequence of compact support $(\diamond)$, justify the exactness of the *Gysin long exact sequence*

$$\xrightarrow{\delta[-1]} H(F)[d_F - d_M] \xrightarrow{i_*[-d_M]} H(M) \xrightarrow{j^*} H(U) \xrightarrow{\delta} \qquad (\diamond\diamond)$$

where $i : F \to N$ and $j : U \to N$ are the canonical injections and $\delta$ is adjoint to the shift of the connecting morphism $c$ in $(\diamond)$.

ii) Show that the connecting morphism $\delta : H(U) \to H(F)[-(d_M - d_F - 1)]$ is simply the restriction to $\mathbb{S}_\epsilon$ followed by integration along fibers of $\pi$

$$\delta(\alpha) = \int_{\mathbb{S}^{d_M - d_F - 1}} \alpha|_{\mathbb{S}_\epsilon} .$$

**1.11.2. Lefschetz Fixed Point Theorem.** Let $M$ be a oriented manifold. Denote by $\delta : M \to M \times M$ the diagonal embedding $x \mapsto (x,x)$ and let $\Delta := \mathrm{Im}(\delta)$. Given $f : M \to M$, denote $\mathrm{Gr}(f) : M \to M \times M$ *the graph map* $x \mapsto (f(x),x)$. The *Lefschetz class of* $f$ is by definition

$$L(f) := \mathrm{Gr}(f)^*(\delta_!(1)) \in H^{d_M}(M),$$

and its *Lefschetz number* is $\Lambda_f := \int_M L(f)$.

a) Explain de following the equalities

$$\Lambda_f := \int_M \mathrm{Gr}(f)^*(\delta_!(1)) = \int_{M\times M} \delta_!(1) \cup \mathrm{Gr}(f)_!(1) = (-1)^{d_M} \int_M \delta^*(\mathrm{Gr}(f)_!(1)). \quad (\diamond)$$

b) Assuming that $f$ has no fixed points, show that the Gysin morphism

$$\mathrm{Gr}(f)_! : H(M)[d_M] \to H(M \times M)[2d_M]$$

factorizes through $H_c(M \times M \backslash \Delta)$ so that $\Lambda_f = 0$.

c) From now on suppose that $M$ is orientable. Let $\mathscr{B} := \{e_i\}_{i\in I}$ be a graded basis of $H(M)$ and let $\mathscr{B}' := \{e'_i\}_{i\in I}$ denote the Poincaré dual basis of $\mathscr{B}$, i.e. such that $e_i \cup e'_j = \delta_{i,j}[\zeta]$, where $[\zeta]$ denotes the fundamental class of $M$. Using the projection formula for $\delta : M \to M \times M$ show that

$$\delta_*(1) = \sum\nolimits_{i\in I} (-1)^{\deg(e_i)}\, e_i \otimes e'_i ,$$

Prove the equality: $\displaystyle\int_M \delta_!(1)|_\Delta = \sum_{k\in\mathbb{N}} (-1)^k \dim\big(H^k(M)\big).$

d) Combining $(\diamond)$ with the last result, show the *Lefschetz fixed point formula*

$$\Lambda_f = \sum\nolimits_{k\in\mathbb{N}} (-1)^k \,\mathrm{Tr}\big(f^* : H^k(M) \to H^k(M)\big)$$

In particular, if this alternating sum doesn't vanish, $f$ has fixed points !

**1.12. Conclusion.** We have reached the end of the preliminaries on Poincaré duality and Gysin morphism in the nonequivariant setting and using the de Rham model for the cohomology.

As shown, the key ingredient is the Poincaré pairing so that, in order to extend everything to the context of $G$-manifolds and equivariant cohomology following exactly the same approach, we will need to begin recalling the equivariant de Rham model for equivariant cohomology, and then introduce the Poincaré pairing and the corresponding adjunctions, in a way that we gan give a sense and prove the equivariant analog to Poincaré duality. Section 4 is entirely devoted to this. In section 5, the $G$-equivariant Gysin morphisms associated with equivariant maps will then be defined formally following the same procedures described in 1.10.

But, before entering into those subjects, we first do a quick review of the origins of equivariant cohomology theory.

## 2. Equivariant Cohomology Background

**1950 Cartan's ENS Seminar.** In May/June 1950, Henri Cartan gave lectures n° 19/20 of the *Séminaire Cartan* at the *Ecole Normale Supérieure de Paris* ([6]). The talks, which concerned a principal fiber bundle $p : \mathscr{E} \to \mathscr{B}$ of base space a manifold $\mathscr{B}$, and fiber space a compact connected Lie group $G$ (of Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$), focused on setting an algebraic framework for the study of the relationship between the cohomologies of $\mathscr{E}$, $\mathscr{B}$ and $G$, and incorporating characteristic classes through an algebraized approach of the Chern-Weil homomorphism $\mathrm{ch} : S(\mathfrak{g})^G \to H(\mathscr{B})$.

In the first lecture, Cartan introduces the algebraic analog to the universal fiber bundle $\mathbb{E}G$ of $G$, *the Weil algebra* $W(\mathfrak{g})$, as an object of the category of $\mathfrak{g}$-differential graded algebras representing the functor "*algebraic connections*", in the same way the classifying space $\mathbb{B}G$ co-represents the *$G$-principal fiber bundles* functor.

$$\begin{tikzcd}
\mathrm{Mor}_{\mathfrak{g}\text{-adg}}(W(\mathfrak{g})),\Omega(\mathscr{E})) \arrow[r, equal, "\text{Weil}"] \arrow[d, "\text{restriction}"', "\text{to basic subcomplexes}"] & \big\{\text{algebraic connexions on } \Omega(\mathscr{E})\big\} \\
\big\{\mathrm{ch} : S(\mathfrak{g})^G \to H(\mathscr{B})\big\} & \big\{\text{infinitesimal connections on } \Omega(\mathscr{E})\big\} \arrow[l, "\text{Chern}"'] \arrow[u, hook] \arrow[d, two heads] \\
\mathrm{Hot}(\mathscr{B},\mathbb{B}G) \arrow[u] \arrow[r, equal, "\text{Steenrod}"] & \big\{G\text{-principal fiber bundle } p : \mathscr{E} \to \mathscr{B}\big\}
\end{tikzcd}$$

It is in his second lecture that Cartan studies different ways to relate the cohomologies of $\mathscr{E}$, $\mathscr{B}$ and $G$. Of these, the most interesting to us is the construction of a differential graded algebra whose cohomology is that of $\mathscr{B}$, taking as its main ingredients the de Rham complex of the total space $(\Omega(\mathscr{E}),d)$ and something else related to the Lie group $G$. For that, Cartan recalls that, through the pullback $p^* : \Omega(\mathscr{B}) \to \Omega(\mathscr{E})$, the complex $\Omega(\mathscr{B})$ is identified with the subcomplex $\Omega(\mathscr{E})^{\mathrm{bas}}$ of *basic* elements of $\Omega(\mathscr{E})$ viewed as a $\mathfrak{g}$-differential graded algebra. But this wouldn't really help, as $\Omega(\mathscr{E})$ is lost. Instead, Cartan pursues his previous idea and introduces the complex $W(\mathfrak{g})\otimes\Omega(\mathscr{E})$ as a candidate for the 'de Rham complex' of the *topological* space $\mathbb{E}G \times \mathscr{E}$, and replaces the previous pullback with the map $\Omega(\mathscr{B}) \to W(\mathfrak{g}) \otimes \Omega(\mathscr{E})$, $\omega \mapsto 1 \otimes p^*(\omega)$, the image of which lies in the subcomplex of basic elements of $W(\mathfrak{g}) \otimes \Omega(\mathscr{E})$, denoted by

$$\Omega_{\mathfrak{g}}(\mathscr{E}) := (W(\mathfrak{g}) \otimes \Omega(\mathscr{E}))^{\mathrm{bas}}.$$

Cartan then states that the resulting map $\Omega(\mathscr{B}) \to \Omega_{\mathfrak{g}}(\mathscr{E})$, which is a homomorphism of differential graded algebras, induces an isomorphism in cohomology, and, moreover, that one has

$$\boxed{\Omega_{\mathfrak{g}}(\mathscr{E}) = ((S(\mathfrak{g}) \otimes \Omega(\mathscr{E}))^{\mathfrak{g}}, d_{\mathfrak{g}})} \tag{$*$}$$

with

$$d_{\mathfrak{g}}(P \otimes \omega) = P \otimes d\omega + \sum\nolimits_i Pe^i \otimes c(e_i)\omega, \tag{$**$}$$

[6] Lecture 19 on May 15, and lecture 20 in two sessions: May 23 and June 19. The contents of these lectures where also presented at the *Colloque de topologie (espaces fibrés)*, held at Brussels on June 5 to 8.

where $d$ is the differential in $\Omega(\mathscr{E})$, $\{e_i\}$ is a basis of $\mathfrak{g}$ of dual basis $\{e^i\}$, and $c(e_i)$ is the contraction operator associated with the vector field generated by the infinitesimal action of $e_i$ on $\mathscr{E}$ ([7]). The differential graded algebra $\Omega_{\mathfrak{g}}(\mathscr{E})$, nowadays commonly known as *the Cartan's complex*, met Cartan's requirements perfectly.

At this point, it is worth emphasizing that the construction of $\Omega_{\mathfrak{g}}(M)$ made sense whether or not the action of $G$ on $M$ is free. Although clear, this was out of focus at the time of Cartan's lectures, where research was mostly concentrated on manifolds and principal fiber bundles rather than on general $G$-manifolds, and still less on general topological $G$-spaces.

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**1960 Borel's IAS Seminar.** Some years later, in 1958-59, Armand Borel, who had been an active participant in the Cartan Seminar and in the Leray's courses at the *Collège de France* since his arrival in Paris in 1949, held his *Seminar on transformation groups* at the Institute for Advanced Study in Princeton ([B$_2$]). There, Borel drew attention to the advantages of considering for *any locally compact* $G$-space $X$, the orbit space of $\mathbb{E}G \times X$ under the diagonal action of $G$:

$$X_G := \mathbb{E}G \times_G X\,,$$

as the homotopically best-suited substitute for the orbit space $X/G$ *in whatever way the group $G$ (continuously) acts on $X$*! It was best-suited, mainly because the space $X_G$ bundles together not only the space $X$ and the group $G$, but also orbits and classifying spaces.

Indeed, $X_G$ appears as the total space of the following two maps

$$\begin{tikzcd}
 & \mathbb{B}G \\
X_G := \mathbb{E}G \times_G X \arrow[ur, two heads, "\text{fiber}=X", "\pi"'] \arrow[dr, two heads, "p", "\text{fiber}=\mathbb{B}G_x"'] & \\
 & X/G,
\end{tikzcd}$$

- $\pi : X_G \twoheadrightarrow \mathbb{B}G$, $\overline{(w,x)} \mapsto \overline{w}$, a locally trivial fibration of fiber space $X$, and
- $p : X_G \twoheadrightarrow X/G$, $\overline{(w,x)} \mapsto \overline{x}$, where the fibers are the classifying spaces $\mathbb{B}G_x$ of the different stability groups $G_x$ for $x \in X$.

As Borel says in its introduction: *It allows us to tie together the cohomology groups of $X$, $X/G$, and the fixed point set $F$, with those of the classifying spaces of the stability groups and of $G$.*

The space $X_G$, which Borel called *twisted product*, is known today as *the homotopy quotient*, *the homotopy orbit space* or more frequently *the Borel construction*.

[7] All these statements, that are not difficult to prove, were given without any justification by Cartan. Later Michel André in his Ph.D. thesis ([A], 1962) directed by Claude Chevalley gave complete proofs for Cartan's lectures statements.

Excerpt from [$B_2$] (A. Borel, IV-§3 , p. 55, 1960), where, for the first time, a reference to the Borel construction appears [8].

> 3.9. REMARK. All our discussion will center around the space $X_G$, and the remarks 3.6, 3.7 will be basic. Similar arguments have been used by Conner [5] when G is a circle, in rational cohomology. For an algebraic analogue when G is discrete, see [7, Chap. V]. The space $X_G$ and the embedding $F(X; G) \times B_G \subset X_G$ were also mentioned to the author by A. Shapiro. The proof of Smith's theorem 4.3 is also related to that of [2].

Beyond its immediate aim of the homological study of the set of fixed points $F := X^G$, the seminar laid most of the foundations of what would later be known as *the equivariant cohomology of locally compact $G$-spaces*. Orbit types, slice theorems, spectral sequences, fixed points theorems, were already present, if not yet in their final form, at least at a level that would appeal to other mathematicians for further development.

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**1968 Atiyah-Segal: Equivariant *K*-theory.** The restriction map

$$H(X_G) \to H(F_G) \tag{†}$$

appears in almost every section of applications of the Borel Seminar. Conditions are often given to ensure it is an isomorphism, but they are quite restricting. In addition, in the case of the circle group $G := T^1$ action, although it is clear that Borel was aware of the fact that (†) is an isomorphism modulo $H(\mathbb{B}G)$-torsion, he never states it in these terms. It is only ten years later, in 1968, when Atiyah and Segal introduce the *equivariant $K$-theory for locally compact $G$-spaces* that the following enhancement of $(*)$ appears for the first time.

*Localization theorem ([AS, S], 1968). The localized restriction map*

$$K_G(X)_{\mathfrak{P}} \to K_G(G.X^S)_{\mathfrak{P}} \tag{‡}$$

*where $\mathfrak{P}$ is a prime ideal of $K_G(\bullet)$, $S$ is minimal among the subgroups of $G$ such that $\mathfrak{P}$ is the inverse image of a prime ideal of $K_S(\bullet)$, and $X^S$ is the set of $S$-fixed points, is an isomorphism.*

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**1971 Quillen: Equivariant cohomology.** As $K$-theory has a cohomological behavior, the equivariant *cohomology* theory soon came to light. Daniel Quillen did it in [Q] (1971), when he explicitly merged the ideas of Atiyah-Segal and Borel. The *equivariant cohomology of a $G$-space $X$ with coefficients in a ring $A$*, denoted by $H_G(X)$, is then defined as the ordinary cohomology of Borel's construction $X_G$, i.e.

$$H_G(X) := H(X_G; A).$$

Quillen proves the localization theorem for the case where $G$ is an elementary $p$-group, and for the case where $G$ is a torus $T$.

[8] The references '[2]', '[5]' and '[7]' correspond respectively to [$B_1$], [Co] and [Gr].

*Theorem. Assume either $X$ is compact or paracompact with* $\dim_{\mathrm{ch}}(X) < +\infty$ *and that the set of identity components of the isotropy groups of points of $X$ is finite. Then the inclusion of $X^T$ in $X$ induces an isomorphism*

$$H_T(X)\big[\big(H_T^1(\bullet) - 0\big)^{-1}\big] \to H_T(XT)\big[\big(H_T^1(\bullet) - 0\big)^{-1}\big]. \tag{‡‡}$$

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**1975.** In this year, Wu-yi Hsiang's book "*Cohomology theory of topological transformation groups*" ([H]) appeared, in which the third chapter promptly introduces the reader to the foundations of equivariant cohomology for locally compact $G$-spaces. It includes a version of the localization theorem more in the vein of Atiyah-Segal that Hsiang calls the *Borel-Atiyah-Segal localization theorem*. It is stated as follows:

*Theorem. Assume either $X$ is compact or paracompact with* $\dim_{\mathrm{ch}}(X) < +\infty$ *and that the set of identity components of the isotropy groups of points of $X$ is finite. For a multiplicative system $S \subseteq H_G(\bullet)$ $(= H(\mathbb{B}G))$, put*

$$X^S = \big\{x \in X \bigm| \textit{no element of } S \textit{ maps to zero in } H(\mathbb{B}G) \to H(\mathbb{B}G_x)\big\}.$$

*Then, the localized restriction map*

$$S^{-1}H_G(X) \to S^{-1}H_G(X^S),$$

*is an isomorphism.*

For example, if $G$ is a torus $T$, and $S = H_T(\bullet)\backslash\{0\}$, one has $X^S = X^F$ so that we again see that the kernel and cokernel of the restriction map

$$H_T(X) \to H_T(X^T)$$

are torsion $H_T(\bullet)$-modules as stated in Quillen's (‡‡).

**Comment.** At this point, it should be noted that Hsiang's introduction to equivariant cohomology would have sufficed to introduce equivariant Poincaré duality, equivariant Gysin morphisms and localisation theorems. In this regard, we could have chosen to work with singular or sheaf cohomology with coefficients in arbitrary field and characteristic, and, in many cases, even in the ring of integers (see section 8 for details).

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**1980. Atiyah-Bott, Berline-Vergne: The equivariant differential forms.** The reader may have noticed that the Cartan's complex played no role in the previous paragraphs. This is because, at the time, Borel, Quillen, Hsiang, ... were mostly interested in applying equivariant cohomology (often with coefficients in fields of positive characteristic) to find conditions for the existence of fixed points in locally compact $G$-spaces and to infer cohomological properties of the fixed point sets from those of the ambient space $X$ (for example being a cohomological manifold when $X$ is such).

In the early 1980s the whole theory underwent an unexpected development when N. Berline and M. Vergne succeeded in proving the Duistermaat-Heckman formula on the pushforward of the Liouville measure on a symplectic manifold under the moment map ([DH], 1982) by a new fixed point theorem for $G$-manifolds inspired by an old paper of R. Bott ([Bo], 1966).

Let $G$ be a compact Lie group and $M$ a $G$-manifold. For $X \in \mathfrak{g} := \mathrm{Lie}(G)$, let $X^*$ be the vector field over $M$ generated by the infinitesimal action of $X$, and denote by $c(X)$ and $\mathcal{L}(X)$ respectively the contraction and the Lie derivative operators associated with $X^*$, acting on the differential algebra of complex de Rham differential forms $(\mathbb{C} \otimes \Omega(M), d)$. The vector $X$ is called *nondegenerate* if, for $m \in M$ fixed by the one-parameter group $\exp(tX)$, the linear operator $L_m(X)$ on the tangent space $T_m(M)$ induced by the Lie derivative $\mathcal{L}(X)$, is invertible. (Notice that this condition implies that $T_m(M)$ is even-dimensional.)

In their work, Berline-Vergne introduce the linear operator on $\mathbb{C} \otimes \Omega(M)$:

$$d_X := d - 2\pi i c(X^*). \tag{$\diamond$}$$

It verifies $d_X^2 = -2\pi i \mathcal{L}(X)$, so that if one denotes by $\Omega(M)^X$ the sub-algebra of de Rham differential forms on $M$ invariant under $\exp(tX)$, the pair $(\Omega(M)^X, d_X)$ is a ($\mathbb{Z}/2\mathbb{Z}$-graded) differential algebra. Berline-Vergne denote by $H_X(M)$ its cohomology, and prove the following fixed point theorem.

*Theorem ([BV$_1$, BV$_2$, BV$_3$]). Let $G$ be a compact Lie group and $M$ an oriented compact $G$-manifold (of even dimension). Then, if $X \in \mathfrak{g}$ is nondegenerate and $\mu \in H_X(M)$, one has:*

$$\int_M \mu = \sum_{m \in M^X} \frac{\mu(m)}{\mathrm{Pf}(L_m(X))}$$

*where $\mu(m)$ is the restriction of $\mu$ to the singleton $\{m\}$, $M^X$ is the fixed point set (necessarily finite) of $\exp(tX)$ and $\mathrm{Pf}(L_m(X))$ is the Pfaffian of $L_m(X)$.*

At about the same time the Atiyah-Bott paper ([AB], 1984) appeared. Motivated by the same work of Duistermaat-Heckman, as well as a recent work of E. Witten ([W]), it introduced a de Rham model for the equivariant cohomology of manifolds and states the corresponding localization theorems. In *loc.cit.* (th. 4.13) Atiyah-Bott, taking finite dimensional approximations of $\mathbb{E}G$, shows that the cohomology of the Cartan's complex $(\Omega_{\mathfrak{g}}(M), d_{\mathfrak{g}})$ is the ordinary cohomology of the topological space $M_G$. In this way the original, and somehow neglected, Cartan's complex $(\Omega_{\mathfrak{g}}(M), d_{\mathfrak{g}})$ turned out to have been an excellent model for the equivariant cohomology of manifolds. The elements of $\Omega_{\mathfrak{g}}(M)$ have since become known as the *$G$-equivariant (de Rham) differential forms*.

When one compares the Berline-Vergne operator $d_X$ ($\diamond$) to Cartan's operator $d_{\mathfrak{g}}$ (p. 21), one immediately understands that the map

$$\mathrm{ev}_X : \Omega_{\mathfrak{g}}(M) \longrightarrow \Omega(M)^X, \quad P \otimes \mu \longmapsto P(-2\pi i X)\mu$$

commutes with differentials inducing the map between cohomologies

$$\mathbb{C} \otimes H_G(M) \xrightarrow{\mathrm{ev}_X} H_X(M).$$

If $T \subseteq G$ is the torus topologically generated by $X \in \mathfrak{g}$, we have $M^X = M^T$ and the commutative diagram of restrictions to fixed point sets:

$$\begin{tikzcd} \mathbb{C} \otimes H_T(M) \arrow[r, "\mathrm{ev}_X"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\oplus"] & H_X(M) \arrow[d, "\simeq"] \\ \mathbb{C} \otimes H_T(M^T) \arrow[r, two heads, "\mathrm{ev}_X"'] & H_X(M^X) \end{tikzcd} \tag{$*$}$$

where the left vertical arrow is an isomorphism modulo $H_T$-torsion after Quillen.

Now, the proof of the Berline-Vergne fixed-point theorem proves also that the right vertical arrow in $(*)$ is a *true* isomorphism ([9]). As a consequence, the map $\mathrm{ev}_X : \mathbb{C} \otimes H_T(M) \to H_X(M)$ is surjective and the Berline-Vergne fixed point theorem could also be justified through the Atiyah-Bott's de Rham version of the localization theorem. Indeed, the equivariant integration map $\int_M$ gives rise to the commutative diagram

$$\begin{tikzcd} \mathbb{C} \otimes H_T(M) \arrow[r, "\mathrm{ev}_X"] \arrow[d, "\int_M"'] \arrow[dr, phantom, "\oplus"] & H_X(M) \arrow[d, "\int_M"] \\ H_T(\bullet) \arrow[r, "\mathrm{ev}_X"'] & \mathbb{C} \end{tikzcd}$$

with, in the second line, $\mathrm{ev}_X(P) = P(-2\pi i X)$. Then, by the localization theorem for $H_T(M)$, we see that for all $\mu \in H_X(M)$ and every $\tilde{\mu} \in H_T(M)$ such that $\mathrm{ev}_X(\tilde{\mu}) = \mu$, one has:

$$\int_M \mu = \Big(\int_M \tilde{\mu}\Big)(-2\pi i X) = \sum_{m \in M^T} \frac{\mu(m)}{\mathrm{Eu}_T(m,M)(-2\pi i X)},$$

where $\mathrm{Eu}_T(m,M)$ is the equivariant Euler class of $m \in M$, as introduced by Atiyah-Bott in (2.19)-*loc.cit.*

The Berline-Vergne and Atiyah-Bott works stimulated renewed interest in equivariant cohomology, in particular because of its applications to Lie group representation theory. What happened next goes well beyond the aim of this work. Interested readers should read the excellent account of equivariant cohomology theory for manifolds that can be found in chapters 6 and 7 of the book [BGV] (1992), and for singular spaces in the [GKM] (1998) article, which also reviews the equivariant intersection cohomology following R. Joshua ([Jo], 1987) as well as the Poincaré duality in equivariant intersection cohomology following J-.L. Brylinski ([Br], 1992).

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## 2.1. Category of Cochain $\mathfrak{g}$-Complexes

**2.1.1. Field in Use.** Lie groups and Lie algebras, vector spaces, complexes of vector spaces, linear maps, tensor products and related stuff, are defined over the field of real numbers $\mathbb{R}$.

**2.1.2. $\mathfrak{g}$-modules.** Let $\mathfrak{g}$ be a real *Lie algebra*. A *representation of* $\mathfrak{g}$, also called *a* $\mathfrak{g}$*-module*, will be a real vector space $V$ together with a Lie algebra homomorphism $\rho_V : \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(V)$. For simplicity, the notation "$Y \cdot v$" will frequently replace "$\rho_V(v)$" when the representation is understood.

The *trivial representation of* $\mathfrak{g}$ *on a vector space* $V$, is the one where $\rho_V = 0$.

[9] This results from the fact that, thanks to the Poincaré lemma for Berline-Vergne cohomology stating that the pullback map $H_X(M) \to H_X(\mathbb{R} \times M)$ is an isomorphism, it is easy to check that one has a long exact sequence of Berline-Vergne cohomologies:

$$\to H_{X,c}(M \backslash M^X) \to H_X(M) \to H_X(M^X) \to$$

where $H_{X,c}(M \backslash M^X) = 0$, after the original proof of the Berline-Vergne fixed point theorem.

Given $\mathfrak{g}$-modules $V$ and $W$, a *$\mathfrak{g}$-module morphism from $V$ to $W$* is a linear map $\lambda : V \to W$ s.t. $\lambda \circ \rho_V(Y) = \rho_W(Y) \circ \lambda$ for all $Y \in \mathfrak{g}$. We denote by $\mathrm{Hom}_{\mathfrak{g}}(V,W)$ the subspace of $\mathrm{Hom}_{\mathbb{R}}(V,W)$ of such maps.

A $\mathfrak{g}$-module $V$ is said to be:

- *simple or irreducible*, if it is nonzero and has no nontrivial submodules;
- *semisimple*, if it is a direct sum of irreducible $\mathfrak{g}$-modules;
- *reducible* if it is a direct sum of two nonzero $\mathfrak{g}$-modules;
- *completely reducible* if it is a direct sum of irreducible modules;

The $\mathfrak{g}$-modules and their morphisms constitute a category, the *category of $\mathfrak{g}$-modules* denoted by $\mathrm{Mod}(\mathfrak{g})$.

**2.1.3. Exercise.** Let $V$ be a $\mathfrak{g}$-module. Show the equivalence of:

a) $V$ is completely reducible.

b) $V$ is a sum of irreducible modules.

c) If $W$ is a submodule of $V$ then $V = V' \oplus W$ for some submodule $V'$.

**2.1.4. Exercise.** Given a $\mathfrak{g}$-module $V$, denote by $V^{\mathfrak{g}}$ the subspace of *$\mathfrak{g}$-invariant vectors of $V$*, i.e. of $v \in V$, such that $Y \cdot v = 0$ for all $Y \in \mathfrak{g}$.

a) Show that for all $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(V,W)$, $\varphi(V^{\mathfrak{g}}) \subseteq W^{\mathfrak{g}}$. Derive the fact that the correspondence $V \rightsquigarrow V^{\mathfrak{g}}$, $\varphi \rightsquigarrow \varphi|_{V^{\mathfrak{g}}}$ is functorial from $\mathrm{Mod}(\mathfrak{g})$ intito $\mathrm{Vec}(\mathbb{R})$.

b) Endow $\mathbb{R}$ with the trivial action of $\mathfrak{g}$. Show that the map

$$\mathrm{Hom}_{\mathfrak{g}}(\mathbb{R},V) \to V^{\mathfrak{g}}, \quad \varphi \mapsto \varphi(1),$$

is a natural isomorphism of functors $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{R},\_) \to (\_)^{\mathfrak{g}}$. In particular, $(\_)^{\mathfrak{g}}$ is left exact but not necessarily exact.

**2.1.5. Differential Graded $\mathfrak{g}$-Complexes.** A *differential graded $\mathfrak{g}$-complex*, a *$\mathfrak{g}$-complex* in short, is a quadruple $(C,d,\boldsymbol{\theta},c)$ where:

- $(C,d)$ is a complex in $\mathrm{DGM}(\mathbb{R})$ (cf. 1.1.5);
- $\boldsymbol{\theta} : \mathfrak{g} \to \mathit{End}_{\mathrm{GV}(\mathbb{R})}(C)$ is a Lie algebra morphism, the *$\mathfrak{g}$-derivation* ([10]);
- $c : \mathfrak{g} \to \mathit{Mor}_{\mathrm{GV}(\mathbb{R})}(C,C[-1])$ is a linear map, the *$\mathfrak{g}$-contraction*;

such that, for all $X,Y \in \mathfrak{g}$

$$\begin{cases} \text{i) } c(X) \circ c(Y) + c(Y) \circ c(X) = 0 \\ \text{ii) } d \circ c(X) + c(X) \circ d = \boldsymbol{\theta}(X) \\ \text{iii) } \boldsymbol{\theta}(Y) \circ c(X) - c(X) \circ \boldsymbol{\theta}(Y) = c([Y,X]) \end{cases} \tag{$\diamond$}$$

**2.1.6. Remark.** From ($\diamond$)-(ii), one immediately obtains $d \circ \boldsymbol{\theta}(\_) = \boldsymbol{\theta}(\_) \circ d$ which implies that $\boldsymbol{\theta}$ naturally induces an action of $\mathfrak{g}$ on the cohomology of

[10] Recall that given two $\mathbb{Z}$-graded vector spaces $C$ and $D$, we denote by $\mathit{Mor}_{\mathrm{GV}(\mathbb{R})}(C,D)$ the group of graded homomorphisms of degree zero from $C$ into $D$. The terminology *derivation* comes from the fact that in the main case where $(C,d)$ is the de Rham complex of a $G$-manifold, the group $G$ acts on $(C,d)$ by differential graded algebra automorphisms, so that the infinitesimal action of its Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ will be by differential graded algebra derivations.

$(C,d)$. However, that same condition shows that $c(X)$ is a homotopy for $\boldsymbol{\theta}(X)$, so that this induced action is in fact trivial.

**2.1.7. Morphisms of $\mathfrak{g}$-Complexes.** A *morphism of graded* $\mathfrak{g}$*-complexes*, or *morphism of* $\mathfrak{g}$*-complexes* in short, $\boldsymbol{\alpha} : (C,d,\boldsymbol{\theta},c) \to (D,d,\boldsymbol{\theta},c)$, is a morphism of complexes $\boldsymbol{\alpha} : (C,d) \to (D,d)$ commuting with derivations and contractions, i.e. such that $\boldsymbol{\alpha} \circ \boldsymbol{\theta} = \boldsymbol{\theta} \circ \boldsymbol{\alpha}$ and $\boldsymbol{\alpha} \circ c = c \circ \boldsymbol{\alpha}$.

**2.1.8. Category of $\mathfrak{g}$-Complexes.** The $\mathfrak{g}$-complexes $(C,d,\boldsymbol{\theta},c)$ and their morphisms constitute the *category of* $\mathfrak{g}$*-complexes* denoted by $\mathrm{DGM}(\mathfrak{g},\mathbb{R})$.

In the sequel, a $\mathfrak{g}$-complex $(C,d,\boldsymbol{\theta},c)$ may be denoted by $(C,d)$ and even simply $C$, whenever the remaining data are understood.

**2.1.9. Split $\mathfrak{g}$-Complexes.** Given an inclusion of $\mathfrak{g}$-modules $N \subseteq M$, we will use the notation "$N|M$" to express that the natural map

$$\mathrm{Hom}_{\mathfrak{g}}(V,M) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(V,M/N) \tag{‡}$$

is **surjective** for all **finite** dimensional $\mathfrak{g}$-module $V$.

**Exercise.** Show that the condition $N|M$ is equivalent to the fact that for every $\mathfrak{g}$-submodule $M' \subseteq M$ such that $N \subseteq M'$ is of finite codimension, there exists a $\mathfrak{g}$-submodule $H \subseteq M'$ such that $M' = H \oplus N$.

**Definition.** For a $\mathfrak{g}$-complex $(C,d)$, let $B^i := \mathrm{im}(d_{i-1})$ and $Z^i := \ker(d_i)$ respectively be *the* $\mathfrak{g}$*-submodules of* $i$*-coboundaries and* $i$*-cocycles of* $(C,d)$. The $\mathfrak{g}$-complex $(C,d)$ will be called $\mathfrak{g}$*-split* whenever one has

$$B^i|Z^i|C^i, \qquad \text{for all } i \in \mathbb{Z}.$$

**2.1.10. Lemma.** *Keep the above notations and prove the following,*

a) *If $N|M$, the natural map* $\dfrac{M^{\mathfrak{g}}}{N^{\mathfrak{g}}} \longrightarrow \left(\dfrac{M}{N}\right)^{\mathfrak{g}}$ *is an isomorphism.* $\quad (\diamond)$

b) *The condition $B^i|Z^i$ is equivalent to the fact that $(Z^i)^{\mathfrak{g}} \to (Z^i/B^i)$ is surjective, and it is also equivalent to the existence of a $\mathfrak{g}$-submodule $H^i$ of $Z^i$ such that $Z^i = B^i \oplus H^i$, in which case $H^i$ is a trivial $\mathfrak{g}$-module isomorphic to $Z^i/B^i$.*

c) *A $\mathfrak{g}$-complex $(C,d)$ such that each $C^i$ is completely reducible, is $\mathfrak{g}$-split.*

*Proof*

a) After 2.1.4, the functor $(\_)^{\mathfrak{g}}$ is isomorphic to $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{R};\_)$ and the sequence $\mathbf{0} \to N^{\mathfrak{g}} \to M^{\mathfrak{g}} \to (M/N)^{\mathfrak{g}}$ is left exact. The split condition ensures it is also right exact.

b) Recall that $\mathcal{H}^i := Z^i/B^i$ is a trivial $\mathfrak{g}$-module (see 2.1.6). Following (a), the split condition immediately gives the surjection $(Z^i)^{\mathfrak{g}} \twoheadrightarrow (\mathcal{H}^i)^{\mathfrak{g}} = \mathcal{H}^i$. Conversely, one clearly has $\mathrm{Hom}_{\mathfrak{g}}(\mathcal{H}^i,\_) = \mathrm{Hom}_{\mathbb{R}}(\mathcal{H}^i,(\_)^{\mathfrak{g}})$ and, thereafter, the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{g}}(\mathcal{H}^i,Z^i) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathfrak{g}}(\mathcal{H}^i,\mathcal{H}^i) \arrow[d, equal] \\
\mathrm{Hom}_{\mathbb{R}}(\mathcal{H}^i,(Z^i)^{\mathfrak{g}}) \arrow[r, two heads] & \mathrm{Hom}_{\mathbb{R}}(\mathcal{H}^i,\mathcal{H}^i)
\end{tikzcd}$$

where the surjectivity of the second line implies the surjectivity of the first one. In particular, there exists $\sigma \in \mathrm{Hom}_{\mathfrak{g}}(\mathcal{H}^i, Z^i)$ such that $\pi \circ \sigma = \mathrm{id}$ where $\pi : Z^i \twoheadrightarrow \mathcal{H}^i$ denotes the canonical projection. Setting $H^i := \mathrm{Im}(\sigma)$ completes de proof.

c) Clear from exercise 2.1.3. □

**2.1.11. Proposition.** *Let $(C,d)$ be a $\mathfrak{g}$-split $\mathfrak{g}$-complex.*

a) *The inclusion $C^{\mathfrak{g}} \subseteq C$ is a quasi-isomorphism*

b) *If $V$ is a finite dimensional **semi-simple** $\mathfrak{g}$-module, the inclusions*

$$\begin{array}{ccccc} V^{\mathfrak{g}} \otimes C & \supseteq & V^{\mathfrak{g}} \otimes C^{\mathfrak{g}} & \subseteq & (V \otimes C)^{\mathfrak{g}} \\ Hom^\bullet_{\mathbb{R}}(V^{\mathfrak{g}},C) & \supseteq & Hom^\bullet_{\mathbb{R}}(V^{\mathfrak{g}},C^{\mathfrak{g}}) & \subseteq & Hom^\bullet_{\mathfrak{g}}(V,C) \end{array}$$

*are quasi-isomorphisms.*

*Proof*

a) Immediate from (2.1.10-(a)).

b) Let us first show that if $W$ is a simple $\mathfrak{g}$-module different from $\mathbb{R}$, the complexes $(W \otimes C)^{\mathfrak{g}}$ and $Hom^\bullet_{\mathfrak{g}}(W,C)$ are acyclic.

It suffices to treat only the $Hom^\bullet$ case, since one has

$$Hom^\bullet_{\mathfrak{g}}(W,C) = Hom^\bullet_{\mathbb{R}}(W,C)^{\mathfrak{g}} = (W^\vee \otimes C)^{\mathfrak{g}}\,.$$

An $i$-cocycle of $Hom^\bullet_{\mathfrak{g}}(W,C)$ is a $\mathfrak{g}$-module morphism $\lambda : W \to C^i$ such that $d \circ \lambda = 0$, i.e. such that $\mathrm{im}(\lambda) \subseteq Z^i$. But the composition of $\lambda$ with the surjection $Z^i \twoheadrightarrow Z^i/B^i$ is null since $\mathfrak{g}$ acts trivially on cohomology, so that in fact $\mathrm{im}(\lambda) \subseteq B^i$. Now, thanks to the fact that $Z^i|C^i$, we can lift $\lambda : W \to B^i$ to $\mu : W \to C^{i-1}$ and we have thus proved that $\lambda = d \circ \mu$, i.e. that $\lambda$ is a coboundary.

If $V$ is a semisimple $\mathfrak{g}$-module, one decomposes $V$ as $V^{\mathfrak{g}} \oplus W$, where $W$ is a direct sum of simple $\mathfrak{g}$-modules different from $\mathbb{R}$. Then

$$Hom^\bullet_{\mathfrak{g}}(V,C) = Hom^\bullet_{\mathfrak{g}}(V^{\mathfrak{g}},C) \oplus Hom^\bullet_{\mathfrak{g}}(W,C)$$

is quasi-isomorphic to $Hom^\bullet_{\mathfrak{g}}(V^{\mathfrak{g}},C)$ after the previous paragraph. But

$$Hom^\bullet_{\mathfrak{g}}(V^{\mathfrak{g}},C) = Hom^\bullet_{\mathfrak{g}}(V^{\mathfrak{g}},C^{\mathfrak{g}}) = Hom^\bullet_{\mathbb{R}}(V^{\mathfrak{g}},C^{\mathfrak{g}}),$$

so that $Hom^\bullet_{\mathbb{R}}(V^{\mathfrak{g}},C^{\mathfrak{g}}) \subseteq Hom^\bullet_{\mathfrak{g}}(V,C)$ is clearly a quasi-isomorphism.

Finally, that the inclusion $Hom^\bullet_{\mathbb{R}}(V^{\mathfrak{g}},C^{\mathfrak{g}}) \subseteq Hom^\bullet_{\mathbb{R}}(V^{\mathfrak{g}},C)$ is a quasi-isomorphism results from (a) since $V^{\mathfrak{g}} \simeq \mathbb{R}^r$ and the inclusion being considered becomes simply $\prod_{1\leqslant i\leqslant r} C^{\mathfrak{g}} \subseteq \prod_{1\leqslant i\leqslant r} C$. □

## 2.2. Equivariant Cohomology of $\mathfrak{g}$-Complexes

**2.2.1. The symmetric Algebra of $\mathfrak{g}^\vee$.** Let $S(\mathfrak{g})$ be the ring of polynomial maps from $\mathfrak{g}$ to $\mathbb{R}$, graded by twice the polynomial degree and denote by $S^d(\mathfrak{g})$ the subspace of elements of degree $d$, in particular $S^2(\mathfrak{g}) = \mathfrak{g}^\vee$ and $S^m(\mathfrak{g}) = 0$ for every odd integer $m$. Let $\boldsymbol{\theta} : \mathfrak{g} \to \mathrm{Der}_{\mathbb{R}}(S(\mathfrak{g}))$ denote the Lie algebra homomorphism induce by coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^\vee$.

Fix for later use a vector space basis $\{e_i\}$ of $\mathfrak{g}$, of dual basis $\{e^i\}$.

**2.2.2. Cartan Complexes.** Given a $\mathfrak{g}$-complex $(C,d,\boldsymbol{\theta},c)$, we are interested in the polynomial maps $\omega : \mathfrak{g} \ni Y \mapsto \omega(Y) \in C$, i.e. the elements $\omega \in S(\mathfrak{g}) \otimes C$. The Lie algebra $\mathfrak{g}$ acts on each $S^a(\mathfrak{g}) \otimes C^b$ by the formula

$$\boldsymbol{\theta}(Y)(P \otimes \mu) := \boldsymbol{\theta}(Y)(P) \otimes \mu + P \otimes \boldsymbol{\theta}(Y)(\mu), \qquad \forall Y \in \mathfrak{g}.$$

A polynomial map $Y \mapsto \omega(Y)$ is then $\mathfrak{g}$-invariant if and only if it satisfies the equality

$$\theta(X)(\omega(Y)) + \omega([X,Y]) = 0,$$

for all $X,Y \in \mathfrak{g}$. Put

$$\boxed{C_{\mathfrak{g}} := (S(\mathfrak{g}) \otimes C)^{\mathfrak{g}} = \bigoplus\nolimits_{k\in\mathbb{Z}} C_{\mathfrak{g}}^k} \tag{$C_{\mathfrak{g}}$}$$

where $C_{\mathfrak{g}}^k := \sum_{a+b=k} \left(S^a(\mathfrak{g}) \otimes C^b\right)^{\mathfrak{g}}$. The $S(\mathfrak{g})$-linear map $d_{\mathfrak{g}} : C_{\mathfrak{g}} \to C_{\mathfrak{g}}$,

$$\boxed{d_{\mathfrak{g}}(1 \otimes \omega) = 1 \otimes d\omega + \sum\nolimits_i e^i \otimes c(e_i)\omega} \tag{$d_{\mathfrak{g}}$}$$

is a morphism of graded spaces of degree $+1$. It verifies $d_{\mathfrak{g}}^2 = \sum_i e^i \otimes \boldsymbol{\theta}(e_i)$, so that, over $C_{\mathfrak{g}}$, one has

$$d_{\mathfrak{g}}^2 = \sum_i e^i \boldsymbol{\theta}(e_i) \otimes \mathrm{id}.$$

But $\Xi := \sum_i e^i \boldsymbol{\theta}(e_i)$ is the null operator on $S(\mathfrak{g})$. Indeed, since it acts as a derivation on $S(\mathfrak{g})$, it suffices to show that it vanishes on any $\lambda \in \mathfrak{g}^\vee$, i.e. that $\Xi(\lambda)(e_j) = 0$ for all $j$, which comes from the straightforward computation

$$\Xi(\lambda)(e_j) = \Big(\sum_i e^i \theta(e_i)(\lambda)\Big)(e_j) = \sum_i e^i(e_j)\lambda([e_i,e_j]) = \lambda([e_j,e_j]) = 0.$$

Hence, $d_{\mathfrak{g}}^2 = 0$ in $C_{\mathfrak{g}}$. This $d_{\mathfrak{g}} \in \mathit{Endgr}^1_{S(\mathfrak{g})^{\mathfrak{g}}}(C_{\mathfrak{g}})$ is the *Cartan differential*.

**2.2.3. Definition.** The pair $\left(C_{\mathfrak{g}}, d_{\mathfrak{g}}\right)$ is a complex. It is *the Cartan (equivariant) complex associated with the* $\mathfrak{g}$*-complex* $(C,d,\boldsymbol{\theta},c)$, and the cohomology of $\left(C_{\mathfrak{g}}, d_{\mathfrak{g}}\right)$ is its $\mathfrak{g}$*-equivariant cohomology*, denoted in the sequel by

$$\boxed{H_{\mathfrak{g}}(C) := h\big(C_{\mathfrak{g}}, d_{\mathfrak{g}}\big)}$$

**2.2.4. *Important Remark.*** The graded space $C_{\mathfrak{g}}$ is an $S(\mathfrak{g})^{\mathfrak{g}}$-graded module (4.1.3), the differential $d_{\mathfrak{g}}$ is $S(\mathfrak{g})^{\mathfrak{g}}$-linear, and the cohomology $H_{\mathfrak{g}}(C)$ is an $S(\mathfrak{g})^{\mathfrak{g}}$-graded module.

**2.2.5** Any morphism of $\mathfrak{g}$-complexes $\boldsymbol{\alpha} : (C,d,\boldsymbol{\theta},c) \to (D,d,\boldsymbol{\theta},c)$ induces a canonical $S(\mathfrak{g})$-linear morphism of complexes $\alpha_{\mathfrak{g}} : C_{\mathfrak{g}} \to D_{\mathfrak{g}}$ by the formula $\boldsymbol{\alpha}_{\mathfrak{g}} = \mathrm{id} \otimes \boldsymbol{\alpha}$.

**2.2.6. Theorem.** *With the above notations one has,*

a) *The correspondence* $(C,d,\boldsymbol{\theta},c) \rightsquigarrow (C_{\mathfrak{g}},d)$ *and* $\boldsymbol{\alpha} \rightsquigarrow \boldsymbol{\alpha}_{\mathfrak{g}}$ *is a covariant functor from* $\mathrm{DGM}(\mathfrak{g},\mathbb{R})$ *into* $\mathrm{DGM}(\mathbb{R})$.
b) *For every* $\mathfrak{g}$*-complex* $(C,d,\boldsymbol{\theta},c)$*, there exists a spectral sequence converging to* $H_{\mathfrak{g}}(C)$ *with*

$$\left(\mathbb{E}_0^{p,q} = \left(S^p(\mathfrak{g}) \otimes C^q\right)^{\mathfrak{g}},\ d_0 = 1 \otimes d\right) \Rightarrow H_{\mathfrak{g}}^{p+q}(C).$$

c) *Let $G$ be a compact Lie group, $\mathfrak{g} := \mathrm{Lie}(G)$ and $C$ end $D$ two **$\mathfrak{g}$-split** $\mathfrak{g}$-complexes (2.1.9).*

i) *The $(\mathbb{E}_2,d_2)$ spectral sequence term in (b) is given by*

$$\Big(\mathbb{E}_2^{p,q} = S^p(\mathfrak{g})^{\mathfrak{g}} \otimes H^q(C),\ d_2 = \textstyle\sum_i e^i \otimes c(e_i)\Big) \Rightarrow H_{\mathfrak{g}}^{p+q}(C).$$

ii) *If $H^m(C) = 0$ for all odd (or for all even) $m$, then*

$$H_{\mathfrak{g}}(C) = S(\mathfrak{g})^{\mathfrak{g}} \otimes h(C).$$

iii) *If $\boldsymbol{\alpha} : C \to D$ is a quasi-isomorphism of $\mathfrak{g}$-complexes, $\boldsymbol{\alpha}_{\mathfrak{g}} : C_{\mathfrak{g}} \to D_{\mathfrak{g}}$ is a quasi-isomorphism.*

d) *Let $G$ be a **commutative** compact Lie group and $\mathfrak{g} := \mathrm{Lie}(G)$,*

i) *For every $\mathfrak{g}$-complex $(C,d,\boldsymbol{\theta},c)$, the subcomplex $(C^{\mathfrak{g}},d)$ is stable under $\boldsymbol{\theta}$ and $c$, i.e. $(C^{\mathfrak{g}},d,\boldsymbol{\theta},c)$ is a well defined $\mathfrak{g}$-complex.*

ii) *If $j : C^{\mathfrak{g}} \hookrightarrow C$ denotes the inclusion map, $j_{\mathfrak{g}}$ is a quasi-isomorphism.*

iii) *The $(\mathbb{E}_2,d_2)$ spectral sequence term in (b) is given by*

$$\Big(\mathbb{E}_2^{p,q} = S^p(\mathfrak{g}) \otimes H^q(C^{\mathfrak{g}}),\ d_2 = \textstyle\sum_i e^i \otimes c(e_i)\Big) \Rightarrow H_{\mathfrak{g}}^{p+q}(C)$$

iv) *If $H^m((C)^{\mathfrak{g}}) = 0$ for all odd (or for all even) $m$, then*

$$H_{\mathfrak{g}}(C) = S(\mathfrak{g}) \otimes h(C^{\mathfrak{g}}).$$

v) *If $\boldsymbol{\alpha} : C^{\mathfrak{g}} \to D^{\mathfrak{g}}$ is a quasi-isomorphism, $\boldsymbol{\alpha}_{\mathfrak{g}}$ is a quasi-isomorphism.*

*Proof*

a) Clear.

b) For $m \in \mathbb{Z}$, let $K_m = \big(S^{\geqslant m}(\mathfrak{g}) \otimes C\big)^{\mathfrak{g}}$. Each $K_m$ is clearly a sub-complex of $(C_{\mathfrak{g}},d_{\mathfrak{g}})$ and $\big(C_{\mathfrak{g}} = K_0 \supseteq K_1 \supseteq \cdots\big)$ is a *regular* decreasing filtration of $(C_{\mathfrak{g}},d_{\mathfrak{g}})$ (see [Go] §4 pp. 76-) giving rise to the stated spectral sequence.

c) i) The assumption that $G$ is compact ensures that each (finite dimensional) $\mathfrak{g}$-module $S^p(\mathfrak{g})$ is semisimple. Proposition 2.1.11-(b) may be used, and $\big(S^p(\mathfrak{g}) \otimes C)^{\mathfrak{g}},1 \otimes d\big)$ is quasi-isomorphic to $\big(S^p(\mathfrak{g})^{\mathfrak{g}} \otimes C,1 \otimes d\big)$. Consequently $(\mathbb{E}_0,d_0)$ in (b) is quasi-isomorphic to $\big(S(\mathfrak{g})^{\mathfrak{g}} \otimes C,1 \otimes d\big)$ and $\mathbb{E}_1^{p,q} = S^p(\mathfrak{g})^{\mathfrak{g}} \otimes H^q(C)$. But the differential $d_1 : \mathbb{E}_1^{p,q} \to \mathbb{E}_1^{p+1,q}$ is null since the $S(\mathfrak{g})$ vanishes in odd degrees, therefore $\mathbb{E}_1 = \mathbb{E}_2$, which completes the proof of the claim.

ii) Since the differential $d_r$ is of total degree 1 and that $\mathbb{E}_r^{p,q} = 0$ if $p$ or $q$ is odd for all $r \geqslant 2$, one has $d_r = 0$ for $r \geqslant 2$, and $\mathbb{E}_2 = \mathbb{E}_\infty$.

iii) Follows immediately from (c-i).

d) i) We must check that $\boldsymbol{\theta}(Y)c(X)C^{\mathfrak{g}} = 0$ for all $X,Y \in \mathfrak{g}$, but, on $C^{\mathfrak{g}}$ one has $\boldsymbol{\theta}(Y)c(X) = \boldsymbol{\theta}(Y)c(X) + c(X)\boldsymbol{\theta}(Y) = c([Y,X]) = c(0)$ since $\mathfrak{g}$ is abelian and from property (iii) of $\mathfrak{g}$-complexes (see 2.1.5-($\diamond$)).

ii,iii,iv,v) Left to the reader. □

**2.2.7. Split $G$-Complexes.** It's worth noting that the proof of 2.2.6-(c) makes use of the split condition (2.1.9) *only* for the finite dimensional sub-$\mathfrak{g}$-modules $V \in S(\mathfrak{g})$, whose $\mathfrak{g}$-module structure is obtained by differentiating its natural $G$-module structure.

The split condition 2.1.9 can easily be adapted to the context of $G$-modules. For any inclusion of $G$-modules $N \subseteq M$ one writes "$N|M$" whenever the natural map

$$\mathrm{Hom}_G(V,M) \longrightarrow \mathrm{Hom}_G(V,M/N) \tag{‡}$$

is **surjective** for all **finite dimensional** $G$-module $V$.

**2.2.8. Definition.** A complex of $G$-modules $(C,d)$ is said to be $G$-*split* whenever $B^i|Z^i|C^i$, for all $i \in \mathbb{Z}$.

The proof of the following proposition is the same as 2.1.11.

**2.2.9. Proposition.** *Let $(C,d)$ be a $G$-split complex of $G$-modules such that the natural action of $G$ in cohomology is trivial. Then,*

a) *The inclusion $C^G \subseteq C$ is quasi-isomorphism.*

b) *If $V$ is a **semisimple** finite dimensional $G$-module, the inclusions*

$$\begin{array}{ccccc} V^G \otimes C & \supseteq & V^G \otimes C^G & \subseteq & (V \otimes C)^G \\ Hom^\bullet_{\mathbb{R}}(V^G,C) & \supseteq & Hom^\bullet_{\mathbb{R}}(V^G,C^G) & \subseteq & Hom^\bullet_G(V,C) \end{array}$$

*are quasi-isomorphisms.*

# 3. Equivariant Cohomology of $G$-Manifolds

## 3.1. Equivariant Differential Forms

**3.1.1. Fields in Use.** Unless otherwise stated, manifolds, Lie groups and Lie algebras, vector spaces, complexes of vector spaces, linear maps, tensor products and related stuff, will be defined over the field of real numbers $\mathbb{R}$.

**3.1.2. $G$-Derivations and Contractions.** Let $G$ be a **connected** Lie group. Denote by $\mathfrak{g} := \mathrm{Lie}(G) = T_eG$ the Lie algebra of $G$ endowed with the adjoint action. As in 2.2.1, let $S(\mathfrak{g})$ be the ring of polynomial maps from $\mathfrak{g}$ to $\mathbb{R}$, graded by twice the polynomial degree.

Let $M$ be a $G$-manifold. Each $Y \in \mathfrak{g}$ defines a vector field on $M$ by setting

$$\vec{Y}(m) := \frac{d}{dt}\Big(t \mapsto \exp(tY)\cdot m\Big)_{t=0}$$

Let $\vec{Y}\cdot\omega$ denote the *contraction* of the differential form $\omega \in \Omega(M)$ by the vector field $\vec{Y}$. The map $c(Y) : \Omega(M) \to \Omega(M)$, $\omega \mapsto \vec{Y}\cdot\omega$, is then an *antiderivation* of degree $-1$ and the map $c : \mathfrak{g} \to \mathit{Mor}_{\mathrm{GV}(\Bbbk)}(\Omega(M),\Omega(M)[-1])$ verifies the condition (i) for $\mathfrak{g}$-complexes (see 2.1.5-($\diamond$)).

The Lie derivative with respect to the vector field $\vec{Y}$, gives a Lie algebra representation $\boldsymbol{\theta} : \mathfrak{g} \to \mathit{End}_{\mathrm{GV}(\Bbbk)}(\Omega(M))$ by *algebra derivations*.

Both of the operators $\boldsymbol{\theta}(Y)$ and $c(Y)$, resp. *the $G$-derivations and the $G$-contractions* stabilizes the subcomplex of compact support differential forms, and $(\Omega(M),d,\boldsymbol{\theta},c)$ and $(\Omega_{\mathrm{c}}(M),d,\boldsymbol{\theta}c)$ become $\mathfrak{g}$-complexes in the sense of 2.1.5.

**3.1.3. Definition.** Let $G$ be a compact connected Lie group. The *complex of $G$-equivariant differential forms, resp. with compact support, of $M$*, are the following Cartan complexes (2.2.3)

$$\begin{aligned} \big(\Omega_G(M),d_G\big) &:= \big(\Omega(M)_{\mathfrak{g}},d_{\mathfrak{g}}\big) = \Big(\big(S(\mathfrak{g})\otimes\Omega(M)\big)^G,d_{\mathfrak{g}}\Big) \\ \textit{resp.}\ \big(\Omega_{G,\mathrm{c}}(M),d_G\big) &:= \big(\Omega_{\mathrm{c}}(M)_{\mathfrak{g}},d_{\mathfrak{g}}\big) = \Big(\big(S(\mathfrak{g})\otimes\Omega_{\mathrm{c}}(M)\big)^G,d_{\mathfrak{g}}\Big). \end{aligned}$$

Their cohomology, denoted by $H_G(M)$, resp. $H_{G,\mathrm{c}}(M)$, are the *$G$-equivariant cohomology, resp. with compact support, of $M$*.

In the case where $M = \{\bullet\}$, we have $H_G(\{\bullet\}) = S(\mathfrak{g})^G = S(\mathfrak{g})^{\mathfrak{g}}$. The notation "$H_G$" stands for "$H_G(\{\bullet\})$".

The Cartan complexes $\Omega_G(M)$,$\Omega_{G,\mathrm{c}}(M)$ and the equivariant cohomology spaces $H_G(M)$ and $H_{G,\mathrm{c}}(M)$ are $H_G$-graded modules (*cf.* 4.1.3).

**3.1.4. Proposition.** *Let $G$ be a compact connected Lie group.*

a) *The complexes of $G$-modules $(\Omega(M),d)$ and $(\Omega_{\mathrm{c}}(M),d)$ are $G$-split (2.2.7). In particular, if $C$ denotes $(\Omega(M),d)$ or $(\Omega_{\mathrm{c}}(M),d)$, the inclusions*

$$S(\mathfrak{g})^G\otimes C \supseteq S(\mathfrak{g})^G\otimes C^G \subseteq (S(\mathfrak{g})\otimes C)^G$$

*are quasi-isomorphisms.*

b) *The correspondence $M \rightsquigarrow (\Omega(M),d,\boldsymbol{\theta},c)$, $f \rightsquigarrow f^*$ is a contravariant functor from the category of $G$-manifolds into the category of $G$-split $\mathfrak{g}$-complexes.*

c) *The correspondence* $M \rightsquigarrow (\Omega_c(M), d, \boldsymbol{\theta}, c)$, $f \rightsquigarrow f^*$ *is a contravariant functor from the category of* $G$*-manifolds and* ***proper*** *maps to the category of* $G$*-split* $\mathfrak{g}$*-complexes.*
d) *There exists a* ***functor*** *on the category* $G$-Man *of* $G$*-manifolds and* $G$*-equivariant maps that assigns to every* $G$*-manifold* $M$ *a spectral sequence that converges to its equivariant cohomology*

$$\mathbb{E}_2^{p,q} = S^p(\mathfrak{g})^{\mathfrak{g}} \otimes H^q(M) \Rightarrow H_G^{p+q}(M).$$

e) *There exists a* ***functor*** *on the category* $G$-Man$_{\rm pr}$ *of* $G$*-manifolds and* $G$*-equivariant* ***proper*** *maps that assigns to every* $G$*-manifold* $M$ *a canonical spectral sequence that converges to its equivariant cohomology with compact support*

$$\mathbb{E}_2^{p,q} = S^p(\mathfrak{g})^{\mathfrak{g}} \otimes H_c^q(M) \Rightarrow H_{G,c}^{p+q}(M).$$

*Proof*

a) For $i \in \mathbb{N}$, the *pushforward action* of $G$ on $\Omega^i(M)$ is defined as $g_*(\omega) := (g^{-1})^*(\omega)$ for all $g \in G$ and $\omega \in \Omega^i$, so that $(g_1 g_2)_* = g_{1*} \circ g_{2*}$.

If $V$ be is a (smooth) finite dimensional representation of $G$ over $\mathbb{C}$, we make the group $G$ act on $\mathrm{Hom}(V,\Omega^i(M))$ by the formula

$$(g\cdot\lambda)(v) = g_*\big(\lambda(g^{-1}v)\big), \quad \forall \lambda \in \mathrm{Hom}(V,\Omega^i(M)),$$

so that $\lambda$ is a $G$-module morphism if and only if $g\cdot\lambda = \lambda$. We claim that there exists a "symmetrization" operator

$$\Sigma : \mathrm{Hom}(V,\Omega^i(M)) \to \mathrm{Hom}(V,\Omega^i(M))^G$$

such that $\Sigma^2 = \Sigma$ and $\Sigma(\lambda) = \lambda$ if and only if $\lambda$ is a $G$-module morphism.

Indeed, let $\lambda$ be a linear map from $V$ to $\Omega^i(M)$. For every $i$-tuple of vector fields $\{\chi_1,\ldots,\chi_i\}$ over $M$ and each $v \in V$, the real function

$$M \ni x \mapsto \Big(\int_G g_*\big(\lambda(g^{-1}v)\big)(x)\big(\chi_1(x),\ldots,\chi_i(x)\big)\, dg\Big) \in \mathbb{R}$$

where $dg$ is a $G$-invariant form of top degree on $G$, such that $1 = \int_G dg$, is a smooth function **because $V$ is finite dimensional**, and it depends linearly on $v \in V$, and multilinearly and antisymmetrically on the $\chi_*$. We therefore have an $i$-differential form which we denote by

$$\Sigma(\lambda)(v) := \int_G g_*\big(\lambda(g^{-1}v)\big)\, dg, \tag{$*$}$$

and whose fundamental properties are

- $\Sigma(d\circ\lambda) = d\circ\Sigma(\lambda)$;
- $\Sigma(\lambda) : V \to \Omega^i(M)$ is a $G$-module morphism;
- $\Sigma(\lambda) = \lambda$ if $\lambda$ is already a $G$-module morphism.

We can now resume the proof that $Z^i(M)|\Omega^i(M)$. Given a $G$-module morphism $\mu \in \mathrm{Hom}_G(V, B^{i+1}(M))$, there always exists a linear map $\lambda : V \to \Omega^i(M)$ lifting $\mu$, i.e. such that $\mu = d\circ\lambda$, but then one applies the symmetrization operator $\Sigma$ and one gets $\mu = \Sigma(\mu) = \Sigma(d\circ\lambda) = d\circ\Sigma(\lambda)$, which shows that the $G$-module morphism $\Sigma(\lambda)$ lifts $\mu$.

For $Z^i_c(M)|\Omega^i_c(M)$, note that, since $V$ is finite dimensional, the supports of the elements in $\lambda(V)$ are contained in one and the same compact subset $C \subseteq M$, but then the supports of the $g_*(\lambda(g^{-1}v))$ in $(*)$ are contained in $G{\cdot}C$ which is obviously compact. Therefore, given $\lambda : V \to \Omega_c(M)$, one gets a linear map $\Sigma(V) : V \to \Omega_c(M)$ which is a $G$-module morphism, and the preceding arguments apply to the compactly supported case.

To prove that $B^i(M)|Z^i(M)$, il suffices, from 2.1.10-(b), to show that every cocycle is cohomologous to a $G$-invariant cocycle. But before doing so, let us recall a general homotopy argument. Given a smooth map $\varphi : \mathbb{R}{\times}M \to N$, if $\omega \in \Omega^i(N)$ the pullback $\varphi^*\omega$ belongs to $\Omega^i(\mathbb{R}{\times}M)$, i.e. is a section of the exterior algebra bundle of the cotangent bundle $T^*(\mathbb{R}{\times}M)$ of $\mathbb{R}{\times}M$. Now, the canonical decomposition $T^*(\mathbb{R}{\times}M)$ as the direct sum of cotangent bundles $T^*(\mathbb{R}) \oplus T^*(M)$, gives rise to a canonical decomposition of the $i$-th exterior power of the cotangent bundle

$$\bigwedge^i T^*(\mathbb{R}{\times}M) = \bigwedge^i (T^*M) \oplus \Big(T^*(\mathbb{R}) \otimes \bigwedge^{i-1}(T^*M)\Big).$$

Consequently, the pullback $\varphi^*(\omega)$ canonically decomposes as

$$\varphi^*(\omega)(t,x) = \alpha(t,x) + dt \wedge \beta(t,x),$$

where $\alpha$ (resp. $\beta$) is a section of the vector bundle $\bigwedge^i T^*(M)$ (resp. $\bigwedge^{i-1} T^*(M)$) over the base space $\mathbb{R}{\times}M$.

When $\omega$ is in addition a cocycle, so is $\varphi^*(\omega)$ and, in view of the previous decomposition, this amounts to the following two conditions

$$d\alpha(t,x) = 0, \qquad \frac{\partial}{\partial t}\alpha(t,x) = d\beta(t,x),$$

where $d$ is the coboundary in $\Omega(M)$ ($t$ is then assumed constant). In particular, if $\varphi_t : M \to N$ denotes the map $x \mapsto \varphi(t,x)$, we get

$$\begin{aligned} \varphi_t^*(\omega) - \varphi_0^*(\omega) &= \alpha(t) - \alpha(0) \\ &= \int_0^t \frac{\partial}{\partial t}\alpha(t)\,dt = \int_0^t d\beta(t)\,dt = d\Big(\int_0^t \beta(t)\,dt\Big), \end{aligned} \tag{$**$}$$

and the cocycles $\varphi_t^*(\omega)$ are all cohomologous to $\varphi_0^*(\omega)$.

At this point it is worth noting that this process also gives a canonical element $\varpi(x) = \int_0^1 \beta(t,x)\ dt \in \Omega^{i-1}(M)$, depending on $\omega$ and such that $\varphi_1^*(\omega) - \varphi_0^*(\omega) = d\varpi$.

Under the hypothesis of our proposition, a first consequence of these notes, is that if $\omega \in Z^i(M)$ then $g^*\omega$ is cohomologous to $\omega$ for all $g \in G$. Indeed, since $G$ is connected, there is a smooth path $\gamma : \mathbb{R} \to G$ such that $\gamma(0) = e$ and $\gamma(1) = g$, and then taking $\varphi : \mathbb{R}{\times}M \to M$, $(t,x) \mapsto \gamma(t){\cdot}x$, one concludes that $g^*\omega = \gamma_1^*(\omega) \sim \gamma_0^*(\omega) = \omega$.

More generally, given any diffeomorphism $\phi : \mathbb{R}^{d_G} \to G$ onto an open subset $U \subseteq G$, one defines a smooth multiplicative action of $\mathbb{R}$ over $U$ by setting $t \star g := \phi(t \cdot \phi^{-1}(g))$ for all $t \in \mathbb{R}$ and $g \in U$, and considers, for each

$g \in U$, the map $\varphi_g : \mathbb{R}{\times}M \to M$, $\varphi_g(t,x) = (t \star g)x$. After that, if $\omega$ is a cocycle of $\Omega^i(M)$ we will have

$$g^*\omega - g_0^*\omega = d\Big(\int_0^1 \beta(t,g)\,dt\Big), \tag{$***$}$$

with $g_0 := \phi(0)$ and where $\beta(t,g)$ denotes a family of elements of $\Omega^{i-1}(M)$ depending smoothly on $(t,g) \in \mathbb{R}{\times}U$, i.e. for any $(i-1)$-tuple $(\chi_1,\dots,\chi_{i-1})$ of vector fields over $M$, the following map is smooth:

$$\mathbb{R}{\times}U{\times}M \ni (t,g,x) \mapsto \beta(t,g,x)(\chi_1(x),\dots,\chi_{i-1}(x)) \in \mathbb{R}.$$

We now come to a key point. If in addition, one has a compactly supported function $\rho : U \to \mathbb{R}$, then, for any top degree form $dg$ on $G$, one has

$$\begin{aligned}\int_G \rho(g)g^*\omega\,dg &= \int_G \rho(g)\big(g^*\omega - g_0^*\omega\big)\,dg + \Big(\int_G \rho(g)\,dg\Big)g_0^*\omega\\ &= d\Big(\int_G\int_0^1 \rho(g)\beta(t,g)\,dg\Big) + \Big(\int_G \rho(g)\,dg\Big)g_0^*\omega\end{aligned}$$

where $\int_G\int_0^1 \rho(g)\beta(t,g)\,dg$ is a **smooth** differential form over $M$. But, as we already show that $g_0^*\omega \sim \omega$, since $G$ is connected, we may conclude that

$$\int_G \rho(g)g^*\omega\,dg \sim \Big(\int_G \rho(g)\,dg\Big)\omega,$$

something that is satisfied by any compactly supported function $\rho : G \to \mathbb{R}$ whose support is contained in any open subset of $M$ diffeomorphic to $\mathbb{R}^d_G$.

If we now make use of the fact that $G$ is compact (which we haven't done so far), we can choose the form $dg$ to be $G$-invariant such that $\int_G dg = 1$, and we can fix a smooth partition of unity $\{\rho_i\}$ subordinate to a finite good cover (*cf.* ([3])) of $G$. Then

$$\begin{aligned}\Sigma(\omega) := \int_G g^*\omega\,dg = \int_G \sum_i \rho_i(g)g^*\omega\,dg = \sum_i \int_G \rho_i(g)g^*\omega\,dg\\ \sim \Big(\sum_i \int_G \rho_i(g)\,dg\Big)\omega = \Big(\int_G \sum_i \rho_i(g)\,dg\Big)\omega = \omega\end{aligned}$$

where, obviously, $\Sigma(\omega)$ is a $G$-**invariant** cocycle, which completes the poof that $B^i(M)|Z^i(M)$ as $G$-modules.

If we denote by $|\_|$ the support of a differential form, we see in what precedes that for $t \in [0,1]$ and $g \in G$ one has

$$\begin{cases} |\beta(t)| \subseteq \gamma([0,1])\cdot|\omega| & \text{in } (**)\\ |\rho(g)\beta(t,g)| \subseteq ([0,1] \star |\rho|)|\omega| & \text{in } (***)\end{cases}$$

so that if $|\omega|$ is compact, the previous arguments show that $\Sigma(\omega) - \omega$ is in fact the differential of a compactly supported differential form, i.e. we have also proved that $B_c^i(M)|Z_c^i(M)$.

b,c,d) Follow by (a) and 2.2.6 by interchanging $\mathfrak{g}$ and $G$, by 2.2.7 and 2.2.9. □

**3.1.5. Exercise and remarks.** Show that the conclusion in 3.1.4-(a) does not change if we weaken the connectedness hypothesis of $G$ to simply require the action of $G$ on $C$ to be homotopically trivial. Show that this arrives in particular when, $G$ being connected, one is interested in $H_H(M)$ where $H$ is a closed subgroup of $G$, connected or not. In that case, if $H_\circ$ denotes the connected component of $1 \in H$, one has $H_H(M) = H_{H_\circ}(M)^W$ and $H_{H,\mathrm{c}}(M) = H_{H_\circ,\mathrm{c}}(M)^W$, where $W = H/H_\circ$.

## 3.2. The Borel Construction

**3.2.1. The Classifying Space.** Let $G$ be a compact connected Lie group and $\mathbb{E}G$ a *universal fiber bundle for* $G$. Recall that this topological space is the limit of an inductive system in the category of (right) $G$-manifolds $\{\mathbb{E}G(n) \to \mathbb{E}G(n+1)\}_{n\in\mathbb{N}}$, where $\mathbb{E}G(n)$ is compact, connected, oriented, $n$-acyclic and, moreover, the action of $G$ on $\mathbb{E}G(n)$ is free. A *classifying space of* $G$ is then the quotient manifold $\mathbb{B}G = \mathbb{E}G/G$, limit of the inductive system in the category of manifolds $\{\mathbb{B}G(n) \to \mathbb{B}G(n+1)\}$ where each $\mathbb{B}G(n) := \mathbb{E}G(n)/G$ is compact, simply connected since $G$ is connected, and oriented.

**3.2.2** Given a $G$-manifold $M$, the quotient $M/G$ may lack good differentiability properties since the action of $G$ is not, in general, a free action. A key idea to deal with this issue, dating to the 1950s, is to replace the $G$-manifold $M$ by the product $\mathbb{E}G \times M$ endowed with the *diagonal action* of $G$, $g\cdot(e,x) := (eg^{-1},gx)$. Now, because $\mathbb{E}G$ is "contractible", the topological space $\mathbb{E}G \times M$ has the same homotopy type as $M$ and moreover has the advantage that $G$ acts freely on it. The quotient space is denoted, following Armand Borel ([11]):

$$\boxed{M_G := (\mathbb{E}G \times M)/G}$$

The natural fibration of fiber $M$ (*cf.* page 22):

$$\begin{array}{ccc} M_G := \mathbb{E}G \times_G M & \xrightarrow{\pi} & \mathbb{E}G/G =: \mathbb{B}G \\ [e,x] & \longmapsto & [e] \end{array}$$

establishes an important link between the three spaces $M, M_G, \mathbb{B}G$. Finally, if $f : M \to N$ is a $G$-equivariant map, the induced map $f_G : M_G \to N_G$, $[e,m] \mapsto [e,f(m)]$, is well defined and the diagram

$$\begin{tikzcd} M_G \arrow[r, "f_G"] \arrow[d, two heads, "\pi"'] & N_G \arrow[d, two heads, "\pi"] \\ \mathbb{B}G \arrow[r, equal] & \mathbb{B}G \end{tikzcd}$$

is clearly commutative.

**3.2.3. Definition.** The functor $M \rightsquigarrow M_G$, $f \rightsquigarrow f_G$, from the category of $G$-manifolds to the category of fiber spaces over the classifying space $\mathbb{B}G$, is called *the Borel construction*

**3.2.4** Although the topological space $M_G$ is not a manifold, it is the limit of an inductive system of such. Indeed, for each $n \in \mathbb{N}$, since the compact group

[11] Confer §3 of chapter IV in [$\mathrm{B}_2$], especially the remark §3.9, reproduced at page 23 of these notes, where Borel cites previous works of Conner and of Shapiro using this construction in some special cases.

$G$ acts freely on the product manifold $\mathbb{E}G(n) \times M$, the topological quotient $M_G(n) = \mathbb{E}G(n) \times_G M$ has a natural manifold structure, canonically oriented whenever $M$ is so. One gets an inductive system in the category of manifolds $\{\mu_n : M_G(n) \to M_G(n+1)\}_{n\in\mathbb{N}}$ with $M_G = \varinjlim M_G(n)$, and even an inductive system in the category of fibrations with fiber $M$ and **compact** base spaces

$$\begin{tikzcd} \cdots \arrow[r, dashed] & M_G(n) \arrow[r, "\mu_n"] \arrow[d, two heads, "\pi_n"'] & M_G(n+1) \arrow[r, dashed] \arrow[d, two heads, "\pi_{n+1}"'] & \cdots & M_G \arrow[d, two heads, "\pi"'] \\ \cdots \arrow[r, dashed] & \mathbb{B}G(n) \arrow[r, "\beta_n"] & \mathbb{B}G(n+1) \arrow[r, dashed] & \cdots & \mathbb{B}G \end{tikzcd}$$

giving rise to the projective system of de Rham complexes Rham

$$\big\{\Omega^*(M_G(n+1)) \xrightarrow{\mu_n^*} \Omega^*(M_G(n))\big\}_{n\in\mathbb{N}}$$

and the projective system of the Rham cohomology

$$\big\{H^d(M_G(n+1)) \xrightarrow{H^d(\mu_n^*)} H^d(M_G(n))\big\}_{n\in\mathbb{N}}$$

for each $d \in \mathbb{N}$, which has the remarkable property that, for a given $d$, the system is stationary, i.e. $H^d(\mu_n^*)$ is bijective for sufficiently large $n$.

The same remarks hold for the compact support case since the maps $\mu_n$ are proper. One then has the projective system of de Rham complexes

$$\big\{\Omega_c^*(M_G(n+1)) \xrightarrow{\mu_n^*} \Omega_c^*(M_G(n))\big\}_{n\in\mathbb{N}}$$

and; for each $d \in \mathbb{N}$, the stationary projective systems of the Rham cohomology

$$\big\{H_c^d(M_G(n+1)) \xrightarrow{H^d(\mu_n^*)} H_c^d(M_G(n))\big\}_{n\in\mathbb{N}}.$$

**3.2.5. Remark.** One can show that in both cases $H^d(\mu_n^*)$ is bijective for all $n > d+1$. The projective limit of $\{H^d(M_G(n))\}_{n\in\mathbb{N}}$ identifies then canonically with the $d$-th **singular** cohomology $H^d(M_G;\mathbb{R})$, and the projective limit of $\{H_c^d(M_G(n))\}_{n\in\mathbb{N}}$ with the $d$-th **singular** cohomology of **vertical compact support** $H_{cv}^d(M_G;\mathbb{R})$ ([12]). Using Cartan's results in [$\mathrm{Ca}_1$, $\mathrm{Ca}_2$] one obtains canonical isomorphisms

$$H_G(M) \simeq H(M_G;\mathbb{R}) \quad \text{and} \quad H_{G,c}(M) \simeq H_{\text{c.v.}}(M_G;\mathbb{R}).$$

**3.2.6. Serre Spectral Sequences.** The fibrations $\pi_n$ in 3.2.4 are Serre fibrations and as such, give rise to a projective system of spectral sequences

$$\begin{cases} \mathbb{E}_2^{p,q}(M_G(n)) := H^p(\mathbb{B}G(n)) \otimes H^q(M) \Rightarrow H^{p+q}(M_G(n)) \\ \mathbb{E}_{c,2}^{p,q}(M_G(n)) := H^p(\mathbb{B}G(n)) \otimes H_c^q(M) \Rightarrow H_c^{p+q}(M_G(n)) \end{cases}$$

whose limits are the *(Serre) spectral sequence associated with* $\pi : M_G \to \mathbb{B}G$.

$$\begin{cases} \mathbb{E}_2^{p,q}(M_G) := H^p(\mathbb{B}G) \otimes H^q(M) \\ \mathbb{E}_{c,2}^{p,q}(M_G) := H^p(\mathbb{B}G) \otimes H_c^q(M) \end{cases} \tag{$\mathbb{E}(M_G)$}$$

[12] The notation $H_{cv}(\_)$, borrowed from [BT] p. 61, means *compact vertical cohomology.*

**3.2.7. Proposition.** *The Serre spectral sequences $(\mathbb{E}(M_G))$ associated with the fibration $\pi : M_G \to \mathbb{B}G$ canonically identifies with the spectral sequences already met in 3.1.4-(d).*

*Proof.* Implicit in [$\mathrm{Ca}_1$, $\mathrm{Ca}_2$]. □

**3.2.8. Exercise.** Let $f : M \to N$ be a $G$-equivariant map between oriented $G$-manifolds. For each $n \in \mathbb{N}$, as in 3.2.4, denote by $f_G(n) : M_G(n) \to N_G(n)$ the corresponding induced map over $\mathbb{B}G(n)$.

a) Show that the following diagrams are cartesian with $\mu(n)$ and $\nu(n)$ proper.

$$\begin{array}{ccc} M_G(n) & \xrightarrow{\mu(n)} & M_G(n+1) \\ {\scriptstyle f_G(n)}\downarrow & & \downarrow{\scriptstyle f_G(n+1)} \\ N_G(n) & \xrightarrow{\nu(n)} & N_G(n+1) \end{array}$$

b) Prove the following equalities

$$\begin{cases} \nu(n)^* \circ f_G(n+1)_* = f_G(n)_* \circ \mu(n)^* \\ f_G(n+1)^* \circ \nu(n)_! = \mu(n)_! \circ f_G(n)^* \end{cases}$$

c) When $f : M \to N$ is moreover a closed embedding, one defines the *equivariant cohomology with support in $M$* by

$$H_{G,M}(N) := H_{M_G}(N_G).$$

Show that there exists a convergent spectral sequence $(\mathbb{E}_{M\subseteq N,r}, d_r)$

$$\mathbb{E}^{p,q}_{M\subseteq N,2} := H^p(\mathbb{B}G) \otimes H^q_M(N) \Rightarrow H^{p+q}_{G,M}(N).$$

# 4. Equivariant Poincaré Duality

## 4.1. Differential Graded Modules over a Graded Algebra

**4.1.1. Graded Algebras.** A *graded algebra* is a graded vector space $A \in \mathrm{GV}(\mathbb{R})$ with a family of bilinear maps $\cdot : A^a \times A^b \to A^{a+b}$ such that the triple $(A,0,+,\cdot)$ is an $\mathbb{R}$-algebra.

**4.1.2. Examples**

a) For a graded vector space $N \in \mathrm{GV}(\mathbb{R})$, the space of graded endomorphisms $(\mathit{End}^{\bullet}_{\mathbb{R}}(N),0,+,\mathrm{id},\circ)$ (1.1.3) is a noncommutative graded algebra.

b) $\Omega_G := S(\mathfrak{g})^{\mathfrak{g}}$ is a positively and evenly graded commutative algebra.

c) $\Omega(M)$ and $\Omega_G(M)$ are positively graded anticommutative algebras.

d) $\Omega_{\mathrm{c}}(M)$ and $\Omega_{G,\mathrm{c}}(M)$ are positively graded anticommutative algebras, with no unit element whenever $M$ is not compact.

**4.1.3. Graded Modules.** An $\Omega_G$*-graded module*, $\Omega_G$*-gm* in short, is a graded space $V \in \mathrm{GV}(\mathbb{R})$ together with a homomorphism $\Omega_G \to \mathit{Endgr}^0_{\mathbb{R}}(V)$ of graded algebras of degree 0. Given two $\Omega_G$-gm's $V$ and $W$, a *graded homomorphism of* $\Omega_G$*-gm's of degree* $d$ *from* $V$ *to* $W$ is a graded homomorphism of graded spaces $\boldsymbol{\alpha} : V \to W$ of degree $d$ (1.1.3), which is compatible with the action of $\Omega_G$, i.e. $\boldsymbol{\alpha}(P\cdot v) = P\cdot\boldsymbol{\alpha}(v)$ for all $P \in \Omega_G$ and $v \in V$. We denote by $\mathit{Homgr}^d_{\Omega_G}(V,W)$ the space of such homomorphisms and by

$$\mathit{Hom}^{\bullet}_{\Omega_G}(V,W) = \left\{\mathit{Homgr}^d_{\Omega_G}(V,W)\right\}_{d\in\mathbb{Z}}$$

the graded space of *graded homomorphisms of* $\Omega_G$*-gm's*.

When $d = 0$, we may write $\mathit{Homgr}_{\Omega_G}(V,W)$ instead of $\mathit{Homgr}^0_{\Omega_G}(V,W)$.

**4.1.4. Example.** Examples 4.1.2-(c,d) are examples of $\Omega_G$-graded modules.

**4.1.5** The *category* $\mathrm{GM}(\Omega_G)$ *of* $\Omega_G$*-graded modules* is the category whose objects are the $\Omega_G$-gm and whose *morphisms* are the graded homomorphisms of degree 0. We will equivalently write $\mathit{Mor}_{\mathrm{GM}(\Omega_G)}(V,W)$ and $\mathit{Homgr}_{\Omega_G}(V,W)$ the set of morphisms from $V$ to $W$.

**4.1.6** A direct sum $\bigoplus\limits_{\mathfrak{i}\in\mathfrak{I}} \Omega_G[m_{\mathfrak{i}}]$, with $m_{\mathfrak{i}} \in \mathbb{Z}$, is called *a free* $\Omega_G$*-graded module*.

**4.1.7. Proposition**

a) *An object* $V \in \mathrm{GM}(\Omega_G)$ *is* projective (resp. injective) *if and only if the functor* $\mathit{Hom}^{\bullet}(V,\_) : \mathrm{GM}(\Omega_G) \rightsquigarrow \mathrm{GM}(\Omega_G)$ (resp. $\mathit{Hom}^{\bullet}(\_,V)$) *is exact.*

b) *The category* $\mathrm{GM}(\Omega_G)$ *is an abelian category with enough injective and projective objects. The cohomological dimension of* $\mathrm{GM}(\Omega_G)$ *is finite and equals the rank of* $G$.

*Proof.* (a) is an immediate consequence of the direct decomposition of functors

$$\mathit{Hom}^{\bullet}_{\Omega_G}(\_,\_) = \bigoplus\nolimits_{m\in\mathbb{Z}} \mathit{Homgr}_{\Omega_G}(\_,\_[m]) = \bigoplus\nolimits_{m\in\mathbb{Z}} \mathit{Homgr}_{\Omega_G}(\_[-m],\_).$$

(b) *Projectivity.* Let $\{v_\mathfrak{i}\}_{\mathfrak{i}\in\mathfrak{I}}$ be a family of *homogeneous* generators for $V \in \mathrm{GM}(\Omega_G)$ and consider, for each $\mathfrak{i} \in \mathfrak{I}$, the map $\gamma_\mathfrak{i} : \Omega_G[-d_\mathfrak{i}] \to V$, $x \mapsto xv_\mathfrak{i}$ which is clearly a morphism in $\mathrm{GM}(\Omega_G)$. The sum

$$\sum\nolimits_{\mathfrak{i}\in\mathfrak{I}} \gamma_\mathfrak{i} : \bigoplus\nolimits_{\mathfrak{i}\in\mathfrak{I}} \Omega_G[-d_\mathfrak{i}] \twoheadrightarrow V \tag{$\diamond$}$$

represents $V$ as the quotient in $\mathrm{GM}(\Omega_G)$ of a free, and thus projective, $\Omega_G$-gm.

*Injectivity.* We reproduce the proof of theorem 1.2.2 in [Go] §1.4 in the context of graded rings.

The correspondence $\quad V \rightsquigarrow \widehat{V} := \mathit{Hom}^\bullet_\mathbb{Z}(V,(\mathbb{Q}/\mathbb{Z})[0])$ $\hfill (\diamond\diamond)$

is an additive contravariant functor from the category of *left* (resp. *right*) $\Omega_G$-gm to the category of *right* (resp. *left*) $\Omega_G$-gm ([13]), and is exact, by (a), since

$$\mathit{Homgr}_\mathbb{Z}(\_,(\mathbb{Q}/\mathbb{Z})[0]) = \mathrm{Hom}_\mathbb{Z}((\_)^0,\mathbb{Q}/\mathbb{Z})$$

and since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

*Lemma 1. The map $\nu(V) : V \to \widehat{\widehat{V}}$, $v \mapsto (\gamma \mapsto \gamma(v))$ is an* injective *morphism.*

*Proof of lemma 1.* Because $\nu(V)$ is clearly a morphism of graded modules, it is injective if and only if it doesn't kill any homogeneous nonzero element. If $0 \neq v \in V^d$, the subgroup $\mathbb{Z}{\cdot}v \subseteq V^d$ is isomorphic to some $\mathbb{Z}/n\mathbb{Z}$ for $n \neq \pm 1$, and there exists a nonzero homomorphism $\gamma'' : \mathbb{Z}{\cdot}v \to \mathbb{Q}/\mathbb{Z}$ (exercise), restriction of some $\gamma' : V^d \to \mathbb{Q}/\mathbb{Z}$ (thanks to the injectivity of $\mathbb{Q}/\mathbb{Z}$). Extend this $\gamma'$ to the whole of $V$, assigning zero on the homogeneous factors $V^e$ when $e \neq d$. This last extension, denoted by $\gamma : V \to \mathbb{Q}/\mathbb{Z}$, is a graded morphism of degree $-d$ and verifies $\nu(V)(v)(\gamma) = \gamma(v) \neq 0$ by construction, so that $\nu(V)(v) \neq 0$, which completes the proof of lemma 1. ⊟

*Lemma 2. For any free* right *$\Omega_G$-gm $F$, the* left *$\Omega_G$-gm $\widehat{F}$ is injective.*

*Proof of lemma 2.* We recall (*cf.* [Bk] Chap. II, §4, Prop. 1) that for any left $\Omega_G$-dgm $N$, the maps

$$\begin{array}{rcl} \mathit{Hom}^\bullet_{\Omega_G}(N,\mathit{Hom}^\bullet_\mathbb{Z}(\Omega_G,(\mathbb{Q}/\mathbb{Z})[0])) & \rightleftarrows & \mathit{Hom}^\bullet_\mathbb{Z}(N,(\mathbb{Q}/\mathbb{Z})[0]) \\ \gamma & \longmapsto & \big(v \mapsto \gamma(v)(1)\big) \\ \big(v \mapsto (x \mapsto \xi(xv))\big) & \longleftarrow\!\shortmid & \xi \end{array}$$

are isomorphisms of graded vector spaces each inverse to the other. It follows that $\mathit{Hom}^\bullet_\mathbb{Z}(\Omega_G,(\mathbb{Q}/\mathbb{Z})[0]))$ is an injective left $\Omega_G$-gm if and only if the functor $\mathit{Hom}^\bullet_\mathbb{Z}(\_,(\mathbb{Q}/\mathbb{Z})[0])$ is exact, but this is equivalent, by (a), to the exactness of the functor $\mathit{Homgr}_\mathbb{Z}(\_,(\mathbb{Q}/\mathbb{Z})[0]) = \mathrm{Hom}_\mathbb{Z}((\_)^0,\mathbb{Q}/\mathbb{Z})$, which is clear since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. ⊟

Now, if $V$ is a left $\Omega_G$-gm, fix some epimorphism of right $\Omega_G$-gm $\pi : F \twoheadrightarrow \widehat{V}$ where $F$ is free as in ($\diamond$). The morphism $\widehat{\pi} : \widehat{\widehat{V}} \to \widehat{F}$ is injective and composed with $\nu(V) : V \to \widehat{\widehat{V}}$, injective by lemma 1, we get an injective morphism $V \hookrightarrow \widehat{F}$

[13] If $N$ is a *right* $\Omega_G$-gm, the structure of *left* $\Omega_G$-module of $\mathit{Hom}^\bullet_\mathbb{Z}(N,(\mathbb{Q}/\mathbb{Z})[0])$ is given by $(x{\cdot}\gamma)(y) := \gamma(yx)$ for all $x \in \Omega_G$ and $y \in N$. If $N$ is a *left* $\Omega_G$-gm, the structure of *right* $\Omega_G$-module of $\mathit{Hom}^\bullet_\mathbb{Z}(N,(\mathbb{Q}/\mathbb{Z})[0])$ is given by $(\gamma{\cdot}x)(y) := \gamma(xy)$ for all $x \in \Omega_G$ and $y \in N$.

of left $\Omega_G$-gm, where $\widehat{F}$ is an injective left $\Omega_G$-gm by lemma 2. This completes the proof of the existence of enough injective objects in $\mathrm{GM}(\Omega_G)$.

The statement about $\dim_{\mathrm{ch}}(\mathrm{GM}(H_T))$ results from the fact (Chevalley's theorem) that $\Omega_G$ is a polynomial algebra in $\mathrm{rk}(G)$ variables. One may then refer to Hilbert's Syzygy Theorem (*cf.* [J] p. 385, and Ex. 2, p. 387). $\square$

**4.1.8. Exercise.** Let $A$ be a graded $\mathbb{R}$-algebra which is an integral domain.

a) Show that $S^{-1}A$, where $S$ denotes the multiplicative system of homogeneous nonzero elements of $A$, is an injective object of $\mathrm{GM}(A)$. Also, prove that the canonical inclusion $A \hookrightarrow S^{-1}A$ is an injective envelope for $A$.

b) Show that when $\mathrm{rk}(G) > 0$, the degrees of a non trivial injective object of $\mathrm{GM}(\Omega_G)$ cannot be bounded below ([14]).

c) Show that if $V \in \mathrm{GM}(\Omega_G)$ is positively graded, it admits projective resolutions in $\mathrm{GM}(\Omega_G)$ all of whose terms are positively graded.

The next two sections are straightforward generalizations of sections 1.1.5 and 1.1.8 from graded vector spaces to $\Omega_G$-graded modules.

**4.1.9. Differential Graded Modules.** An $\Omega_G$*-differential graded module*, $\Omega_G$*-dgm* in short, is a pair $(V,d)$ with $V \in \mathrm{GM}(\Omega_G)$ and $d \in \mathit{Endgr}^1_{\Omega_G}(V)$, called *differential*, is such that $(V,d)$ is a complex, i.e. $d^2 = 0$. A *morphism of* $\Omega_G$*-dgm* $\boldsymbol{\alpha} : (V,d) \to (V',d')$ is a morphism of $\Omega_G$-gm's which is also a morphism of complexes, i.e. $d' \circ \boldsymbol{\alpha} = \boldsymbol{\alpha} \circ d$. The $\Omega_G$-dgm's and their morphisms constitute the *category* $\mathrm{DGM}(\Omega_G)$ *of* $\Omega_G$*-differential graded modules*. The category $\mathrm{DGM}(\Omega_G)$ is an abelian category.

**4.1.10. The $\boldsymbol{Hom}^\bullet(\_,\_)$ and $(\_\otimes\_)^\bullet$ Bi-functors.** Given two $\Omega_G$-dgm's $(V,d)$ and $(V',d')$, we recall the definition of the $\Omega_G$-dgm's

$$\big(\mathit{Hom}^\bullet_{\Omega_G}(V,V'),D_\bullet\big) \quad \text{and} \quad \big((V \otimes_{\Omega_G} V')^\bullet,\boldsymbol{\Delta}_\bullet\big).$$

As $\Omega_G$-graded modules they are defined by

$$m \mapsto \begin{cases} \mathit{Hom}^m_{\Omega_G}(V,V') := \mathit{Homgr}^m_{\Omega_G}(V,V') \\ (V \otimes_{\Omega_G} V')^m := \pi(V \otimes_{\mathbb{R}} V')^m \end{cases}$$

where $\pi : V \otimes_{\mathbb{R}} V' \twoheadrightarrow V \otimes_{\Omega_G} V'$, $v \otimes v' \mapsto [v \otimes v']$, is the canonical (graded) surjection (see remark 4.1.12). The differentials $D_\bullet$ and $\boldsymbol{\Delta}_\bullet$ are:

$$\begin{cases} D_m(f) = d' \circ f - (-1)^m f \circ d \\ \boldsymbol{\Delta}_m([v \otimes v']) = [d(v) \otimes v'] + (-1)^{|v|}[v \otimes d'(v')] \end{cases}$$

where $v \otimes v' \in V^{|v|} \otimes V'^{|v'|}$ and $|v| + |v'| = m$.

The fact that $D$ and $\boldsymbol{\Delta}$ are $\Omega_G$-linear results from the fact that $\Omega_G$ is graded only by **even** degrees (!).

[14] A graded space $V$ is said to be *bounded below (resp. above)*, if there exists $N \in \mathbb{Z}$ such that $V^i = 0$ for all $i < N$ (resp. $i > N$). The graded algebra $\Omega_G$ is bounded below by 0.

**4.1.11** These constructions are natural w.r.t. each side entry, which means that they define two bi-functors

$$\begin{aligned} Hom^{\bullet}_{\Omega_G}((\_),(\_)) &: \mathrm{DGM}(\Omega_G) \times \mathrm{DGM}(\Omega_G) \rightsquigarrow \mathrm{DGM}(\Omega_G) \\ ((\_) \otimes_{\Omega_G} (\_))^{\bullet} &: \mathrm{DGM}(\Omega_G) \times \mathrm{DGM}(\Omega_G) \rightsquigarrow \mathrm{DGM}(\Omega_G) \end{aligned}$$

which are bi-additive and have the usual variances and exactnesses. For example, the first is contravariant and left exact on the left entry, and covariant and left exact on the right entry, while the second is bi-covariant and right exact.

**4.1.12. Remark.** Some care must be taken with the tensor product since it hides some deep subtleties. A good way to understand it is to note that $V \otimes_{\Omega_G} V'$ is the quotient of the graded space $V \otimes_{\mathbb{R}} V'$ by the subspace $W$ spanned by the tensors $Pv \otimes v' - v \otimes Pv'$ with $P \in \Omega_G$ and $(v,v') \in V \times V'$ both *homogeneous*. One then shows that $W$ is a graded subcomplex of $(V \otimes_{\mathbb{R}} V', \mathbf{\Delta})$, so that the canonical surjection $\pi : (V \otimes_{\mathbb{R}} V', \mathbf{\Delta}) \twoheadrightarrow (V \otimes_{\mathbb{R}} V', \mathbf{\Delta})/W$ is an epimorphism of graded complexes, therefore inducing over $V \otimes_{\Omega_G} V'$ a structure of $\Omega_G$-dgm. Again, a key point is that $\Omega_G$ is graded only by **even** degrees.

**4.1.13. The $\boldsymbol{\Omega_G}$-Dual of a Complex.** In section 1.1.11, we introduced the basic duality functor $Hom^{\bullet}_{\Bbbk}(\_,\Bbbk) : \mathrm{DGM}(\Bbbk) \rightsquigarrow \mathrm{DGM}(\Bbbk)$ and noted that it was an exact functor (1.1.13). In the framework of $\Omega_G$-dgm's, the corresponding functor is the *basic $\Omega_G$-duality functor*

$$Hom^{\bullet}_{\Omega_G}(\_,\Omega_G) : \mathrm{DGM}(\Omega_G) \rightsquigarrow \mathrm{DGM}(\Omega_G)$$

which is generally **not** exact, **nor does** it respect quasi-isomorphisms.

**4.1.14. The Forgetful Functor.** If we disregard differentials, $\Omega_G$-dgm's simply appear as $\Omega_G$-gm's, and likewise for morphisms. Forgetting the complex structure gives the *forgetful functor* $o : \mathrm{DGM}(\Omega_G) \rightsquigarrow \mathrm{GM}(\Omega_G)$ which is exact and will often be implicit in some of our considerations.

## 4.2. Deriving Functors

**4.2.1. Deriving Functors Defined on the Category GM($\boldsymbol{\Omega_G}$).** We have already shown (4.1.7) that the abelian category $\mathrm{GM}(\Omega_G)$ has enough projective and injective objects ([15]). We will now recall the definition of the *left and right derived functors* associated with an additive functor $F : \mathrm{Ab}' \to \mathrm{Ab}$ between abelian categories where $\mathrm{Ab}'$ has enough projective and injective objects.

The *left and right derived functors*, respectively $\mathbb{L}_\star F : \mathrm{Ab}' \rightsquigarrow \mathcal{K}_\star(\mathrm{Ab})$ and $\mathbb{R}^\star F : \mathrm{Ab}' \rightsquigarrow \mathcal{K}^\star(\mathrm{Ab})$ ([16]) applied to an object $V \in \mathrm{Ab}'$ are defined by the following steps. First, choose an injective and a projective resolution of $V$,

$$\begin{aligned} &\mathbf{0} \longrightarrow V \xrightarrow{\ \epsilon\ } \mathcal{I}^0 \xrightarrow{d_0} \mathcal{I}^1 \xrightarrow{d_1} \mathcal{I}^2 \xrightarrow{d_2} \cdots \\ &\cdots \xrightarrow{d_{-2}} \mathcal{P}^{-2} \xrightarrow{d_{-1}} \mathcal{P}^{-1} \xrightarrow{d_0} \mathcal{P}^0 \xrightarrow{\ \epsilon\ } V \longrightarrow \mathbf{0}. \end{aligned}$$

[15] See also Grothendieck [Gr], chapter **I**, Thm. 1.10, p. 135.

[16] $\mathcal{K}^\star(\mathrm{Ab})$ (resp. $\mathcal{K}_\star(\mathrm{Ab})$) is the category of cochain (resp. chain) complexes of Ab whose morphisms are the morphisms of complexes modulo homotopy.

Next, let $\mathcal{I}^\star V$ stand for the truncated complex $\big(0 \to \mathcal{I}^0 \xrightarrow{d_0} \mathcal{I}^1 \xrightarrow{d_1} \cdots\big)$, and $\mathcal{P}^\star V$ for $\big(\cdots \xrightarrow{d_{-1}} \mathcal{P}^{-1} \xrightarrow{d_0} \mathcal{P}^0 \to \mathbf{0}\big)$, and set

$$\begin{cases} \mathbb{L}^\star F(V) := F(\mathcal{P}^\star V) \\ \mathbb{R}^\star F(V) := F(\mathcal{I}^\star V) \end{cases} \tag{$*$}$$

One proves that the complexes $(*)$ are homotopically independent of the chosen resolutions, so that are well defined objects of $\mathcal{K}^*(\mathrm{Ab})$.

As the targets of the derived functors $\mathbb{R}^\star F$ and $\mathbb{L}^\star F$ are complexes, one is interested in their cohomologies. Their classical notations are

$$\begin{cases} (\mathbb{R}^i F)(V) := H^i(\mathbb{R}^\star(\_)) \\ (\mathbb{L}^i F)(\_) := H^i(\mathbb{L}^\star F(\_)). \end{cases}$$

It is easily seen from the above definitions that the *augmentation morphisms* of complexes $\epsilon : V[0] \to \mathcal{I}^\star$ and $\epsilon : \mathcal{P}_\star \to V[0]$, give rise to natural morphisms of complexes $F(V[0]) \to (\mathbb{R}^\star F)(V)$ and $(\mathbb{L}^\star F)(V) \to F(V[0])$, inducing canonical morphisms

$$F(V) \to (\mathbb{R}^0 F)(V) \quad \text{and} \quad (\mathbb{L}^0 F)(V) \to V.$$

These are isomorphisms whenever $F$ is respectively left and right exact.

**4.2.2. Simple Complex Associated with a Double Complex.** The category $\mathcal{C}^\natural(\mathrm{Ab})$ of (cochain) complexes of an abelian category Ab is again an abelian category so that we can look at the category $\mathcal{C}^{\star,\natural}(\mathrm{Ab}) := \mathcal{C}^\star(\mathcal{C}^\natural(\mathrm{Ab}))$ of (cochain) complexes of $\mathcal{C}^\natural(\mathrm{Ab})$ also called *double (cochain) complexes of* Ab. A double complex $N^{\star,\natural} := (N^{\star,\natural}, \delta_{\star,\natural}, d_{\star,\natural}) \in \mathcal{C}^{\star,\natural}(\mathrm{Ab})$ is generally represented as a two dimensional ladder all of whose subdiagrams are commutative.

$$\begin{array}{ccccccc}
 & \uparrow{\scriptstyle d_{i-1,j+1}} & & \uparrow{\scriptstyle d_{i,j+1}} & & \uparrow{\scriptstyle d_{i+1,j+1}} & \\
\xrightarrow{\delta_{i-2,j+1}} & N^{i-1,j+1} & \xrightarrow{\delta_{i-1,j+1}} & N^{i,j+1} & \xrightarrow{\delta_{i,j+1}} & N^{i+1,j+1} & \xrightarrow{\delta_{i+1,j+1}} \\
 & \uparrow{\scriptstyle d_{i-1,j}} & & \uparrow{\scriptstyle d_{i,j}} & & \uparrow{\scriptstyle d_{i+1,j}} & \\
\xrightarrow{\delta_{i-2,j}} & N^{i-1,j} & \xrightarrow{\delta_{i-1,j}} & N^{i,j} & \xrightarrow{\delta_{i,j}} & N^{i+1,j} & \xrightarrow{\delta_{i+1,j}} \\
 & \uparrow{\scriptstyle d_{i-1,j-1}} & & \uparrow{\scriptstyle d_{i,j-1}} & & \uparrow{\scriptstyle d_{i+1,j-1}} & \\
\xrightarrow{\delta_{i-2,j-1}} & N^{i-1,j-1} & \xrightarrow{\delta_{i-1,j-1}} & N^{i,j-1} & \xrightarrow{\delta_{i,j-1}} & N^{i+1,j-1} & \xrightarrow{\delta_{i+1,j-1}} \\
 & \uparrow{\scriptstyle d_{j-2}} & & \uparrow{\scriptstyle d_{j-2}} & & \uparrow{\scriptstyle d_{j-2}} &
\end{array}$$

The *simple (or total) complex* associated with$N^{\star,\natural}$ is the complex $(Tot^\circ(N^{\star,\natural}), D_\circ)$, where, for all $m \in \mathbb{Z}$,

$$\begin{cases} Tot^m(N^{\star,\natural}) := \bigoplus_{m=a+b} N^{i,j} \\ D_m(n_{i,j}) := d_{i,j}(n_{i,j}) + (-1)^j \delta_{i,j}(n_{i,j}) \end{cases} \qquad \begin{array}{ccc} N^{i,j+1} & & \\ {\scriptstyle d_{i,j}}\uparrow & & \\ N^{i,j} & \xrightarrow{(-1)^j \delta_{i,j}} & N^{i+1,j} \end{array}$$

In this way, one obtains an additive exact functor

$$Tot^\circ := \mathcal{C}^{\star,\natural}(\mathrm{Ab}) \rightsquigarrow \mathcal{C}^\circ(\mathrm{Ab}).$$

**4.2.3. Spectral Sequences Associated with Double Complexes.** The double complex $N^{\star,\natural}$ is said to be *of the first quadrant* if $\{(i,j) \mid N^{i,j} \neq \mathbf{0}\} \subseteq \mathbb{N} \times \mathbb{N}$. As explained in [Go] (§4.8, p. 86), one assigns to this kind of double complex, two *regular* decreasing filtrations of $(Tot^\circ(N^{\star,\natural}), D_\circ)$. The first is relative to the *line* $\natural$*-filtration* $Tot^\circ(N^{\star,\natural})_\ell := Tot^\circ(N^{\star,\natural\geq\ell})$, and the second to the *column* $\star$*-filtration* $Tot^\circ(N^{\star,\natural})_c := Tot^\circ(N^{\star\geq c,\natural})$. Each filtration gives rise to a spectral sequence converging to the cohomology of $(Tot^\circ N^{\star,\natural}, D_\circ)$, respectively

$$\begin{cases} {}'\mathbb{E}_2^{p,q} := H^p_\natural H^q_\star(N^{\star,\natural}) \Rightarrow H^{p+q}_\circ(Tot^\circ N^{\star,\natural}, D_\circ), \\ {}''\mathbb{E}_2^{p,q} := H^p_\star H^q_\natural(N^{\star,\natural}) \Rightarrow H^{p+q}_\circ(Tot^\circ N^{\star,\natural}, D_\circ), \end{cases}$$

where $H_\star$ (resp. $H_\natural$) is the cohomology w.r.t. $\delta_\star$ (resp. $d_\natural$).

**4.2.4. The $\mathbb{R}^\star\,Hom^\bullet_{\Omega_G}(\_,\_)$ and $(\_)\otimes^{\mathbb{L}^\star}_{\Omega_G}(\_)$ Bi-functors.** Given two $\Omega_G$-graded modules $V$, $W \in \mathrm{GM}(\Omega_G)$, we may consider the four functors

$$Hom^\bullet_{\Omega_G}(V,\_), \quad Hom^\bullet_{\Omega_G}(\_,W), \quad V \otimes_{\Omega_G} (\_), \quad (\_) \otimes_{\Omega_G} W,$$

where the first two are left exact and the other two are right exact.
In order to simplify notations we shall often write '$Hom^\bullet$' for '$Hom^\bullet_{\Omega_G}$'.

Now, given projective resolutions $\mathcal{P}^\natural(V) \to V$, $\mathcal{P}^\natural(W) \to W$ and an injective resolution $W \to \mathcal{I}^\star(W)$ in $\mathrm{GM}(\Omega_G)$, we have natural morphisms of double complexes ([17])

$$\begin{matrix} Hom^\bullet(\mathcal{P}^\natural(V), W[0]^\star) & \longrightarrow & Hom^\bullet(\mathcal{P}^\natural(V), \mathcal{I}^\star W) & \longleftarrow & Hom^\bullet(V[0]^\natural, \mathcal{I}^\star W) \\ \mathcal{P}^\natural(V) \otimes W[0]^\star & \longrightarrow & \mathcal{P}^\natural(V) \otimes \mathcal{P}^\star(W) & \longleftarrow & V[0]^\natural \otimes \mathcal{P}^\star(W) \end{matrix}$$

giving rise to canonical morphisms of complexes on $\Omega_G$-gm

$$\begin{matrix} Hom^\bullet(\mathcal{P}^\natural(V), W) & \longrightarrow & Tot^\circ\, Hom^\bullet(\mathcal{P}^\natural(V), \mathcal{I}^\star W) & \longleftarrow & Hom^\bullet(V, \mathcal{I}^\star W) \\ \mathcal{P}^\natural(V) \otimes W & \longrightarrow & Tot^\circ\big(\mathcal{P}^\natural(V) \otimes \mathcal{P}^\star(W)\big) & \longleftarrow & V \otimes \mathcal{P}^\star(W) \end{matrix} \tag{$\ddagger$}$$

The following proposition is classical (*loc. cit.*).

**4.2.5. Proposition.** *The morphisms* ($\ddagger$) *are quasi-isomorphisms.* ([18])

*Sketch of the proof.* For the first line of ($\ddagger$), one notes that the morphisms of complexes are compatible with line and column filtrations of double complexes of the first quadrant. In the case of

$$Hom^\bullet(\mathcal{P}^\natural(V), W) \to Tot^\circ\, Hom^\bullet(\mathcal{P}^\natural(V), \mathcal{I}^\star W),$$

since for each $i \in \mathbb{Z}$ the map $Hom^\bullet(\mathcal{P}^i(V), W) \to Tot^\circ\, Hom^\bullet(\mathcal{P}^i(V), \mathcal{I}^\star W)$ is a quasi-isomorphism, the induced map on the $''\mathbb{E}$ terms of the associated spectral sequences (4.2.3) is an isomorphism and we conclude. The case of

$$Tot^\circ\, Hom^\bullet(\mathcal{P}^\natural(V), \mathcal{I}^\star W) \longleftarrow Hom^\bullet(V, \mathcal{I}^\star W)$$

[17] By $W[0]^\bullet$ we denote the complex satisfying $W[0]^0 = W$ and $W[0]^i = \mathbf{0}$ for $i \neq 0$.

[18] In fact they are homotopic equivalences, but we won't need to be so precise.

is almost the same except that now we must consider the line filtration and use the $'\mathbb{E}$ spectral sequence.

The second line in (‡) is dealt with in the same way after observing that 4.2.3 also applies (dually) to double complexes of the *third* quadrant. ⊡

As a consequence of 4.2.5, in each line of (‡) the complexes represent the *same objet* in the derived category $\mathcal{D}^\star(\mathrm{GM}(\Omega_G))$. They are classically denoted by $\mathbb{R}^\star\, Hom^\bullet_{\Omega_G}(V,W)$ and $V \otimes^{\mathbb{L}^\star}_{\Omega_G} W$. The constructions are natural w.r.t. each entry so that we get two bi-functors

$$\begin{aligned} \mathbb{R}^\star\, Hom^\bullet_{\Omega_G}((\_),(\_)) &: \mathrm{GM}(\Omega_G)\times\mathrm{GM}(\Omega_G) \rightsquigarrow \mathcal{D}^\star(\mathrm{GM}(\Omega_G)) \\ ((\_)\otimes^{\mathbb{L}^\star}_{\Omega_G}(\_))^\bullet &: \mathrm{GM}(\Omega_G)\times\mathrm{GM}(\Omega_G) \rightsquigarrow \mathcal{D}^\star(\mathrm{GM}(\Omega_G)) \end{aligned} \qquad (\diamond)$$

which are bi-additive and have the usual variances and exactnesses. They clearly extend the bi-functors in 4.1.11 from $\mathrm{GM}(\Omega_G)$ to $\mathcal{D}^\star(\mathrm{GM}(\Omega_G))$.

**4.2.6. The *Ext*$^\bullet$ and *Tor*$^\bullet$ Bi-functors.** Given $V,W\in\mathrm{GM}(\Omega_G)$, one defines for $i\in\mathbb{Z}$

$$\begin{cases} Ext^{i,\bullet}_{\Omega_G}(V,W) := H^i_\star\big(\mathbb{R}^\star\, Hom^\bullet_{\Omega_G}(V,W)\big) \\ Tor^\bullet_{\Omega_G,i}(V,W) := H^i_\star\big(V\otimes^{\mathbb{L}^\star}_{\Omega_G} W\big) \end{cases}$$

Where $H^i_\star$ is the $i$'th cohomology functor on $\mathcal{D}^\star(\mathrm{GL}(\Omega_G))$.

**4.2.7. Defining $\mathbb{R}^\star\, Hom^\bullet_{\Omega_G}(\_,\Omega_G)$ on $\mathrm{DGM}(\Omega_G)$.** We proceed as in 4.2.4 except that we will consider only injective resolutions of $\Omega_G$ in $\mathrm{GM}(\Omega_G)$. ([19])

Let $V := (V,d)\in\mathrm{DGM}(\Omega_G)$. For any $N\in\mathrm{GM}(\Omega_G)$, we already endowed the $\Omega^\bullet_G$-graded module $Hom^\bullet_{\Omega_G}(V,N)$ with a canonical structure of $\Omega^\bullet_G$-differential graded module $(Hom^\bullet_{\Omega_G}(V,N),D_\bullet)$ (*cf.* 4.1.11) in such a way that we obtain a left exact functor

$$Hom^\bullet_{\Omega_G}(V,\_) : \mathrm{GM}(\Omega_G)\rightsquigarrow \mathrm{DGM}(\Omega_G).$$

It follows that the target of the functor $\mathbb{R}^\star\, Hom^\bullet_{\Omega_G}(\_,\Omega_G) := Hom^\bullet_{\Omega_G}(\_,\mathcal{I}^\star N)$ is the category $\mathcal{C}^\star(\mathrm{DGM}(\Omega_G))$. The functor transforms homotopies to identities, and respects quasi-isomorphisms, it therefore induces a contravariant functor

$$\mathbb{R}^\star\, Hom^\bullet_{\Omega_G}(\_,\Omega_G) : \mathrm{DGM}(\Omega_G)\rightsquigarrow \mathcal{D}^\star(\mathrm{DGM}(\Omega_G)), \qquad (\diamond)$$

which we will call *the derived duality functor*. This functor, composed with the $i$'th cohomology functor $H^i_\star : \mathcal{D}^\star(\mathrm{DGM}(\Omega_G))\rightsquigarrow \mathrm{DGM}(\Omega_G)$ gives the *$i$'th extension functor*

$$Ext^{i,\bullet}_{\Omega_G}(\_,\Omega_G) := H^i_\star\big(\mathbb{R}^\star\, Hom^\bullet_{\Omega_G}(\_,\Omega_G)\big) : \mathrm{DGM}(\Omega_G)\rightsquigarrow \mathrm{DGM}(\Omega_G).$$

[19] The good notion of projective resolution for an $\Omega_G$-dgm $(V,d)$ is the one of *simultaneous* projective resolutions. These are resolutions $\cdots\to\mathcal{P}^2\to\mathcal{P}^1\to\mathcal{P}^0\to V\to 0$ $(*)$ in $\mathrm{DGM}(H_T)$ where $\mathcal{P}^i$ is a projective $\Omega_G$-gm's, and such that the *derived sequence* $\cdots\to h\mathcal{P}^2\to h\mathcal{P}^1\to h\mathcal{P}^0\to hV\to 0$ $(**)$ is a projective resolution in $\mathrm{GM}(\Omega_G)$. When the graded space $V$ is bounded below (*cf.* ([14])), the complex $(V,d)$ always admits *simultaneous projective resolutions* and the derived functor $\mathbb{R}^\star\, Hom^\bullet((V,d),\Omega_G)$ may as well be defined as $Hom^\bullet(\mathcal{P}^\natural,\Omega_G)$, as in the case of $\Omega_G$-gm's, but we won't use this point of view in these notes.

The family (indexed by $\star \in \mathbb{Z}$) of all these functors

$$Ext^{\star,\bullet}_{\Omega_G}(\_,\Omega_G) := H^\star\big(I\!\!R^\star\, Hom^\bullet_{\Omega_G}(\_,\Omega_G)\big) : \mathrm{DGM}(\Omega_G) \rightsquigarrow \mathrm{DGM}^{\star,\bullet}(\Omega^\bullet_G) \quad (\diamond)$$

where $\mathrm{DGM}^{\star,\bullet}(\Omega^\bullet_G)$ is the category of $\star$-graded $\Omega^\bullet_G$-dgm (the action of $\Omega^\bullet_G$ does not affect the $\star$-degree), constitutes a $\partial$-functor in $\mathcal{K}^\star(\mathrm{DGM}(\Omega_G))$.

**4.2.8. Spectral Sequences Associated with $I\!\!R^\star\, Hom^\bullet_{\Omega_G}(\_,\Omega_G)$.** In the last paragraph we defined the derived duality functor $(\diamond)$ for any $\Omega_G$-dgm $(V,d)$ as the complex of $\Omega_G$-dgm's

$$I\!\!R^\star\, Hom^\bullet_{\Omega_G}((V,d),\Omega_G) := Hom^\bullet_{\Omega_G}((V,d),(\mathcal{I}^\star\Omega_G,\delta_\star)), \quad (\dagger)$$

that will be represented as a double complex with lines indexed by '$\bullet$' and columns by ''$\star$'. The differentials $d$ and $\delta_\star$ commute, $d$ increases de $\bullet$-degree and leaves unchanged the $\star$-degree, while $\delta_\star$ does the opposite.

**4.2.9. Proposition** *Let $(V,d) \in \mathrm{DGM}(\Omega_G)$.*

a) *There exist convergent spectral sequences*

$$\begin{cases} {}'\!E^{p,q} := H^p_\bullet(Ext^{q,\bullet}_{\Omega_G}(V,\Omega_G)) \Rightarrow H^{p+q}_\circ\big(Tot^\circ\, I\!\!R^\star\, Hom^\bullet_{\Omega_G}(V,\Omega_G),D_\circ\big) \\ {}''\!E^{p,q} := \mathrm{Ext}^{p,q}_{\Omega_G}(hV,\Omega_G)) \quad\ \Rightarrow H^{p+q}_\circ\big(Tot^\circ\, I\!\!R^\star\, Hom^\bullet_{\Omega_G}(V,\Omega_G),D_\circ\big) \end{cases}$$

b) *If $V$ is projective as $\Omega_G$-gm ([20]), then the following morphism of complexes induced by the augmentation $\epsilon : \Omega_G \to \mathcal{I}^\star$ is a quasi-isomorphism:*

$$Hom^\bullet_{\Omega_G}((V,d),\Omega_G) \xrightarrow{(\epsilon)} Tot^\circ\, I\!\!R^\star\, Hom^\bullet_{\Omega_G}((V,d),\Omega_G).$$

c) *If $hV$ is projective as $\Omega_G$-gm, then the following natural morphisms of $\Omega_G$-gm's are isomorphisms:*

$$Hom^\bullet_{\Omega_G}(hV,\Omega_G) \xrightarrow{(\epsilon)} \mathrm{Tot}^\circ\, Hom^\bullet_{\Omega_G}(hV,\mathcal{I}^\star) \longrightarrow h\big(\mathrm{Tot}^\circ\, Hom^\bullet_{\Omega_G}(V,\mathcal{I}^\star)\big)$$

*Proof.* (a) By 4.1.7 we can fix an injective resolution $\Omega_G \to (\mathcal{I}^\star\Omega_G,\delta_\star)$ of $\Omega_G$-gm of *finite length*, whereby the line $\bullet$-filtration and the column $\star$-filtration are both regular for the total order '$\bullet + \star$'. We have

$$({}'\!E^{p,\star}_0,d_0) = (Homgr^p(V,\mathcal{I}^*\Omega_G),\delta_\star)$$
$$({}''\!E^{p,\bullet}_0,d_0) = Hom^\bullet((V,d),\mathcal{I}^p\Omega_G)$$

and the proposition follows. (b,c) are straightforward consequences of (a). □

**4.2.10. Proposition.** *Let $(V,d)$ be an $\Omega_G$-dgm.*

a) *For any $N \in \mathrm{GM}(\Omega_G)$, there exists a natural morphism of $\Omega_G$-modules*

$$\xi(V,N) : h(Hom^\bullet_{\Omega_G}((V,d),N)) \to Hom^\bullet_{\Omega_G}(hV,N)$$

b) *If $V$ and $hV$ are projective (free) $\Omega_G$-gm, then $\xi(V,\Omega_G)$ is an isomorphism.*

c) *Let $(V,d)$ and $(V',d')$ be $\Omega_G$-dgm's where $V$ and $V'$ are projective (free) $\Omega_G$-gm's. If $\boldsymbol{\alpha} : (V,d) \to (V',d')$ is a quasi-isomorphism of $\Omega_G$-dgm's, the following induced morphism of $\Omega_G$-dgm's is a quasi-isomorphism:*

$$Hom^\bullet_H(\boldsymbol{\alpha},\Omega_G) : Hom^\bullet_H((V',d'),\Omega_G) \to Hom^\bullet_H((V,d),\Omega_G).$$

[20] A projective $\Omega_G$-gm is always free, *cf.* in [J] the corollary of theorem 6.21, p. 386.

*Proof.* In order to simplify notations we write '$\mathit{Hom}^\bullet$' for '$\mathit{Hom}^\bullet_{\Omega_G}$'.

Let $Z \subseteq V$, resp. $B \subseteq V$, denote the $\Omega_G$-graded submodules of cocycles, resp. coboundaries, of $(V,d)$, and let $N$ be any $\Omega_G$-graded module.

(a) Applying the functor $\mathit{Hom}^\bullet(\_,N)$ to the short exact sequence:

$$\mathbf{0} \to Z \longrightarrow V \xrightarrow{d} B[1] \to \mathbf{0}, \tag{†}$$

one gets the left exact sequence of $\Omega_G$-complexes

$$\mathbf{0} \to \mathit{Hom}^\bullet(B,N)[-1] \xrightarrow{\alpha} \mathit{Hom}^\bullet(V,N) \xrightarrow{\beta} \mathit{Hom}^\bullet(Z,N),$$

and the short exact sequence of $\Omega_G$-complexes

$$\mathbf{0} \to \mathit{Hom}^\bullet(B,N)[-1] \xrightarrow{\alpha} \mathit{Hom}^\bullet(V,N) \xrightarrow{\beta} Q^\bullet(Z,N) \to \mathbf{0}, \tag{$*$}$$

where $Q^\bullet(Z,N)$ denotes the image of $\beta$. Note that the left and right complexes in $(*)$ have null differentials so that they coincide with their cohomology.

The cohomology sequence associated with $(*)$ is the long exact sequence

$$\xrightarrow{c_{i-1}} \mathit{Hom}^i(B,N) \xrightarrow{a_i} h^i\,\mathit{Hom}^\bullet(V,N) \xrightarrow{b_i} Q^i(Z,N) \xrightarrow{c_i} \mathit{Hom}^i(B,N) \xrightarrow{a_{i+1}},$$

where one easily verifies that $c_i$ is the restriction to $Q^i(V,N)$ of the natural map $\mathit{Hom}^\bullet(Z,N) \to \mathit{Hom}^\bullet(B,N)$ induced by the inclusion $B \subseteq Z$. In this way we obtain the exact triangle of $\Omega_G$-graded modules

$$h\mathit{Hom}^\bullet(V,N) \xrightarrow{b} Q^\bullet(Z,N) \xrightarrow{c} \mathit{Hom}^\bullet(B,N) \xrightarrow{a[+1]}. \tag{$**$}$$

On the other hand, if we apply $\mathit{Hom}^\bullet(\_,N)$ to the short exact sequence

$$\mathbf{0} \to B \subseteq Z \to hV \to \mathbf{0}, \tag{††}$$

we obtain the left exact sequence

$$\mathbf{0} \to \mathit{Hom}^\bullet(hV,N) \xrightarrow{b'} \mathit{Hom}^\bullet(Z,N) \xrightarrow{c'} \mathit{Hom}^\bullet(B,N),$$

which, joined to $(**)$, gives rise to the following commutative diagram with exact horizontal lines:

$$\begin{tikzcd}
 & h\mathit{Hom}^\bullet(V,N) \arrow[r, "b"] \arrow[d, dashed, "\xi(V,N)"] & Q^\bullet(V,N) \arrow[r, "c"] \arrow[d, "\subseteq"] \arrow[dr, phantom, "\bigoplus"] & \mathit{Hom}^\bullet(B,N) \arrow[d, "="] \\
\mathbf{0} \arrow[r] & \mathit{Hom}^\bullet(hV,N) \arrow[r, "b'"] & \mathit{Hom}^\bullet(Z,N) \arrow[r, "c'"] & \mathit{Hom}^\bullet(B,N)
\end{tikzcd} \tag{$\mathcal{D}$}$$

establishing the existence of $\xi(V,N)$.

(b) If $hV$ is projective, the morphism $c'$ is surjective and

$$I\!R^i\,\mathit{Hom}^\bullet_{\Omega_G}(Z,N) = I\!R^i\,\mathit{Hom}^\bullet_{\Omega_G}(B,N), \quad \forall i \geq 1. \tag{$\diamond$}$$

It follows that $\xi(V,N)$ is bijective if and only if $Q^\bullet(V,N) = \mathit{Hom}^\bullet(Z,N)$, which is equivalent, as $V$ is projective, to $I\!R^1\,\mathit{Hom}^\bullet_{\Omega_G}(B[1],N) = \mathbf{0}$, and to

$$I\!R^1\,\mathit{Hom}^\bullet_{\Omega_G}(Z[1],N) = \mathbf{0}, \tag{$\diamond\diamond$}$$

thanks to $(\diamond)$. Let us prove this equality.

For each $\ell \in \mathbb{Z}$, the projectivity of $V[\ell]$ and the exactness of (†), implies that

$$I\!R^i\, Hom^\bullet_{\Omega_G}(Z[\ell],N) \simeq I\!R^{i+1}\, Hom^\bullet_{\Omega_G}(B[\ell{+}1],N), \quad \forall i \geq 1,$$

and, again by (⋄),

$$I\!R^i\, Hom^\bullet_{\Omega_G}(Z[\ell],N) \simeq I\!R^{i+1}\, Hom^\bullet_{\Omega_G}(Z[\ell{+}1],N), \quad \forall i \geq 1,$$

so that we have, for all $m \geq 1$

$$I\!R^1\, Hom^\bullet_{\Omega_G}(Z[1],N) \simeq I\!R^{1+m}\, Hom^\bullet_{\Omega_G}(Z[1{+}m],N),$$

and (⋄⋄) follows from the fact that $\dim_{\mathrm{ch}}(\mathrm{GM}(\Omega_G)) < +\infty$ (4.1.7-(b)).

(c) Consider the exact triangle in $\mathrm{DGM}(\Omega_G)$

$$(V,d) \xrightarrow{\boldsymbol{\alpha}} (V',d') \xrightarrow{p_1} (\hat{c}(\boldsymbol{\alpha}),\Delta) \xrightarrow[{[+1]}]{p_2}$$

where $(\hat{c}(\boldsymbol{\alpha}),\Delta)$ denotes the *cone* of $\boldsymbol{\alpha}$, i.e. the $\Omega_G$-gm $\hat{c}(\boldsymbol{\alpha}) := V' \oplus V[1]$ with differential $\Delta(v',w) := (d'v' + \boldsymbol{\alpha}\omega, -d\omega)$. Note that $h(\hat{c}(\boldsymbol{\alpha})) = 0$ since by the universal property of the cone construction, $\hat{c}(\boldsymbol{\alpha})$ is acyclic if and only if $\boldsymbol{\alpha}$ is a quasi-isomorphism. Now, since additive functors respect exactness of triangles and cones, the morphism $Hom^\bullet_{\Omega_G}(\boldsymbol{\alpha},\Omega_G)$ is a quasi-isomorphism if and only if the complex $\hat{c}(Hom^\bullet_{\Omega_G}(\boldsymbol{\alpha},\Omega_G)) = Hom^\bullet_{\Omega_G}(\hat{c}(\boldsymbol{\alpha}),\Omega_G)$ is acyclic. This is indeed the case following (b) because, $\hat{c}(\boldsymbol{\alpha})$ and $h(\hat{c}(\boldsymbol{\alpha}))$ being both projective $\Omega_G$-gm, we have $h(Hom^\bullet_{\Omega_G}(\hat{c}(\boldsymbol{\alpha}),\Omega_G)) = Hom^\bullet_{\Omega_G}(h(\hat{c}(\boldsymbol{\alpha})),\Omega_G) = 0$. □

**4.2.11. Remarks**

a) If one disregards the morphism $\xi(V,\Omega_G)$ in 4.2.10-(a), then the fact that $h(Hom^\bullet_{\Omega_G}(V,\Omega_G))$ and $Hom^\bullet_{\Omega_G}(hV,\Omega_G)$ are isomorphic $\Omega_G$-gm's when $V$ and $hV$ are projectives is an immediate result of 4.2.9-(b,c).

b) The statement 4.2.10-(c) is a straightforward consequence of 4.2.9-(b). Indeed, it claims that the restriction of the functor $Hom^\bullet_{\Omega_G}(\_,\Omega_G)$ to the full subcategory of $\Omega_G$-dgm's $(V,d)$ with $V$ projective (free) as $\Omega_G$-gm is a derived functor, so that, as such, it preserves quasi-isomorphisms.

c) In 4.2.10 the projectivity assumption on the $\Omega_G$-gm $W \in \{V,hV\}$ can be weaken to just assume that

$$I\!R^i\, Hom^\bullet_{\Omega_G}(W,\Omega_G) = 0, \qquad (\forall i > 0).$$

*Exercice: Check that the proofs of 4.2.10-(b,c) remain valid under this assumption.*

**4.2.12. Exercise.** Prove that $V$ and $hV$ are projective (free) $\Omega_G$-gm if and only if $Z$ and $B$ are. *Hint. Follow the ideas in the proof of 4.2.10-(b).*

### 4.3. Equivariant Integration

Let $G$ be a **compact connected** Lie group and $M$ an oriented $G$-manifold of dimension $d_M$.

Extend the $\mathbb{R}$-linear integration map $\int_M : \Omega_c(M) \to \mathbb{R}$ by $S(\mathfrak{g})$-linearity to the map

$$\int_M : S(\mathfrak{g}) \otimes \Omega_c(M) \to S(\mathfrak{g}), \quad P \otimes \omega \mapsto P \int_M \omega \tag{⋄}$$

As $G$ acts on $S(\mathfrak{g})\otimes\Omega_{\mathrm{c}}(M)$ by $g\cdot(P\otimes\omega) := g\cdot P\otimes g\cdot\omega$, the above integration map is $G$-equivariant since one has $\int_M g\cdot\omega = \int_M \omega$, as a consequence of the connectedness of $G$ (see proof 3.1.4-(a)). Therefore, the restriction of ($\diamond$) to the subspace of $G$-equivariant differential forms with compact support

$$\Omega_{G,\mathrm{c}}(M) := \big(S(\mathfrak{g})\otimes\Omega_{\mathrm{c}}(M)\big)^G = \big(S(\mathfrak{g})\otimes\Omega_{\mathrm{c}}(M)\big)^{\mathfrak{g}}$$

takes values in $\Omega_G := S(\mathfrak{g})^G$ (3.1.3). We denote this restriction by

$$\int_M : \Omega_{G,\mathrm{c}}(M) \to \Omega_G \tag{$\diamond\diamond$}$$

and call it *the equivariant integration*, which is clearly a morphism of $\Omega_G$-graded modules of degree $-d_M$.

Now, the graded algebra $\Omega_{G,\mathrm{c}}(M)$ has already been endowed with the differential $d_G(P\otimes\omega) = P\otimes d\omega + \sum_i Pe^i\otimes c(e_i)\omega$ (see 3.1.3, 2.2.2-($d_{\mathfrak{g}}$)), and a homogeneous equivariant form $\zeta\in\Omega^d_{G,\mathrm{c}}(M)$ of total degree $d$ decomposes in a unique way as a sum

$$\zeta = \sum_{0\le i\le d/2}\Big(\sum_{Q\in\mathscr{B}(i)} Q\otimes\omega_Q\Big)$$

where $\mathscr{B}(i)$ denotes a vector space basis of $S^i(\mathfrak{g})$ and $\omega_Q\in\Omega^{d-2\deg Q}(M)$. As a consequence, one easily checks that if $\zeta$ is an equivariant coboundary, the terms $Q\otimes\omega_Q$ in the above decomposition such that $\omega_Q\in\Omega_{\mathrm{c}}(M)$ is of top degree $d_M$ are already coboundaries, i.e. $\omega_Q\in B^{d_M}_{\mathrm{c}}(M)$, and consequently $\int_M\zeta = 0$. We have thereby proved the following lemma.

**4.3.1. Lemma.** $\displaystyle\int_M d_G(\Omega_{G,\mathrm{c}}(M)) = 0.$

Therefore, the equivariant integration ($\diamond\diamond$) induces a morphism of $\Omega_G$-graded modules of degree $[-d_M]$ in cohomology:

$$\int_M : H_{G,\mathrm{c}}(M) \to H_G \tag{$\diamond\diamond\diamond$}$$

also called *equivariant integration*.

**4.3.2. Equivariant Integration vs. Integration Along Fibers.** As we already pointed out in 3.2.7, $G$-equivariant cohomology is canonically isomorphic to the projective limit of the de Rham cohomologies of the fibered spaces $\pi_n : M_G(n) = \mathbb{E}G(n)\times_G M \twoheadrightarrow \mathbb{B}G(n)$ (3.2.4). Moreover, for each fixed $d\in\mathbb{N}$ the projective system $\{H^d(M_G(n))\}_n$ is stationary and converges to $H^d(M_G)$. Now, each $\pi_{M,n} : M_G(n)\to\mathbb{B}_G(n)$ is a fibration with oriented base space, whose fiber is the oriented manifold $M$. The operation of integration along $M$ is then well defined $\int_M : H_{\mathrm{c}}(M_G(n))[d_M]\to H_{\mathrm{c}}(\mathbb{B}_G(n))$ (see 1.9.5) and induces a limit map

$$\pi_{M,*} : H_{G,\mathrm{c}}(M)[d_M] \to H(\mathbb{B}_G) = \Omega_G(\{\bullet\})$$

**Proposition.** *The map $\pi_{M,*}$ coincides with the equivariant integration.*

*Proof.* Left to the reader. □

### 4.4. Equivariant Poincaré Pairing

Equivariant integration is what we need to mimic the nonequivariant Poincaré pairing (1.3) in the equivariant framework.

**4.4.1** The composition of the $\Omega_G$-bilinear map $\Omega_G(M)\otimes\Omega_{G,\mathrm{c}}(M)\to\Omega_{G,\mathrm{c}}(M)$, $\alpha\otimes\beta\mapsto\alpha\wedge\beta$, with equivariant integration $\int_M:\Omega_{G,\mathrm{c}}(M)\to\Omega_G$, gives rise to a nondegenerate pairing

$$\mathbb{P}_{M,G}:\Omega_G(M)\otimes\Omega_{G,\mathrm{c}}(M)\longrightarrow\Omega_G\,,\quad \alpha\otimes\beta\longmapsto\int_M\alpha\wedge\beta \tag{$\mathbb{P}_G$}$$

inducing the *Poincaré pairing in equivariant cohomology*

$$\mathscr{P}_{G,M}:H_G(M)\otimes H_{G,\mathrm{c}}(M)\longrightarrow H_G\,,\quad [\alpha]\otimes[\beta]\longmapsto\int_M[\alpha]\cup[\beta] \tag{$\mathscr{P}_G$}$$

**4.4.2** The left adjoint map associated with $\mathbb{P}_G$ (see 1.3) is now the map

$$\begin{aligned}\mathbb{D}_{G,M}:\Omega_G(M)&\longrightarrow Hom^\bullet_{\Omega_G}\big(\Omega_{G,\mathrm{c}}(M),\Omega_G\big)\\ \alpha\quad&\longmapsto \mathbb{D}_{G,M}(\alpha):=\Big(\beta\mapsto\int_M\alpha\wedge\beta\Big)\end{aligned} \tag{$\mathbb{D}_G$}$$

and one has, following lemma 4.3.1, for $\alpha$ homogeneous

$$\begin{aligned}\mathbb{D}_G\big((-1)^{d_M}d_G\alpha\big)(\beta)&=\int_M(-1)^{d_M}d_G\alpha\wedge\beta\\ &=\int_M(-1)^{d_M+|\alpha|+1}\alpha\wedge d_G\beta=(-1)^{|\beta|}\mathbb{D}_G(\alpha)(d_G\beta),\end{aligned}$$

Hence, following the conventions in 1.1.7 and 4.1.11, $\mathbb{D}_{G,M}$ is a morphism of $\Omega_G$-graded complexes from $\Omega_G(M)[d_M]$ to $Hom^\bullet_{\Omega_G}(\Omega_{G,\mathrm{c}}(M),\Omega_G)$.

**4.4.3** The right adjoint map associated with $\mathbb{P}_G$ (see 1.7.1) is the map

$$\begin{aligned}\mathbb{D}'_{G,M}:(\Omega_{G,\mathrm{c}}(M),d_G)&\longrightarrow\big(Hom^\bullet_{\Omega_G}\big(\Omega_G(M),\Omega_G\big),-D\big)\\ \beta\quad&\longmapsto\mathbb{D}'_{G,M}(\beta):=\Big(\alpha\mapsto\int_M\alpha\wedge\beta\Big)\end{aligned} \tag{$\mathbb{D}'_{G,}$}$$

which is also a morphism of $\Omega_G$-graded complexes.

**4.4.4. Exercise.** Verify that ($\mathbb{P}_G$) is a nondegenerate pairing and that $\mathbb{D}'_{G,M}$ is a morphism of $\Omega_G$-graded complexes.

### 4.5. *G*-Equivariant Poincaré Duality Theorem

**4.5.1. Theorem.** *Let $G$ be a compact connected Lie group, and $M$ an oriented $G$-manifold of dimension $d_M$. Then,*

a) *The $\Omega_G$-graded morphism of complexes*

$$\boxed{\mathbb{D}_{G,M}:\Omega_G(M)[d_M]\longrightarrow Hom^\bullet_{\Omega_G}\big(\Omega_{G,\mathrm{c}}(M),\Omega_G\big)}$$

*is a quasi-isomorphism.*

b) *The morphism* $I\!D_{G,M}$ *induces* the "Poincaré morphism in $G$-equivariant cohomology" *(see 4.2.10-(a))*

$$\boxed{\mathscr{D}_{G,M} : H_G(M)[d_M] \longrightarrow \mathrm{Hom}^{\bullet}_{H_G}\big(H_{G,\mathrm{c}}(M), H_G\big)}$$

*If* $\mathrm{Extgr}^i_H(H_{G,\mathrm{c}}(M)) = 0$ *for all* $i > 0$*, for example if* $H_{G,\mathrm{c}}(M)$ *is a free* $H_G$*-module, then* $\mathscr{D}_{G,M}$ *is an isomorphism.*

c) *There are natural spectral sequence converging to* $H_G(M)[d_M]$

$$\begin{cases} I\!E_2^{p,q}(M) = \big(\mathrm{Ext}^p_{H_G}\big(H_{G,\mathrm{c}}(M), H_G\big)\big)^q \Rightarrow H_G^{p+q}(M)[d_M] \\ I\!F_2^{p,q}(M) = H_G^p \otimes_{\mathbb{R}} \mathit{Hom}^{\bullet}_{\mathbb{R}}(H^q_{\mathrm{c}}(M), \mathbb{R}) \Rightarrow H_G^{p+q}(M)[d_M] \end{cases}$$

*where, in the first one,* $q$ *denotes the graded vector space degree.*

d) *Moreover, if* $M$ *is of finite type, the* $H_G$*-graded morphism of complexes*

$$\boxed{I\!D'_{G,M} : \Omega_{G,\mathrm{c}}(M)[d_M] \longrightarrow \mathrm{Hom}^{\bullet}_{H_G}\big(\Omega_G(M), H_G\big)}$$

*is a quasi-isomorphism, and mutatis mutandis for* (b) *and* (c).

*Proof*

a) We recall the filtration of the Cartan complex we already used in the proof of 2.2.6-(b): An equivariant form in $(\Omega_G(M), d_G)$ is said to be *of index* $m$ if it belongs to the subspace

$$\Omega_G(M)_m := \big(S^{\geqslant m}(\mathfrak{g}) \otimes \Omega(M)\big)^G .$$

One easily checks that each $\Omega_G(M)_m$ is stable under the Cartan differential $d_G$, that $\Omega_G(M) = \Omega_G(M)_m$ for all $m \leqslant 0$ and that one has a decreasing filtration

$$\Omega_G(M) = \Omega_G(M)_0 \supseteq \Omega_G(M)_1 \supseteq \Omega_G(M)_2 \supseteq \cdots \qquad (*)$$

Furthermore, $\Omega^i_G(M) \cap \Omega_G(M)_m = 0$ whenever $m > i$, so that $(*)$ is a *regular filtration* (see [Go] §4 pp. 76-).

In a similar way, $\lambda \in \mathit{Hom}^{\bullet}_{\Omega_G}(\Omega_G(M), \Omega_G)$ is said to be *of index* $m$ whenever

$$\lambda\Big(\big(S^a(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(M)\big)^G\Big) \subseteq \Omega_G^{\geqslant a+m}, \qquad \forall a \in \mathbb{N},$$

and we denote $\mathit{Hom}^{\bullet}_{\Omega_G}(\Omega_{G,\mathrm{c}}(M), \Omega_G)_m$ the subspace of such maps. As before, each of these spaces is is a subcomplex of $(\mathit{Hom}^{\bullet}_{\Omega_G}(\Omega_G(M)), D)$ and the decreasing filtration

$$\cdots \supseteq \mathit{Hom}^{\bullet}_{\Omega_G}(\Omega_{G,\mathrm{c}}, \Omega_G)_m \supseteq \mathit{Hom}^{\bullet}_{\Omega_G}(\Omega_{G,\mathrm{c}}, \Omega_G)_{m+1} \supseteq \cdots \qquad (**)$$

verifies for each $\lambda$ homogeneous of degree $i$

$$a + \dim M + i \geqslant \deg \lambda\big((S^a(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(M))^G\big) \geqslant a + i, \qquad \forall a \in \mathbb{N},$$

so that $(**)$ is also regular.

An immediate verification shows that $I\!D_{G,M}$ is a morphism of graded filtered modules, i.e.

$$I\!D_{G,M}\big(\Omega_G(M)[d_M]_m\big) \subseteq \mathit{Hom}^{\bullet}_{\Omega_G}\big(\Omega_{G,\mathrm{c}}(M), \Omega_G\big)_m, \qquad \forall m \in \mathbb{Z},$$

giving rise, therefore, to a morphism between the associated spectral sequences (see [Go], §4 Thm. 4.3.1, p. 80) whose $\mathbb{E}_0$ terms are

$$\begin{cases} \big((S(\mathfrak{g}) \otimes \Omega(M))^G, 1 \otimes d\big)[d_M] \\ Hom^\bullet_{\Omega_G}\big(\big((S(\mathfrak{g}) \otimes \Omega_c(M))^G, 1 \otimes d\big), \Omega_G\big) \end{cases}$$

and which are respectively quasi-isomorphic to

$$\begin{cases} \Omega_G \otimes \big(\Omega(M), d\big)[d_M] \\ Hom^\bullet_{\Omega_G}\big(\Omega_G \otimes (\Omega(M), d), \Omega_G\big) \end{cases}$$

Indeed, the first one is just 3.1.4-(a), and the second one results from the fact that, since $G$ is compact, there is a canonical isomorphism $S(\mathfrak{g}) = \Omega_G \otimes_{\mathbb{R}} \mathcal{H}$, where $\mathcal{H}$ denotes the (graded) subspace of $G$-harmonic polynomials of $S(\mathfrak{g})$ (see [Dx], thm. **7.3.5** p. 241, §**8** pp. 277-), so that

$$\big((S(\mathfrak{g}) \otimes_{\mathbb{R}} \Omega_c(M))^G, 1 \otimes d_M\big) = \Omega_G \otimes_{\mathbb{R}} \big((\mathcal{H} \otimes \Omega_c(M))^G, 1 \otimes d_M\big),$$

and the quasi-isomorphisms of 3.1.4-(a)

$$\Omega_G \otimes \big(\Omega_c(M), d\big) \supseteq \Omega_G \otimes \big(\Omega_c(M)^G, d\big) \subseteq \big((S(\mathfrak{g}) \otimes \Omega_c(M))^G, 1 \otimes d\big)$$

are morphisms of complexes of ***free*** $\Omega_G$-graded modules. Consequently, the induced morphisms on the corresponding $\Omega_G$-dual complexes will still be quasi-isomorphisms (*cf.* 4.2.10-(c)).

Putting together these observations, the induced morphism on the $\mathbb{E}_1$ terms of the concerned spectral sequences by $\mathbb{D}_{G,M}$, is simply

$$\begin{array}{ccc} H_G \otimes H(M)[d_M] & \xrightarrow{\mathbb{1} \otimes \mathscr{D}_M} & H_G \otimes Hom^\bullet_{\mathbb{R}}(H_c(M), \mathbb{R}) \\ & & \| \\ & & \mathrm{Hom}_{H_G}\big((H_G \otimes H_c(M), H_G)\big) \end{array}$$

where one recognizes in $\mathbb{1} \otimes \mathscr{D}_M$ the classical Poincaré duality 1.3.2.

b) This is an application of proposition 4.2.10 and 4.2.11-(c), since, as we noted in the previous paragraphs, $\Omega_G := \Omega_G(M)$ is a free $\Omega_G$-gm.

c) The first spectral sequence $\mathbb{E}(M)$ is just the $''\mathbb{E}$ spectral sequence of 4.2.9 converging to the right-hand side of ($\star$). On the other hand, the spectral sequence, $\mathbb{F}_2^{p,q}(M)$, is the one we used in the proof of (a).

d) Left to the reader. □

**4.5.2. Torsion-freeness, Freeness and Reflexivity.** Proposition 4.5.1-(b,d) shows that the freeness of equivariant cohomology as $H_G$-gm is a sufficient condition for equivariant Poincaré duality to hold. The question then arises whether some weaker condition could be equivalent to duality. Apart from freeness, two other properties have been thoroughly study in Allday-Franz-Puppe [AFP$_1$].

- Torsion-freeness. An $H_G$-gm $V$ is said to be *torsion-free* if, for all $v \in V$,
$$\mathrm{Ann}(v) := \{P \in H_G \mid P \cdot v = 0\} = 0.$$
The torsion-freeness of equivariant cohomology is clearly a necessary condition for duality as the modules $Hom^\bullet_{H_G}(\_, H_G)$ are torsion-free. It is also

a sufficient condition for the injectivity of the Poincaré morphism (Prop. 5.9 [AFP$_1$], see ex. 7.1.3), but it is not for duality, as the explicit examples of Franz-Puppe [FP] (2006) show.

- Reflexivity. An $H_G$-gm $V$ is said to be *reflexive* if the natural map
$$V \to \mathit{Hom}^\bullet_{H_G}(\mathit{Hom}^\bullet_{H_G}(V, H_G), H_G)$$
is an isomorphism.

  For a finite type manifold $M$, while the reflexivity of $H_G(M)$ and $H_{G,\mathrm{c}}(M)$ are clearly necessary conditions to duality, the converse, which is also true, is more subtle. The equivalence between duality and reflexivity has been established in [AFP$_1$] (Prop. 5.10) for $G$ abelian, and in Franz [F$_2$] (Cor. 5.1) for general $G$.

The following diagram illustrates the relationship between the different kinds of nontorsions in equivariant cohomology and significant properties of the equivariant Poincaré pairing.

$$\begin{array}{ccccc}
\{\texttt{free}\} \subseteq & \{\texttt{reflexive}\} & \subseteq & \{\texttt{torsion-free}\} \\
& \Updownarrow & & \Updownarrow \\
& \left\{\begin{array}{c}\textsf{Perfect}\\ \textsf{Poincaré pairing}\end{array}\right\} & \subseteq & \left\{\begin{array}{c}\textsf{Nondegenerate}\\ \textsf{Poincaré pairing}\end{array}\right\}
\end{array}$$

It is worth noting that in [F$_1$] (2015), Franz gives the first known examples of compact manifolds having reflexive but nonfree equivariant cohomology.

### 4.6. $T$-Equivariant Poincaré Duality Theorem

When $G$ is a **compact connected torus** $T = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, we have:

$$\begin{cases} \Omega_T = S(\mathfrak{t}) \\ \Omega_T(M) = S(\mathfrak{t}) \otimes_{\mathbb{R}} \Omega(M)^T \\ \Omega_{T,\mathrm{c}}(M) = S(\mathfrak{t}) \otimes_{\mathbb{R}} \Omega_{\mathrm{c}}(M)^T \end{cases}$$

so that

$$\begin{aligned} \mathit{Hom}_{\Omega_T}(\Omega_{T,\mathrm{c}}, \Omega_T) &= \mathit{Hom}_{S(\mathfrak{t})}\big(S(\mathfrak{t}) \otimes \Omega_{\mathrm{c}}(M)^T, S(\mathfrak{t})\big) \\ &= \mathit{Hom}_{\mathbb{R}}\big(\Omega_{\mathrm{c}}(M)^T, S(\mathfrak{t})\big) \\ &= S(\mathfrak{t}) \otimes_{\mathbb{R}} \mathit{Hom}_{\mathbb{R}}\big(\Omega_{\mathrm{c}}(M)^T, \mathbb{R}\big) \end{aligned}$$

The left adjoint map $I\!\!D_T(M)$ associated with the $T$-equivariant Poincaré pairing $I\!\!P_T$ (see 4.4.2) identifies naturally to $\mathbb{1} \otimes I\!\!D_M$,

$$\boxed{\begin{array}{ccc} S(\mathfrak{t}) \otimes \Omega(M)^T[d_M] & \xrightarrow[\mathbb{1}\otimes I\!\!D_M]{I\!\!D_T(M)} & S(\mathfrak{t}) \otimes \mathit{Hom}_{\mathbb{R}}\big(\Omega_{\mathrm{c}}(M)^T, \mathbb{R}\big) \\ P \otimes \alpha & \longmapsto & P \otimes \left(\beta \mapsto \displaystyle\int_M \alpha \wedge \beta\right) \end{array}}$$

and the right adjoint map (see 4.4.3) to

$$\boxed{\begin{array}{ccc} S(\mathfrak{t}) \otimes \Omega_{\mathrm{c}}(M)^T[d_M] & \xrightarrow[\mathbb{1}\otimes I\!\!D'_M]{I\!\!D'_T(M)} & S(\mathfrak{t}) \otimes \mathit{Hom}_{\mathbb{R}}\big(\Omega(M)^T, \mathbb{R}\big) \\ P \otimes \beta & \longmapsto & P \otimes \left(\alpha \mapsto \displaystyle\int_M \alpha \wedge \beta\right) \end{array}}$$

The following theorem is a particular case of 4.5.1.

**4.6.1. Theorem.** *Let $T$ be a compact connected torus, and $M$ an oriented $T$-manifold of dimension $d_M$.*

a) *The $H_T$-graded morphism of complexes*

$$\boxed{\mathbb{D}_T(M) : \Omega_T(M)[d_M] \longrightarrow \mathrm{Hom}_{H_T}\big(\Omega_{T,\mathrm{c}}(M),\Omega_T\big)}$$

*is a quasi-isomorphism.*

b) *The morphism $\mathbb{D}_T(M)$ induces* the "Poincaré morphism in $T$-equivariant cohomology" *(see 4.2.10-(a))*

$$\boxed{\mathscr{D}_{T,M} : H_T(M)[d_M] \longrightarrow \mathrm{Hom}_{H_T}\big(H_{T,c}(M),H_T\big)}$$

*If $H_{T,\mathrm{c}}(M)$ is a free $H_T$-module, $\mathscr{D}_{T,M}$ is an isomorphism.*

c) *There are natural spectral sequences converging to $H_T(M)[d_M]$*

$$\begin{cases} \mathbb{E}_2^{p,q}(M) = \big(\mathrm{Ext}^p_{H_T}\big(H_{T,\mathrm{c}}(M),H_T\big)\big)^q \Rightarrow H_T^{p+q}(M)[d_M] \\ \mathbb{F}_2^{p,q}(M) = H_T^p \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathbb{R}}(H_{\mathrm{c}}^q(M),\mathbb{R}) \Rightarrow H_T^{p+q}(M)[d_M] \end{cases}$$

*where, in the first one, $q$ denotes the graded vector space degree.*

d) *Moreover, if $M$ is of finite type, the $\Omega_G$-graded morphism of complexes*

$$\boxed{\mathbb{D}'_T(M) : \Omega_{T,\mathrm{c}}(M)[d_M] \longrightarrow \mathrm{Hom}_{H_T}\big(\Omega_T(M),H_T\big)}$$

*is a quasi-isomorphism, and mutatis mutandis for* (b) *and* (c).

*Proof* (a) Since we have the identification $\mathbb{D}_T(M) = \mathbb{1} \otimes \mathbb{D}_M$, we may conclude using 2.2.6-(d). Statements (b,c,d) are particular cases of 4.5.1. $\square$

**4.6.2. Remark.** Recall that $H_{T,\mathrm{c}}(M)$ is a free $H_T$-module whenever $M$ has no odd (or no even) degree cohomology with compact support (2.2.6-(d)-(iv)). Obviously, though not very interesting, this is also the case when $T$ acts trivially on $M$, since then $c(Y) = \boldsymbol{\theta}(Y) = 0$, $\forall Y \in \mathfrak{t}$, and $H_{T,\mathrm{c}}(M) = H_T \otimes_{\mathbb{R}} H_{\mathrm{c}}(M)$.

# 5. Equivariant Gysin Morphism

We now follow the steps in section 1.10 for the construction of the Gysin morphisms in the equivariant framework.

## 5.1. $G$-Equivariant Gysin Morphism for General Maps

**5.1.1. Equivariant Finite de Rham Type Coverings.** Recall that after 1.6, if $G$ is a compact Lie Group, a $G$-manifold $M$ is the union of a countable ascending chain $\mathscr{U} := \{U_0 \subseteq U_1 \subseteq \cdots\}$ of $G$-stable open subsets of finite type.

The following theorem, equivariant analog of 1.7.8, is a simple corollary of the $G$-equivariant Poincaré duality theorem 4.5.1. The proof is left to the reader.

**5.1.2. Theorem.** *Let $G$ be a compact connected Lie group, and $M$ an oriented $G$-manifold of dimension $d_M$. Then, for every filtrant cover $\mathscr{U}$ of $M$ by $G$-stable open subsets, the natural morphism $\varinjlim_{U\in\mathscr{U}} \Omega_{G,c}(U) \to \Omega_{G,c}(M)$ is an isomorphism, and the analog to 1.7.8-(b) is a well defined morphism of complexes:*

$$\mathbb{D}'_{G,\mathscr{U}} : (\Omega_{G,c}(M), d_G)[d_M] \longrightarrow \varinjlim_{U\in\mathscr{U}} \big(Hom_{H_G}(\Omega_G(U), H_G), -D\big)$$

*If in addition the open sets in $\mathscr{U}$ are of finite type, $\mathbb{D}'_{G,\mathscr{U}}$ is a quasi-isomorphism.*

**5.1.3** We now closely follow the instructions of section 1.10.2 for the construction of Gysin morphisms.

Let $f : M \to N$ be a $G$-equivariant map between oriented $G$-manifolds. To $\beta \in \Omega_{G,c}(M)$ we assign the linear form on $\Omega_G(N)$ defined by $\mathbb{D}'_{G,f}(\beta) : \alpha \mapsto \int_M f^*\alpha \wedge \beta$. In this way we obtain the diagram

$$\begin{tikzcd} \Omega_{G,c}(M)[d_M] \arrow[r, dotted, "f_*"] \arrow[dr, "\mathbb{D}'_{G,f}"'] & \Omega_{G,c}(N)[d_N] \arrow[d, "\mathbb{D}'_N\ \left(\substack{\text{quasi-iso if } N \text{ is of}\\ \text{finite type}}\right)"] \arrow[dl, phantom, "\oplus"] \\ & Hom_{H_G}(\Omega_G(N), H_G) \end{tikzcd}$$

which may be closed in cohomology whenever $N$ is of **finite type**, since $\mathbb{D}'_{G,N}$ is then a quasi-isomorphism (4.5.1-(d)).

When $N$ is not of finite type, one fixes any equivariant cover $\mathscr{U}$ of $N$ made up of open finite type subsets (5.1.1), and replaces $\mathbb{D}'_{G,N}$ by the morphism $\mathbb{D}'_{G,\mathscr{U}}$ of theorem 5.1.2. The diagram

$$\begin{tikzcd} \Omega_{G,c}(M)[d_M] \arrow[r, dotted, "f_*"] \arrow[dr, "\mathbb{D}'_{G,f,\mathscr{U}}"'] & \Omega_{G,c}(N)[d_N] \arrow[d, "\mathbb{D}'_{G,\mathscr{U}}\ \text{(quasi-iso)}"] \arrow[dl, phantom, "\oplus"] \\ & \varinjlim_{U\in\mathscr{U}} Hom_{H_G}(\Omega_G(U), H_G) \end{tikzcd}$$

where $\mathbb{D}'_{G,f,\mathscr{U}}$ is defined as in 1.10.2, may be closed in cohomology since $\mathbb{D}'_{G,\mathscr{U}}$ is a quasi-isomorphism. The closing arrow

$$\boxed{f_* : H_{G,c}(M)[d_M] \to H_{G,c}(N)[d_N]}$$

*the equivariant Gysin morphism associated with* $f$, is therefore defined as

$$f_* := (\mathscr{D}'_{G,\mathscr{U}})^{-1} \circ h(I\!\!D'_{G,f,\mathscr{U}}).$$

**5.1.4. Theorem and definitions.** *With the above notations,*

a) *The equality*

$$\int_M f^*[\alpha] \cup [\beta] = \int_N [\alpha] \cup f_*[\beta] \qquad (\diamond\diamond)$$

*holds for all* $[\alpha] \in H_G(N)$ *and* $[\beta] \in H_{G,\mathrm{c}}(M)$.

b) *Furthermore,* $f_*$ *is a morphism of* $H_G(N)$*-modules, i.e. the equality, called the* equivariant projection formula,

$$f_*\big(f^*[\alpha] \cup [\beta]\big) = [\alpha] \cup f_*([\beta]) \qquad (\diamond\diamond\diamond)$$

*holds for all* $[\alpha] \in H_G(N)$ *and* $[\beta] \in H_{G,\mathrm{c}}(M)$.

c) *The correspondence*

$$(\_)_* : G\text{-Man}^{\mathrm{or}} \rightsquigarrow \mathrm{GM}(H_G) \quad \text{with} \quad \begin{cases} M \rightsquigarrow M_* := H_{G,\mathrm{c}}(M)[d_M] \\ f \rightsquigarrow f_* \end{cases}$$

*is a covariant functor. We will refer to it as* the equivariant Gysin functor for general maps.

d) *Suppose that* $M$ *and* $N$ *are manifolds of finite type. If the pullback morphism* $f^* : H_G(N) \to H_G(M)$ *is an isomorphism, then the Gysin morphism* $f_* : H_{G,\mathrm{c}}(M)[d_M] \to H_{G,\mathrm{c}}(N)[d_N]$ *is also.*

*Proof.* (a) Immediate from de definition of the Gysin morphism.

(b) Unlike the proof of the nonequivariant statement 1.7.4-(b), this claim is no longer a formal consequence of (a) because equivariant cohomology may have torsion elements, something that doesn't affect equivariant integration. Instead, when $N$ is of finite type and since then $I\!\!D'_N$ is a quasi-isomorphism, we can check that the following equality holds at the *cochain* level,

$$I\!\!D'_{G,f}(f^*(\alpha) \cup \beta) = \text{“}I\!\!D'_N(\alpha \cup f_*(\beta))\text{”} = I\!\!D'_{G,f}(\beta) \circ \mu_{\mathrm{r}}(\alpha), \qquad (\dagger)$$

where the central term is there for purely heuristic reasons and where we denote $\mu_{\mathrm{r}}(\alpha) : \Omega_G(N) \to \Omega_G(N)$ the right multiplication by $\alpha$, i.e. $\mu_{\mathrm{r}}(\alpha)(\_) = (\_) \cup \alpha$. The identification of the left and right terms in (†) is then a straightforward verification from the definition of $I\!\!D'_{G,f}$. When $N$ is not of finite type, we follow the same arguments with $I\!\!D'_{G,f,\mathscr{U}}$ instead of $I\!\!D'_{G,f}$.

(c) is clear. (d) as $f^* : \Omega_G(N) \to \Omega_G(M)$ is a quasi-isomorphism, the induced map $\mathit{Hom}^\bullet_{H_G}(\Omega_G(N), H_G) \to \mathit{Hom}^\bullet_{H_G}(\Omega_G(M), H_G)$ is also, following 4.2.10-(c), and one concludes, since $I\!\!D'_{G,M}$ and $I\!\!D'_{G,N}$ are quasi-isomorphisms. □

**5.1.5. Exercise.** Prove the following enhancement of the statement 5.1.4-(d). If $\pi : V \to B$ is a vector bundle over an oriented manifold $B$, the map $\pi$ is of finite type (1.4.1), and $\pi^* : H_G(B) \to H_G(V)$ and $\pi_* : H_{G,\mathrm{c}}(V)[d_V] \to H_{G,\mathrm{c}}(B)[d_B]$ and both isomorphisms (*cf.* 1.4.2-(c)).

## 5.2. $G$-Equivariant Gysin Morphism for Proper Maps

**5.2.1** Following 1.10.1, let $f : M \to N$ be a **proper** $G$-equivariant map between oriented $G$-manifolds. To $\alpha \in \Omega_G(M)$ we assign the $H_G$-linear form on $\Omega_{G,\mathrm{c}}(N)$ defined by $I\!D'_{G,f}(\alpha) : \beta \mapsto \int_M f^*\beta \wedge \alpha$. In this way we obtain the diagram

$$\begin{tikzcd} \Omega_G(M)[d_M] \arrow[r, dotted, "f_!"] \arrow[dr, "I\!D'_{G,f}"'] & \Omega_G(N)[d_N] \arrow[d, "I\!D'_{G,N}\ \text{(quasi-iso)}"] \arrow[dl, phantom, "\oplus"] \\ & \Omega_{G,\mathrm{c}}(N)^\vee \end{tikzcd}$$

which may be closed in cohomology because $I\!D'_{G,N}$ is a quasi-isomorphism, as shown in 4.5.1-(a). The closing arrow:

$$\boxed{f_! : H_G(M)[d_M] \to H_G(N)[d_N]},$$

*the equivariant Gysin morphism associated with a proper map* $f$, is therefore defined as

$$f_! := (\mathscr{D}'_{G,\mathscr{U}})^{-1} \circ h(I\!D'_{G,f}).$$

**5.2.2. Theorem and definitions.** *With the above notations,*

a) *The equality*
$$\int_M f^*[\beta] \cup [\alpha] = \int_N [\beta] \cup f_![\alpha] \qquad (\star\star)$$
*holds for all* $[\alpha] \in H_G(M)$ *and* $[\beta] \in H_{G,\mathrm{c}}(N)$.

b) *Furthermore,* $f_!$ *is a morphism of* $H_{G,\mathrm{c}}(N)$*-modules, the equality, called the* equivariant projection formula for proper maps,
$$f_!\big(f^*[\beta] \cup [\alpha]\big) = [\beta] \cup f_![\alpha] \qquad (\star\star\star)$$
*holds for all* $[\beta] \in H_{G,\mathrm{c}}(N)$ *and* $[\alpha] \in H_G(M)$.

c) *The correspondence*
$$f_! : G\text{-}\mathrm{Man}^{\mathrm{or}}_{\mathrm{pr}} \rightsquigarrow \mathrm{GM}(H_G) \qquad \textit{with} \qquad \begin{cases} M \rightsquigarrow M_! := H_G(M)[d_M] \\ f \rightsquigarrow f_! \end{cases}$$
*is a covariant functor. We will refer to it as* the equivariant Gysin functor for proper maps.

d) *If the pullback morphism* $f^* : H_{G,\mathrm{c}}(N) \to H_{G,\mathrm{c}}(M)$ *is an isomorphism, the Gysin morphism* $f_! : H_G(M)[d_M] \to H_G(N)[d_N]$ *is also an isomorphism.*

e) *The natural map* $\phi(\_) : H_{G,\mathrm{c}}(\_)[d\_] \to H_G(\_)[d\_]$ (1.2.3) *is a homomorphism between the two equivariant Gysin functors* $(\_)_* \to (\_)_!$ *over the category* $G\text{-}\mathrm{Man}^{\mathrm{or}}_{\mathrm{pr}}$*, i.e. the diagrams*

$$\begin{array}{ccc} H_{G,\mathrm{c}}(M) & \xrightarrow{\phi(M)} & H_G(M) \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_!} \\ H_{G,\mathrm{c}}(N) & \xrightarrow{\phi(N)} & H_G(N) \end{array}$$

*are naturally commutative.*

*Proof.* Same as the proof of 5.1.4, left to the reader. □

## 5.3. Comparison Theorems

The next theorem establishes a close connexion between the nonequivariant and the equivariant Gysin morphisms. It is a basic tool for the generalization of known properties of classical Gysin morphisms into the equivariant framework.

**5.3.1. Theorem.** *Let $G$ be a compact connected Lie group and $f : M \to N$ a $G$-equivariant map between oriented $G$-manifolds. There exists a natural morphism of the spectral sequences $I\!F$ of theorem 4.5.1-(c) converging to the Gysin morphism $f_* : H_{G,\mathrm{c}}(M)[d_M] \to H_{G,\mathrm{c}}(N)[d_N]$,*

$$\begin{array}{ccc} I\!F_{\mathrm{c},2}(M) = H_G \otimes H_{\mathrm{c}}(M)[d_M] & \Rightarrow & H_{G,\mathrm{c}}(M)[d_M] \\ \Big\downarrow{\scriptstyle 1\otimes f_*} & & \Big\downarrow{\scriptstyle f_*} \\ I\!F_{\mathrm{c},2}(N) = H_G \otimes H_{\mathrm{c}}(N)[d_N] & \Rightarrow & H_{G,\mathrm{c}}(M)[d_N] \end{array}$$

*and in the proper case to $f_! : H_G(M)[d_M] \to H_G(N)[d_N]$,*

$$\begin{array}{ccc} I\!F_{2}(M) = H_G \otimes H(M)[d_M] & \Rightarrow & H_{G}(M)[d_M] \\ \Big\downarrow{\scriptstyle 1\otimes f_!} & & \Big\downarrow{\scriptstyle f_!} \\ I\!F_{2}(N) = H_G \otimes H(N)[d_N] & \Rightarrow & H_{G}(M)[d_N] \end{array}$$

*Proof.* Clear from the proof of 4.5.1 and the definition of Gysin morphisms. □

## 5.4. Universal Property of the equivariant Gysin Morphism

**Proposition** *Let $f : M \to N$ be a $G$-equivariant map between oriented $G$-manifolds.*

a) *A morphism of complexes $\varphi_* : \Omega_{G,\mathrm{c}}(M)[d_M] \to \Omega_{G,\mathrm{c}}(N)[d_N]$ induces the equivariant Gysin morphism*

$$f_* : H_{G,\mathrm{c}}(M)[d_M] \to H_{G,\mathrm{c}}(N)[d_N],$$

*if and only if*

$$\int_M f^*\alpha \wedge \beta = \int_N \alpha \wedge \varphi_*\beta, \quad \forall\alpha \in \Omega_G(N),\ \forall\beta \in \Omega_{G,\mathrm{c}}(M).$$

b) *If $f$ is a proper, a morphism of complexes $\varphi_! : \Omega_G(M)[d_M] \to \Omega_G(N)[d_N]$ induces the equivariant Gysin morphism*

$$f_! : H_G(M)[d_M] \to H_G(N)[d_N],$$

*if and only if*

$$\int_M f^*\beta \wedge \alpha = \int_N \beta \wedge \varphi_!\alpha, \quad \forall\alpha \in \Omega_G(M),\ \forall\beta \in \Omega_{G,\mathrm{c}}(N).$$

**5.4.1. Remark.** The last proposition is a simple consequence of the definition of the Gysin morphism. But one must beware that, unlike the nonequivariant case (1.9.1), it is generally not true that the equivariant Gysin morphism is characterized by the equality of **cohomology classes**:

$$\int_M f^*[\alpha] \cup [\beta] = \int_N [\alpha] \cup f_*[\beta], \quad \forall[\alpha] \in H_G(N),\ \forall[\beta] \in H_{G,\mathrm{c}}(M). \qquad (\diamond\diamond)$$

(or ($\star\star$) for $f_!$ in the proper case). For example, the uniqueness of $f_*$ satisfying the relation ($\diamond\diamond$), results only from the injectivity of the map:

$$\begin{array}{rcl} \mathbb{D}_{G,N} : H_{G,\mathrm{c}}(N) & \longrightarrow & \mathrm{Hom}_{H_G}\big(H_G(N), H_G\big) \\ {[\beta]} & \longrightarrow & \Big([\alpha] \to \int_N [\alpha] \wedge [\beta]\Big) \end{array}$$

a property that is not always satisfied. Indeed, let $T$ be a torus and $N$ a compact oriented $T$-manifold without fixed points. We know from the localization theorem, that $H_T(N)$ is a torsion $H_T$-module and consequently that $Hom_{H_T}(H(N), H_T) = 0$, so that $\mathbb{D}_{G,N}$ is null, although $H_T(N) \neq 0$.

**Exercise.** Let $T = \mathbb{S}^1 \times \mathbb{S}^1$ act on $N = \mathbb{S}^1$ by $(t,u)(v) = uv$.

a) $H_T = \mathbb{R}[X,Y]$, $H_T(N) = \mathbb{R}[Y]$, $End_{H_T}(H_T(N)) = \mathbb{R}[Y]$.

b) For any map $f : N \to N$ and any $\lambda \in End_{H_G}(H_G(N))$ one has

$$\int_N f^*[\alpha] \cup [\beta] = \int_N [\alpha] \cup \lambda[\beta], \quad \forall [\alpha], [\beta] \in H_T(N).$$

c) Let $N$ be any oriented $G$-manifold such that $H_{G,\mathrm{c}}(N)$ is an $H_G$-free module. Show that condition ($\diamond\diamond$) (resp. ($\star\star$) for proper maps) of theorem 5.1.4 (resp. 5.2.2) completely characterizes Gysin morphisms for maps $f : M \to N$.

## 5.5. Group Restriction

Let $G$ be a compact *connected* Lie group. For any closed subgroup $H \subseteq G$, *connected or not*, and for any $G$-manifold $M$, the canonical projection of Borel constructions $\mathbb{E}G \times_H M \twoheadrightarrow \mathbb{E}G \times_G M$ which is a locally trivial fibration with fiber $G/H$, induces by inverse image the *restriction homomorphism* of equivariant cohomology rings

$$\mathrm{Res}_H^G : H_G(M) \to H_H(M).$$

At this point, one could react against the possible lack of connectedness of $H$ in so far as this property has been everywhere required in these notes. However, a careful examination shows that connectedness is only needed to ensure that the action of $G$ on $M$ is homotopically trivial, a property that is clearly inherited by any subgroup $H$ of a connected group $G$, whether the subgroup is connected or not (*cf.* 3.1.5). In that case if $H_\circ$ denotes the connected component of 1 in $H$ and $W_H = H/H_\circ$, we have

$$H_H = S(\mathfrak{h})^H \quad \text{and} \quad H_H(M) = H_{H_\circ}(M)^{W_H}.$$

**5.5.1. Theorem.** *For any closed subgroup $H \subseteq G$ and any equivariant map $f : M \to N$ between oriented $G$-manifolds, the following diagrams of Gysin morphisms are commutative:*

$$\begin{array}{ccc} H_G(M) & \xrightarrow{f_*} & H_G(N) \\ {\scriptstyle \mathrm{Res}_H^G}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_H^G} \\ H_H(M) & \xrightarrow{f_*} & H_H(N) \end{array} \qquad \begin{array}{ccc} H_{G,\mathrm{c}}(M) & \xrightarrow{f_!} & H_{G,\mathrm{c}}(N) \\ {\scriptstyle \mathrm{Res}_H^G}\downarrow & (f \text{ is proper}) & \downarrow{\scriptstyle \mathrm{Res}_H^G} \\ H_{H,\mathrm{c}}(M) & \xrightarrow{f_!} & H_{H,\mathrm{c}}(N) \end{array}$$

*Proof.* For a general map $f : M \to N$ the diagram of induced maps between Borel constructions

$$\begin{tikzcd} \mathbb{E}G \times_H M \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & \mathbb{E}G \times_H N \arrow[d, "\pi"] \\ \mathbb{E}G \times_G M \arrow[r, "f"] & \mathbb{E}G \times_G M \end{tikzcd}$$

is *cartesian* and if we endow $G/H$ with an orientation, the integration along the fibers of $\pi$ enters in the *commutative* diagram of complexes:

$$\begin{tikzcd} \Omega_{H,\mathrm{c}}(N) \arrow[r, "f^*"] \arrow[d, "\int_{G/H}"'] \arrow[dr, phantom, "\oplus"] & \Omega_{H,\mathrm{c}}(M) \arrow[d, "\int_{G/H}"] \\ \Omega_{G,\mathrm{c}}(N) \arrow[r, "f^*"] & \Omega_{G,\mathrm{c}}(M) \end{tikzcd}$$

We may then conclude thanks to 5.4-(a) and that $\int_{G/H}$ is adjoint to $\mathrm{Res}_H^G$.

The case where $f : M \to N$ is proper follows in the same way. □

### 5.6. Explicit Constructions of Equivariant Gysin Morphisms

Although we gave a universal definition for the equivariant Gysin morphism in the last section, it is worth recalling alternative constructions for some particular maps where there exist explicit morphisms of Cartan complexes inducing the Gysin morphism, just as in the nonequivariant case (1.9).

**5.6.1. Constant Map.** Let $M$ be an oriented $G$-manifold. The constant map $c_M : M \to \{\bullet\}$ is $G$-equivariant, $H_G(\{\bullet\}) = H_T$ is free and $I\!D_{G,\{\bullet\}}$ is bijective. Therefore, the cohomological adjunction 5.4.1-($\diamond\diamond$) uniquely determines the Gysin morphism and we have, for all $\beta \in \Omega_{T,\mathrm{c}}(M)$:

$$c_{M*}(\beta) = \Big(\int_{\{\bullet\}} 1 \cup c_{M*}[\beta]\Big) = \int_M \beta .$$

**5.6.2. Equivariant Open Embedding.** Let $M$ be an oriented $G$-manifold. If $U$ is a $G$-invariant open set in $M$, denote by $\iota : U \subseteq M$ the injection and endow $U$ with the induced orientation. One has a natural inclusion of Cartan complexes $\iota_G : \Omega_{G,\mathrm{c}}(U) \to \Omega_{G,\mathrm{c}}(M)$, and the elementary equality

$$\int_U \iota_G^*(\alpha) \wedge \beta = \int_M \alpha \wedge \iota_{G,*}(\beta), \quad \forall \alpha \in \Omega_G(M),\ \forall \beta \in \Omega_{G,\mathrm{c}}(U),$$

shows immediately that the following induced map is the equivariant gysin map:

$$H(\iota_{G*}) : H_{G,\mathrm{c}}(U)[d_U] \longrightarrow H_{G,\mathrm{c}}(M)[d_M].$$

**5.6.3. Equivariant Projection.** Given two oriented $G$-manifolds $M$,$N$, denote by $\mathrm{pr} : M \times N \twoheadrightarrow N$, the projection $(x,y) \mapsto y$.

The map

$$\begin{array}{ccc} \Omega_{\mathrm{c}}(M) \otimes \Omega_{\mathrm{c}}(N) & \xrightarrow{\varphi_*} & \Omega_{\mathrm{c}}(N) \\ \nu \otimes \mu & \xmapsto{\varphi_*} & \big(\int_M \nu\big)\mu \end{array}$$

is a morphism of $H_G$-gm's commuting with $G$-derivations ($G$ is connected), and with $G$-contractions since

$$\begin{aligned}\varphi_*\big(c(X)(\nu\otimes\mu)\big) &= \varphi_*\big(c(X)(\nu)\otimes\mu\big)+(-1)^{\deg\nu}\varphi_*\big(\nu\otimes c(X)(\mu)\big)\\ &= (-1)^{d_M}\varphi_*\big(\nu\otimes c(X)(\mu)\big)=(-1)^{d_M}c(X)\big(\varphi_*(\nu\otimes\mu)\big),\end{aligned}$$

as $\int_M \iota(X)\nu=0$. The morphism $\varphi_*$ may then be naturally extended to a morphism of Cartan complexes $\varphi_{G*}:\big(S(\mathfrak{g})\otimes\Omega_c(M)\otimes\Omega_c(N)\big)^G\to\big(S(\mathfrak{g})\otimes\Omega_c(N)\big)^G$ satisfying

$$\int_{M\times N}\mathrm{pr}^*(\alpha)\wedge\beta=\int_N\alpha\wedge\varphi_{G*}(\beta),\quad \forall\beta\in\Omega_{G,c}(M)\times_{H_G}\Omega_{G,c}(N),\,\forall\alpha\in\Omega_G(N).$$

On the other hand, since the natural map $\Omega_c(M)\otimes\Omega_c(N)\to\Omega_c(M\times N)$ is a quasi-isomorphism (Künneth [BT] p. 50), the induced map

$$(S(\mathfrak{g})\otimes\Omega_c(M)\otimes\Omega_c(N))^G\to(S(\mathfrak{g})\otimes\Omega_c(M\times N))^G=\Omega_{G,c}(M\times N)$$

is also a quasi-isomorphism and one may conclude that

$$\mathrm{pr}_{G,*}:H^*_{G,c}(M\times N)[d_M]\longrightarrow H^*_{G,c}(N)$$

induced by $\varphi_{G,*}$ is the equivariant Gysin map associated with pr.

**5.6.4. Equivariant Fiber Bundle.** Let $(\pi,V,B)$ be an oriented $G$-equivariant fiber bundle with fiber $F$. Integration along fibers (see [BT] **I**§6 pp. 61-63) gives a morphism of complexes $\int_F:\Omega_c(V)\to\Omega_c(B)$ such that if $\psi:V\to V$ is an isomorphism exchanging fibres, then $\int_F\circ\psi^*=\psi^*\circ\int_F$, consequently $\int_F$ is $G$-equivariant. On the other hand, $\int_F$ commutes with the contractions $c(X)$. Indeed, since these are local operators, it suffices (modulo unit partitions if necessary) to verify the claim over a trivializing open subset of $V$, i.e. over $\pi^{-1}(U)$ for $U$ s.t. $\pi^{-1}U\sim F\times U$, where we are in the case of a projection already discussed in 5.6.3.

Now, the map $\int_F:S(\mathfrak{g})\otimes\Omega_c(V)\to S(\mathfrak{g})\otimes\Omega_c(B)$, given by $\int_F P\otimes\omega:=P\otimes\int_F\omega$ restricts naturally to $\int_F:\Omega_{G,c}(V)[d_F]\to\Omega_{G,c}(B)$ as a morphism of Cartan complexes satisfying $\int_V\pi_*\alpha\wedge\beta=\int_B\big(\alpha\wedge\int_F\beta\big)$ since it is so in the nonequivariant case 1.9.5-($*$).

### 5.6.5. Zero Section of an Equivariant Vector Bundle

**The Equivariant Thom Class.** Let $(\pi,V,B)$ be a $G$-equivariant oriented vector bundle. In 5.1.5, we pointed out that the Gysin morphism for compact supports $\pi_*:H_c(V)[d_F]\to H_c(B)$ is an **isomorphism**, so that, in particular:

$$H^i_c(V)=0,\qquad \text{for all } i<d_F. \tag{$\diamond$}$$

**5.6.6. Proposition and definition.** *Assume $G$ is compact and connected.*

a) *There exist homogeneous $G$-equivariant cocycles of total degree $d_F$*

$$\Phi_G=\Phi^{[d_F]}+\Phi^{[d_F-2]}+\Phi^{[d_F-4]}+\cdots$$

*with* $\Phi^{[i]} \in \big(S(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}^{i}(V)\big)^G$ *where* $\Phi^{[d_F]} \in \Omega_{\mathrm{c}}^{d_F}(V)^G$ *represents the Thom class of* $(B,V)$ (see 1.9.7). *Two such cocycles are cohomologous. The map*

$$\begin{array}{ccc} (S(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(B))^G & \xrightarrow{\varphi_{G,*}} & (S(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(V))^G[d_F] \\ \nu & \longmapsto & \pi^*\nu \wedge \tilde{\Phi} \end{array}$$

*is a morphism of Cartan complexes, and the same with '$\Omega$' instead of '$\Omega_{\mathrm{c}}$'.*

b) *The zero section* $\sigma : B \hookrightarrow V$ *of the vector bundle* $\pi : V \to B$ *is a proper* $G$*-equivariant map. The* equivariant Gysin morphisms

$$\begin{cases} \sigma_* : H_{G,\mathrm{c}}(B))[d_B] \to H_{G,\mathrm{c}}(V)[d_V] \\ \sigma_! \ : \ H_G(B))[d_B] \ \to \ H_G(V)[d_V] \end{cases}$$

*are both induced by the morphism of complexes* $\varphi_{G,*}$ *of (a).*

*Proof.* (a) Let $n = d_F$. Since $G$ is connected and compact, there exists $\Phi^{[n]} \in \Omega_{\mathrm{c}}^n(V)^G$ representing the Thom class of $V$. We have

$$d_G(\Phi^{[n]}) = d(\Phi^{[n]}) + c(X)\Phi^{[n]} = c(X)\Phi^{[n]},$$

where $c(X)\Phi^{[n]} \in (S(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}^{n-1}(V))^G$ and $d\big(c(X)\Phi^{[n]}\big) = L(X)\Phi^{[n]} = 0$. But then $c(X)\Phi^{[n]}$ is a coboundary of compact support following ($\diamond$) and, again thanks to the connectedness of $G$, there exists $\Phi^{[n-2]} \in (S(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}^{n-2}(V))^G$ s.t. $c(X)\Phi^{[n]} = d\Phi^{[n-2]}$. The iteration of this procedure, possible because of the vanishing condition ($\diamond$), leads to the $G$-equivariant cocycle $\Phi_G$. The cohomological uniqueness is proved in a similar way. The fact that $\varphi_{G,*}$ is compatible with differentials is obvious as $\Phi_G$ is a cocycle.

(b) By the universal property of the equivariant Gysin morphisms 5.4, it suffices to verify the equality

$$\int_B \sigma^*\alpha \wedge \beta = \int_V \alpha \wedge \varphi_{G,*}(\beta), \qquad \forall \alpha \in \Omega_G(V),\ \forall \beta \in \Omega_{G,\mathrm{c}}(B).$$

Since $\pi^* : H(B) \to H(V)$ is an isomorphism, the same is true in equivariant cohomology following 3.1.4-(d), so that there exists $\alpha' \in \Omega_G(B)$ s.t. $\alpha \sim \pi^*\alpha'$. We are thus lead to verify that

$$\int_B \alpha' \wedge \beta = \int_V \pi^*(\alpha' \wedge \beta) \wedge \Phi_G, \qquad \forall \alpha' \in \Omega_G(B),\ \forall \beta \in \Omega_{G,\mathrm{c}}(B),$$

and this follows from the universal property of the nonequivariant Thom class (1.9.7) that states that one has: $\int_B \omega|_B = \int_V \omega \wedge \Phi, \quad \forall \omega \in H(V)$. □

### 5.7. Exercises

1) Restate and solve exercise 1.11 in the equivariant framework. In particular:
   - If $i : B \hookrightarrow M$ is a *closed equivariant embedding* of oriented $G$-manifolds, denote by $j : U := M \backslash B \hookrightarrow M$ the complementary open embedding, and justify the existence of the following triangles where the left arrows are Gysin morphisms and the right ones are restriction morphisms.
     i) The equivariant compact support cohomology triangle

$$H_{G,\mathrm{c}}(U)[d_U] \xrightarrow{j_*} H_{G,\mathrm{c}}(M)[d_M] \xrightarrow{i^*} H_{G,\mathrm{c}}(B)[d_B] \xrightarrow[{[+1]}]{} . \qquad (\diamond)$$

ii) The equivariant Gysin triangle

$$H_G(B)[d_B] \xrightarrow{i!} H_G(M)[d_M] \xrightarrow{j^*} H_G(U)[d_U] \xrightarrow[{[+1]}]{} \qquad (\diamond\diamond)$$

- In the equivariant version of the Lefschetz fixed point exercise (1.11.2) you will define the *$G$-equivariant Lefschetz class of $f$* by

$$L_G(f) := \mathrm{Gr}(f)^*(\delta_!(1)) \in H_G^{d_M}(M),$$

and its *equivariant Lefschetz number* $\Lambda_{G,f} := \int_M L_G(f)$. Prove that

$$\begin{cases} \mathrm{Res}_1^G L_G(f) = L(f) \in H^{d_M}(M) \\ \Lambda_{G,f} = \Lambda_f \end{cases}$$

and conclude that the *equivariant Lefschetz number* coincides with the nonequivariant one. In particular, if $H_G(M)$ is a torsion module (7.2.1), the Euler characteristic of $M$ is zero.

2) Show that if $f : B \to M$ is $G$-equivariant between oriented $G$-manifolds, the projective limit (see 3.2.5) of nonequivariant Gysin morphisms

$$\varprojlim_n \big(f(n)_* : H_c(B_G(n))[d_B] \to H_c(M_G(n))[d_M]\big)$$

is well defined and coincides with the equivariant Gysin morphism

$$f_* : H_{G,c}(B)[d_B] \to H_{G,c}(M)[d_M].$$

And *mutatis mutandis* for the proper case.

3) i) Show that in 5.6.5, the restriction of the equivariant Thom class to the complement of the zero section, is an equivariant coboundary. (*Hint: remark that* $[\Phi_G] = \sigma_!(1)$ *and use* (1)-($\diamond\diamond$)).

ii) (**) Show that the multiplication by $[\Phi_G]$ defines a map from $H_G(B)$ to the equivariant cohomology of $V$ with supports in $B$:

$$(\_) \wedge [\Phi_G] : H_G(B) \to H_{G,B}(V).$$

Show next that this map is an isomorphism. (*Hint: use the spectral sequence of exercise* 3.2.8-(c) *to reduce to the nonequivariant case*).

iii) (**) Extend (ii) to the case of a closed embedding $B \hookrightarrow M$ of oriented manifolds. (*Hint: show that $B$ may be seen as the zero section of a tubular $G$-stable neighborhood $B_\epsilon$ and use (and justify) the fact that the restriction map $H_{G,B}(M) = H_{G,B}(B_\epsilon)$ is an isomorphism*).

# 6. Equivariant Euler Classes

The reference for this section is Atiyah-Bott's paper [AB], notably §2 and §3.

## 6.1. The Nonequivariant Euler Class

Given a pair of oriented manifolds $(N,M)$ with $N \subseteq M$, we denote by $N_\epsilon$ a tubular neighborhood of $N$ in $M$. As the inclusion $N \subseteq N_\epsilon$ has the same nature as the inclusion of the zero section of a vector bundle $\sigma : B \subseteq V$ (1.9.7), we may define the *Thom class* $[\Phi(N,M)]$ *of the pair* $(N,M)$ following the same principle, that is, by means of the Gysin morphism associated with the closed embedding $i : N \subseteq M$. We thus set :

$$\boxed{[\Phi(N,M)] := i_!(1) \in H^{d_M - d_N}(M)}$$

**6.1.1. Definition.** The *Euler class* $\mathrm{Eu}\big(N,M\big)$ *of the pair* $(N,M)$ is the restriction of the Thom class to $H(N)$ ([21]), i.e. :

$$\mathrm{Eu}\big(N,M\big) := i^* i_!(1) = [\Phi(N,M)]|_N \in H^{d_M - d_N}(N). \tag{$\diamond$}$$

## 6.2. $G$-Equivariant Euler Class

The generalization of the concept of Euler class to the equivariant framework is straightforward thanks to the equivariant Gysin morphism formalism: Given a pair of oriented $G$-manifolds $(N,M)$ with $N \subseteq M$, we denote by $i_G : N \subseteq M$ the inclusion map and define the *$G$-equivariant Euler class* $\mathrm{Eu}_G\big(N,M\big)$ *of the pair* $(N,M)$ by the same formula $(\diamond)$:

$$\boxed{\mathrm{Eu}_G\big(N,M\big) := i_G^* i_{G!}(1) = [\Phi_G(N,M)]|_N \in H_G^{d_M - d_N}(N)}.$$

where $i_{G!} : H_G(N)[d_N] \to H_G(M)[d_M]$ is now the equivariant Gysin morphism.

**6.2.1. Exercise.** Given oriented $G$-manifolds $L \subseteq N \subseteq M$, prove the folowing formula for nested equivariant Euler classes

$$\boxed{\mathrm{Eu}_G\big(L,M\big) = \mathrm{Eu}_G\big(L,N\big) \cup \mathrm{Eu}_G\big(N,M\big)|_L}$$

*Hint: Use the projection formula for Gysin morphisms.*

### 6.2.2. $G$-Equivariant Euler Class of Discrete Fixed Point Sets

In the sequel, we denote by $M^G$ the subspace of $G$-fixed points of $M$.

When $N$ is a discrete subspace of $M^G$, one has

$$\mathrm{Eu}_G\big(N,M\big) \in H_G^{d_M}(N) = \prod\nolimits_{b \in N} S^{d_M}(\mathfrak{g})^G,$$

and $\mathrm{Eu}_G\big(N,M\big)$ is simply the family of invariant polynomials

$$\mathrm{Eu}_G\big(N,M\big) = \big\{\mathrm{Eu}_G\big(b,M\big) \in S^{d_M}(\mathfrak{g})^G\big\}_{b \in N}.$$

[21] *Cf.* formula (2.19), p. 5, in *loc.cit.*

**6.2.3. Proposition.** *If $N$ is a finite subset of $M^G$, one has*

$$\sum\nolimits_{b\in N} \mathrm{Eu}_G\big(b,M\big) = \int_M \Phi_G(N,M)\cup\Phi_G(N,M) \quad \textit{and} \quad |N| = \int_M \Phi_G(N,M).$$

*Proof.* The constant function $\mathbb{1}_N$ and, *a fortiori*, the Thom class $\Phi_G(N,M)$, are both of compact supports. The equalities then immediately follow from the adjoint property of the Gysin morphism $i_* : H_{G,\mathrm{c}}(N) \to H_{G,\mathrm{c}}(M)$ which gives:

$$\sum\nolimits_{b\in N} \alpha|_b = \int_M i_*(\mathbb{1}_N)\cup\alpha, \quad \forall\alpha\in H_G(M). \qquad \square$$

### 6.3. *T*-Equivariant Euler Classes of a Fixed Point

Let $T$ be the maximal torus of the compact connected Lie group $G$ and denote by $T' := N_G(T)$ the *normalizer* of $T$ in $G$. We have $T \subseteq T' \subseteq G$ and if we choose $\mathbb{E}G$ as *universal fiber bundle* for any of these groups, we can easily compare the corresponding Borel constructions for a given $G$-manifold $M$. In this way we obtain a natural commutative diagram of locally trivial fibrations:

$$\begin{tikzcd} M_T := \mathbb{E}G\times_T M \arrow[r, two heads, "p"] \arrow[d, two heads] & M_{T'} := \mathbb{E}G\times_{T'} M \arrow[r, two heads, "q"] \arrow[d, two heads] & M_G := \mathbb{E}G\times_G M \arrow[d, two heads] \\ \mathbb{B}T \arrow[r, two heads, "p"] & \mathbb{B}T' \arrow[r, two heads, "q"] & \mathbb{B}G \end{tikzcd} \qquad (\diamond)$$

– The *Weyl group of* $(G,T)$, i.e. the finite group $W := N_G(T)/T$, acts on the *right* of $M_T$ and $p$ is the orbit map for this action. In particular, the map $p^* : H_{T'}(M) \to H_T(M)^W$ is an isomorphism.

– The fibers of $q$ are isomorphic to $G/T'$ which is *acyclic in rational cohomology*. Indeed, this space is the orbit space of $G/T$ for the right action of $W$ and we know from an old result of Leray ([22]) that, under this action, $H(G/T)$ is the regular representation of $W$. In particular $H(G/T') = H(G/T)^W = \mathbb{k}$, which implies that

$$q^* : H_G(M) \to H_{T'}(M)$$

is an isomorphism.

This is a consequence of the general fact that if $q : X \to Y$ is a locally trivial fibration with acyclic fiber $F$ between manifolds, then $q^* : H(Y) \to H(X)$ is an isomorphism. Indeed, if $\mathscr{U} = \{U\}$ is an good cover of $Y$ (*cf.* ([3])) such that $q : f^{-1}(U) \to U$ is a trivial fibration for all $U \in \mathscr{U}$, then $f^{-1}(U) = U\times F$ and the cover $f^{-1}(\mathscr{U}) := \{f^{-1}(U)\}$ will be also good for $X$. In that case, $q^*$ establishes an *isomorphism* of Čech cohomologies $q^* : \check{H}(\mathscr{U};Y) \to \check{H}(f^{-1}\mathscr{U};X)$ which are known to be canonically isomorphic to de Rham cohomologies. By this result, the maps $q^* : H(\mathbb{E}G(m)\times_G M) \to H(\mathbb{E}G(m)\times_{T'} M)$ are bijective for all finite dimensional approximation $\mathbb{E}G(m)$ of $\mathbb{E}G$, which suffices to our

[22] The statement appears as the Lemma 27.1 in the Ph.D. thesis of A. Borel, defended at La Sorbonne (with Leray as president) in 1952, ([$B_3$], lemme 27.1, p. 193). Borel attributes the result to J. Leray ([L]).

purposes as equivariant cohomology is the projective limit of the cohomologies of these approximations (3.2.5). ([23])

– Summing up, we have the following two canonical isomorphisms

$$\boxed{H_G(M) \xrightarrow{q^*} H_{T'}(M) \xrightarrow{p^*} H_T(M)^W} \tag{‡}$$

– When $M = \{\bullet\}$, we obtain a commutative diagram of Chern-Weil homomorphisms

$$\begin{tikzcd} S(\mathfrak{g})^G \arrow[r, dashed, "\mathrm{Chv}"] \arrow[d, dashed, "\simeq"] & S(\mathfrak{t})^W \arrow[r, "\subseteq"] \arrow[d, dashed, "\simeq"] & S(\mathfrak{t}) \arrow[d, dashed, "\simeq"] \\ H_G \arrow[r, "q^*", "\simeq"'] & H_{T'} \arrow[r, tail, "p^*"] & H_T \end{tikzcd} \tag{‡‡}$$

where $\mathrm{Chv} : S(\mathfrak{g})^G \to S(\mathfrak{t})^W$ is the map that associates a symmetric polynomial function on $\mathfrak{g}$ with its restriction to the subspace $\mathfrak{t}$. The diagram already shows that Chv is an isomorphism, a claim known as the *Chevalley isomorphism* following the celebrated, much more general, *Chevalley's restriction theorem*.

At this point it is worth noting that for each $b \in M^G$ the group $G$ acts naturally on the tangent space $T_b(M)$ through a *linear representation*. Now, if we endow $M$ with a $G$-invariant riemannian metric, the exponential map $\exp : T_b(M) \to M$ is a $G$-equivariant diffeomorphism between $T_b(M)$ and an open neighborhood of $b$ in $M$, so that the computation of equivariant Euler classes on fixed points may be greatly simplified by linearizing the data. The following proposition deals with the linear case.

**6.3.1. Proposition.** *Let $V$ be a linear representation of a compact connected Lie group $G$ with maximal torus $T$.*

a) *The equivariant Euler class $\mathrm{Eu}_T\big(0,V\big)$ belongs to $S(\mathfrak{t})^W$ and the Chevalley isomorphism $\mathrm{Chv} : S(\mathfrak{g})^G \to S(\mathfrak{t})^W$ exchanges $\mathrm{Eu}_G\big(0,V\big)$ and $\mathrm{Eu}_T\big(0,V\big)$.*

b) *If $V := V_1 \oplus V_2$ as $G$-module, then $\mathrm{Eu}_G\big(0,V\big) = \mathrm{Eu}_G\big(0,V_1\big)\mathrm{Eu}_G\big(0,V_2\big)$.*

c) *Denote by $\mathbb{C}(\alpha)$ the complex vector space $\mathbb{C}$ endowed with the representation of $T$ corresponding to the (nonzero)* weight *$\alpha \in \mathfrak{t}^\vee$, i.e. $\exp(tx)(z) = e^{it\alpha(x)}z$, for all $t \in \mathbb{R}$ and $z \in \mathbb{C}$. If the decomposition of $V$ in irreducible representations of $T$ is $V = \mathbb{R}^{\mu_0} \oplus \bigoplus_\alpha \mathbb{C}(\alpha)^{\mu(\alpha)}$, then*

$$\mathrm{Eu}_T\big(0,V\big) = 0^{\mu_0} \prod\nolimits_\alpha \alpha^{\mu(\alpha)}.$$

d) *$\mathrm{Eu}_G\big(0,V\big) \neq 0$ if and only if $V^T = \{0\}$.*

[23] In [AB], p. 4, the interested reader will find partial indications to a different justification, that seems to rely on [Gt].

*Proof.* (a) After the natural isomorphism of functors $H_G(\_) \simeq H_{T'}(\_)$ of (‡), it suffices to justify the commutativity of the following diagram:

$$\begin{tikzcd}
H_{T'}(0) \arrow[r, "i_!"] \arrow[d, "p^*"] \arrow[dr, phantom, "(\mathrm{I})"] & H_{T'}(V) \arrow[r, "i^*"] \arrow[d, "p^*"] \arrow[dr, phantom, "(\mathrm{II})"] & H_{T'}(0) \arrow[d, "p^*"] \\
H_T(0) \arrow[r, "i_!"'] & H_T(V) \arrow[r, "i^*"'] & H_T(0)
\end{tikzcd}$$

The commutativity of the subdiagram (II) is obvious. For (I) we check its dual, the diagram

$$\begin{tikzcd}
H_{T,\mathrm{c}}(V) \arrow[r, "i^*"] \arrow[d, "p_!"] \arrow[dr, phantom, "(\mathrm{I}^\vee)"] & H_T(0) \arrow[d, "p_!"] \\
H_{T',\mathrm{c}}(V) \arrow[r, "i^*"'] & H_{T'}(0)
\end{tikzcd}$$

where $p_! = \int_W$, which is also clearly commutative.

(b,c,d) left to the reader. *Hint for* (c). Following (b), it suffices to show that $\mathrm{Eu}_T\big(0,\mathbb{C}(\alpha)\big) = \alpha$. Taking polar coordinates $(\rho,\theta) \in \mathbb{R}_+ \times [0,2\pi]$ in $\mathbb{C}$, the nonequivariant Thom class $\Phi(0,\mathbb{C})$ is of the form

$$\Phi^{[2]} = \lambda(\rho)\, \rho d\rho \wedge d\theta,$$

where $\lambda : \mathbb{R} \to \mathbb{R}$ is a nonnegative differential function with compact support equal to 1 in a neighborhood of 0 and such that $\int_0^\infty \lambda(\rho)\, \rho d\rho = 1/2\pi$. As it is clear that $\Phi^{[2]}$ is invariant under the action of the unit circle action, it is $T$-invariant. We can thus use this differential 2-form to construct an equivariant Thom class following the procedure described in the proof of 5.6.6-(a). We have

$$(d_{\mathrm{t}}\Phi^{[2]})(X) = c(X)\, \lambda(\rho)\, \rho d\rho \wedge d\theta = -2\pi\alpha(X)\, \lambda(\rho)\, \rho d\rho,$$

and $\Phi^{[0]}(X)$ is necessarily equal to

$$\Phi^{[0]}(\rho,\theta)(X) = -2\pi\alpha(X)\Big(\int_0^\rho \lambda(\rho)\, \rho d\rho - \int_0^{+\infty} \lambda(\rho)\, \rho d\rho\Big),$$

since it must be of compact support. In this way we obtain

$$\mathrm{Eu}_T\big(0,\mathbb{C}(\alpha)\big) = \Phi_T(0,\mathbb{C}(\alpha))|_0 = \Phi^{[0]}(0)(X) = \alpha(X). \qquad \square$$

**6.3.2. Exercise.** If $G$ is the *special orthogonal group* $\mathrm{SO}(3)$ of the euclidean space $\mathbb{R}^3$, show that $\mathrm{Eu}_G\big(0,\mathbb{R}^3\big) = 0$. Conclude that isolated $G$-fixed points may have a null equivariant Euler class when $G$ is nonabelian, contrary to the abelian case.

# 7. Localizations

## 7.1. The Localization Functor

Denote by $Q_G$ the field of fractions of $\Omega_G = H_G$. The *localization functor* is the base change functor ([24])

$$Q_G \otimes_{\Omega_G} (\_) : \mathrm{GM}(\Omega_G) \rightsquigarrow \mathrm{Vec}(Q_G)\,.$$

General results of commutative algebra state for any $\Omega_G$-module $N$, the $H_G$-module $Q_G \otimes_{H_G} N$ is flat and injective (as in appendix §9). The localization functor is exact and when applied to Cartan complexes, we obtain the *localized Cartan complexes*

$$\begin{cases} Q\Omega_G(M) := \big(Q_G \otimes_{\Omega_G} \Omega_G(M), \mathrm{id} \otimes d_G\big) \\ Q\Omega_{G,\mathrm{c}}(M) := \big(Q_G \otimes_{\Omega_G} \Omega_{G,\mathrm{c}}(M), \mathrm{id} \otimes d_G\big) \end{cases}$$

whose cohomologies, the *localized equivariant cohomologies*, respectively denoted $QH_G(M)$ and $QH_{G,\mathrm{c}}(M)$, satisfy :

$$QH_G(M) = Q_G \otimes_{H_G} H_G(M) \quad \text{and} \quad QH_{G,\mathrm{c}}(M) = Q_G \otimes_{H_G} H_{G,\mathrm{c}}(M)\,.$$

**7.1.1. Localized Equivariant Poincaré Duality.** The localized equivariant cohomology is very close to the non equivariant cohomology in that the Poincaré duality pairings are perfect. The following, analog of 4.5.1, simply results from the fact that $Q_G$ is a flat and injective $\Omega_G$-module (details are left to the reader).

**7.1.2. Theorem.** *Let $G$ be a compact connected Lie group, and $M$ an oriented $G$-manifold of dimension $d_M$.*

a) *The morphism of (nongraded) complexes*

$$I\!D_{G,M} : Q\Omega_G(M)[d_M] \to \mathit{Hom}^{\bullet}_{Q_G}\big(Q\Omega_{G,\mathrm{c}}(M), Q_G\big)$$

*induces an isomorphism*

$$\boxed{\mathscr{D}_{G,M} : QH_G(M)[d_M] \to \mathit{Hom}^{\bullet}_{Q_G}\big(QH_{G,\mathrm{c}}(M), Q_G\big)}$$

b) *Moreover, if $M$ is of finite type, the morphism of complexes*

$$I\!D'_{G,M} : Q\Omega_{G,\mathrm{c}}(M)[d_M] \to \mathit{Hom}^{\bullet}_{Q_G}\big(Q\Omega_G(M), Q_G\big)$$

*induces an isomorphism*

$$\boxed{\mathscr{D}'_{G,M} : QH_{G,\mathrm{c}}(M)[d_M] \to \mathit{Hom}^{\bullet}_{Q_G}\big(QH_G(M), Q_G\big)}$$

**7.1.3. Exercise.** Let $M$ be of finite type. Prove that the torsion-freeness (4.5.2) of $H_G(M)$ (resp. $H_{G,\mathrm{c}}(M)$) is a necessary and sufficient condition for

$$\mathscr{D}_{G,M} : H_G(M)[d_M] \to \mathrm{Hom}^{\bullet}_{H_G}\big(H_{G,\mathrm{c}}(M), H_G\big)$$

[24] Note that we loose grading in considering this kind of localization. From this point of view, it would have been more clever to tensor by the ring $L_G := S^{-1}H_G$, where $S$ denotes the multiplicative system of nonzero homogeneous elements of $H_G$. As appendix 9 explains, if $N$ is an $H_G$-gm, the $H_G$-module $L_G \otimes_{H_G} (\_)$ is graded, flat and injective, which is what we really need about localization.

(resp. $\mathscr{D}'_{G,M}$) to be injective. Discuss the case where $M$ is not of finite type.

*Hint: Let $M$ be $H_G$-gm. Show that the canonical map $M \to Q_G \otimes_{H_G} M$ is injective if and only if $M$ is torsion-free. Show that if $M$ is also of finite type the natural map $\mathrm{Hom}^\bullet_{H_G}(M,H_G) \to \mathrm{Hom}^\bullet_{H_G}(M,Q_G)$ induces an isomorphism $Q_G \otimes_{H_G} \mathrm{Hom}^\bullet_{H_G}(M,H_G) \simeq \mathrm{Hom}^\bullet_{Q_G}(Q_G \otimes_{H_G} M,Q_G)$. Apply 7.1.2.*

**7.1.4. Localized Equivariant Gysin Morphisms.** As a consequence of theorem 7.1.2, if $f : M \to N$ is a map between oriented $G$-manifolds, the localized Gysin morphisms

$$\begin{cases} f_* : QH_{G,\mathrm{c}}(M) \to QH_{G,\mathrm{c}}(N) \\ f_! : QH_G(M) \to QH_G(N), \quad \text{if } f \text{ is proper,} \end{cases}$$

are *catacterized*, as in the nonequivariant framework, by the adjoint equalities,

$$\begin{cases} \displaystyle\int_M f^*[\beta] \cup [\alpha] = \int_N [\beta] \cup f_*[\alpha] \\ \displaystyle\int_M f^*[\beta] \cup [\alpha] = \int_N [\beta] \cup f_![\alpha], \quad \text{if } f \text{ is proper.} \end{cases}$$

## 7.2. Torsions in Equivariant Cohomology Modules

**7.2.1. Torsions.** The *annihilator of an element* $v$ of an $H_G$-gm $V$, is the ideal

$$\mathrm{Ann}(v) := \{P \in H_G \mid P \cdot v = 0\}.$$

One says that $v$ is a *torsion element* if $\mathrm{Ann}(v) \neq 0$, otherwise $v$ is a *torsion-free element*. The $H_G$-gm $V$ is called a *torsion module* if all its elements are torsion elements, it is called a *torsion-free module* if zero is its the only torsion element, otherwise, it is called a *nontorsion module*.

**7.2.2. Exercise.** Given an $H_G$-gm $V$, let $\tau(V)$ be the subset of its torsion elements. Show that

1) $\tau(V)$ is a torsion module and the quotient $\varphi(V) := V/\tau(V)$ is torsion-free. The natural map: $Q_G \otimes_{H_G} V \to Q_G \otimes_{H_G} \varphi(V)$ is an isomorphism.
2) $Q_G \otimes_{H_G} V = 0$ if and only if $V$ is torsion.
3) $\mathit{Homgr}_{H_G}(V,Q_G) = 0$ if and only if $V$ is torsion.
4) An inductive limit of torsion modules is a torsion module.
5) A projective limit of torsion modules may be a nontorsion module.

**7.2.3. Exercise.** The *annihilator of an $H_G$-module* is the ideal

$$\mathrm{Ann}(V) := \{P \in H_G \mid P \cdot V = 0\} = \bigcap\nolimits_{v\in V} \mathrm{Ann}(v).$$

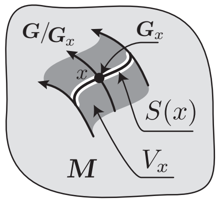
1) Show that if $\mathrm{Ann}(V) \neq 0$, then $V$ is torsion, but the converse may fail. ([25])
2) Show that if $V$ is an $H_T$-algebra with unit element, then $\mathrm{Ann}(V) = \mathrm{Ann}(1)$.
3) Let $\{U_1 \subseteq U_2 \subseteq \cdots U_n \cdots\}$ be an increasing family of $G$-stable open subsets of a $G$-manifold $M$ such that $M = \bigcup_n U_n$. Suppose that $H_G(U_n)$ and

[25] *Hint: For $P \in H_G$, let $W(P) := H_G/(H_G \cdot P)$ and take $V := \bigoplus_{P\in H_G} W(P)$.*

$H_{G,c}(U_n)$ are torsion for all $n \in \mathbb{N}$. Show that $H_{G,c}(M)$ is torsion, whereas $H_G(M)$ may fail to be torsion.

4) In (3) show that $\{\mathrm{Ann}(H_G(U_n))\}_n$ is a decreasing sequence of ideals and that
$$\mathrm{Ann}(H_G(M)) = \bigcap\nolimits_{n\in\mathbb{N}} \mathrm{Ann}(H_{G,c}(U_n)).$$
In particular, if the set $\{\mathrm{Ann}(H_G(U_n))\}$ is finite, then $H_G(M)$ is torsion.

**7.2.4. The slice theorem.** Given a $G$-manifold $M$, the *slice theorem* ([26]) claims that, through every point $x \in M$ a submanifold $S(x)$ passes, *a slice through* $x$, such that the map $G/G_x$
$$G \times_{G_x} S(x) \to M\,, \quad [(g,x)] \mapsto g \cdot x\,,$$
where $G_x$ denotes the *isotropie group of* $x$, is a diffeomorphism onto a $G$-stable neighborhood $V_x$ of $x$. Then
$$H_G(V_x) = H(\mathbb{E}G \times_G G \times_{G_x} S(x)) = H_{G_x}(S(x))\,,$$
as well as $H_{G,c}(V_x) = H_{G_x,c}(V_x)$ (exercise). As a consequence, the $H_G$-module structures of $H_G(V_x)$ and $H_{G,c}(V_x)$ factorise through the natural ring homomorphism $\rho_x : H_G \to H_{G_x}$.




**7.2.5. Proposition.** *Let $T$ be a torus. For every point $x$ in a $T$-manifold $M$, the following equivalences hold.*

a) *$\rho_x : H_T \to H_{T_x}$ is injective if and only if $x \in M^T$. The $H_T$-modules $H_G(V_x)$ and $H_{G,c}(V_x)$ are torsion if and only if $x \notin M^T$*

b) *If $x \in M^T$, then $\mathrm{Eu}_T\big(x,M\big) \neq 0$ if and only if $x$ is an isolated point of $M^T$.*

*Proof.* (a) If $T_x \neq T$, there exist closed subtorus $H \subseteq T$ such that $T = H \times T_x$ and $\dim(H) > 0$, in which case $\ker(\rho_x) = H_H^+ \otimes H_{T_x} \neq 0$. (b) is 6.3.1-(d). $\square$

**7.2.6. Remark.** The interesting point of this proposition is that it faithfully translates topological properties of a point in a $T$-manifold into algebraic properties of $H_T$-modules, opening the way to the algebraic study of the topology of $T$-spaces. When $G$ is no longer abelian, both claims may fail. For (a), if $T$ is a maximal torus for $G$ and $M = G/T$, the isotropie group of $x = g[T] \in M$ is the maximal torus $G_x = gTg^{-1}$, and $\rho_x$ is the inclusion $H_G = (H_T)^W \subseteq H_{G_x}$. Thus, $\rho_x$ is injective although $x$ is not a $G$-fixed point. Exercise 6.3.2 gives a counterexample for (b).

**7.2.7. Orbit Type of $T$-Manifolds.** The torsions of the $H_T$-modules $H_{T,c}(M)$ and $H_T(M)$ play a central role in the *fixed point theorem*.

When $M^T = \emptyset$, the slice theorem and 7.2.5-(a) show that $M$ may be covered by a family of $T$-stable open subspaces $V_x$ where $H_T(V_x)$ is killed by the elements of the nontrivial kernel $\rho_x : H_T \to H_{T_x}$. But then any finite union of those subspaces will also have torsion equivariant cohomology thanks to Mayer-Vietoris sequences, and, if $M$ is compact, we can already say that $H_T(M)$ is torsion. When $M$ is not compact we may not be able to conclude the same (*cf.* 7.2.3-(3)) unless we have some kind of finiteness condition on the kernels of

[26] See Hsiang [H] §I.3, p. 11.

$\rho_x$. As shown in exercise 7.2.3-(4), such condition may be the finiteness of the set those kernels, or, what amounts to the same, the set of the isotropy groups $\mathcal{O}_T(M) := \{T_x \mid x \in X\}$ which is called the *orbit type of the $T$-space $M$* ([27]).

**Definition.** A $T$-manifold $M$ is said *of finite orbit type* if $\mathcal{O}_T(M)$ is finite.

**7.2.8. Exercise.** Show that a $T$-manifold $M$ is always locally of finite orbit type. In particular, if $M$ is compact, it is of finite orbit type.

*Hint. Use the slice theorem. If $x \notin M^T$, show that the slice $S(x)$ is a strict submanifold of $M$ stable under $G_x$ and that $\mathcal{O}_G(G \cdot S(x)) = \mathcal{O}_{G_x}(S(x))$, then conclude by induction on $\dim(M)$. Otherwise, if $x \in M^T$, linearize the action as in 6.3.1 and conclude showing that there is a one-to-one correspondence between isotropy groups in the $T$-space $T_xM$ and subsets of the set of nonzero weights of the linear representation of $T$ on $T_xM$.*

**7.2.9. Proposition.** *If $M^T = \emptyset$ and $M$ is of finite orbit type, then*

$$H_{T,\mathrm{c}}(M) \otimes_{H_T} Q_T = H_T(M) \otimes_{H_T} Q_T = 0.$$

*Proof. – Torsion of $H_{T,\mathrm{c}}(M)$.* Let $(\mathscr{U},\subseteq)$ be the set of $G$-stable open subspaces $U \subseteq M$, such that $H_{T,\mathrm{c}}(U)$ is torsion, partially ordered by set inclusion. The set $\mathscr{U}$ is non empty as it contains every slice neighborhood $V_x$ (7.2.4) and it is an inductive poset by exercise 7.2.3-(3), so that Zorn lemma can be applied. Let $U$ be a maximal element in $\mathscr{U}$. For any $y \in M$, let $V_y$ be a slice neighborhood of $y$. By the exactness of the Mayer-Vietoris sequence for compact supports:

$$\cdots \to H^0_{G,\mathrm{c}}(U\cap V_y) \to H^0_{G,\mathrm{c}}(U)\oplus H^0_{G,\mathrm{c}}(V_y) \to H^0_{G,\mathrm{c}}(U\cup V_y) \to H^0_{G,\mathrm{c}}(U\cap V_y)[1] \to,$$

we easily conclude that $H^0_{G,\mathrm{c}}(U\cup V_y)$ is torsion. Then $U \supseteq V_y$, by the maximality of $U$, hence $U = M$.

*– Torsion of $H_T(M)$.* We cannot use the same argument as in the compact support case because a projective limit of torsion modules is not necessarily torsion. The finiteness assumption on the set of orbit types will now be crucial.

Let $I$ be the intersection of all the ideals $\ker(\rho_x : H_T \to H_{T_x})$ for $x \in M$. The finiteness of the orbit type of $M$ ensures that $I \neq 0$. Let $(\mathscr{U},\subseteq)$ be the set of $G$-stable open subspaces $U \subseteq M$, such that $I \subseteq \mathrm{Ann}(H_T(U))$, partially ordered by set inclusion. The set $\mathscr{U}$ is non empty as it contains every slice neighborhood $V_x$ (7.2.4) and it is an inductive poset by exercise 7.2.3-(4), so that Zorn lemma can be applied. Let $U$ be a maximal element in $\mathscr{U}$. For any $y \in M$, let $V_y$ be a slice neighborhood of $y$. Thanks to the exactness of the first terms of the Mayer-Vietoris sequence: $0 \to H^0_G(U \cup V_y) \to H^0_G(U) \oplus H^0_G(V_y) \to H^0_G(U \cap V_y) \to$, we easily see that $1 \in H^0_G(U \cup V_y)$ is killed by $I$. Then $I \subseteq \mathrm{Ann}(H_G(U \cup V_y))$ by 7.2.3-(2) and $U \supseteq V_y$, by the maximality of $U$, hence $U = M$. □

## 7.3. Localization Theorems

Given a $T$-manifold $M$ and a *closed* subgroup $H \subseteq T$, the fixed point set $M^H := \{x \in M \mid h \cdot x = x \ \forall h \in H\}$ is a submanifold whose connected com-

[27] See [H] chap IV §2, p. 54, for the general definition nottably for non abelian groups.

ponents (not necessarily of equal dimensions) are stable under the action of $T$, and, in addition, orientable if $M$ is so ([28]).

**Terminology.** An homomorphism of $H_T$-modules $\alpha : L \to L'$ will be called an *isomorphism modulo torsion* if its kernel and cokernel are both torsion $H_T$-modules, i.e. if the induced homomorphism of $Q_T$-modules

$$\alpha \otimes_{H_T} \mathrm{id} : L \otimes_{H_T} Q_T \to L' \otimes_{H_T} Q_T$$

is an isomorphism.

**7.3.1. Proposition.** *Let $M$ be an oriented $T$-manifold of finite orbit type. For any $H$ closed subgroup of $T$, denote by $\iota_H : M^H \hookrightarrow M$ the set inclusion. The following morphisms of $H_T$-gm ([29]) are isomorphisms modulo torsion.*

$$\textit{Gysin morphisms}\begin{cases}\iota_{H!} : \ H_T(M^H)[d_{M^H}] \ \to \ H_T(M)[d_M] \\ \iota_{H*} : H_{T,\mathrm{c}}(M^H)[d_{M^H}] \to H_{T,\mathrm{c}}(M)[d_M]\end{cases}$$

$$\textit{Restriction morphisms}\begin{cases}\iota_H^* : H_{T,\mathrm{c}}(M) \to H_{T,\mathrm{c}}(M^H) \\ \iota_H^* : \ H_T(M) \ \to \ H_T(M^H)\end{cases}$$

*Proof.* The kernel and cokernel of the restriction $\iota_H^* : H_{T,\mathrm{c}}(M) \to H_{T,\mathrm{c}}(M^H)$ lay within $H_{T,\mathrm{c}}(U)$, where $U := M \backslash M^H$. Now, as the isotropy groups of the points of $U$ are *strict* subgroups of $T$, there are no $T$-fixed points, i.e. $U^T = \emptyset$, and we can conclude that $H_{T,\mathrm{c}}(U)$ is an $H_T$-torsion module by 7.2.9. In particular, any submodule of $H_{T,\mathrm{c}}(U)$, viz. the kernel and the cokernel of $\iota_H^*$, is a torsion $H_T$-module. By duality the same is true for $\iota_{H,!} : H_T(M^H) \to H_T(M)$.

The other restriction $\iota_H^* : H_T(M) \to H_T(M^H)$ is a little more tricky as its kernel and cokernel lay within $H_{T,U}(X)$ which we have not yet proved is an $H_T$-torsion module. For that, recall that since one has short exact sequences of local section functors over open subspaces

$$0 \to \mathit{\Gamma}_{U_1 \cap U_1}(\_) \longrightarrow \mathit{\Gamma}_{U_1}(\_) \oplus \mathit{\Gamma}_{U_2}(\_) \longrightarrow \mathit{\Gamma}_{U_1 \cup U_2}(\_) \to 0$$

where $\mathit{\Gamma}_U(\_)$ denotes the kernel of the restriction $\mathit{\Gamma}(M, \_) \to \mathit{\Gamma}(M \backslash U, \_)$, one may follow a Mayer-Vietoris procedure to approach $H_{T,U}(X)$ by successively adding slice open sets $V_x \subseteq U$ (7.2.4). In this way, to show that $H_{T,U}(M)$ is a torsion module, it suffices to show that each $H_{T,V_x}(M)$ is so. Now, this $H_T$-module occurs in the exact triangle

$$H_{T,V_x}(M) \longrightarrow H(M) \to H_T(M \backslash V_x) \to$$

where $M \backslash V_x$ is $T$-equivariantly homotopic to $M \backslash T{\cdot}x$ since the slice $S(x)$ is a submanifold of $M$, therefore $H_{T,V_x}(M) \simeq H_{T,T\cdot x}(M) \simeq H_T(T{\cdot}x) = H_{T_x}$, which proves that $H_{T,V_x}(M)$ is a torsion $H_T$-module. $\square$

[28] We recall the reason: under the action of $H$, the tangent spaces $T_x(M)$ for $x \in M^H$ split as the direct sum of $T_x(M^H)$ and a sum of $H$-irreducible two dimensional representations $\mathbb{C}(\alpha)$ (*cf.* 6.3.1-(c)), canonically oriented by their character. Therefore, the orientation of $T_x(M^H)$ determines that of $T_x(M)$ and vice versa.

[29] As the submanifold $M^H$ need not be connected nor equidimensional the shift indication in a notation as $H_T(M^H)[d_{M^H}]$ must be understood component-wise.

**7.3.2. Theorem.** *Let $M$ be a $T$-oriented manifold of finite orbit type such that $M^T$ is a discrete subspace of $M$. Then*

a) *For all $\mu \in H_{T,\mathrm{c}}(M)$ the following* "localization formula" *is satisfied:*

$$\int_M \mu = \sum_{x \in M^T} \frac{\mu|_x}{\mathrm{Eu}_T\left(x, M\right)}.$$

b) *If $M$ is compact of positive dimension*

$$0 = \sum_{x \in M^T} \frac{1}{\mathrm{Eu}_T\left(x, M\right)}.$$

*Proof.* (a) From 7.3.1, the morphism $i_{T,*} : H_{T,\mathrm{c}}(M^T) \to H_{T,\mathrm{c}}(M)$ is an isomorphism modulo torsion, so that it suffices to prove the localization formula for the equivariant Thom classes $\Phi_T(x, M)$ for all $x \in M^T$. But we have already shown that $\int_M \Phi_T(x, M) = 1$ (6.2.3) and that $\Phi_T(x, M)|_x = \mathrm{Eu}_T\left(x, M\right)$ by definition.
(b) Apply the localization formula to $1 \in H_T^0(M)$. □

## 8. Changing the coefficients field

As we have already mentioned, the constructions and results described in these notes are available with coefficients on any field $\Bbbk$ of arbitrary characteristic ([30]). In this section, we give a quick outline, following the principles of the Grothendieck-Verdier duality, of a way to prove equivariant Poincaré duality with coefficients on a general field $\Bbbk$. We then end by introducing the corresponding Gysin morphisms.

Besides the general reference books by Kashiwara-Schapira [$\mathrm{KS}_1$, $\mathrm{KS}_2$], on sheaf theory, derived categories and Grothendieck-Verdier duality, we recommend the recent articles of Allday-Franz-Puppe [$\mathrm{AFP}_1$, $\mathrm{AFP}_2$].

### 8.1. Preliminaries

By *space* we mean a Hausdorff, second-countable and locally contractible topological space.

As we recalled in 3.2, the Borel construction is a functor from the category of $G$-spaces to the category of spaces based on the classifying space $\mathbb{B} := \mathbb{B}G$,

$$\left(\begin{tikzcd} X \arrow[rr, "f", "G\text{-equiv.}"'] \arrow[dr] & & Y \arrow[dl] \\ & \bullet & \end{tikzcd}\right) \rightsquigarrow \left(\begin{tikzcd} X_G \arrow[rr, "f_G"] \arrow[dr, "\pi"'] & & Y_G \arrow[dl, "\pi"] \\ & \mathbb{B} & \end{tikzcd}\right)$$

The usual ordinary and the compactly supported cohomologies over a field $\Bbbk$ of the Borel construction $X_G$ associated with a $G$-space $X$ are, by definition, the equivariant cohomologies over $\Bbbk$ of $X$, i.e. : ([31])

$$H_G(X;\Bbbk) := H(X_G;\Bbbk) \quad \text{and} \quad H_{G,\mathrm{c}}(X;\Bbbk) := H_{cv}(X_G;\Bbbk).$$

with the natural structure of $H_G := H(\mathbb{B};\Bbbk)$-modules induced by the projections $\pi$ (*cf.* p. 22). In the language of sheaf cohomology, this amounts to set:

$$H_G(X;\Bbbk) := \mathbb{R}\varGamma(\mathbb{B}, \mathbb{R}\pi_*\underline{\underline{\Bbbk}}_{X_G}) \quad \text{and} \quad H_{G,\mathrm{c}}(X;\Bbbk) := \mathbb{R}\varGamma(\mathbb{B}, \mathbb{R}\pi_!\underline{\underline{\Bbbk}}_{X_G})$$

These rewriting provide the clue to what needs to be replaced in order to transpose our work in the framework of sheaf cohomology over $\Bbbk$.

P-1) The complexes $\Omega_G(X)$ and $\Omega_{G,\mathrm{c}}(X)$ in $\mathcal{D}^+(\mathrm{DGM}(\Omega_G))$ should respectively be replaced by the complexes $\mathbb{R}\pi_*\underline{\underline{\Bbbk}}_{X_G}$ and $\mathbb{R}\pi_!\underline{\underline{\Bbbk}}_{X_G}$ of the derived category of the category of sheaves of $\Bbbk$-vector spaces over $\mathbb{B}$, which we denote by $\mathcal{D}^+(\mathbb{B};\Bbbk)$.

The graded algebra $\Omega_G := S(\mathfrak{g}^\vee)^G$ then corresponds to some *ad hoc* resolution of the constant sheaf $\underline{\underline{\Bbbk}}_{\mathbb{B}}$.

P-2) To fix ideas, for a topological space $Z$, we choose the complex of sheaves *of Alexander-Spanier cochains of* $Z$ as resolution of its constant sheaf $\underline{\underline{\Bbbk}}_Z$ ([32]),

$$\mathbf{0} \to \underline{\underline{\Bbbk}}_Z \xrightarrow{\epsilon} \underline{\underline{\Omega}}^0(Z;\Bbbk) \xrightarrow{d_0} \underline{\underline{\Omega}}^1(Z;\Bbbk) \xrightarrow{d_1} \cdots$$

---

[30] In fact, on any ring, but the increase of technicalities that would impose is not warranted by the advantages of the resulting generality, so we prefer to limit ourselves to fields.

[31] The notation $H_{cv}(\_)$, borrowed from [BT] p. 61, means *compact vertical cohomology.*

[32] [Go], §2.5, example 2.5.2, p. 134, and §3.7, example 3.7.1, p. 157.

There are several advantages to this choice.

i) The complex $\underline{\underline{\Omega}}(Z;\Bbbk) := (\underline{\underline{\Omega}}^\bullet(Z;\Bbbk), d_*)$ has a natural structure of sheaf of differential graded (anticommutative) algebras (*loc.cit.*).

ii) The correspondence $Z \rightsquigarrow \underline{\underline{\Omega}}(Z;\Bbbk)$ is functorial and contravariant on the category Top of topological spaces and continuous maps.

Indeed, given a continuous map $f : Z \to Z'$, the morphisms of sheaves

$$f_i^{\#} : \underline{\underline{\Omega}}^i(Z';\Bbbk) \to f_*\underline{\underline{\Omega}}^i(Z;\Bbbk), \quad \zeta \mapsto \zeta \circ f,$$

where $\zeta$ denotes a local section of $\underline{\underline{\Omega}}^i(Z';\Bbbk)$.

The resulting graded morphism $f^{\#} : \underline{\underline{\Omega}}(Z';\Bbbk) \to f_*\underline{\underline{\Omega}}(Z;\Bbbk)$ is a morphism of complexes which induces the usual pull-back map in ordinary cohomology $f^* : H(Z';\Bbbk) \to H(Z;\Bbbk)$ ([33]).

iii) In the specific case of a $G$-space $X$, the map

$$\pi^{\#} : \underline{\underline{\Omega}}(\mathbb{B};\Bbbk) \to \pi_*\underline{\underline{\Omega}}(X_G;\Bbbk)$$

is compatible with the underlying differential graded algebras structures, and $\pi_*\underline{\underline{\Omega}}(X_G;\Bbbk)$ appears as an $\underline{\underline{\Omega}}(\mathbb{B};\Bbbk)$-dgm.

Similarly, the subcomplex $\pi_!\underline{\underline{\Omega}}(X_G,\Bbbk) \subseteq \pi_*\underline{\underline{\Omega}}(X_G,\Bbbk)$ is also stabilized by the action of $\pi^{\#}$ and is therefore an $\underline{\underline{\Omega}}(\mathbb{B},\Bbbk)$-sub-dgm of $\pi_*\underline{\underline{\Omega}}(X_G,\Bbbk)$.

At this point, the reader will have noticed the parallel with the preliminaries of chapter 4 for the category $\mathrm{DGM}(\Omega_G)$ (*cf.* 4.1.3). To emphasize the similarities further, we introduce the notations:

$$\underline{\underline{\Omega}}_G := \underline{\underline{\Omega}}(\mathbb{B},\Bbbk), \quad \underline{\underline{\Omega}}_G(X) := \pi_*\underline{\underline{\Omega}}(X_G,\Bbbk), \quad \underline{\underline{\Omega}}_{G,\mathrm{c}}(X) := \pi_!\underline{\underline{\Omega}}(X_G,\Bbbk),$$

and denote by $\mathrm{DGM}(\underline{\underline{\Omega}}_G)$ the category of sheaves of differential graded modules over the sheaf of differential graded algebras $\underline{\underline{\Omega}}_G$.

This allows us to rephrase things in a more convenient language. (See 8.3.1 (†) and (†′) for more details.)

▸ The correspondence $X \rightsquigarrow \underline{\underline{\Omega}}_G(X)$ is a contravariant functor from the category $G$-Top of $G$-spaces and equivariant maps to the category $\mathrm{DGM}(\underline{\underline{\Omega}}_G)$

$$\left(\begin{tikzcd} X \arrow[rr, "f", "G\text{-equiv.}"'] \arrow[dr] & & Y \arrow[dl] \\ & \{\bullet\} & \end{tikzcd}\right) \rightsquigarrow \left(\begin{tikzcd} \underline{\underline{\Omega}}_G(Y) \arrow[rr, "\pi_*(f_G^{\#})"] & & \underline{\underline{\Omega}}_G(X) \\ & \underline{\underline{\Omega}}_G \arrow[ul, "\pi^{\#}"] \arrow[ur, "\pi^{\#}"'] & \end{tikzcd}\right)$$

▸ The correspondence $X \rightsquigarrow \big(\underline{\underline{\Omega}}_{G,\mathrm{c}}(X) \subseteq \underline{\underline{\Omega}}_G(X)\big)$ is a contravariant functorial inclusion from the category $G$-$\mathrm{Top}_{\mathrm{pr}}$ of $G$-spaces and equivariant *proper* maps to the category $\mathrm{DGM}(\underline{\underline{\Omega}}_G)$

$$\left(\begin{tikzcd} X \arrow[rr, "\text{proper } f", "G\text{-equiv.}"'] \arrow[dr] & & Y \arrow[dl] \\ & \{\bullet\} & \end{tikzcd}\right) \rightsquigarrow \left(\begin{tikzcd} \underline{\underline{\Omega}}_{G,\mathrm{c}}(Y) \arrow[rr, "\pi_!(f_G^{\#})"] \arrow[d, hook] & & \underline{\underline{\Omega}}_{G,\mathrm{c}}(X) \arrow[d, hook] \\ \underline{\underline{\Omega}}_G(Y) \arrow[rr, "\pi_*(f_G^{\#})"] & & \underline{\underline{\Omega}}_G(X) \\ & \underline{\underline{\Omega}}_G \arrow[ul, "\pi^{\#}"] \arrow[ur, "\pi^{\#}"'] & \end{tikzcd}\right)$$

iv)

[33] And in compact support cohomology $f^* : H_{\mathrm{c}}(Z';\Bbbk) \to H_{\mathrm{c}}(Z;\Bbbk)$, if $f$ is a proper map.

One important feature about $\underline{\underline{\Omega}}_G$-modules is that they are automatically $\Gamma_\Phi$-acyclic, for every paracompactifying family of supports $\Phi$ (*loc.cit.*, example 3.7.1, p. 157). In particular, the sheaves $\underline{\underline{\Omega}}^i_{G,\mathrm{c}}(X)$ and $\underline{\underline{\Omega}}^i_G(X)$ are acyclic for the functors $\Gamma(X,\_)$, $\Gamma_{\mathrm{c}}(X;\_)$ and $\Gamma_Z(X,\_)$, in which case

$$\begin{cases} H^i(X;\Bbbk) = h^i(\Gamma(X;\underline{\underline{\Omega}}^\bullet_G(X))) \\ H^i_{\mathrm{c}}(X;\Bbbk) = h^i(\Gamma_{\mathrm{c}}(X;\underline{\underline{\Omega}}^\bullet_G(X))) \\ H^i_Z(X;\Bbbk) = h^i(\Gamma_Z(X;\underline{\underline{\Omega}}^\bullet_G(X))) \end{cases}$$

P-3) The obvious candidate to replace the basic duality functor $I\!R Hom^\bullet_{\Omega_G}(\_,\Omega_G)$ should be the functor $(\_) \rightsquigarrow I\!R\underline{\underline{Hom}}^\bullet_{\underline{\underline{\Omega}}_G}(\_,\underline{\underline{\Omega}}_G)$ but since we are considering $\mathrm{DGM}(\underline{\underline{\Omega}}_G)$ within $\mathcal{D}^+(I\!B;\Bbbk)$, we can achieve the same thing by taking

$$(\_) \rightsquigarrow I\!R\underline{\underline{Hom}}^\bullet(\_,\underline{\underline{\Bbbk}}_{I\!B}).$$

Indeed, general algebraic arguments show that, for $\mathcal{M},\mathcal{N} \in \mathrm{DGM}(\underline{\underline{\Omega}}_G)$, the usual morphism of complexes of sheaves

$$I\!R\underline{\underline{Hom}}^\bullet(\mathcal{M},\mathcal{N}) \to I\!R\underline{\underline{Hom}}^\bullet_{\underline{\underline{\Omega}}_G}(\underline{\underline{\Omega}}_G \otimes \mathcal{M},\mathcal{N})$$

is an isomorphism in $\mathcal{D}^+(I\!B;\Bbbk)$. The claim then follows since the morphism $\underline{\underline{\Omega}}_G \otimes_{\underline{\underline{\Bbbk}}_{I\!B}} \mathcal{M} \to \mathcal{M}$, $\omega \otimes m \mapsto \omega m$ is a quasi-isomorphism, by germ analysis.

**8.1.1. Translations** To summarize, we have replaced:

$$\begin{aligned} \mathcal{D}^+(\mathrm{DGM}(\Omega_G)) &\leftrightarrow \mathcal{D}^+(I\!B;\Bbbk) \\ \Omega_G := S(\mathfrak{g}^\vee)^G &\leftrightarrow \underline{\underline{\Omega}}_G \simeq \underline{\underline{\Bbbk}}_{I\!B} \\ \Omega_G(X) &\leftrightarrow \underline{\underline{\Omega}}_G(X) := \pi_*\underline{\underline{\Omega}}(X_G) \simeq I\!R\pi_*\underline{\underline{\Bbbk}}_{X_G} \\ \Omega_{G,\mathrm{c}}(X) &\leftrightarrow \underline{\underline{\Omega}}_{G,\mathrm{c}}(X) := \pi_!\underline{\underline{\Omega}}(X_G) \simeq I\!R\pi_!\underline{\underline{\Bbbk}}_{X_G} \\ I\!R Hom^\bullet_{\Omega_G}(\_,\Omega_G) &\leftrightarrow I\!R\underline{\underline{Hom}}^\bullet(\_,\underline{\underline{\Omega}}_G) \end{aligned}$$

Under these substitutions, the sheafification of the Poincaré adjunctions:

$$\begin{aligned} &\text{(i)} \quad I\!D_{G,M} : \Omega_G(M)[d_M] \to I\!R Hom^\bullet_{\Omega_G}\big(\Omega_{G,\mathrm{c}}(M),\Omega_G\big) \quad (\textit{cf.}\ 4.4.2) \\ &\text{(ii)} \quad I\!D'_{G,M} : \Omega_{G,\mathrm{c}}(M)[d_M] \to I\!R Hom^\bullet_{\Omega_G}\big(\Omega_G(M),\Omega_G\big), \quad (\textit{cf.}\ 4.4.3) \end{aligned}$$

should be morphisms in $\mathcal{D}^+(I\!B;\Bbbk)$

$$\begin{aligned} &\text{(I)} \quad \underline{\underline{I\!D}}_{G,M} : \underline{\underline{\Omega}}_G(M)[d_M] \to I\!R\underline{\underline{Hom}}^\bullet\big(\underline{\underline{\Omega}}_{G,\mathrm{c}}(M),\underline{\underline{\Bbbk}}_{I\!B}\big) \\ &\text{(II)} \quad \underline{\underline{I\!D}}'_{G,M} : \underline{\underline{\Omega}}_{G,\mathrm{c}}(M)[d_M] \to I\!R\underline{\underline{Hom}}^\bullet\big(\underline{\underline{\Omega}}_G(M),\underline{\underline{\Bbbk}}_{I\!B}\big). \end{aligned}$$

Now, thanks to Grothendieck-Verdier duality, the right-hand side in (I) verifies

$$I\!R\underline{\underline{Hom}}^\bullet(I\!R\pi_!\underline{\underline{\Bbbk}}_{X_G},\underline{\underline{\Bbbk}}_{I\!B}) = I\!Rp_*\,I\!R\underline{\underline{Hom}}^\bullet(\underline{\underline{\Bbbk}}_X,\pi^!\underline{\underline{\Bbbk}}_{I\!B}) = I\!Rp_*(\pi^!\underline{\underline{\Bbbk}}_{I\!B}),$$

and (I) is a morphism in $\mathcal{D}^+(I\!B;\Bbbk)$:

$$(\underline{\mathrm{I}}) \quad \underline{\underline{I\!D}}_{G,M} : I\!R\pi_*(\pi^{-1}\underline{\underline{\Bbbk}}_{I\!B})[d_M] \to I\!R\pi_*(\pi^!\underline{\underline{\Bbbk}}_{I\!B}), \qquad (\diamond)$$

On the other hand, (II) is the basic dual of (I) composed with the natural morphism

$$(\underline{\mathrm{II}}) \quad \underline{\underline{\Omega}}_{G,\mathrm{c}}(M) \to I\!R\underline{\underline{Hom}}^\bullet\big(I\!R\underline{\underline{Hom}}^\bullet(\underline{\underline{\Omega}}_{G,\mathrm{c}}(M),\underline{\underline{\Bbbk}}_{I\!B}),\underline{\underline{\Bbbk}}_{I\!B}\big) \qquad (\diamond\diamond)$$

so that proving that (I) and (II) are isomorphisms in $\mathcal{D}^+(X;\Bbbk)$ is equivalent to proving that ($\underline{\mathrm{I}}$) and ($\underline{\mathrm{II}}$) are so.

The next proposition proves that this indeed the case, even in a slightly more general situation.

**8.1.2. Proposition.** *Let $\pi : E \to \mathbb{B}$ be any locally trivial fibration with fiber an equidimensional oriented manifold $M$ of dimension $d_M$ ([34]).*

a) *The Poincaré adjunction principle defines a morphism*

$$\pi^{-1}\underline{\underline{\Bbbk}}_{\mathbb{B}}[d_M] \to \pi^{!}\underline{\underline{\Bbbk}}_{\mathbb{B}}\,, \tag{$\diamond$}$$

*which is an isomorphism in $\mathcal{D}^+(E;\Bbbk)$.*

b) *If $\dim_{\Bbbk} H_{\mathrm{c}}(M;\Bbbk) < +\infty$, the natural morphism*

$$\mathbb{R}\pi_!\underline{\underline{\Bbbk}}_E \to \mathbb{R}\underline{\underline{Hom}}^\bullet\big(\mathbb{R}\underline{\underline{Hom}}^\bullet(\mathbb{R}\pi_!\underline{\underline{\Bbbk}}_E,\underline{\underline{\Bbbk}}_{\mathbb{B}}),\underline{\underline{\Bbbk}}_{\mathbb{B}}\big)$$

*is an isomorphism in $\mathcal{D}^+(E;\Bbbk)$.*

*Proof.* (a) We must first construct the morphism. By Grothendieck-Verdier duality, we have the equivalence

$$\mathbb{R}\pi_*\,\mathbb{R}\underline{\underline{Hom}}^\bullet\big((\pi^{-1}\underline{\underline{\Bbbk}}_{\mathbb{B}})[d_M],\pi^{!}\underline{\underline{\Bbbk}}_{\mathbb{B}}\big) = \mathbb{R}\underline{\underline{Hom}}^\bullet\big(\mathbb{R}\pi_!(\pi^{-1}\underline{\underline{\Bbbk}}_{\mathbb{B}})[d_M],\underline{\underline{\Bbbk}}_{\mathbb{B}}\big). \tag{$\dagger$}$$

The sheaf $\pi^{-1}\underline{\underline{\Bbbk}}_{\mathbb{B}}$ is the constant sheaf $\underline{\underline{\Bbbk}}_E$, and since $\pi : E \to \mathbb{B}$ is locally trivial, the complex of sheaves $\mathbb{R}\pi_!(\underline{\underline{\Bbbk}}_E)$ admits a simple description over a trivializing open subset $U \subseteq \mathbb{B}$. Indeed, in that case the open subset $\tilde{U} := \pi^{-1}(U)$ is simply $U \times M$ and since we have $\underline{\underline{\Bbbk}}_{\tilde{U}} = \underline{\underline{\Bbbk}}_U \boxtimes \underline{\underline{\Bbbk}}_M$ ([35]), we get a canonical identification:

$$\mathbb{R}\pi_!(\underline{\underline{\Bbbk}}_{\tilde{U}}) = (\mathrm{id} \boxtimes \mathbb{R}c_{M!})(\underline{\underline{\Bbbk}}_U \boxtimes \underline{\underline{\Bbbk}}_M) = \underline{\underline{\Bbbk}}_U \otimes \mathbb{R}c_{M!}(\underline{\underline{\Bbbk}}_M)$$

where $c_M : M \to \{\bullet\}$ is the constant map. As a consequence, the restriction of complex of sheaves, at the right-hand side of ($\dagger$), to $U$ verifies

$$\begin{aligned}\mathbb{R}\underline{\underline{Hom}}^\bullet\big(\mathbb{R}\pi_!(\pi^{-1}\underline{\underline{\Bbbk}}_{\mathbb{B}})[d_M],\underline{\underline{\Bbbk}}_{\mathbb{B}}\big)|_U &= \mathbb{R}\underline{\underline{Hom}}^\bullet\big(\underline{\underline{\Bbbk}}_{\mathbb{B}} \boxtimes \mathbb{R}c_{M!}(\Bbbk_M)[d_M],\underline{\underline{\Bbbk}}_{\mathbb{B}}\big)\\ &= \underline{\underline{\Bbbk}}_{\mathbb{B}} \otimes Hom_{\Bbbk}^\bullet\big(\mathbb{R}c_{M!}(\underline{\underline{\Bbbk}}_M)[d_M];\Bbbk\big),\end{aligned} \tag{$\ddagger$}$$

where the reader will have noticed the that instead of $\mathbb{R}Hom^\bullet$, we wrote $Hom^\bullet$ in the second line, the reason being that the functor $Hom_{\Bbbk}^\bullet(\_;\Bbbk)$ is already exact in the category $\mathrm{Vec}(\Bbbk)$ of $\Bbbk$-vector spaces.

Now, if $\mathbf{0} \to \underline{\underline{\Bbbk}}_E \to \underline{\underline{\Omega}}^*(E,\Bbbk)$ is the resolution by Alexander-Spanier cochains (see 8.1-(P-2)), the complex of $\Bbbk$-vector spaces

$$\mathbb{R}c_{M!}(\underline{\underline{\Bbbk}}_E) = \Gamma_{\mathrm{c}}(M,\underline{\underline{\Omega}}^*_E),$$

[34] More generally, it is easily seen that the total space $E$ of a locally trivial fibration $\pi : E \to \mathbb{B}$ with non-connected fiber manifold $M$, is a disjoint union of open subspaces $E_i$ on which the restrictions $\pi_i := \pi|_{E_i}$ are locally trivial fibrations with equidimensional fibers. The hypothesis on equidimensionality in the proposition is therefore not really restrictive.

The other hypothesis concerning the orientability of fibers is more subtle. Since de word 'fiber' is generic, it means that we choose an atlas of $E$ made of trivializations $U \times M$, where $U$ is open in $\mathbb{B}$, and such that the transition homeomorphisms induce *oriented* isomorphisms on $M$. Notice that if $\mathbb{B}$ is *simply connected*, and this is the case when $\mathbb{B} := \mathbb{B}G$ since $G$ is connected, such *fiber-oriented atlases* always exist.

[35] Recall that if $p_i : Z_1 \times Z_2 \to Z_i$ is the canonical projection, and $\mathscr{F}_i$ is a sheaf in $Z_i$, the sheaf $\mathscr{F}_1 \boxtimes \mathscr{F}_2$ is, by definition, $\mathscr{F}_1 \boxtimes \mathscr{F}_2 := p_1^{-1}\mathscr{F}_1 \otimes p_2^{-1}\mathscr{F}_2$.

computes the compact support cohomology $H_{\mathrm{c}}(M;\Bbbk)$. On the other hand, an orientation of $M$ is a choice of a generator $\zeta_i \in H_{\mathrm{c}}^{d_M}(M_i)$ on each connected component $M_i$ of $M$, and, as such, it determines a unique linear form taking the value 1 on each $\zeta_i$, which is the familiar integration operator over $M$,

$$\textstyle\int_M : H_{\mathrm{c}}(M;\Bbbk)[d_M] \to \Bbbk. \tag{$\int$}$$

At this point, recall that since the category $\mathcal{D}^+(\mathrm{Vec}(\Bbbk))$ is *split*, i.e. a complex is isomorphic to its cohomology, we have

$$\mathit{Hom}^\bullet_\Bbbk(\mathbb{R}c_{M!}(\underline{\Bbbk}_M)[d_M];\Bbbk) = \mathit{Hom}^\bullet_\Bbbk(H_{\mathrm{c}}(M)[d_M];\Bbbk) \tag{‡‡}$$

in $\mathcal{D}^+(\mathrm{Vec}(\Bbbk))$. Therefore, ($\int$) determines a 0-cycle of the left-hand side term in (‡‡), hence in (‡) and in (†), and, taking global sections over $\mathbb{B}$ and 0-cohomology we get a well defined element in

$$H^0\Gamma\big(\mathbb{B};\mathbb{R}\pi_*\,\mathbb{R}\underline{\underline{Hom}}^\bullet\big(\pi^{-1}\underline{\underline{\Bbbk}}_{\mathbb{B}}[d_M],\pi^!\underline{\underline{\Bbbk}}_{\mathbb{B}}\big)\big) = \mathrm{Mor}_{\mathcal{D}^+(\mathbb{B};\Bbbk)}\big(\pi^{-1}\underline{\underline{\Bbbk}}_{\mathbb{B}}[d_M],\pi^!\underline{\underline{\Bbbk}}_{\mathbb{B}}\big),$$

which is the morphism $\pi^{-1}\underline{\underline{\Bbbk}}_{\mathbb{B}}[d_M] \to \pi^!\underline{\underline{\Bbbk}}_{\mathbb{B}}$ (◇) announced. ([36])

To show that (◇) is an isomorphism in the derived category $\mathcal{D}^+(E;\Bbbk)$, we need only show that the induced morphism at the germs of those complexes at each $x \in E$ are quasi-isomorphisms, and, for convenience, to restrict to the open subsets $\tilde{U}$, as they cover $E$. In that case, if $x \in \tilde{U} := U \times M$, we can write $x = (b,m) \in U \times M$, and

$$\Big(\big(\pi^{-1}\underline{\underline{\Bbbk}}_{\mathbb{B}}[d_M] \to \pi^!\underline{\underline{\Bbbk}}_{\mathbb{B}}\big)_x\Big) \;\simeq\; \Big((\underline{\underline{\Bbbk}}_M)_m[d_M] \to (c_M^!\Bbbk)_m = \underline{\underline{\mathcal{D}}}^\bullet_M(\Bbbk)_m\Big) \tag{$*$}$$

where $\underline{\underline{\mathcal{D}}}^\bullet_M(\Bbbk)$ is the well known *dualizing complex* on $M$, i.e. the complex of sheaves defined by the complex of presheaf

$$\underline{\mathcal{D}}^\bullet_M : V \mapsto \mathit{Hom}^\bullet_\Bbbk(\mathbb{R}c_{V!}\underline{\underline{\Bbbk}}_V;\Bbbk) = \mathit{Hom}^\bullet_\Bbbk(H_{\mathrm{c}}(V);\Bbbk).$$

Likewise, the morphism at the right-hand side of ($*$) is induced by the morphisms of presheaves $\underline{\Bbbk}_M \to \underline{\mathcal{D}}^\bullet_M(\Bbbk)[-d_M]$:

$$\begin{array}{ccc} \Gamma(V;\underline{\Bbbk}) & \longrightarrow & \Gamma(V;\underline{\mathcal{D}}^\bullet_M(\Bbbk))[-d_M] \\ \| & & \| \\ \Bbbk & \longrightarrow & \mathit{Hom}^\bullet_\Bbbk(H_{\mathrm{c}}(V;\Bbbk);\Bbbk)[-d_M] \end{array} \tag{$**$}$$

which assigns to 1 the integration $\int_V$ relative to the orientation of $V$, induced by the orientation of $M$.

Now, as $M$ is a manifold, a point $m \in M$ has a basis of open neighborhoods $V$ homeomorphic to $\mathbb{R}^{d_M}$, in which case $H_{\mathrm{c}}(V;\Bbbk)$ is concentrated in degree $d_M$, with $H_{\mathrm{c}}^{d_M}(V;\Bbbk) = \Bbbk$. As a consequence, the second line in ($**$) is an isomorphism for such $V$'s, which implies that the morphisms in ($*$) are quasi-isomorphisms for all $x \in E$, ending the proof of (a).

(b) A complex of sheaves of $\Bbbk$ vector spaces $\mathscr{F}^* := (\mathscr{F}^*,d_*)$ on a topological space $\mathbb{B}$ is said to be *cohomologically bounded* if its cohomology sheaves $\mathcal{H}^i(\mathscr{F}_*,d_*) = 0$ vanish for $|i|$ big. It is said to be *perfect* if its cohomology

[36] It is worth noting that, following this procedure, we have sheafified le left Poincaré adjunction on the fibers, which was made possible by having a coherent choice of orientations for the fibers of $\pi$. In the whole, this is the underlying sense of what we called in (a) the Poincaré adjunction principle.

sheaves $\mathcal{H}^i(\mathscr{F}_*,d_*)$ are locally trivial sheaves of finite rank, i.e. if for all $i \in \mathbb{Z}$, the space $\mathbb{B}$ can be covered by open subsets $V$ such that $\mathcal{H}^i(\mathscr{F}_*,d_*)|_V \sim \underline{\underline{\Bbbk}}_V^{n(V)}$, for some $n(V) \in \mathbb{N}$.

*Lemma.* If $\mathscr{F}^*$ is cohomologically bounded and perfect, the natural morphism

$$\iota : \mathscr{F}^* \to \mathbb{R}\underline{\underline{Hom}}^\bullet\big(\mathbb{R}\underline{\underline{Hom}}^\bullet(\mathscr{F}^*,\underline{\underline{\Bbbk}}_{\mathbb{B}}),\underline{\underline{\Bbbk}}_{\mathbb{B}}\big)$$

is an isomorphism in $\mathcal{D}^+(\mathbb{B};\Bbbk)$.

First, notice that since $\underline{\underline{Hom}}(\Bbbk_V^n,\underline{\underline{\Bbbk}}_V) \sim \underline{\underline{Hom}}(\Bbbk_V,\underline{\underline{\Bbbk}}_V)^n \sim \underline{\underline{\Bbbk}}_V^n$, we immediately see that if $\mathcal{L}$ is a locally trivial sheaf of finite local ranks, the functor $\underline{\underline{Hom}}(\mathcal{L},\_)$ is exact in the category of sheaves. As a consequence, we have

$$\mathbb{R}\underline{\underline{Hom}}(\mathcal{L},\underline{\underline{\Bbbk}}_{\mathbb{B}}) = \underline{\underline{Hom}}(\mathcal{L},\underline{\underline{\Bbbk}}_{\mathbb{B}}),$$

and again a locally trivial sheaf of finite ranks, so that

$$\iota : \mathcal{L} \to \mathbb{R}\underline{\underline{Hom}}(\mathbb{R}\underline{\underline{Hom}}(\mathcal{L},\Bbbk_{\mathbb{B}}),\underline{\underline{\Bbbk}}_{\mathbb{B}}) = \underline{\underline{Hom}}(\underline{\underline{Hom}}(\mathcal{L},\Bbbk_{\mathbb{B}}),\underline{\underline{\Bbbk}}_{\mathbb{B}}) \qquad (*)$$

is an isomorphism in $\mathcal{D}^+(\mathbb{B};\Bbbk)$.

Next, if $0 \to \mathscr{F}^0 \to \mathscr{F}^1 \to \cdots \to \mathscr{F}^\ell \to 0$ is a perfect complex of sheaves, the truncated complex $\tau_{<\ell}(\mathscr{F}^*)$ ([37]) is still perfect and one has an small exact sequence of perfect complexes of sheaves

$$0 \to \tau_{<\ell}(\mathscr{F}^*) \to \mathscr{F}^* \to \mathcal{H}^\ell(\mathscr{F}) \to 0,$$

giving rise in $\mathcal{D}^+(\mathbb{B},\Bbbk)$ to the morphism of exact triangles

$$\begin{tikzcd}
\tau_{<\ell}(\mathscr{F}^*) \arrow[r] \arrow[d, "\iota"'] & \mathscr{F}^* \arrow[r] \arrow[d, "\iota"'] & \mathcal{H}^\ell(\mathscr{F}^*) \arrow[r] \arrow[d, "\iota"'] & {} \\
(\tau_{<\ell}(\mathscr{F}^*))^{\vee\vee} \arrow[r] & (\mathscr{F}^*)^{\vee\vee} \arrow[r] & (\mathcal{H}^\ell(\mathscr{F}^*))^{\vee\vee} \arrow[r] & {}
\end{tikzcd} \qquad (**)$$

where $(\_)^{\vee\vee} := \mathbb{R}\underline{\underline{Hom}}(\mathbb{R}\underline{\underline{Hom}}(\_,\Bbbk_{\mathbb{B}}),\underline{\underline{\Bbbk}}_{\mathbb{B}})$.

In $(**)$ the right-hand vertical arrow is an isomorphism after $(*)$, hence we can conclude that the central arrow is an isomorphism if and only if the left-hand arrow is so. But since the truncated complex $\tau_{<\ell}(\mathscr{F}^*)$ has less than $\ell$ terms, we can assume by induction on this number, that this is in fact the case. Therefore, the central arrow in $(**)$ is also an isomorphism in $\mathcal{D}^+(\mathbb{B};\Bbbk)$.

To finish the proof of the lemma, we need only recall that in $\mathcal{D}^+(\mathbb{B};\Bbbk)$, a cohomologically bounded complex is always isomorphic, after a shift of indices, to a bounded complex as in the previous case. ⊟

The proof of (b) is now reduced to proving that the complex $\mathbb{R}\pi_!\underline{\underline{\Bbbk}}_E$ is cohomologically bounded and perfect.

This is already the case over a trivializing open subset $U \subseteq \mathbb{B}$. Indeed, we already explained in the poof of (*a*) that one has in $\mathcal{D}^+(U;\Bbbk)$

$$\mathbb{R}\pi_!\underline{\underline{\Bbbk}}_E|_U \simeq \underline{\underline{\Bbbk}}_U \otimes H_c(M;\Bbbk),$$

and immediately we have the perfection property as this is a local property. It also shows that the sheaves $\mathcal{H}^i(\mathbb{R}\pi_!\underline{\underline{\Bbbk}}_E)$ vanish for $i \notin [\![0,d_M]\!]$, which also

[37] This is the subcomplex $\mathscr{G}^* \subseteq \mathscr{F}^*$, such that $\mathscr{G}^i = \mathscr{F}^i$ for $i < \ell$ and $\mathscr{G}^\ell := \mathrm{im}(d_{\ell-1})$. Notice that, by construction, $\mathcal{H}^i(\mathscr{G}^*) = \mathcal{H}^i(\mathscr{F}^*)$ if $i < \ell$ and $\mathcal{H}^i(\mathscr{G}^*) = 0$ otherwise.

results from a local analysis. But, it is because bounds in $[\![0,d_M]\!]$ are independent of the trivializing open subset $U$, that at the end explains that $I\!R\pi_!\underline{\underline{\Bbbk}}_E$ is *globally* cohomologically bounded and that we can apply the lemma and finish the proposition's proof. $\square$

Now we have all the ingredients necessary to extend the validity of the equivariant Poincaré duality to any field of coefficients $\Bbbk$.

## 8.2. Equivariant Poincaré Duality on arbitrary fields

We can now state and prove the exact analog of theorem 4.5.1. Let,

$$H_G := H(I\!BG;\Bbbk)\,.$$

**8.2.1. Theorem.** *Let $G$ be a compact connected Lie group, and let $M$ be an oriented $G$-manifold of dimension $d_M$. Then,*

a) *The sheafification of the left Poincaré adjunction (see 8.1.1)*

$$\underline{\underline{I\!D}}_{G,M} : \underline{\underline{\Omega}}_G(M)[d_M] \longrightarrow I\!R\underline{\underline{Hom}}^{\bullet}_{\underline{\underline{\Omega}}_G}\big(\underline{\underline{\Omega}}_{G,c}(M),\underline{\underline{\Omega}}_G\big)$$

*is an isomorphism in $\mathcal{D}^+(I\!BG;\Bbbk)$.*

b) *The morphism $\underline{\underline{I\!D}}_{G,M}$ induces* the Poincaré morphism in $G$-equivariant cohomology with coefficients in $\Bbbk$*:*

$$\mathscr{D}_{G,M,\Bbbk} : H_G(M,\Bbbk)[d_M] \longrightarrow \mathrm{Hom}^{\bullet}_{H_G}\big(H_{G,c}(M,\Bbbk),H_G\big)\,.$$

*If $\mathrm{Extgr}^i_{H_G}(H_{G,c}(M;\Bbbk)) = 0$, for all $i > 0$, for example if $H_{G,c}(M;\Bbbk)$ is a free $H_G$-module, then $\mathscr{D}_{G,M,\Bbbk}$ is an isomorphism.*

c) *Moreover, if $M$ is of finite type, the sheafification of the right Poincaré adjunction (see 8.1.1)*

$$\underline{\underline{I\!D}}'_{G,M} : \underline{\underline{\Omega}}_{G,c}(M)[d_M] \to I\!R\underline{\underline{Hom}}^{\bullet}_{\underline{\underline{\Omega}}_G}\big(\underline{\underline{\Omega}}_G(M),\underline{\underline{\Omega}}_G\big)$$

*is a isomorphism in $\mathcal{D}^+(I\!BG;\Bbbk)$ and the analog to* (b) *is also verified.*

*Proof.* Statement (a) and the first part of (c) are immediate applications of proposition 8.1.2 to the map $\pi : M_G \to I\!BG$ after 8.1-(P-3) and the comments at the end of section 8.1.1. The first part of (b) concerning $\mathscr{D}_{G,M,\Bbbk}$, and its analog in (c) for $\mathscr{D}'_{G,M,\Bbbk}$, are then obvious.

Second part of (b). Since $I\!BG$ is simply connected, the sheaves in the complex $\underline{\underline{\Omega}}_G(M) := \pi_!\underline{\underline{\Omega}}(M_G;\Bbbk)$, that we have shown to be local systems (*cf.* proof of prop. 8.1.2), are isomorphic, as $\underline{\underline{\Omega}}_G$-gm, to $\underline{\underline{\Omega}}_G \otimes \Omega_c(M;\Bbbk)$. We therefore have:

$$I\!R\Gamma\big(I\!BG;I\!R\underline{\underline{Hom}}_{\underline{\underline{\Omega}}_G}\big(\underline{\underline{\Omega}}_{G,c}(M),\_\big)\big) = I\!R\mathrm{Hom}_{\Omega(I\!BG;\Bbbk)}\big(\Omega(M_G;\Bbbk),I\!R\Gamma(I\!BG,\_)\big)\,,$$

in $\mathcal{D}^+(I\!BG;\Bbbk)$, and taking derived global sections $I\!R\Gamma(I\!BG;\_)$ on the isomorphism $\underline{\underline{I\!D}}_{G,M}$ (after (a)), we get the quasi-morphism of complexes

$$I\!R\Gamma(I\!BG;\underline{\underline{I\!D}}_{G,M}) : \Omega(M_G;\Bbbk) \to I\!R\mathrm{Hom}_{\Omega(I\!BG;\Bbbk)}\big(\Omega_{cv}(M_G;\Bbbk),\Omega(I\!BG,\Bbbk)\big)\,, \qquad (\ddagger)$$

which is another version of the Poincaré left adjunction.

Here, $I\!R\mathrm{Hom}_{\Omega(I\!BG;\Bbbk)}(\_,\Omega(I\!BG;\Bbbk))$ is the derived functor defined in 4.2.7, and because $\Omega_{cv}(M_G;\Bbbk) \sim \Omega(I\!BG;\Bbbk) \otimes \Omega_{\mathrm{c}}(M;\Bbbk)$ as $\Omega(I\!BG;\Bbbk)$-gm, we are lead to a point where we can apply the same argument of the proof of 4.5.1-(b). $\square$

**8.2.2. Remarks**

a) As already recalled in 4.5.2, Allday-Franz-Puppe improved the second part of the statement (b), showing in [AFP$_1$] that for $M$ of finite type, the morphism $\mathscr{D}_{G,M,\Bbbk}$ is an isomorphism if and only if both $H_G(M;\Bbbk)$ and $H_{G,\mathrm{c}}(M;\Bbbk)$ are reflexive $H_G$-modules.

b) The Poincaré adjunction in the proof of 8.2.1-(b):

$$\Omega(M_G;\Bbbk) \to I\!R\mathrm{Hom}_{\Omega(I\!BG;\Bbbk)}\big(\Omega_{cv}(M_G;\Bbbk),\Omega(I\!BG,\Bbbk)\big) \tag{$\ddagger$}$$

is better understood if we replace

$$\begin{cases} \Omega(M_G;\Bbbk) \leftrightarrow \Omega(I\!BG;\Bbbk) \otimes \Omega(M;\Bbbk) \\ \Omega_{cv}(M_G;\Bbbk) \leftrightarrow \Omega(I\!BG;\Bbbk) \otimes \Omega_{\mathrm{c}}(M;\Bbbk) \end{cases}$$

as it becomes simply:

$$\mathrm{id} \otimes I\!D_M : \Omega(I\!BG;\Bbbk) \otimes \Omega(M;\Bbbk) \to \Omega(I\!BG;\Bbbk) \otimes \mathrm{Hom}_\Bbbk(\Omega_{\mathrm{c}}(M;\Bbbk),\Bbbk),$$

where $I\!D_M$ is the usual nonequivariant left Poincaré adjunction for $M$ (1.3).

## 8.3. Equivariant Gysin morphisms on arbitrary fields

Now that we have proved Poincaré duality over any field $\Bbbk$, we can mimic the method in de Rham (equivariant or not) cohomology to introduce Gysin morphisms and introduce these functors over any field $\Bbbk$.

**8.3.1** In 8.1-(P-2iii), we explained how an equivariant map $f : X \to Y$ between $G$-spaces defines a morphism in $\mathcal{D}^+(Y_G;\Bbbk)$

$$f^\# : \underline{\underline{\Omega}}(Y_G;\Bbbk) \to f_* \underline{\underline{\Omega}}(X_G;\Bbbk) \tag{$\diamond$}$$

inducing the usual pull-back $f^* : H_G(Y,\Bbbk) \to H_G(X;\Bbbk)$. Now, if we apply the functor $\pi_{Y,*}$ to ($\diamond$), we get a morphism in $\mathcal{D}^+(I\!BG;\Bbbk)$

$$\pi_{Y,*}(f^\#) : \pi_{Y,*}\underline{\underline{\Omega}}(Y_G;\Bbbk) \to \pi_{Y,*} f_* \underline{\underline{\Omega}}(X_G;\Bbbk) = \pi_{X,*}\underline{\underline{\Omega}}(X_G;\Bbbk)$$

where one recognizes the complexes $\underline{\underline{\Omega}}_G(X)$ and $\underline{\underline{\Omega}}_G(Y)$, and we can better write:

$$\pi_{Y,*}(f^\#) : \underline{\underline{\Omega}}_G(Y) \to \underline{\underline{\Omega}}_G(X) \tag{$\dagger$}$$

If $f$ is, in addition, a proper map, we have $f_* = f_!$ and the same ideas applying $\pi_{Y,!}$ lead to morphism in $\mathcal{D}^+(Y_G;\Bbbk)$

$$\pi_{Y,!}(f^\#) : \underline{\underline{\Omega}}_{G,\mathrm{c}}(Y) \to \underline{\underline{\Omega}}_{G,\mathrm{c}}(X) \tag{$\dagger'$}$$

inducing the usual pull-back $f^* : H_{G,\mathrm{c}}(Y,\Bbbk) \to H_{G,\mathrm{c}}(X;\Bbbk)$.

In the sequel, we assume that $X$ and $Y$ are manifolds respectively denoted by $M$ and $N$, of dimensions $d_M$ and $d_N$.

**8.3.2. Gysin Morphism for General Maps.** We apply the basic duality functor $(\_)^\vee := I\!R\underline{\underline{Hom}}^\bullet(\_,\underline{\underline{\Bbbk}}_{I\!BG})$ to (†), and consider the diagram in $\mathcal{D}^+(I\!BG;\Bbbk)$

$$\begin{tikzcd}
\underline{\underline{\Omega}}_G(M)^\vee \arrow[rr, "(\pi_{Y,*}(f^\#))^\vee"] & & \underline{\underline{\Omega}}_G(N)^\vee \\
\underline{\underline{\Omega}}_{G,c}(M)[d_M] \arrow[u, "\underline{\underline{I\!D}}'_{G,M}"] \arrow[rr, dashed, "f_*"] & & \underline{\underline{\Omega}}_{G,c}(N)[d_N] \arrow[u, "\underline{\underline{I\!D}}'_{G,N}"', "\simeq"]
\end{tikzcd}$$

where the right-hand vertical arrow is an isomorphism when $N$ is of finite type, after 8.2.1-(c). Following the same principle as in 1.10.2 and 5.1.3, we can proceed to defining *the Gysin morphism for a general map in* $\mathcal{D}^+(I\!BG;\Bbbk)$ as the only morphism $f_*$ closing the diagram. The cohomology of global sections then gives *the Gysin morphism for a general map in equivariant cohomology over* $\Bbbk$:

$$f_* : H_{G,c}(M,\Bbbk)[d_M] \to H_{G,c}(N,\Bbbk)[d_N].$$

The case where $N$ is not of finite type is handled as in *loc.cit.* by taking limits over filtrant covers of finite type open subsets of $N$.

**8.3.3. Gysin Morphism for Proper Maps.** The underlying idea is the same as in the previous case, but we now apply the basic duality functor $(\_)^\vee$ to (†′). We then obtain the diagram

$$\begin{tikzcd}
\underline{\underline{\Omega}}_{G,c}(M)^\vee \arrow[rr, "(\pi_{Y,!}(f^\#))^\vee"] & & \underline{\underline{\Omega}}_{G,c}(N)^\vee \\
\underline{\underline{\Omega}}_G(M)[d_M] \arrow[u, "\underline{\underline{I\!D}}_{G,M}", "\simeq"'] \arrow[rr, dashed, "f_!"] & & \underline{\underline{\Omega}}_G(N)[d_N] \arrow[u, "\underline{\underline{I\!D}}_{G,N}"', "\simeq"]
\end{tikzcd}$$

where both vertical arrows are isomorphisms in $\mathcal{D}^+(I\!BG;\Bbbk)$, after 8.2.1-(a). *The Gysin morphism for proper maps in* $\mathcal{D}^+(I\!BG;\Bbbk)$ is then defined, as in 1.10.2 and 5.1.3, as the only morphism $f_!$ that closes the diagram. The cohomology of global sections then gives *the Gysin morphism for proper maps in equivariant cohomology over* $\Bbbk$:

$$f_! : H_G(M,\Bbbk)[d_M] \to H_G(N,\Bbbk)[d_N].$$

## 9. Appendix

We explain the following fact mentioned in footnote ([24]).

*Proposition. Let $A := A^0 \oplus A^1 \oplus \cdots$ be a graded ring. Denote by $S$ the multiplicative system generated by the nonzero graded elements of $A$. The ring $L := S^{-1}A$ is a graded $A$-module such that, for any $A$-graded module $N$, the tensor product $L \otimes_A N$ is flat and injective in the category of graded $A$-modules.*

*Proof.*

• $\boldsymbol{L \otimes N}$ **is flat.** For any graded ideal $I$ of $A$, one has the long exact sequence:

$$\mathbf{0} \to \mathit{Tor}_1^A(L, A/I) \longrightarrow L \otimes I \longrightarrow L \longrightarrow L \otimes (A/I) \to \mathbf{0} \qquad (*)$$

where $A/I$ is a torsion graded $A$-module. The annihilators of the elements of $A/I$ are graded ideals, generated, as such, by invertible elements of $L$. Therefore

$$\mathit{Tor}_1^A(L, A/I) = \mathbf{0}, \quad \forall i \in \mathbb{N},$$

and we have from $(*)$ the equality $L \otimes I = L$ from which, we deduce

$$L \otimes I \otimes N = L \otimes N$$

for any $A$-graded module $N$. The *ideal criterion of flatness* applies, and the $A$-graded module $L \otimes N$ is flat.

• $\boldsymbol{L \otimes N}$ **is injective.** Let $\alpha : M_1 \subseteq M_2$ be a graded inclusion of graded $A$-modules. We must show that any morphism $\lambda : M_1 \to L \otimes N$ of graded $A$-modules can be extended to $M_2$.

$$\begin{tikzcd} M_1 \arrow[r, "\alpha", "\subseteq"'] \arrow[d, "\lambda"'] & M_2 \arrow[dl, dashed, "\lambda'"] \\ L \otimes N & \end{tikzcd}$$

In the contrary, Zorn's lemma will led us to assume that $M_2 \supsetneqq M_1$ and that $\lambda$ may not be further extended. In particular, $A \cdot m \cap M_1 \neq \mathbf{0}$ for any homogeneous $m \in M_2 \setminus M_1$, hence the quotient $M_2/M_1$ is a torsion module. One then has

$$L \otimes M_1 = L \otimes M_2,$$

and a contradiction arises as a consequence of the diagram

$$\begin{tikzcd} \mathit{Homgr}_A(M_2, L \otimes N) \arrow[r] \arrow[d, "\cong"'] & \mathit{Homgr}_A(M_1, L \otimes N) \arrow[d, "\cong"'] \\ \mathit{Homgr}_L(L \otimes M_2, L \otimes N) \arrow[r, "(=)"'] & \mathit{Homgr}_L(L \otimes M_1, L \otimes N) \end{tikzcd}$$

where the horizontal arrows are induced by the inclusion $M_1 \subseteq M_2$ and the vertical arrows are the well-known canonical natural isomorphisms. $\square$

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