Notes on the Cartan Model

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§ 1. The Cartan model

1.1. Let $\boldsymbol{G}$ denote a connected compact Lie Group with universal fibre bundle $\mathbb{E}\boldsymbol{G}$. Let $\mathbb{E}\boldsymbol{G} = \bigcup_{m\in\mathbb{N}} \mathbb{E}\boldsymbol{G}(m)$, where $\mathbb{E}(m) \subseteq \mathbb{E}\boldsymbol{G}$ is a compact $\boldsymbol{G}$-manifold with no cohomology in degrees belonging to the interval $[1, m+1]$.

Let $\boldsymbol{M}$ be a $\boldsymbol{G}$-manifold. Denote $(\Omega^*(\boldsymbol{M}), d_{\boldsymbol{M}})$ its de Rham complex of differential forms, $\boldsymbol{Z}^*(\boldsymbol{M})$ the subring of cocycles, $\boldsymbol{B}^*(\boldsymbol{M})$ its ideal of coboundaries, and finally $H^*_{\mathrm{dR}}(\boldsymbol{M}) = \boldsymbol{Z}^*(\boldsymbol{M})/\boldsymbol{B}^*(\boldsymbol{M})$ its de Rham cohomology ring.

The "*Cartan complex of* $\boldsymbol{M}$" is by definition

$$\big(\Omega_{\boldsymbol{G}}(\boldsymbol{M}), d_{\boldsymbol{G}}\big) := \begin{cases} \Omega_{\boldsymbol{G}}(\boldsymbol{M}) = \big(S(\mathfrak{g}^\vee) \otimes \Omega(\boldsymbol{M})\big)^{\boldsymbol{G}} \\ d_{\boldsymbol{G}}(\omega(X)) = d_{\boldsymbol{M}}(\omega(X)) - \iota(X)(\omega(X)) \end{cases}$$

Théorème (Cartan model). *For any $\boldsymbol{G}$-manifold $\boldsymbol{M}$, there exist a natural isomorphism*

$$\boxed{H^*(\boldsymbol{M}_{\boldsymbol{G}}, \mathbb{R}) \cong H^*(\Omega_{\boldsymbol{G}}(\boldsymbol{M}), d_{\boldsymbol{G}})}$$

where the left hand side denotes the singular cohomology of the Borel construction on $\boldsymbol{M}$, the space $\boldsymbol{M}_{\boldsymbol{G}} := \mathbb{E}\boldsymbol{G} \times_{\boldsymbol{G}} \boldsymbol{M}$.

1.2. The last theorem is a consequence of the three following propositions.

Proposition A. *For each $n \in \mathbb{N}$, the restriction map*

$$H^n(\mathbb{E}\boldsymbol{G} \times_{\boldsymbol{G}} \boldsymbol{M}, \mathbb{R}) \to H^n(\mathbb{E}\boldsymbol{G}(m) \times_{\boldsymbol{G}} \boldsymbol{M}, \mathbb{R})$$

is un isomorphism for all $m \gg 0$.

Proposition B (Cartan). *For each $m \in \mathbb{N}$, one has a natural isomorphism*

$$H^n(\mathbb{E}\boldsymbol{G}(m) \times_{\boldsymbol{G}} \boldsymbol{M}, \mathbb{R}) \cong H^*\big(\Omega_{\boldsymbol{G}}(\mathbb{E}\boldsymbol{G}(m) \times \boldsymbol{M}), d_{\boldsymbol{G}}\big)$$

Proposition C (Homotopic invariance). *Let $f : \boldsymbol{M} \to \boldsymbol{N}$ be a $\boldsymbol{G}$-equivariant map between $\boldsymbol{G}$-manifolds such that $f^* : H^\ell_{\mathrm{dR}}(\boldsymbol{N}) \to H^\ell_{\mathrm{dR}}(\boldsymbol{M})$ is an isomorphism for all $\ell \leqslant 2N$, then the induced pull-back map*

$$f^\ell : H^\ell\big(\Omega_{\boldsymbol{G}}(\boldsymbol{N}), d_{\boldsymbol{G}}\big) \to H^\ell\big(\Omega_{\boldsymbol{G}}(\boldsymbol{M}), d_{\boldsymbol{G}}\big)$$

is an isomorphism for all $\ell < N$.

1.3. The Cartan model theorem then follows from the natural diagram

$$\begin{array}{ccc} \operatorname*{limproj}_m H(\mathbb{E}\boldsymbol{G}(m) \times_{\boldsymbol{G}} \boldsymbol{M}, \mathbb{R}) & \xrightarrow[\text{prop. B}]{\cong} & \operatorname*{limproj}_m H\big(\Omega_{\boldsymbol{G}}(\mathbb{E}\boldsymbol{G}(m) \times \boldsymbol{M}), d_{\boldsymbol{G}}\big) \\ \cong \big\uparrow \text{prop. A} & & \cong \big\uparrow \text{prop. C} \\ H(\mathbb{E}\boldsymbol{G} \times_{\boldsymbol{G}} \boldsymbol{M}, \mathbb{R}) & & H\big(\Omega_{\boldsymbol{G}}(\boldsymbol{M}), d_{\boldsymbol{G}}\big) \end{array}$$

where the "prop. C" arrow is induced by the projections $\mathbb{E}\boldsymbol{G}(m) \times \boldsymbol{M} \twoheadrightarrow \boldsymbol{M}$, $(x, m) \mapsto m$.

The fact that the arrows in this diagram are bijections is a consequence of propositions A, B, C.

§2. Proposition A

The singular cohomologies of $\mathbb{E}\boldsymbol{G}\times_{\boldsymbol{G}}\boldsymbol{M}$ and of $\mathbb{E}\boldsymbol{G}(m)\times_{\boldsymbol{G}}\boldsymbol{M}$ are just the $\boldsymbol{G}$-equivariant cohomologies of $\mathbb{E}\boldsymbol{G}\times\boldsymbol{M}$ and $\mathbb{E}\boldsymbol{G}(m)\times\boldsymbol{M}$, as the group $\boldsymbol{G}$ acts freely in these topological spaces. On the other and, the natural map between the fibred spaces over $\mathbb{B}\boldsymbol{G}$

$$\begin{array}{ccc}
\mathbb{E}\boldsymbol{G}(m)\times\boldsymbol{M} & \xrightarrow[\subseteq]{\qquad} & \mathbb{E}\boldsymbol{G}\times\boldsymbol{M} \\
\downarrow & & \downarrow \\
\mathbb{E}\boldsymbol{G}\times_{\boldsymbol{G}}(\mathbb{E}\boldsymbol{G}(m)\times\boldsymbol{M}) & \xrightarrow[\subseteq]{} & \mathbb{E}\boldsymbol{G}\times_{\boldsymbol{G}}(\mathbb{E}\boldsymbol{G}\times\boldsymbol{M}) \\
\downarrow & & \downarrow \\
\mathbb{B}\boldsymbol{G} & = & \mathbb{B}\boldsymbol{G}
\end{array}$$

induces a morphism of the Leray spectral sequences associated to these fibrations, which coincides with the natural restriction map

$$\rho_2^{p,q}: H^p(\mathbb{B}\boldsymbol{G})\otimes H^q(\mathbb{E}\boldsymbol{G}\times\boldsymbol{M}) \longrightarrow H^p(\mathbb{B}\boldsymbol{G})\otimes H^q(\mathbb{E}\boldsymbol{G}(m)\times\boldsymbol{M})$$

at the $\mathbb{E}_2^{p,q}$ terms.

Standard arguments show then that if m is sufficiently large (in fact if $m\geqslant 2n$), the induced morphisms $\rho_r^{p,q}$, with $n=p+q$, will be isomorphic between the corresponding subsequent terms $\mathbb{E}_r^{p,q}$, for all $r\geqslant 2$. This implies that the induced map $\rho_\infty^{*,*}$ on $\bigoplus_{n=p+q}\mathbb{E}_\infty^{p,q}$ is also bijective, and proposition A follows since this map is the graded morphism induced by the restriction map

$$H^n(\mathbb{E}\boldsymbol{G}\times_{\boldsymbol{G}}(\mathbb{E}\boldsymbol{G}\times\boldsymbol{M}),\mathbb{R}) \longrightarrow H^n(\mathbb{E}\boldsymbol{G}\times_{\boldsymbol{G}}(\mathbb{E}\boldsymbol{G}(m)\times\boldsymbol{M}),\mathbb{R})$$

filtered by a finite decreasing filtration.

§3. Proposition B

This is Cartan's theorem for principal $\boldsymbol{G}$-bundles.

§4. Proposition C

4.1. Symmetrization. A very particular feature concerns the $\boldsymbol{G}$-modules $\Omega^\ell(\boldsymbol{M})$.

Proposition (Symmetrizing operator). *Let $\boldsymbol{G}$ be a compact Lie group endowed with the Haar measure. Let $\boldsymbol{M}$ be a $\boldsymbol{G}$-manifold. Let $\boldsymbol{V}$ be a finite dimensional $\boldsymbol{G}$-module. For $\ell\in\mathbb{N}$, endow $\boldsymbol{V}\otimes\Omega^\ell(\boldsymbol{M})$ with the diagonal action of $\boldsymbol{G}$,* i.e. $g\cdot(v\otimes\omega)=g\cdot v\otimes g\cdot\omega$.

a) *The map* $\mathfrak{S}:\boldsymbol{V}\otimes\Omega^\ell(\boldsymbol{M})\to\boldsymbol{V}\otimes\Omega^\ell(\boldsymbol{M})$

$$\mathfrak{S}(v\otimes\omega)=\int_{\boldsymbol{G}} g\cdot(v\otimes\omega)\,dg$$

is well defined and verifies:

i) $(\mathbf{id}\otimes d_{\boldsymbol{M}})\circ\mathfrak{S}=\mathfrak{S}\circ(\mathbf{id}\otimes d_{\boldsymbol{M}})$.

ii) $\mathfrak{S}(\boldsymbol{V}\otimes\boldsymbol{K}^\ell)=\big(\boldsymbol{V}\otimes\boldsymbol{K}^\ell\big)^{\boldsymbol{G}}$, *where* $\boldsymbol{K}^\ell\in\{\Omega^\ell(\boldsymbol{M}),\boldsymbol{B}^\ell(\boldsymbol{M}),\boldsymbol{Z}^\ell(\boldsymbol{M}),H^\ell_{\mathrm{dR}}(\boldsymbol{M})\}$

iii) $\mathfrak{S}^2=\mathfrak{S}$,

b) *If $\boldsymbol{G}$ is connected, there exist a canonical isomorphism*

$$H\big((\boldsymbol{V}\otimes\Omega^*(\boldsymbol{M}))^{\boldsymbol{G}},\mathbf{id}\otimes d_{\boldsymbol{M}}\big)\cong\boldsymbol{V}^{\boldsymbol{G}}\otimes H^*_{\rm dR}(\boldsymbol{M})$$

Proof. Claim (a) is standard. For (b), let's denote $\Omega^\ell=\Omega^\ell(\boldsymbol{M})$, $\boldsymbol{B}^\ell=\boldsymbol{B}^\ell(\boldsymbol{M})$ and $\boldsymbol{Z}^\ell=\boldsymbol{Z}^\ell(\boldsymbol{M})$. One then has the following sequence of inclusions and surjections which is exact at the Ω's terms:

$$\boldsymbol{B}^{\ell-1}\underset{\subseteq}{\longrightarrow}\boldsymbol{Z}^{\ell-1}\underset{\subseteq}{\longrightarrow}\Omega^{\ell-1}\xrightarrow[\twoheadrightarrow]{d_M}\boldsymbol{B}^\ell\underset{\subseteq}{\longrightarrow}\boldsymbol{Z}^\ell\underset{\subseteq}{\longrightarrow}\Omega^\ell\xrightarrow[\twoheadrightarrow]{d_M},$$

giving rise to the analog sequence of $\boldsymbol{G}$-modules

$$\boldsymbol{V}\otimes\boldsymbol{B}^{\ell-1}\underset{\subseteq}{\longrightarrow}\boldsymbol{V}\otimes\boldsymbol{Z}^{\ell-1}\underset{\subseteq}{\longrightarrow}\boldsymbol{V}\otimes\Omega^{\ell-1}\xrightarrow[\twoheadrightarrow]{\mathbf{id}\otimes d_M}\boldsymbol{V}\otimes\boldsymbol{B}^\ell\underset{\subseteq}{\longrightarrow}\boldsymbol{V}\otimes\boldsymbol{Z}^\ell\underset{\subseteq}{\longrightarrow}\boldsymbol{V}\otimes\Omega^\ell\xrightarrow[\twoheadrightarrow]{\mathbf{id}\otimes d_M}$$

as $\boldsymbol{V}\otimes(\_)$ is an exact functor. Now, if we take $\boldsymbol{G}$-invariants, the map

$$(\boldsymbol{V}\otimes\Omega^{\ell-1})^{\boldsymbol{G}}\xrightarrow[\twoheadrightarrow]{\mathbf{id}\otimes d_M}(\boldsymbol{V}\otimes\boldsymbol{B}^\ell)^{\boldsymbol{G}}\tag{$\diamond$}$$

remains onto. Indeed, if $\sum_\alpha v_\alpha\otimes\omega_\alpha\in(\boldsymbol{V}\otimes\boldsymbol{B}^\ell)^{\boldsymbol{G}}$, choose $\nu_\alpha\in\Omega^{\ell-1}$ such that $\omega_\alpha=d_{\boldsymbol{M}}(\nu_\alpha)$. We then have after (a)

$$(\mathbf{id}\otimes d_{\boldsymbol{M}})\big(\mathfrak{S}(\textstyle\sum_\alpha v_\alpha\otimes\nu_\alpha)\big)=\mathfrak{S}(\mathbf{id}\otimes d_{\boldsymbol{M}})\big(\sum_\alpha v_\alpha\otimes\nu_\alpha\big)=\mathfrak{S}(\sum_\alpha v_\alpha\otimes\omega_\alpha)=\sum_\alpha v_\alpha\otimes\omega_\alpha\,,$$

proving that ($\diamond$) is onto. One gets in this way a sequence of injections and surjections

$$(\boldsymbol{V}\otimes\boldsymbol{B}^{\ell-1})^{\boldsymbol{G}}\underset{\subseteq}{\to}(\boldsymbol{V}\otimes\boldsymbol{Z}^{\ell-1})^{\boldsymbol{G}}\underset{\subseteq}{\to}(\boldsymbol{V}\otimes\Omega^{\ell-1})^{\boldsymbol{G}}\underset{\twoheadrightarrow}{\to}(\boldsymbol{V}\otimes\boldsymbol{B}^\ell)^{\boldsymbol{G}}\underset{\subseteq}{\to}(\boldsymbol{V}\otimes\boldsymbol{Z}^\ell)^{\boldsymbol{G}}\underset{\subseteq}{\to}(\boldsymbol{V}\otimes\Omega^\ell)^{\boldsymbol{G}}\underset{\twoheadrightarrow}{\to}$$

which is again exact at the Ω's terms. This fact immediately gives a canonical isomorphism

$$H^\ell\big((\boldsymbol{V}\otimes\Omega^*)^{\boldsymbol{G}},\mathbf{id}\otimes d_{\boldsymbol{M}}\big)\longrightarrow(\boldsymbol{V}\otimes\boldsymbol{Z}^\ell)^{\boldsymbol{G}}/(\boldsymbol{V}\otimes\boldsymbol{B}^\ell)^{\boldsymbol{G}}.\tag{$*$}$$

On the other hand, and for the same reasons as above, one has the exact sequence

$$\mathbf{0}\to\boldsymbol{V}\otimes\boldsymbol{B}^\ell\underset{\subseteq}{\longrightarrow}\boldsymbol{V}\otimes\boldsymbol{Z}^\ell\underset{\twoheadrightarrow}{\longrightarrow}\boldsymbol{V}\otimes H^\ell\to\mathbf{0}\,,$$

where $H^\ell:=H^\ell_{\rm dR}(\boldsymbol{M})$. And, using the symmetrizing operator $\mathfrak{S}$ over $\boldsymbol{V}\otimes\boldsymbol{Z}^\ell$ as we did in the previous paragraph, we get the exactness of the sequence

$$\mathbf{0}\to(\boldsymbol{V}\otimes\boldsymbol{B}^\ell)^{\boldsymbol{G}}\underset{\subseteq}{\longrightarrow}(\boldsymbol{V}\otimes\boldsymbol{Z}^\ell)^{\boldsymbol{G}}\underset{\twoheadrightarrow}{\longrightarrow}(\boldsymbol{V}\otimes H^\ell)^{\boldsymbol{G}}\to\mathbf{0}\,,$$

which shows that the right hand side of ($*$) is just $(\boldsymbol{V}\otimes H^\ell)^{\boldsymbol{G}}$. We then have

$$H^\ell\big((\boldsymbol{V}\otimes\Omega^*)^{\boldsymbol{G}},\mathbf{id}\otimes d_{\boldsymbol{M}}\big)=(\boldsymbol{V}\otimes H^\ell)^{\boldsymbol{G}}=\boldsymbol{V}^{\boldsymbol{G}}\otimes H^\ell\,,$$

as the action of $\boldsymbol{G}$ on H^ℓ is trivial because $\boldsymbol{G}$ is connected. ∎

4.2. Proof of C. Put $\Omega_{\boldsymbol{G}}(\boldsymbol{M})_i=(S^{\geqslant i}(\mathfrak{g}^\vee)\otimes\Omega)^{\boldsymbol{G}}$, where $S^{\geqslant i}(\mathfrak{g}^\vee)$ denotes the ideal of $S(\mathfrak{g}^\vee)$ generated by the products of i elements of $\mathfrak{g}^\vee$. Each $\Omega_{\boldsymbol{G}}(\boldsymbol{M})_i$ is clearly a $d_{\boldsymbol{G}}$ subcomplex of $\Omega_{\boldsymbol{G}}(\boldsymbol{M})$ and we get a decreasing filtration

$$\Omega_{\boldsymbol{G}}(\boldsymbol{M})=\Omega_{\boldsymbol{G}}(\boldsymbol{M})_0\supseteq\Omega_{\boldsymbol{G}}(\boldsymbol{M})_1\supseteq\cdots\supseteq\Omega_{\boldsymbol{G}}(\boldsymbol{M})_i\supseteq\cdots\tag{$\diamond\diamond$}$$

which is *regular* ([1]) as one has $\Omega^\ell_{\boldsymbol{G}}(\boldsymbol{M})\cap\Omega_{\boldsymbol{G}}(\boldsymbol{M})_i=0$, for $i>\ell$. As usual, this data generates a

[1] Our reference on spectral sequences is: Godement, *Topologie algébrique et théorie des Faisceaux*, pp. 75-89.

spectral sequence $\mathbb{E}(\boldsymbol{M})_*$ whose first term is

$$\mathbb{E}(\boldsymbol{M})_0^{p,q} = (S^p(\mathfrak{g}^\vee)\otimes\Omega^q(\boldsymbol{M}))^{\boldsymbol{G}}, \qquad d_0 = (\mathbf{id}\otimes d_{\boldsymbol{M}}) : \mathbb{E}(\boldsymbol{M})_0^{p,q} \to \mathbb{E}(\boldsymbol{M})_0^{p,q+1},$$

so that one gets

$$\boxed{\mathbb{E}(\boldsymbol{M})_1^{p,q} = S^p(\mathfrak{g}^\vee)^{\boldsymbol{G}}\otimes H^q_{\mathrm{dR}}(\boldsymbol{M})}$$

as a consequence of (b) in the symmetrizing operator theorem.

Now, given a differentiable $\boldsymbol{G}$-equivariant map $f:\boldsymbol{M}\to\boldsymbol{N}$ one gets a morphism of Cartan complexes $f^*:\Omega^*_{\boldsymbol{G}}(\boldsymbol{N})\to\Omega^*_{\boldsymbol{G}}(\boldsymbol{M})$ such that $f^*(\Omega^*_{\boldsymbol{G}}(\boldsymbol{N})_i)\subseteq(\Omega^*_{\boldsymbol{G}}(\boldsymbol{M})_i)$ for all $i\in\mathbb{N}$, inducing thereafter a morphism of spectral sequences

$$f_r^{*,\bullet}:\mathbb{E}(\boldsymbol{N})_r^{*,\bullet}\to\mathbb{E}(\boldsymbol{M})_r^{*,\bullet}$$

which, for $r=1$, takes the value

$$f_1^{*,\bullet} = \mathbf{id}^*\otimes f^\bullet : S^*(\mathfrak{g}^\vee)^{\boldsymbol{G}}\otimes H^\bullet_{\mathrm{dR}}(\boldsymbol{N}) \longrightarrow S^*(\mathfrak{g}^\vee)^{\boldsymbol{G}}\otimes H^\bullet_{\mathrm{dR}}(\boldsymbol{M}).$$

Standard arguments then show that if $f^\bullet : H^\bullet_{\mathrm{dR}}(\boldsymbol{N})\to H^\bullet_{\mathrm{dR}}(\boldsymbol{M})$ is bijective in degrees $\leqslant 2N$, then

$$f_\infty^{p,q}:\mathbb{E}(\boldsymbol{N})_\infty^{p,q}\longrightarrow\mathbb{E}(\boldsymbol{M})_\infty^{p,q}$$

is bijective for $p+q\leqslant N$.

Recall now (*loc.cit.* thm. 4.4.2) that the filtration ($\diamond\diamond$) gives the sequence of morphisms

$$H(\Omega_{\boldsymbol{G}}(\boldsymbol{M})) = H(\Omega_{\boldsymbol{G}}(\boldsymbol{M})_0)\leftarrow H(\Omega_{\boldsymbol{G}}(\boldsymbol{M})_1)\leftarrow\cdots\leftarrow H(\Omega_{\boldsymbol{G}}(\boldsymbol{M})_i)\leftarrow\cdots$$

and that if we denote by $H(\Omega_{\boldsymbol{G}}(\boldsymbol{M}))_i$ the image of $H(\Omega_{\boldsymbol{G}}(\boldsymbol{M})_i)$ in $H(\Omega_{\boldsymbol{G}}(\boldsymbol{M}))$, we get a decreasing filtration

$$H(\Omega_{\boldsymbol{G}}(\boldsymbol{M})) = H(\Omega_{\boldsymbol{G}}(\boldsymbol{M}))_0\supseteq H(\Omega_{\boldsymbol{G}}(\boldsymbol{M}))_1\supseteq\cdots\leftarrow H(\Omega_{\boldsymbol{G}}(\boldsymbol{M}))_i\supseteq\cdots,$$

such that the spectral series $\mathbb{E}(\boldsymbol{M})_*$ converges to $\mathbb{E}_\infty(\boldsymbol{M}) = \mathrm{Gr}\,H(\Omega_{\boldsymbol{G}}(\boldsymbol{M}))_\star$. The conclusion of the last paragraph maybe then restated by saying that the map

$$\mathrm{Gr}\,f^*:\mathrm{Gr}\,H(\Omega_{\boldsymbol{G}}(\boldsymbol{N}))_\star\to\mathrm{Gr}\,H(\Omega_{\boldsymbol{G}}(\boldsymbol{M}))_\star$$

is bijective in total degrees bounded above by N, and the same for

$$f^*:H(\Omega_{\boldsymbol{G}}(\boldsymbol{N}))\to H(\Omega_{\boldsymbol{G}}(\boldsymbol{M})),$$

since the filtrations are regular.

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