On the equivalence of two stability conditions of FB-modules

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May 16, 2018

Abstract. We give a proof of the equivalence of two stability conditions in the work of Thomas Church and Benson Farb on FB-modules ([1]): *representation stability* (RS) and *character polynomiality* (PC). While the implication (RS) ⇒ (PC) is a simple consequence of the Frobenius character formula, the converse seems not to have been documented, which motivated this work.

An FB-module is a countable family $\mathcal{W} := \{W_m\}_{m\in\mathbb{N}}$ of linear representations W_m of the symmetric groups $\mathcal{S}_m$, assumed, in these notes, to be finite dimensional over the field of rational numbers $\mathbb{Q}$, hereafter called *an* $\mathcal{S}_m$*-module*.

• The FB-module $\mathcal{W}$ is said to be *representation stable* (RS in short), if there exists some $N \in \mathbb{N}$, such that, for all $m \geq N$, the decomposition in simple factors of W_m has the *stable* form:

$$W_m \sim \bigoplus\nolimits_\lambda {V_{\lambda[m]}}^{\boldsymbol{n}_\lambda},$$

where

- ▷ $\lambda := (\lambda_1 \geq \ldots \geq \lambda_\ell)$ is a partition verifying $|\lambda| + \lambda_1 \leq N$;
- ▷ $V_{\lambda[m]}$ is the simple $\mathcal{S}_m$-module associated with $\lambda[m] := (m - |\lambda| \geq \lambda_1 \geq \ldots \geq \lambda_\ell)$;
- ▷ for $m \geq N$, the family of natural numbers $\{\boldsymbol{n}_\lambda\}_\lambda$ is *constant.*

The smallest such N is *the rank of* (RS) *of* $\mathcal{W}$, which we will denote by '$\mathrm{rank}_{\mathrm{RS}}(\mathcal{W})$'.

• The FB-module $\mathcal{W}$ is said to *have a polynomial character* (PC in short), if there exists $N \in \mathbb{N}$ and a polynomial $P_\mathcal{W} \in \mathbb{Q}[X_1, \ldots, X_N]$, such that, for all $m \geq N$ and all $g \in \mathcal{S}_m$, one has:

$$\chi_{W_m}(g) = P_\mathcal{W}(X_1(g), \ldots, X_N(g)),$$

where χ_{W_m} is the character of W_m, and $X_i(g)$ is the number of cycles of length i in the decomposition of g in $\mathcal{S}_m$ as a product of disjoint cycles. The polynomial $P_\mathcal{W}$, which is unique, is *the polynomial character of* $\mathcal{W}$. The smallest such N is *the rank of character polynomiality of* $\mathcal{W}$, it will be denoted by '$\mathrm{rank}_{\mathrm{PC}}(\mathcal{W})$'.

Regarding the equivalence of these conditions, we prove the following theorem.

Theorem (4.1.1). *Let* $\mathcal{W}$ *be an* FB-*module.*

a) $\mathrm{rank}_{\mathrm{PC}}(\mathcal{W}) \leq \mathrm{rank}_{\mathrm{RS}}(\mathcal{W})$.

b) *If* $\mathcal{W}$ *is* (PC) *and has polynomial character* $P_\mathcal{W}$*, then*

$$\mathrm{rank}_{\mathrm{RS}}(\mathcal{W}) \leq \max\{\mathrm{rank}_{\mathrm{PC}}(\mathcal{W}), 2\deg_{\boldsymbol{w}}(P_\mathcal{W})\},$$

where $\deg_{\boldsymbol{w}}(P_\mathcal{W})$ *denotes the polynomial degree of* $P_\mathcal{W}$*, when stipulating that* $\deg_{\boldsymbol{w}}(X_i) := i$.

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Comments. (i) Theorem (2.3.4) exhibit FB-modules $\mathcal{W}$ with $\text{rank}_{\text{PC}}(\mathcal{W}) = 0$ and arbitrarily big $\text{rank}_{\text{RS}}(\mathcal{W})$. It also shows that the bounds in (4.1.1) are optimal. (ii) Beyond FB-modules, we also look at FI-modules $\mathcal{W}$, which are FB-modules with additional structure (1.2). There is a corresponding representation stability condition for FI-modules, its rank is denoted by $\text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{W})$ (1.7.1). One has $\text{rank}_{\text{RS}}(\mathcal{W}) \leq \text{rank}_{\text{RS}}^{\text{FI}}(\mathcal{W})$, but there is no equivalence with (PC) stability, in particular thm. 4.1.1-(b) for FI-modules and $\text{rank}_{\text{RS}}^{\text{FI}}$ is false.

An interesting by-product in this work is the fact that the *weight* $\boldsymbol{w}(W_m)$ of an $\mathcal{S}_m$-module W_m (see 1.5) coincides with the lowest possible degree $\deg_{\boldsymbol{w}}(P)$ of a polynomial P expressing the character χ_{W_m} (3.1.1). Moreover, if $\mathcal{W} := \{W_m\}$ is a (PC) FB-module, then $\deg_{\boldsymbol{w}}(P_{\mathcal{W}}) = \boldsymbol{w}(W_m)$, for all $m \geq \text{rank}_{\text{RS}}(\mathcal{W})$. The next theorem follows from these observations.

Proposition 5.2.1-(b). *If $\mathcal{W}_1$ and $\mathcal{W}_2$ are* FB*-modules which are* $(\text{PC})_{m\geq N}$, *then*

$$\boldsymbol{w}(W_{1,m} \otimes W_{2,m}) = \boldsymbol{w}(W_{1,m}) + \boldsymbol{w}(W_{2,m}),$$

for all $m \geq 2(\boldsymbol{w}(W_{1,m}) + \boldsymbol{w}(W_{2,m}))$. In particular,

$$\boldsymbol{w}((\mathcal{W}_1 \otimes \mathcal{W}_2)_{\geq N}) = \boldsymbol{w}((\mathcal{W}_1)_{\geq N}) + \boldsymbol{w}((\mathcal{W}_2)_{\geq N}),$$

where $(\mathcal{W})_{\geq N}$ denotes the truncation *of $\mathcal{W}$ that replaces W_n by 0 for all $n < N$.*

Contents

The first two sections are elementary. They introduce notations and well-known properties of FI and FB-modules, which advanced readers can skip.

1. Preliminaries

1.1. General notations

- Given a group G, we denote by G-mod the category of G-modules, *i.e.* of finite dimensional linear representations of G over the field of rational numbers $\mathbb{Q}$.
- We denote by $\mathcal{S}_m$ the *symmetric group* defined over $[\![1, m]\!]$. Its elements are the bijections of the interval of natural numbers $[\![1, m]\!]$. For $n \le m$, we write $\mathcal{S}_n \subseteq \mathcal{S}_m$ for the inclusion which identifies a permutation g of $[\![1, n]\!]$ with its obvious extension to $[\![1, m]\!]$ that fixes all $i > n$.
- $\mathbb{Q}_m$ and $\epsilon\mathbb{Q}_m$ denote respectively the *trivial* and the *alternating or signature* $\mathbb{Q}$-linear representation of the group $\mathcal{S}_m$.
- $\mathcal{S}_a \boxtimes \mathcal{S}_b$ denotes the subgroup of $\mathcal{S}_{a+b}$ stabilizing the subset $[\![1, a]\!]$, or, equivalently, the subset $[\![a+1, a+b]\!]$. The subgroup $\mathbf{1}_a \boxtimes \mathcal{S}_b \subseteq \mathcal{S}_a \boxtimes \mathcal{S}_b$ is the *fixator* of $[\![1, a]\!]$, *i.e.* the set of permutations fixing every $i \le a$.
- If W_a is an $\mathcal{S}_a$-module and W_b is an $\mathcal{S}_b$-module, we denote by $W_a \boxtimes W_b$ the tensor product $W_a \otimes_{\mathbb{Q}} W_b$ endowed with the $\mathcal{S}_a \boxtimes \mathcal{S}_b$-module structure defined by the componentwise action $(g_a, g_b)(w_a \otimes w_b) := g_a(w_a) \otimes g_b(w_b)$.
- A *partition* of $m \in \mathbb{N}\backslash\{0\}$ is any decreasing sequence of natural numbers $\lambda := (\lambda_1 \ge \ldots \ge \lambda_\ell > 0)$ such that $m = \sum_i \lambda_i$. It is also denoted as the m-tuple $(1^{\boldsymbol{n}_1}, 2^{\boldsymbol{n}_2}, \ldots, m^{\boldsymbol{n}_m})$ where $\boldsymbol{n}_k := \#\{i \mid \lambda_i = k\}$, so that $m = \sum_i i\, \boldsymbol{n}_i$. The notation $\lambda \vdash m$ says that λ is a partition of m, and $|\lambda|$ is used for the number partitioned by λ. The partition λ is *empty* if $|\lambda| = 0$.
- $\mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m)$ denotes the $\mathbb{Q}$-algebra of *class functions* of $\mathcal{S}_m$. These are the functions $f : \mathcal{S}_m \to \mathbb{Q}$ constant along the conjugacy classes of $\mathcal{S}_m$, *i.e.* such that $f(gxg^{-1}) = f(x)$, $\forall g, x \in \mathcal{S}_m$. The scalar product
$$\langle\!\langle \_ \mid \_ \rangle\!\rangle_{\mathcal{S}_m} : \mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m) \times \mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m) \to \mathbb{Q}$$
is defined by
$$\langle\!\langle f_1 \mid f_2 \rangle\!\rangle_{\mathcal{S}_m} := \frac{1}{|\mathcal{S}_m|} \sum_{g \in \mathcal{S}_m} f_1(g)\, f_2(g^{-1}), \quad \forall f_1, f_2 \in \mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m).$$
- If W_m is an $\mathcal{S}_m$-module, $\chi_{W_m} : \mathcal{S}_m \to \mathbb{Q}$ denotes its character. The *Schur's orthogonality relations* state that if V_1 and V_2 are *simple* $\mathcal{S}_m$-modules, then $\langle\!\langle \chi_{V_1} \mid \chi_{V_2} \rangle\!\rangle_{\mathcal{S}_m}$ is equal to 1 if V_1 is isomorphic to V_2, and to 0 otherwise.

Between here and the end of this preliminary section we recall concepts and terminology from the works of Church and Farb (*cf.* [1]).

1.2. The categories of FB and FI modules

- **FB** denotes the category of Finite sets and Bijections. An FB-*module* is, by definition, a *covariant* functor from the category FB to the category $\mathbf{Vec}_f(\mathbb{Q})$ of finite dimensional $\mathbb{Q}$-vector spaces and $\mathbb{Q}$-linear maps:

$$\mathcal{W} : \mathrm{FB} \rightsquigarrow \mathbf{Vec}_f(\mathbb{Q}) .$$

To give an FB-module $\mathcal{W}$ is equivalent to giving the countable collection $\{W_m := \mathcal{W}([\![1, m]\!])\}_{m\in\mathbb{N}}$, where W_m is an $\mathcal{S}_m$-module. A morphism of FB-modules $f : \mathcal{W} \to \mathcal{Z}$ corresponds to a family $\{f_m : W_m \to Z_m\}_{m\in\mathbb{N}}$ of morphisms of $\mathcal{S}_m$-modules. We have a canonical identification:

$$\mathrm{Mor}_{\mathrm{FB}}(\mathcal{W}, \mathcal{Z}) = \prod\nolimits_{m\in\mathbb{N}} \mathrm{Hom}_{\mathcal{S}_m}(W_m, Z_m)$$

Notation. The category of FB-modules will be denoted by FB-mod. It is a semi-simple abelian category.

- **FI** denotes the category of Finite sets and Injections. An FI-*module* is a *covariant functor* from FI to $\mathbf{Vec}_f(\mathbb{Q})$:

$$\mathcal{W} : \mathrm{FI} \rightsquigarrow \mathbf{Vec}_f(\mathbb{Q}) .$$

To give an FI-module is equivalent to giving

FI-1) an FB-module $\mathcal{W} := \{W_m\}_{m\in\mathbb{N}}$;

FI-2) for all $m \in \mathbb{N}$, an *interior* map $\phi(\mathcal{W})_m : W_m \to W_{m+1}$ (in short ϕ_m), which is $\mathbb{Q}$-linear and such that, for all $g \in \mathcal{S}_m \subseteq \mathcal{S}_{m+1}$, one has

$$\phi_m(g \cdot w) = g \cdot \phi_m(w) .$$

FI-3) for all $n \geq m$, the image of $\phi_{n,m} := \phi_{n-1} \circ \cdots \circ \phi_m$ satisfies:

$$\phi_{n,m}(W_m) \subseteq (W_n)^{\mathbf{1}_m \boxtimes \mathcal{S}_{n-m}} .$$

Under this equivalence, a morphism of FI-modules $f : \mathcal{W} \to \mathcal{Z}$ is simply a morphism of FB-modules which is compatible with the interior maps ϕ_m, *i.e.* such that the diagrams

$$\begin{array}{ccc} W_m & \xrightarrow{\phi(\mathcal{W})_m} & W_{m+1} \\ {\scriptstyle f_m}\downarrow & & \downarrow{\scriptstyle f_{m+1}} \\ Z_m & \xrightarrow{\phi(\mathcal{Z})_m} & Z_{m+1} \end{array}$$

are commutative.

Notation. The category of FI-modules will be denoted by FI-mod. It is an abelian category, which is *not* semi-simple.

1.2.1. Comments

a) The category FB-mod is equivalent to the full subcategory of FI-mod of FI-modules whose interior maps are null. The *forgetful functor* associates to an FI-module $\mathcal{W} := \{\phi_m : W_m \to W_{m+1}\}$ the FB-module $\mathcal{W} := \{W_m\}$. It is an additive and exact functor which will be tacitly used.

b) More interesting, the subcategory FI-mod$'$ of FI-modules whose interior maps are *injective* and *(eventually) exhaustive*, *i.e.* such that the image $\phi_m(W_m)$ generates W_{m+1} as $\mathcal{S}_{m+1}$-module for large m. Among these, the FI-modules $\mathcal{V}_\lambda$'s which bind together the simple modules $V_{\lambda[m]}$ (1.6.1). We will see that FB-modules $\mathcal{W}$ which are (PC) or (RS) are asymptotically isomorphic to direct sums of $\mathcal{V}_\lambda$'s (4.1.2).

1.2.2. The truncations

([1]) The functor $(\_)_{\geq \ell}$: FI-mod $\rightsquigarrow$ FI-mod that "*truncates*" an FI-module $\{W_m\}_{m\in\mathbb{N}}$ by replacing by $\mathbf{0}$ its terms W_m for $m < \ell$, is an additive exact functor. The *cokernel* of the natural inclusion $(\_)_{\geq \ell} \hookrightarrow \mathrm{id}_{\mathrm{FI}}$ is the truncation $(\_)_{<\ell}$, it replaces by $\mathbf{0}$ the terms W_m for $m \geq \ell$. We thus have short exact sequences

$$\mathbf{0} \to (\mathcal{W})_{\geq \ell} \rightarrowtail \mathcal{W} \twoheadrightarrow (\mathcal{W})_{<\ell} \to \mathbf{0}\,,$$

which are natural with respect to $\mathcal{W}$. The full subcategory FI-mod$_{\geq \ell}$ of FI-modules $\mathcal{W}$ such that the inclusion $\mathcal{W}_{\geq \ell} \rightarrowtail \mathcal{W}$ is an isomorphism is an abelian subcategory, and similarly for the full subcategory FI-mod$_{<\ell}$ of FI-modules $\mathcal{W}$ such that the quotient $\mathcal{W} \twoheadrightarrow \mathcal{W}_{<\ell}$ is an isomorphism. One has,

$$\mathrm{Ext}^i_{\mathrm{FI}}\big(\text{FI-mod}_{\geq \ell}, \text{FI-mod}_{\leq \ell}\big) = 0\,, \quad \forall i > 0\,.$$

The intersection FI-mod$_{\geq \ell} \cap$ FI-mod$_{\leq \ell}$ is the (semi-simple) category $\mathcal{S}_\ell$-mod.

1.3. Projective FI-modules

An obvious way to construct FI-modules of the type described in 1.2.1-(b) is to start with à representation W_n of some $\mathcal{S}_n$, and then set, for all $m \in \mathbb{N}$:

$$\mathcal{P}(W_n)_m := \begin{cases} \mathbf{0}\,, & \text{if } m < n, \\ \mathrm{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m-n}}^{\mathcal{S}_m}\big(W_n \boxtimes \mathbb{Q}_{m-n}\big)\,, & \text{otherwise.} \end{cases}$$

For each $m \geq n$, the composition of the following natural maps ι_m and κ_{m+1} :

$$\begin{array}{ccc} W_n \boxtimes \mathbb{Q}_{m-n} & \xhookrightarrow{\ \iota_m\ } & W_n \boxtimes \mathbb{Q}_{m+1-n} \\ & \overset{\psi_m}{\searrow} & \big\downarrow \kappa_{m+1} \\ & & \mathrm{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m+1-n}}^{\mathcal{S}_{m+1}}\big(W_n \boxtimes \mathbb{Q}_{m+1-n}\big) \end{array}$$

[1] This truncations are frequently called *stupid* or *brutal*.

gives the map ψ_m whose image is invariant under $(\mathbf{1}_n \boxtimes \mathcal{S}_{m+1-n})$. In particular, ψ_m is $(\mathcal{S}_n \boxtimes \mathcal{S}_{m-n})$-linear, inducing thus the map

$$\phi_m := \mathrm{ind}(\psi_m) : \mathrm{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m-n}}^{\mathcal{S}_m} \left(W_n \boxtimes \mathbb{Q}_{m-n}\right) \to \mathrm{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m+1-n}}^{\mathcal{S}_{m+1}} \left(W_n \boxtimes \mathbb{Q}_{m+1-n}\right),$$

which satisfies the requirements making the family

$$\mathcal{P}(W_m) := \{\phi_m : \mathcal{P}(W_n)_m \to \mathcal{P}(W_n)_{m+1}\}_{m \in \mathbb{N}},$$

an FI-module.

All the previous constructions are natural on the category of $\mathcal{S}_n$-modules. As a consequence, we have defined a functor

$$\mathcal{P} : \mathcal{S}_n\text{-mod} \rightsquigarrow \text{FI-mod},$$

which is clearly additive and exact.

1.3.1. Proposition. *Let W_n be a representation of $\mathcal{S}_n$.*

a) *The interior maps of $\mathcal{P}(W_n)$ are injective. They are exhaustive $\forall m \geq n$.*
b) *There is a natural identification of functors*

$$\mathrm{Mor}_{\mathrm{FI}}(\mathcal{P}(W_n), \_) = \mathrm{Hom}_{\mathcal{S}_n}(W_n, (\_)_n).$$

c) *The* FI-*module $\mathcal{P}(W_n)$ is a projective* FI-*module, which moreover is simple projective if and only if W_n is a simple $\mathcal{S}_n$-module. All simple projective* FI-*modules are of this form.*
d) *The category* FI-mod *has enough projective objects.*

Proof. Left to the reader. □

1.4. Young diagrams and Pieri's rule

We recall the re-parametrization of irreducible representations of the symmetric groups due to Church and Farb.

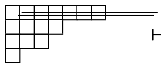
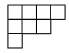
1.4.1. The *socle* and the *weight* of a partition

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ be a non-empty partition of $m := \sum_i \lambda_i$.

- The *socle* of λ is the sub-partition $\underline{\lambda} := (\lambda_2 \geq \ldots \geq \lambda_\ell)$.
- The *weight* of λ is the number $\boldsymbol{w}(\lambda) := |\underline{\lambda}| = \lambda_2 + \cdots + \lambda_\ell$.

Notice that the map $\lambda \mapsto \underline{\lambda}$ is injective from the set of partitions of $m \in \mathbb{N}$, so that it amounts the same, giving $(\lambda \vdash m)$ or giving $(m, \underline{\lambda})$ with $m \geq |\underline{\lambda}| + \underline{\lambda}_1$.

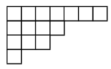
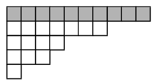
In terms of Young diagrams, in order to get the socle $\underline{\lambda}$ of λ, one simply erases the first row of λ. The picture is thus:

$\lambda :=$  $\mapsto$ $\underline{\lambda} :=$  .

Conversely, given $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell)$ and any $m \geq |\lambda| + \lambda_1$, Church and Farb introduce the notation

$$\boxed{\lambda[m] := (m - |\lambda| \geq \lambda_1 \geq \ldots \geq \lambda_\ell)}$$

which, in terms of Young diagrams, corresponds to adding a first row with as many boxes as is necessary to raise the total number of boxes from $|\lambda|$ to m.

$\lambda :=$  $\mapsto$ $\lambda[m] :=$  .

Notice the following obvious notational facts:

$$\lambda[m] \vdash m\,, \quad \underline{\lambda[m]} = \lambda\,, \quad \lambda = \underline{\lambda}[\,|\lambda|\,]\,.$$

1.4.2. Irreducible representations of the symmetric groups

The irreducible representations of $\mathcal{S}_m$ are parametrized by the partitions $\nu \vdash n$. The simple $\mathcal{S}_m$-module associated with ν is classically denoted by V_ν. The following proposition recalls a fundamental fact about the representations of the symmetric groups (see [2], chap. 4, thm. 4.3, pp. 46–).

1.4.3. Proposition. *The irreducible representations of the symmetric groups over a field k of characteristic zero are defined over the field of rational numbers. If W_m is a $k[\mathcal{S}_n]$-module of finite dimension, the values of the character*

$$\chi_{W_m} : \mathcal{S}_m \to k\,, \quad g \mapsto \mathrm{tr}(g : W_m \to W_m)\,,$$

belong to the ring of integers $\mathbb{Z} \subseteq k$.

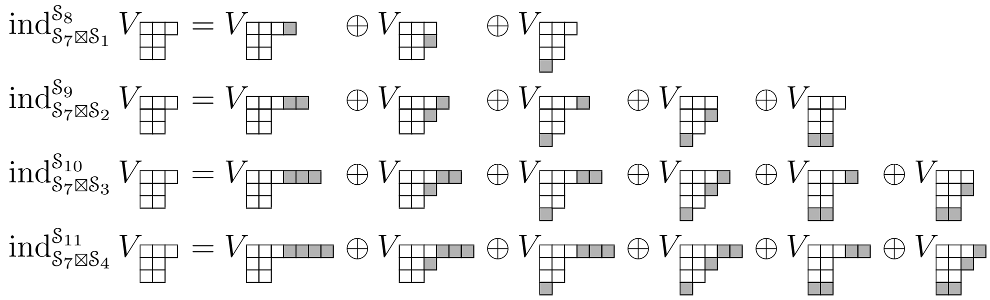
Hint. Because W_m is defined over $\mathbb{Q}$, the trace of $g \in \mathcal{S}_m$ belongs a priori to $\mathbb{Q}$, and because the trace is the sum of the eigenvalues, which are roots of unity, it is also an algebraic integer. It is therefore an integer. □

1.4.4. Pieri's rule

Pieri's rule ([2]) gives the irreducible factors of the terms $\mathcal{P}(V_\nu)_m$, for any given partition $\nu \vdash n \leq m$. The rule says that in the decomposition

$$\mathcal{P}(V_\nu)_m := \operatorname{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m-n}}^{\mathcal{S}_m} \big(V_\nu \boxtimes \mathbb{Q}_{m-n}\big) = \bigoplus_{\mu \vdash m} V_\mu^{\,\boldsymbol{n}_\nu(\mu)}\,, \tag{1}$$

the nonzero multiplicities $\boldsymbol{n}_\nu(\mu)$ are all equal to 1, and the corresponding partitions μ are those obtained from ν by adding $(m-n)$ boxes in *different* columns. For example, if $\nu := (3,2,2)$ and $m \in \{8,9,10,11\}$, we have



where, after a key observation of Church and Farb, for $m \geq |\nu| + \nu_1 (= 10$ in this example), the set of socles of the Young diagrams appearing in the decomposition (1) is constant. The following lemma obviously follows.

1.4.5. Lemma. *Let $\nu \vdash n > 0$.*

a) *For all $\mu \vdash m \geq n$, such that $\mathbf{n}_\nu(\mu) \neq 0$, the weight of μ verifies*

$$|\underline{\nu}| = \boldsymbol{w}(\nu) \leq \boldsymbol{w}(\mu) \leq |\nu|\,,$$

and

$$\begin{cases} \boldsymbol{w}(\mu) = \boldsymbol{w}(\nu) & \iff \mu = \underline{\nu}[m] \\ \boldsymbol{w}(\mu) = |\nu| & \iff \big(m \geq |\nu| + \nu_1\big)\ \&\ \big(\mu = \nu[m]\big)\,. \end{cases}$$

b) *Let $m_0 = |\nu| + \nu_1$ and let $\mathcal{P}_\nu$ be a set of partitions μ such that*

$$\operatorname{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m_0-n}}^{\mathcal{S}_{m_0}} \big(V_\nu \boxtimes \mathbb{Q}_{m_0-n})\big) = \bigoplus\nolimits_{\mu \in \mathcal{P}_\nu} V_{\mu[m_0]}\,.$$

Then, $\nu[m_0] \in \mathcal{P}_\nu$ and $\nu[m_0]_1 = \nu[m_0]_2$.

For all $m > m_0$, we have

$$\operatorname{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m-n}}^{\mathcal{S}_m} \big(V_\nu \boxtimes \mathbb{Q}_{m-n})\big) = \bigoplus\nolimits_{\mu \in \mathcal{P}_\nu} V_{\mu[m]}\,.$$

and, for all $\mu \in \mathcal{P}_\nu$, $\mu[m_0]_1 > \mu[m_0]_2$.

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[2] For a thorough introduction to these rules, read §4.3, p. 54–62, and also Appendix A on the *Littlewood-Richardson rules*, p. 451, both in Fulton-Harris' book [2].

1.5. The *weight* of a representation and of an FB-module

The definition of *weight*, is extended from partitions (1.4.1) to representations and, more generally, to FB-modules.

- The *weight of an* $\mathcal{S}_m$*-module* W_m is the upper bound of the weights of the partitions associated with its irreducible factors, *i.e.* if

$$W_m \sim \bigoplus\nolimits_{\mu\vdash m} V_\tau{}^{\boldsymbol{n}_\mu}\,,$$

  then

$$\boldsymbol{w}(W_m) := \sup\big\{\boldsymbol{w}(\mu) \;\big|\; \boldsymbol{n}_\mu \neq 0\big\}\,.$$

  For example, as a consequence of lemma 1.4.5-(a), we have

$$\begin{cases} |\underline{\nu}| = \boldsymbol{w}(\nu) \leq \boldsymbol{w}(\mathcal{P}(V_\nu))_m \leq |\nu|\,, & \forall m \geq |\nu|\,,\ \text{and} \\ \qquad\qquad\quad \boldsymbol{w}(\mathcal{P}(V_\nu))_m = |\nu|\,, & \forall m \geq |\nu| + \nu_1\,. \end{cases} \qquad (*)$$

- The *weight of an* FB-*module* $\mathcal{W} := (W_m)$ is the upper-bound of the weights of its terms, *i.e.*

$$\boldsymbol{w}(\mathcal{W}) := \sup\big\{\boldsymbol{w}(W_m)\big\}_{m\in\mathbb{N}}\,.$$

- The *weight at infinity of an* FB-*module* $\mathcal{W} := (W_m)$ is

$$\boldsymbol{w}_\infty(\mathcal{W}) := \lim_{N\mapsto+\infty} \boldsymbol{w}(\mathcal{W}_{\geq N})\,.$$

  –It is easy to see that $\boldsymbol{w}_\infty(\mathcal{W}) = \boldsymbol{w}(\mathcal{W}_{\geq N})$, for some $N \gg 0$.
  –We have $\boldsymbol{w}(\mathcal{P}(V_\lambda)) = \boldsymbol{w}_\infty(\mathcal{P}(V_\lambda)) = |\lambda|\,.$

1.5.1. The *weight* truncations

Let $p \in \mathbb{N}$. Given an $\mathcal{S}_m$-module W_m, denote by $(W_m)_{\boldsymbol{w}>p}$ the sum of the irreducible factors of W_m of weight $> p$.

- By Pieri's rule 1.4.5-(a), if $\mathcal{W} := \{\phi_m : W_m \to W_{m+1}\}_{m\in\mathbb{N}}$ is an FI-module, one has $\phi_m\big((W_m)_{\boldsymbol{w}>p}\big) \subseteq \big((W_{m+1})_{\boldsymbol{w}>p}\big)$, in which case, the family

$$\mathcal{W}_{\boldsymbol{w}>p} := \big\{\phi_m(W_m)_{\boldsymbol{w}>p} \to (W_m)_{\boldsymbol{w}>p}\big\}_{m\in\mathbb{N}}$$

  is a sub-FI-module of $\mathcal{W}$.

- Let $\mathcal{W}_{\boldsymbol{w}\leq p} := \mathcal{W}/\mathcal{W}_{\boldsymbol{w}>p}$. The short exact sequence

$$\mathbf{0} \to \mathcal{W}_{\boldsymbol{w}>p} \to \mathcal{W} \to \mathcal{W}_{\boldsymbol{w}\leq p} \to \mathbf{0}$$

  is natural with respect to $\mathcal{W}$.

1.5.2. Remark. The following are easy consequences of the definitions.

a) The full subcategory FI-mod$_{\boldsymbol{w}>p}$ of FI-modules $\mathcal{W}$ such that the inclusion $\mathcal{W}_{\boldsymbol{w}>p} \rightarrowtail \mathcal{W}$ is an isomorphism is an abelian subcategory.
b) The full subcategory FI-mod$_{\boldsymbol{w}\le p}$ of FI-modules $\mathcal{W}$ such that the quotient $\mathcal{W} \twoheadrightarrow \mathcal{W}_{\boldsymbol{w}\le p}$ is an isomorphism is an abelian subcategory.
c) $\mathrm{Ext}^i_{\mathrm{FI}}\big(\mathrm{FI\text{-}mod}_{\boldsymbol{w}>p}, \mathrm{FI\text{-}mod}_{\boldsymbol{w}\le p}\big) = 0$, for all $i \in \mathbb{N}$.

1.6. The FI-module $\mathcal{V}_\lambda$

Given a partition λ, consider the projective FI-module $\mathcal{P}(V_\lambda)$ of 1.3. Lemma 1.4.5-(a) says that for all $m \ge |\lambda|$ the smallest weight of the irreducible components of $\mathcal{P}(V_\lambda)_m$ is exactly $|\underline{\lambda}|$ $(= \boldsymbol{w}(\lambda))$, which is the weight of a unique factor : the simple $\mathcal{S}_m$-module $V_{\underline{\lambda}[m]}$ with multiplicity 1. The following proposition results from this simple observation and the weight filtration 1.5.1.

1.6.1. Proposition. *Let λ be a nonempty partition.*

a) *The terms of the quotient* FI-*module*

$$\mathcal{V}_\lambda := \mathcal{P}(V_\lambda)_{\boldsymbol{w}\le \boldsymbol{w}(\lambda)} = \big\{\phi_m : \mathcal{V}_{\lambda,m} \to \mathcal{V}_{\lambda,m+1}\big\}_{m\in\mathbb{N}}$$

are
$$\mathcal{V}_{\lambda,m} = \begin{cases} \mathbf{0} \ , & \text{for all } m < |\lambda|, \\ V_{\underline{\lambda}[m]} \ , & \text{otherwise.} \end{cases}$$

The interior maps ϕ_m are injective, and are exhaustive for $m \ge |\lambda|$.

b) *If $\mathcal{V}'_\lambda := \{\phi'_m : \mathcal{V}_{\lambda,m} \to \mathcal{V}_{\lambda,m+1}\}_{m\in\mathbb{N}}$ is an* FI-*module such that ϕ'_m is injective and is exhaustive for all $m \ge N$, then $(\mathcal{V}'_\lambda)_{\ge N}$ and $(\mathcal{V}_\lambda)_{\ge N}$ (*cf. *1.2.2) are isomorphic* FI-*modules for $m \ge N$.*

1.7. Representation stability of FI and of FB modules

1.7.1. Representation stable FI-modules

An FI-module $\mathcal{W} = \{\phi_m : W_m \to W_{m+1}\}_{m\in\mathbb{N}}$ is said to be *representation stable for $m \ge N$*, in short $(\mathrm{RS})_{m\ge N}$, if the following conditions are satisfied.

RS-1) The interior maps ϕ_m are injective, and are exhaustive for $m \ge N$.

RS-2) For all $m \ge N$, we have a stable decomposition in simple modules

$$W_m \sim \bigoplus_{|\lambda|\le N} V_{\underline{\lambda}[m]}{}^{\boldsymbol{n}_\lambda},$$

where the $\boldsymbol{n}_\lambda$ are independent of $m \ge N$.

We denote by $\mathrm{rank}^{\mathrm{FI}}_{\mathrm{RS}}(\mathcal{W})$, the smallest such N, and we call it the *rank of representation stability of $\mathcal{W}$.*

1.7.2. Examples

a) $\mathcal{P}(V_\lambda)$ is $(\mathrm{RS})_{m\geq|\lambda|+\lambda_1}$, (lemma 1.4.5-(b)).
b) $\mathcal{P}(W_n)$ $(\mathrm{RS})_{m\geq 2n}$, (consequence of lemma 1.4.5-(b)).
c) $\mathcal{V}_\lambda$ is $(\mathrm{RS})_{m\geq|\lambda|}$, (by definition).

1.7.3. Representation stable FB-modules

An FB-module $\mathcal{W}=\{W_m\}_{m\in\mathbb{N}}$ is said to be *representation stable for* $m\geq N$, in short $(\mathrm{RS})_{m\geq N}$, if the previous condition (RS-2) is satisfied.

In that case, we say that $\mathcal{W}$ and $\bigoplus_{\lambda\in P}\mathcal{V}_\lambda^{\boldsymbol{n}_\lambda}$ are *asymptotically isomorphic as* FB-*modules*, and we may write

$$\mathcal{W}_{\geq N}\sim\bigoplus\nolimits_{|\lambda|\leq N}(\mathcal{V}_\lambda)^{\boldsymbol{n}_\lambda}_{\geq N} \tag{2}$$

We denote by $\mathrm{rank}_{\mathrm{RS}}(\mathcal{W})$, the smallest such N, and we call it the *rank of representation stability of* $\mathcal{W}$.

Warning. For an FI-module the stability rank *as* FB-*module* is, a priori, smaller than the stability rank as FI-module. It is very easy to construct examples where it is strictly smaller, for example take any (RS) FI-module and replace all its interior maps by the zero map.

1.7.4. Proposition

a) $$\mathrm{rank}_{\mathrm{RS}}(\mathcal{P}(V_\lambda))=\mathrm{rank}^{\mathrm{FI}}_{\mathrm{RS}}(\mathcal{P}(V_\lambda))=|\lambda|+\lambda_1.$$

b) *If the trivial representation* $\mathbb{Q}_n$ *is contained in* W_n*, then*

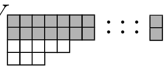
$$\mathrm{rank}_{\mathrm{RS}}(\mathcal{P}(W_n))=\mathrm{rank}^{\mathrm{FI}}_{\mathrm{RS}}(\mathcal{P}(W_n))=2n\,.$$

c) $$\mathrm{rank}_{\mathrm{RS}}(\mathcal{V}_\lambda)=\mathrm{rank}^{\mathrm{FI}}_{\mathrm{RS}}(\mathcal{V}_\lambda)=|\lambda|.$$

d) *If* $\mathcal{W}$ *is an* FB-*module such that* $\mathcal{Z}\subseteq\mathcal{W}$*, where* $\mathcal{Z}$ *is any of the* FI-*modules in the previous claims, then*

$$\mathrm{rank}_{\mathrm{RS}}(\mathcal{W})\geq\mathrm{rank}_{\mathrm{RS}}(\mathcal{Z})\,.$$

Proof. In (c) there is nothing to be proved. Let us denote by $\mathcal{Z}$ any of the other two FI-modules in the claims. We already gave the values of $\mathrm{rank}^{\mathrm{FI}}_{\mathrm{RS}}(\mathcal{Z})$ in 1.7.2. Next, because $\mathrm{rank}_{\mathrm{RS}}(\mathcal{Z})\leq\mathrm{rank}^{\mathrm{FI}}_{\mathrm{RS}}(\mathcal{Z})$, the equality will follow if we show that for $m=\mathrm{rank}^{\mathrm{FI}}_{\mathrm{RS}}(\mathcal{Z})$ the $\mathcal{S}_m$-module Z_m contains an irreducible factor corresponding to a Young diagram having its two first lines of equal lengths:

V  $\subseteq Z_m$ (◇)

Indeed, if $\operatorname{rank}_{\mathrm{RS}}(\mathcal{Z}) < m$, we would have

$$Z_m \sim \bigoplus_{|\lambda|<m} V_{\underline{\lambda}[m]}{}^{\boldsymbol{n}_\lambda}\,, \qquad (*)$$

and the length of the first line of the Young diagram $\underline{\lambda}[m]$ in $(*)$, would be strictly greater than that of the others, *i.e.* $\underline{\lambda}[m]_1 > \underline{\lambda}[m]_2$, which is contrary to our assumption (*cf.* 1.4.5-(b)).

Let us check (◇) in our cases. In (a) this results by Pieri's rule because $m = |\lambda| + \lambda_1$ is the smallest m such that λ is the socle of a diagram $\mu \vdash m$, which means that $\mu_1 = \mu_2$. In (b) the Young diagram corresponding to the trivial module $\mathbb{Q}_n$ has only one line with n boxes, and $\mathcal{P}(\mathbb{Q}_n)_{2n}$ verifies (◇) for $m = 2n$.

(c) W_m contains some irreducible $\mathcal{S}_m$-module of the form (◇). □

2. Character polynomiality of FI-modules

2.1. Character polynomiality

Let $\mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m)$ denote the algebra of *rational class functions* on $\mathcal{S}_m$, *i.e.* functions $f : \mathcal{S}_m \to \mathbb{Q}$ which are constant on each conjugacy class of $\mathcal{S}_m$. Denote by $\mathbb{Q}[\overline{X}]$ the ring of polynomials with rational coefficients and countably many variables $X_1, X_2, \dots,$ endowed with the grading '$\deg_{\boldsymbol{w}}$' that stipulates that:

$$\deg_{\boldsymbol{w}}(X_i) := i\,.$$

2.1.1.The *weight* of a polynomial. To avoid confusion in the sequel with the usual degree $\deg(P)$ of a polynomial $P \in \mathbb{Q}[\overline{X}]$ which stipulates that $\deg(X_i) = 1$, we will call $\deg_{\boldsymbol{w}}(P)$ the *weight* of P.

2.1.2. Proposition. *Denote by $X_{m,i} : \mathcal{S}_m \to \mathbb{N}$ the class function which assigns to $g \in \mathcal{S}_m$, the number $X_{m,i}(g)$ of i-cycles in the decomposition of g as product of disjoint cycles in $\mathcal{S}_m$.*

a) *The map*

$$\begin{aligned} \rho_m : \mathbb{Q}[\overline{X}] &\longrightarrow \mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m) \\ X_i &\longmapsto (g \mapsto X_{m,i}(g)) \end{aligned} \qquad (3)$$

is an homomorphism of algebras whose kernel contains the polynomials

$$\big(X_1 + 2X_2 + \cdots + mX_m - m\big) \quad \textit{and} \quad \big(X_i(X_i - 1)\cdots\big(X_i - \lfloor m/i \rfloor\big)\big)\,. \qquad (4)$$

The restriction

$$\rho_m : \mathbb{Q}[X_1, \ldots, X_{m-1}] \twoheadrightarrow \mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m)$$

is surjective. In particular, the characters of $\mathcal{S}_m$ *are represented by polynomials with rational coefficients and variables* $X_1, \ldots, X_{m-1}$.

b) *For* $n \le m$ *and* $g \in \mathcal{S}_n$, *we have*

$$\text{(i)}\ \rho_m(X_1)(\iota g) = \rho_n(X_1)(g) + (m-n)\,,$$
$$\text{(ii)}\ \rho_m(X_i)(\iota g) = \rho_n(X_i)(g)\,,\ \forall i > 1\,.$$

$$\begin{array}{ccc} \mathcal{S}_n & \xrightarrow{\rho_n} & \\ {\scriptstyle\iota}\downarrow & & A \\ \mathcal{S}_m & \xrightarrow{\rho_m} & \end{array}$$

Hint. (a) That the polynomials (4) belong to $\ker(\rho_m)$ is clear. Next, to see that ρ_m is surjective, we need only show that the characteristic function of a conjugacy class of $\mathcal{S}_m$ can be realized as a polynomial in $X_1, \ldots, X_m$.

For $k \in [\![1, m]\!]$, let $R_k(Z) := Z(Z-1)\cdots\widehat{(Z-k)}\cdots(Z-m)$ and consider

$$D_k(Z) := R_k(Z)/R_k(k) \in \mathbb{Q}[Z]\,.$$

This polynomial has the property that

$$\rho_m(D_k(X_i)(g) = \begin{cases} 1\,, & \text{if } X_i(g) = k\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

So that, if $\sum_i i\,\boldsymbol{n}_i = m$, we get

$$\rho_m(D_{\boldsymbol{n}_1}(X_1)D_{\boldsymbol{n}_2}(X_2)\cdots D_{\boldsymbol{n}_m}(X_m))(g) = \begin{cases} 1\,, & \text{if } g \text{ is of type } (1^{\boldsymbol{n}_1}, \ldots, m^{\boldsymbol{n}_m}) \\ 0\,, & \text{otherwise.} \end{cases}$$

(b) is clear. □

2.1.3. Convention. *In order to alleviate notations, we will write* $X_i(g)$ *for* $X_{m,i}(g)$ *despite the possible ambiguity in '*$X_1(g)$*'* (*cf.* 2.1.2-(b-i)).

2.1.4. Definition. An FB-module $\mathcal{W} := \{W_m\}_{m\in\mathbb{N}}$ is said to have a *polynomial character for* $m \ge N$, in short $(\mathrm{PC})_{m\ge N}$, if there exists a polynomial $P_{\mathcal{W}} \in \mathbb{Q}[\overline{X}]$ such that

$$\chi_{W_m} = \rho_m(P_{\mathcal{W}})\,, \quad \forall m \ge N\,.$$

We denote by $\mathrm{rank}_{\mathrm{PC}}(\mathcal{W})$, the smallest such N, and we call it the *rank of character polynomiality of* $\mathcal{W}$.

2.1.5. Proposition and definition. *The polynomial* $P_{\mathcal{W}}$ *that asymptotically represents the characters of the terms* W_m *of an* FB*-module* $\mathcal{W} := \{W_m\}_{m\in\mathbb{N}}$ *is unique, it is called the* polynomial character of the FB-module $\mathcal{W}$.

Proof. If $P'_{\mathcal{W}}$ were another polynomial representing χ_{W_m} for $m \ggg 0$, the difference $Q := P_{\mathcal{W}} - P'_{\mathcal{W}}$, that we may assume to belong to $\mathbb{Q}[X_1, \ldots, X_N]$, would be a polynomial representing the zero class function for all $m \ggg N$.

If Q is not the null polynomial, we can write it as a polynomial in X_N with coefficients Q_i in $\mathbb{Q}[X_1, \dots, X_{N-1}]$:

$$Q = Q_0 + Q_1 X_N + Q_2 X_N^2 + \cdots + Q_r X_N^r\,, \text{ and } Q_r \neq 0. \qquad (*)$$

Now, for any family $\bar{a} := \{a_1, \dots, a_{N-1}\} \subseteq \mathbb{N}$, and any $i \in \mathbb{N}$, it is easy to find $m_i \ggg N$ and $g_i \in \mathcal{S}_{m_i}$ such that $X_1(g_i) = a_1, \dots, X_{N-1}(g_i) = a_{N-1}$ and $X_N(g_i) > i$. In that case, the polynomial in one variable $Q(\bar{a}, X_N)$ has infinitely many roots and is, therefore, the null polynomial. In particular, in the decomposition $(*)$, we would have $Q_r(\bar{a}) = 0$ for all choices of $\bar{a}$, but this implies that Q_r is the null polynomial in $\mathbb{Q}[X_1, \dots, X_{N-1}]$, contrary to its definition. The polynomial Q must therefore be the null polynomial. ☐

2.2. Frobenius character formula

Given a partition $\lambda := (\lambda_1 \geq \ldots \geq \lambda_\ell) \vdash m$, the celebrated Frobenius formula computes the character χ_{V_λ} of the simple $\mathcal{S}_m$-module V_λ. The important point for us about this formula is that it gives an expression of χ_{V_λ} as a polynomial *depending only on the socle* $\underline{\lambda}$ of λ. Consequently, the same polynomial expresses the characters of all the terms of the FI-module $\mathcal{V}_\lambda = \{V_{\lambda,m}\}_m$, for $m \geq |\lambda|$, which self-explains the property of character polynomiality of $\mathcal{V}_\lambda$.

2.2.1. Frobenius polynomial for $\boldsymbol{\chi_{V_\lambda}}$

Following Macdonald in his book [3] (ex. I.7.14, p. 122), let $y := \{y_1, \dots, y_\ell\}$ be a set of ℓ variables, where $\ell := \ell(\lambda)$. The *discriminant of* y is the antisymmetric homogeneous polynomial

$$\Delta(y) := \prod\nolimits_{i<j} (y_i - y_j)\,.$$

For $d \in \mathbb{N}$, the *d-power sum of* y is the symmetric homogeneous polynomial

$$P_d(y) := y_1^d + \cdots + y_\ell^d\,.$$

The value $\chi_{V_\lambda}(g)$ for $g \in \mathcal{S}_m$, is the coefficient of the monomial

$$y_1^{\lambda_1+(\ell-1)} y_2^{\lambda_2+(\ell-2)} y_3^{\lambda_3+(\ell-3)} \cdots y_\ell^{\lambda_\ell}\,,$$

in the development of the product

$$\Delta(y) \left(\prod\nolimits_{d \geq 1} P_d(y)^{X_d(g)} \right). \qquad (5)$$

This coefficient, denoted by $\mathbf{X}_\lambda$, is a polynomial in $\mathbb{Q}[\overline{X}]$, we call it *the Frobenius polynomial for* χ_{V_λ}.

2.2.2. Proposition. *The Frobenius polynomial* $\boldsymbol{X}_\lambda$ *for* χ_{V_λ} *depends only on the socle* $\underline{\lambda}$ *of* λ*, and belongs to the ring* $\mathbb{Q}[X_1, \ldots, X_{\lambda_2+\ell-2}]$*. Its weight is :*

$$\deg_{\boldsymbol{w}}(\boldsymbol{X}_\lambda) = \boldsymbol{w}(\lambda)\,.$$

The characters of $\mathbb{S}_m$ *can be represented by polynomials in* $\mathbb{Q}[X_1, \ldots, X_{m-1}]$ *of weights* $\leq m-1$*.*

Proof. Because the polynomial (5) is homogeneous, we can make $y_1 = 1$ without loosing information. In that case, $\chi_{V_\lambda}(g)$ is the coefficient in

$$y_2^{\lambda_2+(\ell-2)} y_3^{\lambda_3+(\ell-3)} \cdots y_\ell^{\lambda_\ell}\,,$$

after the development of the product

$$\Delta(\tilde{y}) \Big(\prod\nolimits_{j>1}(1-y_j)\Big) \Big(\prod\nolimits_{d\geq 1}(1+P_d(\tilde{y}))^{X_d(g)}\Big) \tag{‡‡}$$

where $\tilde{y} := \{y_2, \ldots, y_\ell\}$. But, in this product the first factor $\Delta(\tilde{y})$ is already homogeneous of total degree $(\ell-2)+(\ell-3)+\cdots$, so that we must seek, in the development of the remaining factors, terms whose total degree is bounded by $|\underline{\lambda}| = \lambda_2 + \cdots + \lambda_\ell$. But then, since we have

$$\big(1+P_d\big)^{X_d} = 1 + \binom{X_d}{1} P_d + \binom{X_d}{2} P_d^2 + \binom{X_d}{3} P_d^3 + \cdots$$

and because $\deg_{\mathrm{tot}}(P_d^a) = ad$, we conclude that

- If $d > \lambda_2 + \ell - 2$, the factor $\big(1+P_d\big)^{X_d}$ contributes only to χ_{V_λ} with its term 1^{X_d}, so can be ignored. The product symbol $\prod_{d\geq 1}$ in (‡‡) can therefore be replaced by $\prod_{d=1}^{\lambda_2+\ell-2}$.
- The coefficient $\binom{X_d}{j}$ is a polynomial of degree j in X_d and appears attached to monomials in $\tilde{y}$ of total degree jd. We can thus infer that, after development, the expression of χ_{V_λ} is a polynomial in $X_1, \ldots, X_{\lambda_2+\ell-2}$ of *weight* $|\underline{\lambda}| = \boldsymbol{w}(\lambda)$. □

The following corollary of proposition 2.2.2 is now immediate from the definition of representation stable FB-modules 1.7.3.

2.2.3. Corollary *If* $\mathcal{W}$ *is an* FB*-module, then*

$$\mathrm{rank}_{\mathrm{PC}}(\mathcal{W}) \leq \mathrm{rank}_{\mathrm{RS}}(\mathcal{W})\,.$$

Proof. Left to the reader. □

2.3. Basic examples of character polynomiality of FI-modules

2.3.1. The ℓ-cycles of $[\![1,m]\!]$

Given $m,\ell \in \mathbb{N}$, we denote by $\varGamma_\ell^m$ the set of ℓ-tuples $(i_1,\dots,i_\ell)$ of pairwise distinct elements of $[\![1,m]\!]$ modulo cyclic permutation, *i.e.* such that

$$(i_1,\dots,i_\ell) = (i_2,\dots,i_\ell,i_1) = (i_3,\dots,i_\ell,i_1,i_2) = \cdots$$

The symmetric group $\mathcal{S}_m$ acts on $\varGamma_\ell^m$ by

$$g\cdot(i_1,\dots,i_\ell) = (g(i_1), g(i_2),\dots, g(i_\ell))\,. \tag{6}$$

The elements of $\varGamma_\ell^m$ are called *the ℓ-cycles of* $[\![1,m]\!]$.

2.3.2. Comments

a) Given $g \in \mathcal{S}_m$, the set $[\![1,m]\!]$ is decomposed in $\langle g\rangle$-orbits, each of which can be endowed with a cyclic order defined by g. For example, if $x \in [\![1,m]\!]$, we may consider $(x \to g(x) \to g^2(x) \to \cdots \to x)$, which gives the well-known *decomposition of $g \in \mathcal{S}_m$ as product of disjoint cycles.*

b) For $m \geq \ell$, there is a difference between the cases $\ell = 1$ and $\ell > 1$.

For all $\ell \geq 1$, define the *support* of an ℓ-cycle $\gamma := (i_1,\dots,i_\ell)$ the set $\{\!\{\gamma\}\!\}$ of its coordinates:

$$\{\!\{\gamma\}\!\} := \{i_1,\dots,i_\ell\} \subseteq [\![1,m]\!]\,.$$

Then, define $\tilde\gamma \in \mathcal{S}_m$ by:

$$\tilde\gamma := \begin{cases} \tilde\gamma(i_j) := i_{j+1\,(\operatorname{mod}\ell)}\,, & \text{for } i_j \in \{\!\{\gamma\}\!\},\\ \tilde\gamma(x) := x\,, & \text{for } x \notin \{\!\{\gamma\}\!\}.\end{cases}$$

i) For $\ell = 1$, we have $\varGamma_1^m = [\![1,m]\!]$ and the action (6) of $\mathcal{S}_m$ on $\varGamma_1^m$ is the standard action of $\mathcal{S}_m$ on $[\![1,m]\!]$. The map $(\tilde{\_}) : \varGamma_1^m \to \mathcal{S}_m$ is uninteresting, as it is the constant map $\gamma \mapsto \mathrm{id}_{[\![1,m]\!]}$.

ii) For $\ell > 1$, the map $(\tilde{\_}) : \varGamma_\ell^m \subseteq \mathcal{S}_m$ is *injective*, and the action (6) of $\mathcal{S}_m$ on $\varGamma_\ell^m$ is the conjugation action of $\mathcal{S}^m$ on itself, *i.e.*:

$$\widetilde{g\cdot\gamma} = g\,\tilde\gamma\, g^{-1}\,.$$

We will identify γ and $\tilde\gamma$ if no confusion is likely to arise. In this sense, for $g \in \mathcal{S}_m$, the set of fixed points $(\varGamma_\ell^m)^g := \big\{\gamma \in \varGamma_\ell^m \;\big|\; g\cdot\gamma = \gamma\big\}$ is simply:

$$(\varGamma_\ell^m)^g = \big\{\ell\text{-cycles } \gamma \in \mathcal{S}_m \;\big|\; g\gamma = \gamma g\big\}\,. \tag{7}$$

iii) For all $m,\ell > 0$, we have:

$$|\varGamma_\ell^m| = \frac{m(m-1)\cdots(m-(\ell-1))}{\ell}\,.$$

2.3.3. The FI-modules $\mathbb{E}_\nu$

For $m, \ell > 0$, let $\mathbb{E}_\ell^m$ be the vector space spanned by the set Γ_ℓ^m, *i.e.*

$$\mathbb{E}_\ell^m := \bigoplus\nolimits_{\gamma \in \Gamma_\ell^m} \mathbb{Q} \cdot \gamma \,.$$

Endow it with the linear action of $\mathcal{S}_m$ induced by its action on the basis Γ_ℓ^m. Notice that :

- $\mathbb{E}_\ell^m = \mathbf{0}$, for all $m < \ell$.
- For all $m \geq n$, the set Γ_ℓ^n is a subset of Γ_ℓ^m, which is invariant under the action of $\mathbf{1}_n \boxtimes \mathcal{S}_{m-n}$, so that the natural inclusions $\Gamma_\ell^m \subseteq \Gamma_\ell^{m+1}$ induce the interior maps (clearly injective) of an FI-module

$$\boxed{\mathbb{E}_\ell := \big\{\phi(\mathbb{E}_\ell)_m : \mathbb{E}_\ell^m \rightarrowtail \mathbb{E}_\ell^{m+1}\big\}}$$

  where $\phi(\mathbb{E}_\ell)_m$ is exhaustive for $m \geq \ell$, since the natural map between orbit spaces: $\mathcal{S}_m \backslash \Gamma_\ell^m \to \mathcal{S}_{m+1} \backslash \Gamma_\ell^{m+1}$ is bijective for $m \geq \ell$.

More generally, if $\nu := (1^{\boldsymbol{n}_1}, 2^{\boldsymbol{n}_2}, \ldots, N^{\boldsymbol{n}_N}) = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_\ell)$ is a nonempty partition, let

$$\begin{aligned} \mathbb{E}_\nu^m &:= (\mathbb{E}_1^m)^{\otimes \boldsymbol{n}_1} \otimes (\mathbb{E}_2^m)^{\otimes \boldsymbol{n}_2} \otimes \cdots \otimes (\mathbb{E}_N^m)^{\otimes \boldsymbol{n}_N} \,, \\ &:= \mathbb{E}_{\nu_1} \otimes \mathbb{E}_{\nu_2} \otimes \cdots \otimes \mathbb{E}_{\nu_\ell} \,, \end{aligned}$$

and define $\phi(\mathbb{E}_\nu)_m : \mathbb{E}_\nu^m \to \mathbb{E}_\nu^{m+1}$ as the tensor product of interior maps

$$\begin{aligned} \phi(\mathbb{E}_\nu)_m &:= \phi(\mathbb{E}_1)^{\otimes \boldsymbol{n}_1} \otimes \cdots \otimes \phi(\mathbb{E}_N)^{\otimes \boldsymbol{n}_N} \,, \\ &:= \phi(\mathbb{E}_{\nu_1}) \otimes \cdots \otimes \phi(\mathbb{E}_{\nu_\ell}) \,. \end{aligned}$$

The family

$$\boxed{\mathbb{E}_\nu := \big\{\phi(\mathbb{E}_\nu)_m : \mathbb{E}_\nu^m \rightarrowtail \mathbb{E}_\nu^{m+1}\big\}_m}$$

is an FI-module with all interior maps injective, and exhaustive for $m \geq \nu_1$.

2.3.4. Proposition

a) *For $m, \ell \in \mathbb{N}$, the character $\chi_{\mathbb{E}_\ell^m}$ is expressed by the following polynomial of $\mathbb{Q}[X_1, \ldots, X_\ell]$ of weight ℓ and independent of $m \in \mathbb{N}$:*

$$\boldsymbol{E}_\ell = \phi(\ell)\, X_\ell + \sum\nolimits_{ed=\ell\,,\, e \neq 1} \phi(d)\, \frac{d^e}{\ell}\, X_d(X_d - 1) \cdots \big(X_d - (e{-}1)\big)\,, \quad (*_\ell)$$

where ϕ is the Euler's totient function.

The FI-*module* $\mathbb{E}_\ell := \{\mathbb{E}_\ell^m \rightarrowtail \mathbb{E}_\ell^{m+1}\}_m$ *has the following ranks*

$$\mathrm{rank}_{\text{PC}}(\mathbb{E}_\ell) = 0 \quad \textit{and} \quad \mathrm{rank}_{\text{RS}}(\mathbb{E}_\ell) = \mathrm{rank}_{\text{RS}}^{\text{FI}}(\mathbb{E}_\ell) = 2\ell \,.$$

b) *Given an ordered sequence* $\overline{Z} := (Z_1, \ldots, Z_N)$ *of polynomials of* $\mathbb{Q}[\overline{X}]$, *and* $d \in \mathbb{N}$, *denote by* $\mathbb{Q}^{\leq d}[Z_1, \ldots, Z_N]$ *the subspace of polynomials of weight* $\leq d$ *relative to* $\overline{Z}$, i.e. *the subspace spanned by the elements* $Z_1^{a_1} \cdots Z_N^{a_N}$ *where* $\sum_i i\, a_i \leq d$. *Then, for all* $N \in \mathbb{N}$, *the inclusion:*

$$\mathbb{Q}^{\leq d}[\boldsymbol{E}_1, \ldots, \boldsymbol{E}_N] \subseteq \mathbb{Q}^{\leq d}[X_1, \ldots, X_N]$$

is an equality.

c) *For every nonempty partition* $\nu := (1^{\boldsymbol{n}_1}, 2^{\boldsymbol{n}_2}, \ldots, N^{\boldsymbol{n}_N})$, *we have*

i) $\boldsymbol{w}(\mathbb{E}_\nu^m) \leq \min\{m, |\nu|\}$.

ii) *The* FI-*module* $\mathbb{E}_\nu := \{\mathbb{E}_\nu^m \rightarrowtail \mathbb{E}_\nu^{m+1}\}_m$ *has the following ranks*

$$\operatorname{rank}_{\mathrm{PC}}(\mathbb{E}_\nu) = 0 \quad \textit{and} \quad \operatorname{rank}_{\mathrm{RS}}(\mathbb{E}_\nu) = \operatorname{rank}_{\mathrm{RS}}^{\mathrm{FI}}(\mathbb{E}_\nu) = 2|\nu|\,.$$

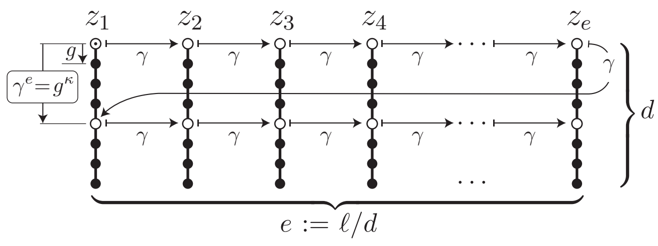
Proof. (a) Since the linear action of $g \in \mathcal{S}_m$ is induced by its action on the basis $\mathit{\Gamma}_\ell^m \subseteq \mathbb{E}_\ell^m$, the trace $\chi_{\mathbb{E}_\ell^m}(g)$ is the cardinality of $(\mathit{\Gamma}_\ell^m)^g$ (2.3.2-(b-ii)).

When $m < \ell$, the set $\mathit{\Gamma}_\ell^m$ is empty and $\mathbb{E}_\ell^m = 0$. And, since in $\mathcal{S}_m$ there is no permutation g such that $\rho_m(X_d)(g) \geq \ell/d$, we have $\rho_m(\boldsymbol{E}_\ell) = 0$, which states (a) when $m < \ell$.

When $m \geq \ell$, following 2.3.2-(b), we consider two cases:

– $\boldsymbol{\ell = 1}$. Then $\mathit{\Gamma}_1^m = [\![1, m]\!]$, $\chi_{\mathbb{E}_1^m}(g) = |[\![1, m]\!]^g| = X_1(g)$, and (a) is obvious.

– $\boldsymbol{\ell > 1}$. The set $(\mathit{\Gamma}_\ell^m)^g$ identifies, after 2.3.2-(b-ii)-(7), with the set of ℓ-cycles $\gamma \in \mathcal{S}_m$ such that $g\gamma = \gamma g$. As a consequence, $g\{\!\{\gamma\}\!\} = \{\!\{\gamma\}\!\}$ and the set $\{\!\{\gamma\}\!\}$ appears endowed with two actions commuting to each other. But then, since the action of γ on $\{\!\{\gamma\}\!\}$ is transitive, the $\langle g\rangle$-orbits share the same cardinality. Denote it by d and set $e := \ell/d$. Then, $g^\kappa = \gamma^e$, for some $\kappa \in [\![1, d]\!]$, relatively prime to d. These observations are gathered in the following image:



If we choose the point '⊙' to stand for the smallest coordinate in γ, this graphic representation becomes one-to-one, and thus helpful for the computation of the cardinality of the set $(\mathit{\Gamma}_\ell^m)^g$. Indeed, it can be used to reconstruct $(\mathit{\Gamma}_\ell^m)^g$ by following the next steps.

1) Take an ordered sequence $(z_1, \dots, z_e)$ of d-cycles in g, among the
$$X_d(g)^{\underline{e}} := X_d(g)\big(X_d(g)-1\big)\cdots\big(X_d(g)-(e{-}1)\big)\,.$$
possible such sequences.
2) Shift to the first position the d-cycle containing the smallest coordinate in $[\![1,m]\!]$. This means that $X_d(g)^{\underline{e}}$ has to be divided by e.
3) Fix the vertical orders of $z_2, \dots, z_e$. There are d^{e-1} such orders.
4) Fix the number κ. There are $\phi(d)$ possibilities.

Therefore,
$$\big|(\mathit{\Gamma}_\ell^m)^g\big| = \sum_{de=\ell} \phi(d)\,\frac{d^e}{\ell}\,X_d(g)^{\underline{e}}\,.$$
In both cases, $\ell = 1$ and $\ell > 1$, the previous expression is independent of $m \in \mathbb{N}$, which means that $\mathrm{rank}_{\mathrm{PC}}(I\!\!E_\ell) = 0$.

To estimate $\mathrm{rank}_{\mathrm{RS}}(I\!\!E_\ell)$, observe that, since the action of $\mathcal{S}_m$ on $\mathit{\Gamma}_\ell^m$ is transitive, $\mathit{\Gamma}_\ell^m = \mathcal{S}_m \cdot (1, \dots, \ell)$, so that, denoting $\gamma := (1, \dots, \ell)$, we have an isomorphism of $\mathcal{S}_m$-spaces:
$$\mathit{\Gamma}_\ell^m = \mathcal{S}_m \cdot \gamma \simeq \mathcal{S}_m \times_{\langle\gamma\rangle \boxtimes \mathcal{S}_{m-\ell}} \{\mathrm{pt}\}\,,$$
because $\mathrm{Stab}_{\mathcal{S}_m}(\gamma) = \langle\gamma\rangle \boxtimes \mathcal{S}_{m-\ell}$. Therefore,
$$I\!\!E_\ell^m = \mathrm{ind}_{\mathcal{S}_\ell \boxtimes \mathcal{S}_{m-\ell}}^{\mathcal{S}_m}\big((\mathrm{ind}_{\langle\gamma\rangle}^{\mathcal{S}_\ell}\mathbb{Q}) \boxtimes \mathbb{Q}_{m-\ell}\big)\,.$$
and $I\!\!E_\ell$ is the projective FI-module $\mathcal{P}\big(\mathrm{ind}_{\langle\gamma\rangle}^{\mathcal{S}_\ell}\mathbb{Q}\big)$ for which we have already shown that it is $(\mathrm{RS})_{m \geq 2\ell}$ (*cf.* 1.7.2-(b)).

We can be a more precise observing that the simple $\mathcal{S}_\ell$-module $\mathbb{Q}_\ell$ appears with multiplicity 1 in $\mathrm{ind}_{\langle\gamma\rangle}^{\mathcal{S}_\ell}\mathbb{Q}$, which is true since:
$$\mathrm{Hom}_{\mathcal{S}_\ell}(\mathrm{ind}_{\langle\gamma\rangle}^{\mathcal{S}_\ell}\mathbb{Q}, \mathbb{Q}_\ell) = \mathrm{Hom}_{\langle\gamma\rangle}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}\,.$$
Thus, $I\!\!E_\ell$ contains $\mathcal{P}(\mathbb{Q}_\ell) = \mathcal{P}(V_{0[\ell]})$ as the sub-FI-module, and, thanks to 1.7.4-(d),
$$\mathrm{rank}_{\mathrm{RS}}(I\!\!E_\ell) = \mathrm{rank}_{\mathrm{RS}}^{\mathrm{FI}}(I\!\!E_\ell) = 2\ell\,.$$
(b) In (a), the equation $(*_\ell)$ shows that $\boldsymbol{E}_\ell = \phi(\ell)\,X_\ell$ modulo a polynomial of weight ℓ in the variables X_d for $d|\ell$ and $d \neq \ell$. (In particular, $\boldsymbol{E}_1 = X_1$.) Now, given $N \geq 1$, the system of equations $(*_\ell)$, for $\ell \leq N$, can be inverted by recursively introducing some polynomials $Q_\ell^m(Z_1, \dots, Z_\ell) \in \mathbb{Q}[\overline{Z}]$ of weight ℓ, such that
$$X_\ell = Q_\ell^m(\boldsymbol{E}_1, \dots, \boldsymbol{E}_\ell)\,, \quad \forall \ell \leq N\,.$$
The equality,
$$\mathbb{Q}[X_1, \dots, X_N] = \mathbb{Q}[\boldsymbol{E}_1, \dots, \boldsymbol{E}_N]\,,$$

but also,

$$\mathbb{Q}^{\leq d}[X_1, \dots, X_N] = \mathbb{Q}^{\leq d}[\boldsymbol{E}_1, \dots, \boldsymbol{E}_N]\,, \quad \forall d \in \mathbb{N}\,,$$

then follows easily.

(c) Since the action of $\mathcal{S}_m$ on $\mathbb{E}_\nu^m := \mathbb{E}_{\nu_1}^m \otimes \cdots \otimes \mathbb{E}_{\nu_\ell}^m$ is induced by its component-wise action on the basis $\varGamma_\nu^m := \varGamma_{\nu_1}^m \times \varGamma_{\nu_2}^m \times \cdots \times \varGamma_{\nu_\ell}^m$, the structure of $\mathbb{Q}[\mathcal{S}_m]$-module of $\mathbb{E}_\nu^m$ follows from the structure of $\mathcal{S}_m$-space of $\varGamma_\nu^m$. Given $\bar\gamma := (\gamma_1, \dots, \gamma_\ell) \in \varGamma_\nu^m$, we have

$$\mathcal{S}_m \cdot \bar\gamma = \mathcal{S}_m / \operatorname{Stab}_{\mathcal{S}_m}(\bar\gamma) = \mathcal{S}_m \times_{\mathcal{S}_{\{\!\{\bar\gamma\}\!\}}} \big(\mathcal{S}_{\{\!\{\bar\gamma\}\!\}} / \operatorname{Stab}_{\mathcal{S}_{\{\!\{\bar\gamma\}\!\}}}(\bar\gamma)\big) \tag{‡}$$

where we set $\{\!\{\bar\gamma\}\!\} := \{\!\{\gamma_1\}\!\} \cup \cdots \cup \{\!\{\gamma_\ell\}\!\}$.

As a consequence the $\mathbb{Q}[\mathcal{S}_m]$-module $\mathbb{E}_\nu^m$ is a direct sum of induced modules of the form:

$$\operatorname{ind}_{\mathcal{S}_{\{\!\{\bar\gamma\}\!\}} \boxtimes \mathcal{S}_{\complement\{\!\{\bar\gamma\}\!\}}}^{\mathcal{S}_m} W_n \boxtimes \mathbb{Q}_{m-n}\,,$$

where we can choose to have $\{\!\{\bar\gamma\}\!\} := [\![1, n]\!]$, for $n := |\{\!\{\bar\gamma\}\!\}| \leq \min\{m, |\nu|\}$, in which case

$$W_n := \mathbb{Q}[\mathcal{S}_n / \operatorname{Stab}_{\mathcal{S}_n}(\bar\gamma)] = \operatorname{ind}_{\operatorname{Stab}_{\mathcal{S}_n}(\bar\gamma)}^{\mathcal{S}_n} \mathbb{Q}_n\,. \tag{‡‡}$$

When $m \geq |\nu|$, the corollary 1.4.5-(a) to Pieri's rule, gives the inequality

$$\boldsymbol{w}(\nu) \leq \boldsymbol{w}(\mathbb{E}_\nu^m) \leq |\nu|\,,$$

and (c-i) follows.

(c-ii) Since $\chi_{\mathbb{E}_\nu^m} = (\chi_{\mathbb{E}_1^m})^{\boldsymbol{n}_1} \cdots (\chi_{\mathbb{E}_N^m})^{\boldsymbol{n}_N}$ and since $\operatorname{rank}_{\text{PC}}(\mathbb{E}_\ell) = 0$, after (a), it follows immediately that $\operatorname{rank}_{\text{PC}}(\mathbb{E}_\nu) = 0$.

To finish, we notice that, on the one hand, the number of terms of the form (‡) is exactly de number of $\mathcal{S}_m$-orbits in $\varGamma_\nu^m$, and, on the other hand, the sequence of inclusions $\mathcal{S}_m \backslash \varGamma_\nu^m \subseteq \mathcal{S}_{m+1} \backslash \varGamma_\nu^{m+1}$ is *strict increasing* until it stabilizes for $m \geq |\nu|$. As a consequence, the truncated FI-module $(\mathbb{E}_\nu)_{\geq|\nu|}$ is isomorphic to the sum of truncated projective FI-modules $\mathcal{P}(W_n)_{\geq|\nu|}$, all of which being $(\text{RS})_{m\geq 2|\nu|}$, after 1.7.2-(b). We therefore have proved that

$$\operatorname{rank}_{\text{RS}}(\mathbb{E}_\nu) \leq \operatorname{rank}_{\text{RS}}^{\text{FI}}(\mathbb{E}_\nu) \leq 2|\nu|\,.$$

To see that these are equalities, it suffices, by 1.7.4, to show that, for $m = |\nu|$, one has:

$$\mathcal{P}(\mathbb{Q}_m) \subseteq \mathbb{E}_\nu\,. \tag{†}$$

But, this is clear if we choose $\bar\gamma \in \varGamma_\nu^m$ such that $|\{\!\{\bar\gamma\}\!\}| = m$ (as in (‡‡)), in which case (†) follows, proceeding as in (a), from the inclusion:

$$\mathbb{Q}_m \subseteq \operatorname{ind}_{\operatorname{Stab}_{\mathcal{S}_m}(\{\!\{\bar\gamma\}\!\})}^{\mathcal{S}_m}(\mathbb{Q}_m)\,. \qquad \square$$

3. Polynomial weight vs. Representation weight

3.1. On the minimal polynomial weight of a character

The Frobenius polynomial $\boldsymbol{X}_\lambda \in \mathbb{Q}[\overline{X}]$ is of weight $\boldsymbol{w}(\lambda)$, and there are infinitely many polynomials $P_\lambda \in \mathbb{Q}[X_1, \ldots, X_{m-1}]$ such that $\rho_m(P_\lambda) = \rho_m(\boldsymbol{X}_\lambda) = \chi_{V_\lambda}$. In this section, we advance the study of the relationship between weights of polynomials and weights of representations. We will see that $\boldsymbol{w}(W_m)$ is the lowest possible weight for a polynomial $P_{W_m} \in \mathbb{Q}[X_1, \ldots, X_{m-1}]$ verifying $\rho_m(P_{W_m}) = \chi_{W_m}$. For example, the polynomial $\boldsymbol{X}_{W_m} := \sum_\lambda \boldsymbol{n}_\lambda \boldsymbol{X}_\lambda$, where $\boldsymbol{X}_\lambda$ is the Frobenius polynomial of an irreducible factor V_λ of multiplicity $\boldsymbol{n}_\lambda$ in W_m. Moreover, we will see that this polynomial $\boldsymbol{X}_{W_m}$ is the only one satisfying these weights conditions, if and only if $\boldsymbol{w}(W_m) \leq m/2$.

3.1.1. Theorem

a) *For $d, m \in \mathbb{N}$, denote by $\rho_{m|d}$ the restriction of $\rho_m : \mathbb{Q}[\overline{X}] \to \mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m)$ to the subspace $\mathbb{Q}^{\leq d}[\overline{X}]$ of polynomials of weight $\leq d$. The image of the map*

$$\rho_{m|d} : \big(\mathbb{Q}^{\leq d}[\overline{X}]\big) \to \mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m)$$

is the subspace

$$\mathbb{Q}_{\mathrm{cl}}^{\leq d}(\mathcal{S}_m) = \big\langle\, \chi_{V_\lambda} \;\big|\; \big(|\lambda| = m\big) \;\&\; \big(\, \boldsymbol{w}(\lambda) \leq d\big) \big\rangle_{\mathbb{Q}}\,, \tag{8}$$

and $\ker(\rho_{m|d}) = 0$ *if and only if $d \leq m/2$.*

b) *Let $P \in \mathbb{Q}[\overline{X}]$.*

   i) *The scalar product $\langle\!\langle\, 1 \mid P \,\rangle\!\rangle_{\mathcal{S}_m}$ is constant for $m \geq \deg_{\boldsymbol{w}}(P)$.*

   ii) *The scalar product $\langle\!\langle\, \chi_{V_\lambda} \mid P \,\rangle\!\rangle_{\mathcal{S}_m}$ vanishes for $\lambda \vdash m$ s.t. $\boldsymbol{w}(\lambda) > \deg_{\boldsymbol{w}}(P)$.*

   iii) *Let W_m be a representation of $\mathcal{S}_m$. If $\rho_m(P) = \chi_{W_m}$, then:*

$$\deg_{\boldsymbol{w}}(P) \geq \boldsymbol{w}(W_m)\,.$$

c) *Let $W_m = \bigoplus_{\lambda \vdash m} {V_\lambda}^{\boldsymbol{n}_\lambda}$ be a representation of $\mathcal{S}_m$. The polynomial*

$$\boldsymbol{X}_{W_m} := \sum\nolimits_{\lambda \vdash m} \boldsymbol{n}_\lambda \boldsymbol{X}_\lambda \in \mathbb{Q}[X_1, \ldots, X_{m-1}]$$

where $\boldsymbol{X}_\lambda$ is the Frobenius polynomial for χ_{V_λ} (2.2.2), verifies

$$\rho_m(\boldsymbol{X}_{W_m}) = \chi_{W_m} \quad \textit{and} \quad \deg_{\boldsymbol{w}}(\boldsymbol{X}_{W_m}) = \boldsymbol{w}(W_m)\,,$$

and it is the only polynomial verifying simultaneously these two equalities, if and only if $\boldsymbol{w}(W_m) \leq m/2$.

Proof. (a) In (8), the inclusion '$\supseteq$' is 2.2.2. For the converse, it suffices to restrict oneself to the case of a monomial $\rho_m(X_1^{\boldsymbol{n}_1} \cdots X_d^{\boldsymbol{n}_d})$ for $\sum_i i\, \boldsymbol{n}_i \leq d$, or, thanks to 2.3.4-(b), what amounts to the same thing, to show that

$$(\chi_{\mathbb{E}_1^m})^{\boldsymbol{n}_1}\cdots(\chi_{\mathbb{E}_d^m})^{\boldsymbol{n}_d}\in\big\langle\chi_{V_\lambda}\;\big|\;\big(|\lambda|=m\big)\;\&\;\big(\,\boldsymbol{w}(\lambda)\le d\,\big)\big\rangle_{\mathbb{Q}}\,.$$

Here, one recognizes, on the left, the character of $\mathbb{E}_\nu^m$, for $\nu:=(1^{\boldsymbol{n}_1},\ldots,d^{\boldsymbol{n}_d})$. Now, since we already know after 2.3.4-(c), that $\boldsymbol{w}(\mathbb{E}_\nu^m)\le|\nu|$, we conclude that $\boldsymbol{w}(\mathbb{E}_\nu^m)\le d$. The irreducible factors of $\mathbb{E}_\nu^m$ are therefore of the form V_λ with $\boldsymbol{w}(\lambda)\le d$, which settles the inclusion '$\subseteq$'.

For the last claim about $\ker(\rho_{m|d})$, notice that the space $\mathbb{Q}^{\le d}[X_1,\ldots,X_d]$ admits as a basis the set of monomials

$$M(d):=\big\{X_1^{\boldsymbol{n}_1}X_2^{\boldsymbol{n}_2}\ldots X_d^{\boldsymbol{n}_d}\;\big|\;\textstyle\sum_{i=1}^d i\,\mathbf{n}_i\le d\,\big\}\,,$$

while the space $\mathbb{Q}_{\mathrm{cl}}^{\le d}(\mathcal{S}_m)$ has a basis indexed by the following set of partitions:

$$L(d):=\big\{\underline{\lambda}:=(1^{\boldsymbol{n}_1},2^{\boldsymbol{n}_2},\ldots,\lambda_2^{\boldsymbol{n}_{\lambda_2}})\,\big|\,\big(\textstyle\sum_{i=1}^{\lambda_2}i\,\mathbf{n}_i\le d\big)\;\&\;\big(\textstyle\sum_{i=1}^{\lambda_2}i\,\mathbf{n}_i+\lambda_2\le m\big)\big\}\,,$$

because $\{\chi_{V_\lambda}\mid\lambda\vdash m\}$ is linearly independent after Schur orthogonality.

Since $\lambda_2\le d$, the map $\xi:L(d)\to M(d)$, which associates $(1^{\boldsymbol{n}_1},2^{\boldsymbol{n}_2},\ldots,\lambda_2^{\boldsymbol{n}_{\lambda_2}})$ with the monomial $X_1^{\boldsymbol{n}_1}X_2^{\boldsymbol{n}_2}\ldots X_{\lambda_2}^{\boldsymbol{n}_{\lambda_2}}$ is well defined and injective. The condition for ξ to be bijective is that λ_2 be able to take the value d, in which case $\sum_{i=1,\ldots,\lambda_2}i\,\mathbf{n}_i=d$, and that condition appears to be simply that $2d\le m$.

(b-i) Again, thanks to 2.3.4-(b), we need only prove that the number:

$$\big\langle\!\big\langle\,1\;\big|\;\chi_{\mathbb{E}_1^m}(g)^{\boldsymbol{n}_1}\chi_{\mathbb{E}_2^m}(g)^{\boldsymbol{n}_2}\cdots\chi_{\mathbb{E}_r^m}(g)^{\boldsymbol{n}_r}\,\big\rangle\!\big\rangle_{\mathcal{S}_m}\,,\qquad(\diamond)$$

is constant for all $m\ge\sum_{i=1}^r i\,\mathbf{n}_i$. But $(\diamond)$ is simply the dimension of the subspace of invariant tensors in the $\mathcal{S}_m$-module

$$\mathbb{E}_\nu^m:=(\mathbb{E}_1^m)^{\otimes\boldsymbol{n}_1}\otimes(\mathbb{E}_2^m)^{\otimes\boldsymbol{n}_2}\otimes\cdots\otimes(\mathbb{E}_r^m)^{\otimes\boldsymbol{n}_r}\,,$$

where $\nu:=(1^{\boldsymbol{n}_1},2^{\boldsymbol{n}_2},\ldots,r^{\boldsymbol{n}_r})$. Let $\nu=(\nu_1\ge\nu_2\ge\cdots\ge\nu_\ell)$. The canonical basis $\mathcal{B}_\nu^m$ of $\mathbb{E}_\nu^m$ is the set of tensors

$$\gamma_1\otimes\gamma_2\otimes\cdots\otimes\gamma_\ell\,,$$

where γ_i is a cycle of length ν_i of $\mathcal{S}_m$. The basis $\mathcal{B}_\nu^m$ is clearly in bijection with the set $\mathcal{T}_\nu^m$ of Young *tableaux* τ of shape ν, with the i'th row filled with elements of $[\![1,m]\!]$ so that they represent a cycle of length ν_i of $\mathcal{S}_m$. The action of $\mathcal{S}_m$ on $\mathcal{B}_\nu^m$ induces in $\mathcal{T}_\nu^m$ the natural action of $\mathcal{S}_m$ on Young tableaux.

Because of these identifications, the dimension of $(\mathbb{E}_\nu^m)^{\mathcal{S}_m}$ is the cardinality of the orbit space $\mathcal{T}_\nu^m/\mathcal{S}_m$, and the stability we are seeking to prove is equivalent to the natural map

$$\mathcal{T}_\nu^m/\mathcal{S}_m \to \mathcal{T}_\nu^{m+1}/\mathcal{S}_{m+1}\,,\ \ [\tau\,(\mathrm{mod}\,\mathcal{S}_m)] \mapsto [\tau\,(\mathrm{mod}\,\mathcal{S}_{m+1})]\,,$$

being a bijection. But the necessary and sufficient condition for this is precisely that the total number of boxes $|\nu|$ be smaller or equal to m, since, in that case, a single permutation $g \in \mathcal{S}_m$ will allow renumbering of all boxes simultaneously with numbers in the interval $[\![1, |\nu|]\!]$. Hence ($\diamond$) is constant for $m \geq |\nu| = \sum_i i\,\boldsymbol{n}_i$.

(b-ii) After (a), $\rho_m(P)$ belongs to the linear span of the characters χ_{V_ν} of $\mathcal{S}_m$, with $\boldsymbol{w}(\nu) \leq \deg_{\boldsymbol{w}}(P)$, and, thanks to Schur orthogonality, these characters are orthogonal to any χ_{V_λ} with $\boldsymbol{w}(\lambda) > \deg_{\boldsymbol{w}}(P)$, hence the claim.

(b-iii) Indeed, if we had $\deg_{\boldsymbol{w}}(P) < \boldsymbol{w}(W_m)$ then, for any irreducible factor V_λ of W_m of weight $\boldsymbol{w}(V_\lambda) = \boldsymbol{w}(W_m)$, we would get, after (b-ii):

$$0 = \langle\!\langle\, \chi_{V_\lambda} \mid P \,\rangle\!\rangle_{\mathcal{S}_m} = \langle\!\langle\, \chi_{V_\lambda} \mid W_m \,\rangle\!\rangle_{\mathcal{S}_m} \neq 0\,,$$

which is a contradiction. Hence $\deg_{\boldsymbol{w}}(P) \geq \boldsymbol{w}(W_m)$.

(c) Immediate consequence of (b-iii) and the study of $\ker(\rho_{m|d})$ in (a). □

3.1.2. Comments

- The proposition 2.3.4-(a) showed that, for fixed $m \geq N$, the character of the tensor product $\mathbb{E}_\nu^m := (\mathbb{E}_1^m)^{\otimes \boldsymbol{n}_1} \otimes (\mathbb{E}_2^m)^{\otimes \boldsymbol{n}_2} \otimes \cdots \otimes (\mathbb{E}_N^m)^{\otimes \boldsymbol{n}_N}$, as an $\mathcal{S}_m$-module, is the polynomial: $\boldsymbol{E}_\nu := \boldsymbol{E}_1^{\boldsymbol{n}_1} \boldsymbol{E}_2^{\boldsymbol{n}_2} \cdots \boldsymbol{E}_N^{\boldsymbol{n}_N}$ of $\mathbb{Q}[X_1, \ldots, X_N]$, whose weight $\deg_{\boldsymbol{w}}(\boldsymbol{E}_\nu) = \sum_i i\,\boldsymbol{n}_i$ can be arbitrarily big. On the other hand, theorem 3.1.1-(c) showed that the same character is given by the polynomial $\boldsymbol{X}_{\mathbb{E}_\nu^m}$ of $\mathbb{Q}[X_1, \ldots, X_{m-1}]$, of weight $\boldsymbol{w}(\mathbb{E}_\nu^m) < m$.

  This disagreement is due to the fact that, while $\boldsymbol{E}_\nu$ expresses a priori all the characters in the family $\{\chi_{\mathbb{E}_\nu^m}\}_{m \geq N}$ *simultaneously*, the Frobenius polynomial $\boldsymbol{X}_{\mathbb{E}_\nu^m}$ expresses a priori only one of them, viz. $\chi_{\mathbb{E}_\nu^m}$.

  Heuristically speaking, the elements in $\ker(\rho_m)$ allow the reduction of the weight of the polynomial $\boldsymbol{E}_\nu$ in order to reach the required bound m. This distinction will become meaningful in the next section.

- It is also worth noting that while the explicit writing of $\boldsymbol{E}_\nu$ is very simple, thanks to formula 2.3.4-(a), the writing of $\boldsymbol{X}_{\mathbb{E}_\nu^m}$ can be quite complex insofar as it entails knowing the set of multiplicities $\boldsymbol{n}_\lambda$ of the irreducible factors V_λ in the decomposition $\mathbb{E}_\nu^m = \bigoplus_{\lambda \vdash m} V_\lambda^{\boldsymbol{n}_\lambda}$, and also in the explicit description of each $\boldsymbol{X}_\lambda$, for $\boldsymbol{n}_\lambda \neq 0$.

The following is a useful corollary to theorem 3.1.1-(c)

3.1.3. Corollary. *Let $\mathcal{W}$ be an* FB*-module which is* $(\mathrm{PC})_{m\geq N}$ *and has polynomial character* $P_{\mathcal{W}}$.

a) $P_{\mathcal{W}} = \mathbf{X}_{W_m}$, *for all* $m \geq \max\{2\deg_{\boldsymbol{w}}(P_{\mathcal{W}}), N\}$.

b) $\boldsymbol{w}(W_m) = \deg_{\boldsymbol{w}}(P_{\mathcal{W}})$, *for all* $m \geq \max\{2\deg_{\boldsymbol{w}}(P_{\mathcal{W}}), N\}$.

c) $\boldsymbol{w}(\mathcal{W}_{\geq N}) = \deg_{\boldsymbol{w}}(P_{\mathcal{W}})$.

Proof. (a,b) Immediate after 3.1.1-(c).

(c) For all $m \geq N$, we have $\rho_m(P_{\mathcal{W}}) = \chi_{W_m}$ so that $\boldsymbol{w}(W_m) \leq \deg_{\boldsymbol{w}}(P_{\mathcal{W}})$, again after 3.1.1-(c). Therefore, $\sup_{m\geq N}\{\boldsymbol{w}(W_m)\} = \deg_{\boldsymbol{w}}(P_{\mathcal{W}})$, after (b). □

4. (PC) versus (RS)

4.1. Equivalence of the stability conditions

We show the equivalence, for an FB-module $\mathcal{W}$, between the properties (RS) and (PC). In this, two more elements will deserve special attention. First, the ranks of validity of the properties, and, second, the weight of the polynomial $P_{\mathcal{W}}$. The complete statement is the following.

4.1.1. Theorem. *Let $\mathcal{W}$ be an* FB*-module.*

a) *If $\mathcal{W}$ is* (RS)*, then $\mathcal{W}$ is* (PC) *and* $\mathrm{rank}_{\mathrm{PC}}(\mathcal{W}) \leq \mathrm{rank}_{\mathrm{RS}}(\mathcal{W})$.

b) *If $\mathcal{W}$ is* (PC) *with polynomial character $P_{\mathcal{W}}$, then $\mathcal{W}$ is* (RS) *and*

$$\mathrm{rank}_{\mathrm{RS}}(\mathcal{W}) \leq \max\{\mathrm{rank}_{\mathrm{PC}}(\mathcal{W}), 2\deg_{\boldsymbol{w}}(P_{\mathcal{W}})\}\,.$$

Proof. (a) is corollary 2.2.3. (b) Set $d := \deg_{\boldsymbol{w}}(P_{\mathcal{W}})$. After 3.1.3-(a), we know that $P_{\mathcal{W}} = \mathbf{X}_{W_m}$ for all $m \geq 2d$, so that, if we decompose W_{2d} in its simple $\mathbb{Q}[\mathcal{S}_{2d}]$-submodules:

$$W_{2d} = \bigoplus\nolimits_{|\lambda|\leq d} (V_{\lambda[2d]})^{\boldsymbol{n}_\lambda}\,,$$

and if we denote $\lambda' := (\lambda_1 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell)$ for $\lambda := (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell)$, then the FI-module

$$\mathcal{W}' := \bigoplus\nolimits_{|\lambda|\leq d} (\mathcal{V}_{\lambda'})^{\boldsymbol{n}_\lambda}\,,$$

which is clearly $(\mathrm{RS})_{m\geq 2d}$, will be such that $P_{\mathcal{W}'} = P_{\mathcal{W}}$, by construction. As a consequence, there exists an isomorphism of FB-modules:

$$\mathcal{W}_{\geq \max\{2d,N\}} \sim \mathcal{W}'_{\geq \max\{2d,N\}}\,, \qquad (\diamond)$$

and $\mathcal{W}$ is (RS) for $m \geq \max\{2d, N\}$, as stated. □

4.1.2. Remark. The last proof shows, in $(\diamond)$, that an FB-module that is (PC) is asymptotically isomorphic, as FB-module, to an (RS) FI-module.

5. Addendum on the weight of a tensor product

5.1. Reinterpretation of the weight of a representation

The theorem 3.1.1-(c) gives an alternative definition of the weight of a representation W_m of $\mathcal{S}_m$ *as the lowest possible weight of the polynomials expressing its character* χ_{W_m}.

5.2. On the weight of the tensor product of representations

5.2.1. Proposition

a) *If* $W_{1,m}$ *and* $W_{2,m}$ *are representations of* $\mathcal{S}_m$*, then*

$$\boldsymbol{w}(W_{1,m} \otimes W_{2,m}) \leq \boldsymbol{w}(W_{1,m}) + \boldsymbol{w}(W_{2,m})\,.$$

b) *If* $\mathcal{W}_1$ *and* $\mathcal{W}_2$ *are* FB*-modules which are* $(\mathrm{PC})_{m\geq N}$*, then*

$$\boldsymbol{w}(W_{1,m} \otimes W_{2,m}) = \boldsymbol{w}(W_{1,m}) + \boldsymbol{w}(W_{2,m})\,,$$

for all $m \geq 2(\boldsymbol{w}(W_{1,m}) + \boldsymbol{w}(W_{2,m}))$*. In particular,*

$$\boldsymbol{w}((\mathcal{W}_1 \otimes \mathcal{W}_2)_{\geq N}) = \boldsymbol{w}((\mathcal{W}_1)_{\geq N}) + \boldsymbol{w}((\mathcal{W}_2)_{\geq N})$$

c) $$\boldsymbol{w}(V_{\lambda[m]} \otimes V_{\mu[m]}) = |\lambda| + |\mu|, \quad \forall m \geq 2(|\lambda| + |\mu|).$$

Proof. (a)As $\chi_{W_1\otimes W_2} = \chi_{W_1}\chi_{W_2}$, we get

$$\boldsymbol{w}(W_1\otimes W_2) \underset{(1)}{\leq} \deg_{\boldsymbol{w}}(\boldsymbol{X}_{W_1}\boldsymbol{X}_{W_2}) = \deg_{\boldsymbol{w}}(\boldsymbol{X}_{W_1}) + \deg_{\boldsymbol{w}}(\boldsymbol{X}_{W_2}) \underset{(2)}{=} \boldsymbol{w}(W_1) + \boldsymbol{w}(W_2),$$

inequality '$\leq_{(1)}$' after 3.1.1-(b-iii), and equality '$=_{(2)}$' after 3.1.1-(c).

(b) Let $d_i := \deg_{\boldsymbol{w}}(P_{\mathcal{W}_i})$. We know after corollary 3.1.3-(a), that $P_{\mathcal{W}_i} = \boldsymbol{X}_{W_{i,m}}$, for $m \geq 2\max\{d_1, d_2\}$. Hence, $\boldsymbol{X}_{W_{1,m}}\boldsymbol{X}_{W_{2,m}} = \boldsymbol{X}_{W_{1,m}\otimes W_{2,m}}$, for $m \geq 2(d_1+d_2)$, after the same corollary. Therefore,

$$\boldsymbol{w}(W_{1,m} \otimes W_{2,m}) = \boldsymbol{w}(W_{1,m}) + \boldsymbol{w}(W_{2,m})\,,$$

for $m \geq 2(d_1 + d_2)$, again after 3.1.1-(c).

(c) Particular case of (b). □

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