Some equivalences of derived exponential functors

Alexander Zakharov*

Abstract

Given a functor $F\colon \mathrm{Ab}^{free,fg} \to \mathrm{Ab}^{flat}$, such that $F(A \oplus B) \simeq F(A) \otimes F(B)$ we will show that the algebra structure on its left derived functor $\mathbb{L}^* F(P)$ is completely determined by the augmented coalgebra structure on $F(\mathbb{Z})$ if $P \in D^{\geq 3}_{perf}(\mathrm{Ab})$. Among other applications we will prove that the Dold-Puppe isomorphism $H_*(K(A,n);\mathbb{Z}) \simeq \mathbb{L}_* \operatorname{Sym} A[n]$ is functorial in $A \in \mathrm{Ab}^{fg}$ if $n \geq 2$.

*xaxa3217 (•_•) gmail com

1 Introduction

1.1 Results

Fix a base ring $\Bbbk$ and let $F\colon (\mathrm{Ab}^{\mathrm{free,fg}}, \oplus) \to (\Bbbk - \mathrm{Mod}^{\mathrm{flat}}, \oplus)$ be a monoidal functor with values in flat $\Bbbk$-modules. Such functor provides $F(\mathbb{Z})$ with a commutative and cocommutative Hopf algebra structure over $\Bbbk$.

In the following we prefer cohomological notation. Let $D^{<-1}_{perf}(\mathrm{Ab})$ denotes complexes in $D(\mathrm{Ab})$ which are bounded and has cohomology concentrated in degress < -1. Denote by $D^{<-1}_{perf}(\mathrm{Ab})^\vee$ the image of $D^{<-1}_{perf}(\mathrm{Ab})$ under $\mathcal{R}Hom(-, \mathbb{Z})$.

In case $P \in D^{<-1}_{perf}(\mathrm{Ab})^\vee$ is represented by a complex of free modules, we can define the *left derived functors* of F by $\mathbb{L}F(P) := N(F(K(P))) \in D(\mathrm{Ab})$, where $N\, \mathrm{cs}\,\mathrm{Ab}\colon \rightarrow\leftarrow \mathrm{Ch}^{\geq 0}(\mathrm{Ab})\colon K$ is the usual Dold-Kan equivalence of categories.

In this work we will prove the following.

Theorem 1.1.1. *For F as above, the algebra structure on $\mathbb{L}F(P) \in D^{\geq 0}(\mathrm{Ab})$ does not depend on the multiplication in $F(\mathbb{Z})$. Namely, if G is another monoidal functor, provided with isomorphism $F(\mathbb{Z}) \simeq G(\mathbb{Z})$ of coalgebras preserving the augmentation $F(\mathbb{Z}) \to \mathbb{Z}$, the we have isomorphism of functors*

$$\mathbb{L}F \simeq \mathbb{L}G\colon D^{<-1}_{perf}(\mathrm{Ab}) \to D(\mathrm{Ab}).$$

To state the other result, as an application of the technique developed in this article, recall an existence of Dold-Puppe isomorphism $DP\colon H_*(K(A, n); \mathbb{Z}) \simeq \mathbb{L}_* \mathrm{Sym}(A[n])$ for a finetely generated abelian group $A \in \mathrm{Ab}^{fg}$. Both parts a functors in A, but the construction of A *a priori* is not canonical. In fact, as shown in [4] there is *no* functorial isomorphism of $H_3(K(A, 1); \mathbb{Z})$ and $\mathbb{L}_3 \mathrm{Sym}(A[1])$. It turns out that this phenomena occurs only on the level of classical group homology:

Theorem 1.1.2. *The Dold-Puppe isomorphism* DP *is functorial for $n \geq 2$.*

Our third result is related to "cohomological" version of Dold theorem given by

Theorem 1.1.3 ([5])**.** *There algebra $H^*(\mathrm{SP}^n X; \mathbb{Z})/Tors$ functorially depends on the algebra $H^*(X; \mathbb{Z})$.*

It is natural to ask if the algebra $H^*(\mathrm{SP}^n X; \mathbb{Z})$ can be recovered by $H^*(X; \mathbb{Z})$. In general this is not true, because the information about torsion coefficients get lost. After providing a simple couter-example based on [7], we prove the following:

Theorem 1.1.4. *For a suspension space $X = \Sigma Y$, the algebra $H^*(\mathrm{SP}^n X; \mathbb{Z})$ functorially depends on $\mathbb{Z}$-module $\bar{H}^*(X; \mathbb{Z})$*

Finally let us note, that this work grew from [6] as an attempt to cover classical topology problems.

2 Preliminaries

2.1 Categories Ab' and Ab''

Consider category Sets_* of finite based sets. We denote its objects by $S_+ = S \sqcup \{\mathrm{pt}\}$, where pt is the base point of S_+.

We have a faithful functor $\mathbb{Z}\langle - \rangle\colon \mathrm{Sets}_* \to \mathrm{Ab}$ given by $S_+ \to \mathbb{Z}\langle S \rangle = \mathbb{Z}^{\oplus S}$.

Definition 2.1.1. The category Ab' is the image of $\mathbb{Z}\langle - \rangle\colon \mathrm{Sets}_* \to \mathrm{Ab}^{\mathrm{free,fg}}$. Define $\mathrm{Ab}'' \subset \mathrm{Ab}^{\mathrm{free,fg}}$ is the image of Ab' under the counter-variant functor $V \to V^\vee$.

Hence Ab' is equivalent to Sets_*, while Ab'' is equivalent to $\mathrm{Sets}_*^{\mathrm{op}}$.

Example 2.1.2. 1. Objects of Ab' are free modules $\mathbb{Z}^{\oplus S}$ with a chosen unordered basis. A morphism $f_* \in \mathrm{Hom}_{\mathrm{Ab}'}(\mathbb{Z}^{S_1}, \mathbb{Z}^{S_2})$ corresponds to a map $f\colon S_1 \to S_2 \sqcup \{\mathrm{pt}\}$, such that $f_* e_s = e_{f(s)}, s \in S_1$ if $f(s) \neq \mathrm{pt}$ and zero otherwise.

2. Similarly a morphism $f^* \in \mathrm{Hom}_{\mathrm{Ab}''}(\mathbb{Z}^{S_1}, \mathbb{Z}^{S_2})$ corresponds to a map $S_2 \to S_1 \sqcup \{\mathrm{pt}\}$, such that $f^* e_s = \sum\limits_{s' \in f^{-1}(s)} e_{s'}$.

Both categories $\mathrm{Ab}', \mathrm{Ab}''$ share morphisms $0 \to \mathbb{Z}$ and $\mathbb{Z} \to 0$. We see that $\mathbb{Z}^{\oplus 2} \xrightarrow{+} \mathbb{Z}$ belongs to Ab', while $\mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}^{\oplus 2}$ belongs to Ab''.

We consider $(\mathrm{Ab}', \oplus), (\mathrm{Ab}'', \oplus)$ as monoidal categories under the direct sum. This corresponds to the connected sum in Sets_* given by $S_{1+} \vee S_{2+} = (S_1 \sqcup S_2)_+$. We will be interested in monoidal functors of the form $(\mathrm{Ab}, \oplus) \to (\Bbbk - \mathrm{Mod}, \otimes)$.

Proposition 2.1.3. *We have following equivalences:*

1. $\{F\colon (\mathrm{Ab}^{\mathrm{free,fg}}, \oplus) \to (\Bbbk - \mathrm{Mod}, \otimes)\} = \{$*Commutative and cocommutative Hopf algebras* $/\Bbbk\}$.
2. $\{F\colon (\mathrm{Ab}', \oplus) \to (\Bbbk - \mathrm{Mod}, \otimes)\} = \{$*Commutative algebras with augmentation* $/\Bbbk\}$.
3. $\{F\colon (\mathrm{Ab}'', \oplus) \to (\Bbbk - \mathrm{Mod}, \otimes)\} = \{$*Cocommutative coalgebras with augmentation* $/\Bbbk\}$.

Provided by evaluation $F(\mathbb{Z})$.

Precisely, given a commutative affine group scheme G over $\Bbbk$ we denote the corresponding monoidal functor by $\mathcal{O}_G^{\otimes -}\colon (\mathrm{Ab}^{free,fg}, \oplus) \to (\Bbbk - \mathrm{Mod}, \otimes)$ defined by $\mathcal{O}_G^{\otimes V} := \mathcal{O}(G \otimes_{\underline{\mathbb{Z}}} V^\vee)$, where $V^\vee$ is a discrete group considered as a constant group scheme over $\mathbb{Z}$. For example $\mathcal{O}_G^{\otimes \mathbb{Z}^n} \simeq G^{\times n}$, so that $\mathbb{Z}^{\oplus 2} \xrightarrow{+} \mathbb{Z}$ induces algebra structure in $\mathcal{O}_G$, while $\mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}^{\oplus 2}$ induces the comultiplication.

2.2 Derived functors

Here we recall some basic facts following [1]. Recall that we have an equivalence of categories and $N\colon \mathrm{cs\,Ab} \rightleftarrows \mathrm{Ch}^{\geq 0}\colon K$ given by the Moore *normalization* functor $N(-)$ and the Kan functor $K(-)$. We have natural isomorphisms $K \circ N \simeq \mathrm{id}$ and $N \circ K \simeq \mathrm{id}$. Also for all categories above we have a notion of homotopy between morphisms, such that the functors N, K preserve homotopies. In particular the equivalence survives on the homotopy level. Let $F\colon \mathrm{Ab}^{\mathrm{free,fg}} \to \Bbbk - \mathrm{Mod}^{\mathrm{flat}}$ be a functor.

Proposition 2.2.1 ([1])**.** *The natural prolongation* $F\colon \mathrm{cs\,Ab}^{\mathrm{free,fg}} \to \mathrm{s}\,\Bbbk - \mathrm{Mod}^{\mathrm{flat}}$ *preserves homotopies.*

This amounts to the following notion

Definition 2.2.2. For F as above, define its *left derived functor* $\mathbb{L}F\colon D^{<-1}_{perf}(\mathrm{Ab})^\vee \to D(\Bbbk - \mathrm{Mod})$ by the formula $\mathbb{L}F(P) := N(F(K(P)))$, where $P \in \mathrm{Ch}^{\geq 0}(\mathrm{Ab})$ is a bounded complex of free $\mathbb{Z}$-modules.

Assume $F\colon (\mathrm{Ab}^{\mathrm{free,fg}}, \oplus) \to (\Bbbk - \mathrm{Mod}, \otimes)$ is monoidal. We have the following (lax) monoidal transformations:

$$\mathrm{AW}\colon N(A) \otimes N(B) \to N(A \otimes B),$$

and

$$\mathrm{EZ}\colon N(A \otimes B) \to N(A) \otimes N(B),$$

where $A, B \in \mathrm{cs\,Ab}$. Recall that the Eilenberg-Zilber morphism or the shuffle map EZ is symmetric, while the Alexander-Whitney or the face-degeneracy map AW is not. In particular the complex $N(F(K(P)))$ is provided with cocommutative comultiplication via EZ, while the multiplication given by AW is only associative. In fact $N(F(K(P)))$ is E_∞-algebra, hence the cohomology $\mathbb{L}^*F(P)$ form a commutative graded algebra.

We finish this section by noting that all results above hold in the simplicial setting by using $\mathrm{s\,Ab}, \mathrm{Ch}^{\leq 0}(\mathrm{Ab})$ and $D^{<-1}_{perf}(\mathrm{Ab})$.

3 Homotopy refinement of the Kan functor K

In this section we consider categories of based topological spaces and simplicial sets $\mathrm{Top}^{\mathrm{pt}}$ and $\mathrm{sSet}^{\mathrm{pt}}$. Given $(X, \mathrm{pt}) \in \mathrm{sSet}^{\mathrm{pt}}$ defines a free abelian group $\mathbb{Z}\langle X\rangle \in \mathrm{s\,Ab}$. The constant simplicial set pt together with morphisms $\mathrm{pt} \to X \to \mathrm{pt}$ defines a splitting $\mathbb{Z}\langle X\rangle \simeq \mathbb{Z}\langle X/\mathrm{pt}\rangle \oplus \mathbb{Z}\langle \mathrm{pt}\rangle$, where $\mathbb{Z}\langle X/\mathrm{pt}\rangle_n \in Ab$ has a unordered basis given by elements of the form $x' := x - \mathrm{pt}$, $x \in X_n \setminus \mathrm{pt}_n$. Clearly we have $\mathbb{Z}\langle X/pt\rangle \in \mathrm{s\,Ab}'$.

We have a functor $\mathbb{Z}\langle -/\mathrm{pt}\rangle \colon \mathrm{sSet}^{\mathrm{pt}} \to \mathrm{s\,Ab}'$.

3.1 Moore spaces

Definition 3.1.1. A Moore space $\mathrm{M}(A,n) \in \mathrm{Top}^{\mathrm{pt}}$ for $A \in \mathrm{Ab}^{\mathrm{fg}}$ and $n > 1$ is a space such that $\pi_i(\mathrm{M}(A,n)) = 0, i = 0,1$ and $\bar{H}_i(\mathrm{M}(A,n);\mathbb{Z}) = 0, i \neq n$, provided with an isomorphism $H_n(\mathrm{M}(A,n);\mathbb{Z}) \simeq A$.

We can represent $\mathrm{M}(A,n)$ as a CW-complex with cells in dimensions $n, n+1$ as follows. If $A = [\mathbb{Z}^R \xrightarrow{f} \mathbb{Z}^{\mathbb{G}}]$ is given by two-term resolution, then one can put $M(A,n) := \mathrm{Cone}(\vee_R S^n \xrightarrow{F} \vee_G S^n)$, where F is such that $H_n(F) = f$. In fact $\mathrm{M}(A,n)$ can be modeled by a finite simplicial complex. Moreover, if $F : (\mathrm{M}(A,n),\mathrm{pt}) \to (\mathrm{M}(A',m),\mathrm{pt})$ is a map of the topological spaces corresponding to simplicial complexes, then the simplicial approximation theorem asserts that F is homotopic to a simplicial map $F^{\varepsilon} : (\mathrm{M}(A,n),\mathrm{pt}) \to (\mathrm{M}(A',m),\mathrm{pt})$ after passing to sufficiently fine barycentric subdivision. Assume $n > 1$. Passing from simplicial complexes to simplicial sets allows to consider Moore simplical sets. Thus any topological statement about maps between Moore space has simplicial couterpart, abusing notation we will denote a simplicial model of the Moore space also by $\mathrm{M}(A,n) \in \mathrm{sSet}^{\mathrm{finite,pt}}$.

We note that though for $n > 1$ the homotopy type of $M(A,n)$ is unique, there is *no* functor $\mathrm{Ab}^{fg} \to \mathrm{Ho(Top)}$ providing a natural realization of Moore space. In fact there are sufficiently many morphisms between Moore spaces. First we note that there are sufficiently many maps between Moore spaces.

Proposition 3.1.2. *For $n > 1$ and $A, A' \in \mathrm{Ab}^{\mathrm{fg}}$ the natural maps*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top}^{\mathrm{pt}})}(\mathrm{M}(A,n),\mathrm{M}(A',n)) \xrightarrow{C_*^{\mathrm{sing}}(-)} \mathrm{Hom}_{D(\mathrm{Ab})}(A[n],A'[n]) = \mathrm{Hom}_{\mathrm{Ab}}(A,A') \tag{3.1.1}$$

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top}^{\mathrm{pt}})}(\mathrm{M}(A,n),\mathrm{M}(A',n+1)) \xrightarrow{C^{\mathrm{sing}_*(-)}} \mathrm{Hom}_{D(\mathrm{Ab})}(A[n],A[n+1]) = \mathrm{Ext}^1(A,A') \tag{3.1.2}$$

are surjective.

Proof. Let $P_\bullet := [\mathbb{Z}^R \overset{m_A}{\to} \mathbb{Z}^G] \to A$ be a free resolution of A. For $n > 1$ the space $\mathrm{M}(A,n)$ is homotopy equivalent to a CW-complex given by a cofiber of $M_A : \vee_R S^n \to \vee_G S^n$ of a map induced by m_A. Let $C_*^A := C_*^{\mathrm{cw}}(M(A,n))$. Similarly for A'.

Given $f \in \mathrm{Hom}_{\mathrm{Ab}}(A,A')$, there is a morphism $\tilde{f} : P_\bullet \to Q_\bullet$ inducing f. Consider the corresponding diagram of maps:

$$\begin{array}{ccccc} \bigvee_R S^n & \xrightarrow{M_A} & \bigvee_G S^n & \longrightarrow & \mathrm{M}(A,n) \\ \downarrow & & \downarrow{\scriptstyle \tilde{F}} & & \downarrow{\scriptstyle g} \\ \bigvee_{R'} S^n & \xrightarrow{M_{A'}} & \bigvee_{G'} S^n & \longrightarrow & \mathrm{M}(A',n) \end{array}$$

If $n > 1$ the left square is homotopy commutative, thus by basic properties of cofiber sequences the dashed arrow exists and produces a map $g : M(A,n) \to M(A',n)$ such that $H_n(g) = f$. This proves surjectivity in (3.1.1). For the second arrow redefine $C_*^{A'} = C_*^{\mathrm{cw}}(M(A',n+1))$ and leave C_*^A as above. Then

$$\mathrm{Ext}^1_{\mathrm{Ab}}(A,A') = \mathrm{Hom}_{C^{\leq 0}}(C_*^A, C_*^{A'})/\sim$$

where $\sim$ is the equivalence relation identifying homotopical morphisms. Since

$$\mathrm{Hom}_{C^{\leq 0}}(C_*^A, C_*^{A'}) = \mathrm{Hom}_{\mathrm{Ab}}(C_{n+1}^A, C_{n+1}^{A'}) = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top}^{\mathrm{pt}})}(M(A,n)/\,\mathrm{Sk}_n\, M(A,n), \mathrm{Sk}_n\, M(A',n+1))$$

naturally maps to $\mathrm{Hom}_{\mathrm{Ho(Top^{pt})}}(M(A,n), M(A', n+1))$ and passes through $\sim$, we obtain:

$$\mathrm{Ext}^1_{\mathrm{Ab}}(A,A') \to \mathrm{Hom}_{\mathrm{Ho(Top^{pt})}}(M(A,n), M(A', n+1)) \overset{C_*^{\mathrm{sing}}(-)}{\longrightarrow} \mathrm{Ext}^1(A,A')$$

This composition is the identity, this proves that the map in 3.1.2 is surjective. □

Remark 3.1.3. For $n = 1$ the surjectivity of (3.1.1) fails. To construct a counterexample consider $\mathbb{R}P^2 \sim \mathrm{M}(A,1)$ for $A = \mathbb{Z}/2$. Assume there is a map $\Delta : \mathbb{R}P^2 \to \mathbb{R}P^2 \vee \mathbb{R}P^2$ such that $\Delta_* \in H_1(\mathbb{R}P^2, \mathbb{Z}) = \mathrm{Hom}_{\mathrm{Ab}}(A, A \oplus A)$ is the diagonal morphism. Denote by $\pi_i : \mathbb{R}P^2 \vee \mathbb{R}P^2 \to \mathbb{R}P^2$ two collapsing maps. Let $\gamma \in H^1(\mathbb{R}P^2, \mathbb{Z}/2)$ be the generator. Then $\pi_i \circ \Delta$ induces identity morphisms on $H_1(\mathbb{R}P^2, \mathbb{Z}/2)$ and hence on its dual $H^1(\mathbb{R}P^2, \mathbb{Z}/2)$. Hence $(\pi_1 \circ \Delta)^*\gamma \cup (\pi_2 \circ \Delta)^*\gamma = \gamma^2 \neq 0 \in H^2(\mathbb{R}P^2, \mathbb{Z}/2)$. On the other hand it iss equal to $\Delta^*(\pi_1^*\gamma \cup \pi_2^*\gamma) = 0$.

Let $P_* \in D^{<-1}_{\mathrm{perf}}(\mathrm{Ab})$ and fix a (non functorial) Moore space construction $\mathrm{M}(-)$. Consider

$$\mathrm{M}(P_*) := \bigvee_i \mathrm{M}(H_i(P_*), i) \in \mathrm{sSet}^{\mathrm{finite,pt}} \tag{3.1.3}$$

One can always assume that $\mathrm{M}(P_*)$ has no non-trivial simplices in degrees ≤ 1, thus by Whitehead's theorem any two such constructions $\mathrm{M}(P_*)$ and $\mathrm{M}'(P_*)$ are homotopy equivalent and are suspensions spaces. In particular $\mathrm{M}(P_*)$ is a comonoid. The natural isomorphism

$$P_* \simeq \bigoplus_i H_i(P_*)[i] \in D(\mathrm{Ab})$$

provides a natural isomorphism

$$N(\mathbb{Z}\langle \mathrm{M}(P_*)/\,\mathrm{pt}\rangle) \simeq P_* \in D(\mathrm{Ab}) \tag{3.1.4}$$

If $F : \mathrm{M}(P_*) \to \mathrm{M}'(Q_*) \in \mathrm{sSet}^{finite,pt}$ then we denote by $\overline{F} : P_* \to Q_* \in D(\mathrm{Ab})$ the morphism $N(\mathbb{Z}\langle F/\,\mathrm{pt}\rangle)$ in terms of (3.1.4).

Recall that $I' : \mathrm{s\,Ab}' \to \mathrm{s\,Ab}$ is the natural embedding. Let $\mathrm{Ho}(\mathrm{s\,Ab}')$ denotes a category obtained by identifying those morphisms in $\mathrm{s\,Ab}'$ which become homotopical in $\mathrm{Ho}(\mathrm{s\,Ab})$ under I'. Thus $\mathrm{Ho}(I') : \mathrm{Ho}(\mathrm{s\,Ab}') \to \mathrm{Ho}(\mathrm{s\,Ab})$ is faithfull.

Lemma 3.1.4. *Given* $P_*, Q_* \in D^{<-1}_{\mathrm{perf}}(\mathrm{Ab})$ *and* $F_i : \mathrm{M}(P_*) \to \mathrm{M}'(Q_*) \in \mathrm{sSet}^{finite,pt}, i = 1,2$ *such that* $\overline{F}_1 = \overline{F}_2$*, we have*

$$\mathbb{Z}\langle F_1/\,\mathrm{pt}\rangle = \mathbb{Z}\langle F_2/\,\mathrm{pt}\rangle \in \mathrm{Ho}(\mathrm{s\,Ab}')$$

.

Proof. More generally, the Dold-Kan correspondence together with (3.1.4), implies the isomorphism

$$\mathrm{Hom}_{\mathrm{Ho(s\,Ab)}}(\mathbb{Z}\langle \mathrm{M}(P_*)/\,\mathrm{pt}\rangle, \mathbb{Z}\langle \mathrm{M}'(Q_*)\rangle) \overset{D(N)}{\longrightarrow} \mathrm{Hom}_{D(\mathrm{Ab})}(P_*, Q_*)$$

By definition $\mathrm{Ho}(I') : \mathrm{Ho}(\mathrm{s\,Ab}') \to \mathrm{Ho}(\mathrm{s\,Ab})$ is faithful, this finish the proof. □

3.2 The construction of K' and K''

The following observation provides a refinement of the Dold-Kan functor $D(K) : D(\mathrm{Ab}) \to \mathrm{Ho}(\mathrm{s\,Ab})$ and serves our main construction.

Theorem 3.2.1. *There is a monoidal functor* $K' : D^{<-1}_{\mathrm{perf}}(\mathrm{Ab}) \to \mathrm{Ho}(\mathrm{s\,Ab}')$*, defined up to a natural isomorphism, such that* $\mathrm{Ho}(I') \circ K'$ *is naturally isomorphic to* $D(K)$*.*

Proof. Let $P_*, Q_* \in D^{<-1}_{\text{perf}}(\text{Ab})$ be complexes and $f \in \text{Hom}_{D^{<-1}_{\text{perf}}(\text{Ab})}(P_*, Q_*)$.. Fix a (non functorial) Moore space constructions $\text{M}_1(-)$ and $\text{M}_2(-)$.

We have a natural decomposition

$$\text{Hom}_{D^{<-1}_{\text{perf}}(\text{Ab})}(P_*, Q_*) = \bigoplus_i \text{Hom}_{\text{Ab}}(H_i(P_*), H_i(Q_*)) \oplus \bigoplus_j \text{Ext}^1_{\text{Ab}}(H_j(P_*), H_{j+1}(Q_*))$$

The set $\text{Hom}_{\text{Ho}(\text{sSet}^{finite,pt})}(\text{M}_1(P_*), \text{M}_2(Q_*))$ forms an abelian group, since the argument is a comonoid. Then by Proposition 3.1.2 there is $F \in \text{Hom}_{\text{Ho}(\text{sSet}^{\text{finite,pt}})}(\text{M}_1(P_*), \text{M}_2(Q_*))$ such that $\overline{F} \sim f$. By repeating the argument we can form a diagram in $\text{sSet}^{\text{finite,pt}}$:

$$\begin{array}{ccc} \text{M}_1(P_*) & \xrightarrow{F_1} & \text{M}_1(Q_*) \\ {\scriptstyle I_P}\downarrow & & \downarrow{\scriptstyle I_Q} \\ \text{M}_2(P_*) & \xrightarrow{F_2} & \text{M}_2(Q_*) \end{array} \tag{3.2.1}$$

The vertical arrows in (3.2.1) correspond to maps, provided by Proposition 3.1.2, such that the induced $H_*(I_P)$ and $H_*(I_Q)$ are identity maps. Note that in general arrows in the diagram (3.2.1) are not canonical and the square does not commute up to homotopy.

On the other hand applying $N(\mathbb{Z}\langle -, \text{pt}\rangle)$ in terms of the natural identification (3.1.4) we get

$$\begin{array}{ccc} P_* & \xrightarrow{f} & Q_* \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id} \\ P_* & \xrightarrow{f} & Q_* \end{array}$$

Then lemma 3.1.4 implies that the following square commutes and all arrows are canonical in $\text{Ho}(\text{s}\,\text{Ab}')$:

$$\begin{array}{ccc} \mathbb{Z}\langle \text{M}_1(P_*)/\,\text{pt}\rangle & \xrightarrow{\overline{F_1}} & \mathbb{Z}\langle \text{M}_1(Q_*)/\,\text{pt}\rangle \\ {\scriptstyle \overline{I_P}}\downarrow & & \downarrow{\scriptstyle \overline{I_Q}} \\ \mathbb{Z}\langle \text{M}_2(P_*)/\,\text{pt}\rangle & \xrightarrow{\overline{F_2}} & \mathbb{Z}\langle \text{M}_2(Q_*)/\,\text{pt}\rangle \end{array} \tag{3.2.2}$$

Thus, if be define $K'(P_*) := \mathbb{Z}\langle \text{M}_1(P_*)/\,\text{pt}\rangle$ and $K'(f) := \overline{F_1} = \mathbb{Z}\langle F_1/\,\text{pt}\rangle$ we obtain a genuine functor with the desired properties. Moreover, passing to a different construction $\text{M}_2(-)$ produces a naturally isomorphic functor.

It remains to check that K' is monoidal, i.e. there is a natural isomorphism

$$K'(P_* \oplus Q_*) \xrightarrow{\sim} K'(P_*) \oplus K'(Q_*) \in D(\text{Ab})$$

This follows by the similar argument: there is an equivalence

$$F : \text{M}(P_* \oplus Q_*) \sim \text{M}(P_*) \vee \text{M}(Q_*)$$

and $\overline{F} \in \text{Ho}(\text{s}\,\text{Ab}')$ is canonically defined. This proves the claim. □

Let $D^{<-1}_{\text{perf}}(\text{Ab})^\vee$ denotes the image of $D^{<-1}_{\text{perf}}$ under monoidal functor $\mathcal{RH}om_{D(\text{Ab})}(-, \mathbb{Z})$. Note that $D^{>2}_{\text{perf}}(\text{Ab}) \subset D^{<-1}_{\text{perf}}(\text{Ab})^\vee$. On the other hand, since the functor $\text{Hom}_{\text{Ab}}(-, \mathbb{Z})$ provides an equivalence of $\text{Ab}' \subset \text{Ab}$ with $\text{Ab}'' \subset \text{Ab}$ in Ab, as a direct corollary we obtain:

Corollary 3.2.2. *There is a monoidal functor* $K'' : D^{<-1}_{\text{perf}}(\text{Ab})^\vee \to \text{Ho}(\text{cs}\,\text{Ab}'')$, *defined up to a natural isomorphism, such that* $\text{Ho}(I'') \circ K''$ *is naturally isomorphic to* $D(K)$.

□

3.3 On $\mathbb{L}F \simeq \mathbb{L}G$

Recall that $I'' : \mathrm{Ab}'' \to \mathrm{Ab}$ is the natural embedding. Denote by $\Bbbk - \mathrm{Mod}^{flat} \subset \mathrm{Ab}$ the subcategory of flat $\Bbbk$-modules. If $F : \mathrm{Ab}^{\mathrm{fg,free}} \to \mathrm{Ab}$ is a functor, then it is possible to define $\mathbb{L}F : D_{\mathrm{perf}}(\mathrm{Ab}) \to D(\mathrm{Ab})$. Above we introduced $D^{<-1}_{\mathrm{perf}}(\mathrm{Ab})^{\vee}$ to be the full subcategory $D_{\mathrm{perf}}(\mathrm{Ab})$ formed by the image of $\mathcal{RH}om_{D(\mathrm{Ab})}(-,\mathbb{Z})$ applied to $D^{-1}_{\mathrm{perf}}(\mathrm{Ab})$. Note in particular $D^{>2}_{\mathrm{perf}}(\mathrm{Ab}) \subset D^{<-1}_{\mathrm{perf}}(\mathrm{Ab})^{\vee}$.

Theorem 3.3.1. *Assume $F, G : \mathrm{Ab}^{\mathrm{fg,free}} \to \Bbbk - \mathrm{Mod}^{\mathrm{flat}}$ are two functors:*

1. *provided with an isomorphism $\phi'' : F'' \simeq G''$. Then there is a natural isomorphism of functors*

$$\phi_{F,G} : \mathbb{L}F \simeq \mathbb{L}G$$

from $D^{<-1}_{\mathrm{perf}}(\mathrm{Ab})^{\vee}$ to $D(\Bbbk - \mathrm{Mod})$. In addition, if $F, G : (\mathrm{Ab}^{\mathrm{fg,free}}, \oplus) \to (\Bbbk - \mathrm{Mod}^{\mathrm{flat}}, \otimes)$ and ϕ'' are monoidal, then so is ϕ.

2. *provided with an isomorphism $\phi' : F' \simeq G'$. Then there is a natural isomorphism of functors*

$$\phi_{F,G} : \mathbb{L}F \simeq \mathbb{L}G$$

from $D^{<-1}_{\mathrm{perf}}(\mathrm{Ab})$ to $D(\Bbbk - \mathrm{Mod})$. In addition, if $F, G : (\mathrm{Ab}^{\mathrm{fg,free}}, \oplus) \to (\Bbbk - \mathrm{Mod}^{\mathrm{flat}}, \otimes)$ and ϕ' are monoidal, then so is ϕ.

Proof. Thanks to theorem 3.2.2 there is a natural isomorphism $\mathbb{L}F = D(N) \circ F \circ D(K) \simeq D(N) \circ F'' \circ K''$. If F is monoidal, then so is the isomorphism. Thus $\mathbb{L}F$ restricted to $D^{<-1}_{\mathrm{perf}}(\mathrm{Ab})^{\vee}$ depends only on F'', this proves the second claim. The first clame is literally the same. □

Corollary 3.3.2. *For monoidal functors F, G as above, for any $P_* \in D^{-1}_{\mathrm{perf}}(\mathrm{Ab})^{\vee}$ the isomorphism $\phi_{F,G}$: $\mathbb{L}F(P_*) \simeq \mathbb{L}G(P_*)$ is a natural (in P_*) isomorphism of monoids in $D(\Bbbk - \mathrm{Mod})$.*

Proof. The claim is a tautological corollary of monoidality of ϕ in theorem 3.3.1. The algebra structure on $\mathbb{L}F(P_*)$ is induced by the sum morphism $+ : P_* \oplus P_* \to P_*$, thus the monoid structure

$$\mathbb{L}F(P_*) \otimes \mathbb{L}F(P_*) \simeq \mathbb{L}F(P_* \oplus P_*) \overset{\mathbb{L}F(+)}{\to} \mathbb{L}F(P_*)$$

is preserved by monoidal functor ϕ. □

Remark 3.3.3. The assumption on cohomology of P is sharp. In the following section we will introduce monoidal functors Bin and Γ. So that $\mathrm{Bin}'' \simeq \Gamma''$ and $\mathbb{L}\,\mathrm{Bin}(\mathbb{Z}/2[-2]) \overset{L}{\otimes} \mathbb{Z}/2 \simeq H^*(K(\mathbb{Z}/2,1);\mathbb{Z}/2)$ is not isomorphic to $\mathbb{L}\Gamma(\mathbb{Z}/2[-2]) \overset{L}{\otimes} \mathbb{Z}/2 \simeq H^*(K(\mathbb{Z},1);\mathbb{Z}/2) \otimes H^*(K(\mathbb{Z},2);\mathbb{Z}/2)$.

Remark 3.3.4. In fact one can prove an isomorphism $\mathbb{L}F(\mathbb{Z}[-n]) \simeq \mathbb{L}G(\mathbb{Z}[-n])$ directly. Namely $K(\mathbb{Z}[-n]) \in \mathrm{cs\,Ab}''$ and the complex $N(F(K(\mathbb{Z}[-n])))$ uses only coalgebra structure of $F(\mathbb{Z})$, so we get a tautological additive isomorphism of $\mathbb{L}F(\mathbb{Z}[-n])$ and $\mathbb{L}G(\mathbb{Z}[-n])$. Then the straighforward computation shows that the multiplication given by Alexander-Whitney formula for AW does not use the multiplication at all! What makes the theorem above remarkable is that we do not have even a comparsion morphism on the nose already in case $P = \mathbb{Z}/2[-n]$.

4 Binomial algebras

Definition 4.0.1. For $V \in \mathrm{Ab}^{free,fg}$, the free algebra of divided powers $\Gamma(V) \subset \mathrm{Sym}(V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the algebra generated by expressions $v^{[n]} := \frac{v^n}{n!}$ for all $n \geq 0$, $v \in V$.

Definition 4.0.2. For $V \in \mathrm{Ab}^{free,fg}$ The free binomial algebra $\mathrm{Bin}(V)$ is algebra of all integer-valued polynomial functions on $V^\vee$.

By naturality this defines a monoidal functor $\mathrm{Bin}(-)\colon (\mathrm{Ab}^{free,fg}, \oplus) \to (\mathrm{Ab}^{flat}, \otimes)$. In particular $\mathrm{Bin}(\mathbb{Z})$ is spanned by binoms $\binom{x}{n}$. The comultiplication is given by:

$$\Delta\binom{x}{n} = \binom{x_1+x_2}{n} = \sum_{i+j=n} \binom{x_1}{i}\binom{x_2}{j}.$$

We denote by $\mathrm{Bin}^{\leq k}(-)$ and $\Gamma^{\leq k}(-)$ the multiplicative filtrations corresponding to the degree of polynomial functions. We have an isomorphism of monoidal functors $\mathrm{Gr}\,\mathrm{Bin}(V) \simeq \Gamma(V)$ given by $\binom{v}{n} \to v^{[n]}$. The comultiplication formula provides an isomorphism of coalgebras $\mathrm{Bin}(\mathbb{Z}) \simeq \Gamma(\mathbb{Z})$. More precisely we have:

Proposition 4.0.3. *The monoidal functors* $\mathrm{Bin}'' := \mathrm{Bin}|_{\mathrm{Ab}''}$ *and* $\Gamma'' := \Gamma|_{\mathrm{Ab}''}$ *with target* $(\mathrm{Ab}, \otimes)$ *are isomorphic.*

Apart from the canonical *increasing* filtration $\mathrm{Bin}^{\leq n}$, we have a *decreasing* filtration $\mathrm{IBin}''_{>n} \subset \mathrm{Bin}''$ defined as follows:

$$\mathrm{IBin}''_{>n}(\mathbb{Z}^S) = \mathbb{Z}\langle \prod_{s\in S} \binom{e_s}{n_s} \mid \sum_{s \in S} n_s > n \rangle.$$

It is easy to see that in general a morphism $f\colon \mathbb{Z}^{S_1} \to \mathbb{Z}^{S_2}$ restricts to a map $\mathrm{IBin}''_{>n}(\mathbb{Z}^{S_1}) \to \mathrm{IBin}''_{>n}(\mathbb{Z}^{S_2})$ only in case $f \in \mathrm{Ab}''$.

Definition 4.0.4. Let $\mathrm{Sym}'\colon (\mathrm{Ab}', \oplus) \to (\mathrm{BiAlg}, \otimes)$ be a monoidal functor determined by the bialgebra $\mathrm{Sym}'(\mathbb{Z})$ defined as follows:

1. The multiplication is given by the identification $\mathrm{Sym}'(\mathbb{Z}) = \mathbb{Z}[t]$.
2. The comultiplication is given by $\Delta t = t_1 + t_2 + t_1 t_2 = t \otimes 1 + 1 \otimes t + t \otimes t$.
3. The unit and the augmentation are given by $1 \in \mathbb{Z}[t]$ and $t\mathbb{Z}[t]$ respectively.

So $\mathrm{Spec}\,\mathrm{Sym}'(\mathbb{Z})$ is the "polynomial" version of $\widehat{\mathbb{G}}_m$. Consider now $\mathrm{Bin}''\colon (\mathrm{Ab}'', \oplus) \to (\mathrm{BiAlg}, \otimes)$. Since $\mathrm{Bin}''(\mathbb{Z}^S) \in \mathrm{BiAlg}$ has a preferred unordered basis given by binoms $\prod_{s \in S} \binom{e_s}{n_s}$, one can form a dual $\mathrm{Bin}''(\mathbb{Z}^S)^\vee$ of the same cardinality as $\mathrm{Bin}''(\mathbb{Z}^S)$. Setting $\mathrm{Bin}''\vee(V) := (\mathrm{Bin}''(V^\vee))^\vee$ defines a functor $(\mathrm{Ab}', \oplus) \to (\mathrm{BiAlg}, \otimes)$

Proposition 4.0.5. *We have an isomorphism* $\mathrm{Bin}''^\vee \simeq \mathrm{Sym}'$ *of functors* $(\mathrm{Ab}', \oplus) \to (\mathrm{BiAlg}, \otimes)$. *Under this duality we have* $\mathrm{IBin}''_{>n}{}^\perp \simeq \mathrm{Sym}'^{\leq n}$.

Proof. The claim reduces to the isomorphism of bialgebras $\mathrm{Bin}''(\mathbb{Z})^\vee \simeq \mathrm{Sym}'(\mathbb{Z})$. If we have a bialgebra E with basis e_i, let f^i denote its duals in $F := E^\vee$. Set $u(e, f) := \sum e_i f^i$, then the structure constants of F are determined by: $\Delta_e u(e, f) = m_f(u(e', f), u(e'', f))$ and $m_e(u(e, f'), u(e, f'')) = \Delta_f u(e, f)$. Our case corresponds to the formal function $u(x, t) := (1+t)^x = \sum_n \binom{x}{n} t^n$. The claim is equivalent to checking identities $\Delta_x(1+t)^x = m_t((1+t)^{x_1}, (1+t)^{x_2})$ and $m_x((1+t_1)^x, (1+t_2)^x) = \Delta_t(1+t)^x$. By the definition $\mathrm{IBin}''_{>n}(\mathbb{Z})$ is spanned by $\binom{x}{i}, i > n$, which are duals of t^i, so that $\mathrm{IBin}_{>n}(\mathbb{Z})^\perp$ is spanned by $t^j, j \leq n$. □

In particular the product in Bin restricts to a morphism $\mathrm{IBin}''_{>n} \otimes \mathrm{IBin}''_{>m} \to \mathrm{IBin}''_{>\max(n,m)}$, hence $\mathrm{IBin}''_{>m}$ provides a non-multiplicative decresing filtration. Since $\mathrm{IBin}''_{\geq 0} = \mathrm{Bin}''$, $\mathrm{IBin}''_{>n}$ is an ideal for each n.

By technical reasons we want to introduce another monoidal functor corresponding to the formal group law $\widehat{\mathbb{G}}_m$. For a motivation note that $\operatorname{Spec}\operatorname{Sym}'(\mathbb{Z})$ is a monoid scheme $(\mathbb{A}^1, \cdot)$ with coordinate t in the neighborhood of $1 \in \mathbb{A}^1$, so $\operatorname{Sym}'(\mathbb{Z})$ is not a Hopf algebra. Clearly, passing to the completion at (t) we obtain $\hat{\operatorname{Spec}_0}\operatorname{Sym}'(\mathbb{Z}) \simeq \widehat{\mathbb{G}}_m$. The Cartier dual $\mathbb{H}^\vee = \widehat{\mathbb{G}}_m$ gives rise to a monoidal functor

$$\widehat{\operatorname{Sym}'}\colon (\mathrm{Ab}^{free,fg}, \oplus) \to (\mathrm{Ab}^{flat}, \otimes).$$

In previous notation $\widehat{\operatorname{Sym}'}(-) = \mathcal{O}_{\widehat{\mathbb{G}}_m}^{\otimes -}$.

It is easy to describe its behavior beond the morphisms in Ab'. For example the map induced by $\Delta\colon \mathbb{Z} \to \mathbb{Z}^{\oplus 2}$ induce a morphism $\widehat{\operatorname{Sym}'}(t) \to \widehat{\operatorname{Sym}'}(t_1, t_2)$ is given by $t \to t_1 + t_2 + t_1 t_2$. Similarly $\mathbb{Z} \xrightarrow{a\cdot} \mathbb{Z}$ is given by $t \to (1+t)^a - 1$.

5 Symmetric powers

In this section we revise the Dold-Thom isomorphism and relate $C^*(\mathrm{SP}^\infty X)$ with $\mathbb{L}\operatorname{Bin}(\bar{H}^*(X;\mathbb{Z}))$. A less direct approach, but with other goals was developed in [2]. Below we will assume that (X, pt) is based simplicial set, which is connected, i.e. $X_0 = \{\mathrm{pt}\}$ and its cohomology are finitely generated.

5.1 A splitting of $\mathbb{Z}\langle \mathrm{SP}^\infty(X, \mathrm{pt})\rangle$

We denote by $\mathrm{pt} \in \mathrm{sSet}$ the constant simplicial set corresponding to a point.

Notation 5.1.1. Denote by $\mathrm{sSet}^{\text{finite,pt}}$ the category of connected based finite simplicial sets such that for $(X, \mathrm{pt}) \in \mathrm{sSet}^{\text{finite,pt}}$, $X_0 = \mathrm{pt}_0$.

For X as above, $\mathbb{Z}\langle X\rangle / \mathbb{Z}\langle \mathrm{pt}\rangle$ splits of $\mathbb{Z}\langle X\rangle$ via the projection $p : \mathbb{Z}\langle X\rangle \to \mathbb{Z}\langle X\rangle$ defined by $p(x_n) = x_n - \mathrm{pt}_n$, where $x \in X_n$.

Definition 5.1.2. Let

$$\mathbb{Z}\langle X/\mathrm{pt}\rangle := \mathbb{Z}\langle X\rangle/\langle \mathrm{pt}\rangle \simeq \mathbb{Z}^{\oplus X\setminus\{\mathrm{pt}\}} \in \mathrm{s\,Ab}'$$

denotes the natural in (X, pt) lifting of $\mathrm{Im}[p] = \mathbb{Z}\langle X\rangle/\mathbb{Z}\langle \mathrm{pt}\rangle \in \mathrm{s\,Ab}$ in $\mathrm{s\,Ab}'$.

Recall that $\mathrm{SP}^\infty(X, \mathrm{pt}) = \varinjlim_n \mathrm{SP}^n(X, \mathrm{pt})$ is the union of symmetric powers under the natural inclusions $\mathrm{SP}^n(X, \mathrm{pt}) \to \mathrm{SP}^{n+1}(X, \mathrm{pt})$. Since $\mathrm{SP}^\infty(X, \mathrm{pt})$ is a monoid in sSet, $\mathbb{Z}\langle \mathrm{SP}^\infty(X, \mathrm{pt})\rangle$ is a simplicial bialgebra.

The following is an appropriate form of an observation due to Dold:

Theorem 5.1.3. *For (X, pt) as above the natural morphism of monoids $\mathrm{SP}^\infty X \to \mathbb{Z}\langle X/\mathrm{pt}\rangle$ is a quasi-isomorphism. There is a natural isomorphism of simplicial bialgebras*

$$\mathbb{Z}\langle \mathrm{SP}^\infty(X, \mathrm{pt})\rangle \simeq \operatorname{Sym}'(\mathbb{Z}\langle X/\mathrm{pt}\rangle) \in \mathrm{s\,BiAlg}\,.$$

This identification corresponds to a quasi-isomorphism of bialgebras

$$\operatorname{Sym}'(\mathbb{Z}\langle X/\mathrm{pt}\rangle) \to \mathbb{Z}\langle \mathbb{Z}\langle X/\mathrm{pt}\rangle\rangle.$$

Proof. The morphism of monoids $\mathrm{SP}^\infty X \to \mathbb{Z}\langle X/\mathrm{pt}\rangle$ is determined by mapping $\mathrm{pt} \to 0$ and $x \to x'$. It is quasi-isomorphism the classical Dold-Thom theorem.

Fix a natural number n. Then a q-simplex $\mathrm{SP}^n(X, \mathrm{pt})$ is a symmetric expression

$$x_N = x_1 \cdot \ldots \cdot x_n \in \mathrm{SP}^n(X, \mathrm{pt})$$

for $x_i \in X_q$ and $N = \{1, \ldots, n\}$. For any $x \in X_q$ put $x' := x - pt$ for $x \notin pt_q$ and $pt'_q := pt_q$.

Thus $x_N = \sum_{I \subset N} x'_I \cdot pt_q^{n-|I|} \in \mathbb{Z}\langle \mathrm{SP}^n(X, \mathrm{pt})\rangle$ Mapping pt_q to 1 gives a natural identification $\mathbb{Z}\langle \mathrm{SP}^N(X, \mathrm{pt})\rangle \simeq \bigoplus_{i \leq n} \mathrm{Sym}'^i(\mathbb{Z}\langle X/\mathrm{pt}\rangle)$ of simiplicial groups. It is easy to see that the isomorphism above is consistent with the inclusions $\mathrm{SP}^n(X, \mathrm{pt}) \to \mathrm{SP}^{n+1}(X, \mathrm{pt})$, given by $x_N \to x_N \cdot pt_q$ on the LHS, and with the natural inclusions $\mathrm{Sym}'^{\leq n} \to \mathrm{Sym}'^{\leq n+1}$.

Therefore we obtain an isomorphsim of simplicial groups $\mathbb{Z}\langle \mathrm{SP}^\infty(X, \mathrm{pt})\rangle \to \mathrm{Sym}'(\mathbb{Z}\langle X/\mathrm{pt}\rangle)$. Clearly it is a morphism of simplicial algebras and the image of pt_q is $1 \in \mathrm{Sym}'(\mathbb{Z}\langle X/\mathrm{pt}\rangle)$ for all q.

Let us compute the comultiplication on the LHS. The diagonal

$$\Delta : \mathrm{SP}^n(X, \mathrm{pt}) \to \mathrm{SP}^n(X, \mathrm{pt}) \times \mathrm{SP}^n(X, \mathrm{pt})$$

is a multiplicative map satisfying $\Delta(x) = x \times x$ for $x \in X_\bullet$.

Thus the induced morphism $\Delta : \mathbb{Z}\langle \mathrm{SP}^n(X, \mathrm{pt})\rangle \to \mathbb{Z}\langle \mathrm{SP}^n(X, \mathrm{pt})\rangle \otimes \mathbb{Z}\langle \mathrm{SP}^n(X, \mathrm{pt})\rangle$ is a map of algebras satisfying the formula

$$\Delta(x'_q) = x'_q \otimes pt_q + pt_q \otimes x'_q + x'_q \otimes x'_q \tag{5.1.1}$$

for all $x_q \in X_q \setminus pt_q$. Clearly $\Delta(pt_q) = pt_q \otimes pt_q$.

The formal multiplicative group comultiplication in the definition 4.0.4 of Sym' immediatly implies that our isomorphism is an isomorphsim of bialgebras.

The remaining part asserts that given $A = \mathbb{Z}\langle X/\mathrm{pt}\rangle \in \mathrm{s\,Ab}'$ we have a natural morphism $\mathrm{Sym}'(A) \to \mathbb{Z}\langle A\rangle$. Here the multiplication and the comultiplication on the RHS corresponds to the group algebra structure. The formula is given by mapping e_s to e, where $e \in \mathbb{Z}\langle A\rangle$ is neutral element given by $0 \in A$. It is straighforward to check that this morphism respects bialgebra structure. □

Remark 5.1.4. By the above the object $\mathbb{Z}\langle \mathrm{SP}^\infty(X, \mathrm{pt})\rangle \in \mathrm{s\,Ab}$ admits an increasing and the decreasing filtrations. Note that the diagonal map does not respect the increasing filtration, meaning that the induced coalgebra structure on the normalized complex $N(\mathrm{Sym}'(\mathbb{Z}\langle X/\mathrm{pt}\rangle))$ is not filtered.

Corollary 5.1.5. *For each n, the natural isomorphisms of algebras*

$$\mathbb{Z}^{\mathrm{SP}^n(X,\mathrm{pt})} \simeq \mathrm{Bin}'' / \mathrm{IBin}''_{>n}(\mathbb{Z}\langle X/\mathrm{pt}\rangle^\vee),$$

and coalgebras:

$$\mathbb{Z}\langle \mathrm{SP}^n(X, \mathrm{pt})\rangle \simeq \mathrm{Sym}'^{\leq n}(\mathbb{Z}\langle X/\mathrm{pt}\rangle),$$

are compatible with n and functorial in X.

Since $\mathbb{Z}\langle \mathrm{SP}^\infty(X, \mathrm{pt})\rangle = \oplus_i \mathrm{Sym}^i \mathbb{Z}\langle X/\mathrm{pt}\rangle$ splits, as a well-known corollary we have

Corollary 5.1.6. *For any coeffcient ring, we have a canonical decomposition:*

$$H_*(\mathrm{SP}^\infty X) \simeq \oplus_i H_*(\mathrm{SP}^n X/\mathrm{SP}^{n-1} X) \simeq H_*(X^{\wedge n}/\Sigma_n),$$

which is compatible with the Pontryagin algebra structure on the LHS. The groups $H^*(\mathrm{SP}^n X)$ satisfy Mittag-Leffler condition, hence $H^*(\mathrm{SP}^\infty X) \simeq \varprojlim_n H^*(\mathrm{SP}^n X)$.

Remark 5.1.7. In fact we have an isomorphism $\mathbb{Z}^{\mathrm{SP}^\infty X} \simeq \overline{\mathrm{Bin}}(\mathbb{Z}\langle X/\mathrm{pt}\rangle^\vee)$, where $\overline{\mathrm{Bin}}(\mathbb{Z}^S) = \varprojlim_n \mathrm{Bin}'' / \mathrm{IBin}''_{>n}(\mathbb{Z}^S)$ is the "completion" of $\mathrm{Bin}(\mathbb{Z}^S)$, i.e. infinite series of binomials. The Mittag-Leffrer condition justify the reason why we can consider binomial polynomials instead of binomial series to describe cochains $\mathbb{Z}^{\mathrm{SP}^\infty X}$. In other words, the natural map $\mathrm{Bin}(\mathbb{Z}\langle X/\mathrm{pt}\rangle^\vee) \simeq \varinjlim_n \mathbb{Z}^{\mathrm{SP}^n X} \to \mathbb{Z}^{\mathrm{SP}^\infty X}$ is a quasi-isomorphism.

We remark the following

Proposition 5.1.8. *For X as above, the natural morphism $\mathbb{L}\,\mathrm{Sym}'(\mathbb{Z}\langle X/\mathrm{pt}\rangle) \to \mathbb{L}\widehat{\mathrm{Sym}'}(\mathbb{Z}\langle X/\mathrm{pt}\rangle)$ is a quasi-isomorphism.*

Proof. By décalage isomorphism we have $\mathbb{L}^n \operatorname{Sym}^d(P) \simeq \mathbb{L}^{n+d}\Lambda^d(P[-1])$. In our case $P := N(\mathbb{Z}\langle X/\operatorname{pt}\rangle)$ can be represented by a finite complex of free modules with bounded ranks. Hence $\mathbb{L}^n \operatorname{Sym}^d(P) = 0$ for any given n and sufficiently large d. Hence $\bigoplus_i \mathbb{L}\operatorname{Sym}^i(P) \to \prod_i \mathbb{L}\operatorname{Sym}^i(P)$ is a quasi-isomorphism, which is equivalent to the statement. □

5.2 Derived functors of Bin and Sym

Now we restate the previous results in terms of derived functors and relate them with Eilenberg-Maclane spaces. Recall that Dold-Thom theorem provides a canonical, up to homotopy, equivalence $\operatorname{SP}^\infty X$ and $\prod_i K(\bar{H}_i(X;\mathbb{Z}), i) =: K(\bar{H}_*(X;\mathbb{Z}))$, where $\bar{H}_*(X;\mathbb{Z}) := \bigoplus_i \bar{H}_i(X;\mathbb{Z})[i]$ is equivalent to $\bar{C}_*(X;\mathbb{Z})$ in $D(\mathrm{Ab})$. Simplicially it is given by the morphism of monoids $\operatorname{SP}^\infty X \to \mathbb{Z}\langle X/\operatorname{pt}\rangle$, such that the geometric realization of the RHS is $K(\bar{H}_*(X;\mathbb{Z}))$.

Definition 5.2.1. The Dold-Puppe isomorphism DP is the composition of isomorphisms in $D(\mathrm{Ab})$:

$$C_*(K(\bar{H}_*(X;\mathbb{Z}))) \simeq C_*(\operatorname{SP}^\infty X) \simeq \mathbb{L}\operatorname{Sym}(\bar{H}_*(X;\mathbb{Z})).$$

Similarly we define $\mathrm{DP}\colon C^*(K)(\bar{H}_*(X;\mathbb{Z})) \simeq \mathbb{L}\operatorname{Bin}(\bar{H}^*(X;\mathbb{Z}))$. This isomorphisms are functorial in (X, pt).

On the other hand one can consider $C_*(K(A_*))$ and $\mathbb{L}\operatorname{Sym}(A_*)$ as functors in $A_* \in D^{\leq -1}(\mathrm{Ab})$ with values in $D(\mathrm{Ab})$. As we pointed out in the introduction this functors are *not* isomorphic in general.

Remark 5.2.2. Given (X, pt) with prescribed homology groups $A_* := \bar{H}_*(X;\mathbb{Z})$, provides via the Dold-Pupped isomorpism a splitting filtration on $H_*(K(A_*))$. The filtration is multiplicative with respect to the Pontragin product. The corresponding splitting filtration on $H^*(K(A_*))$ is *not* multiplicative in general. Moreover, the filtration provides a non-trivial invariant of a homotopy type with prescribed homology groups.

An example is provided by setting $X_1 = S^2 \wedge S^4$ and $X_2 = \mathbb{C}P^2$. Let y_2, y_4 be generators of $H^*(X_1;\mathbb{Z})$ and t is a generator of $H^2(X_2;\mathbb{Z})$. Let $E = K(\mathbb{Z}[2] \oplus \mathbb{Z}[4]) = \mathbb{C}P^2 \times K(\mathbb{Z}, 4)$. Pick the fundamental classes $x_2, x_4 \in H^*(E;\mathbb{Z})$. By Dold-Thom theorem we have maps $f_i\colon X_i = \operatorname{SP}^1 X_i \to K(\mathbb{Z}[2] \oplus \mathbb{Z}[4])$, which are determined by pull-backs $f_1^* x_2 = y_2, f_1^* x_4 = y_4$ and $f_2^* x_2 = t, f_2^* x_4 = t^2$. Denote by $F_1^{>n}, F_2^{>n}$ the decreasing filtrations on $H^*(E)$ corresponding to $\operatorname{IBin}''_{>n}(\mathbb{Z}\langle X_i/\operatorname{pt}\rangle)$. Then $\mathbb{Q}\otimes F_1^{>1}$ is an ideal generated by x_2^2, x_4^2, x_2x_4, while $\mathbb{Q}\otimes F_2^{>1}$ is generated by $x_2^2 = x_4, x_2^3$. Hence $F_1^{>-1} \neq F_2^{>-1}$.

5.3 A functorial Dold-Puppe isomorphism

There are no functorial isomorphism $H_*(K(A,1);\mathbb{Z})$ and $\mathbb{L}_*\operatorname{Sym}(A[-1])$. Here we will establish the functoriality of the Dold-Puppe isomorphism in range $D^{<-1}_{perf}(\mathrm{Ab})$ using the previous consuction of DP by restrictig to simply-connected Moore spaces.

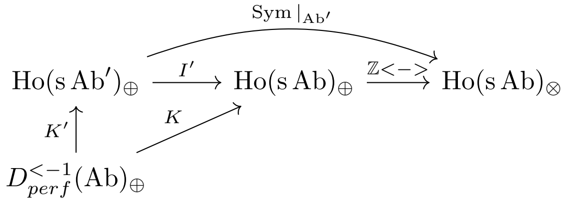
Recall that the Eilenberg-Maclane space $K(P) = \prod_i K(H_i(P), i)$ can be described functorially in $P \in \mathrm{Ch}^{<0}(\mathrm{Ab})$ as geometric realization $|\mathbb{Z}\langle K(P)\rangle|$. Then, the monoidal functor $\mathbb{Z}\langle -\rangle\colon (\mathrm{Ab}, \oplus) \to (\mathrm{Ab}^{flat}, \otimes)$ allows to describe the chains $C_*(K(P))$ as $\mathbb{L}\mathbb{Z}\langle -\rangle(P)$.

Proposition 5.3.1. *There is a natural isomorphism*

$$C_*(K(P)) = \mathbb{L}\mathbb{Z}\langle -\rangle(P) \simeq \mathbb{L}\operatorname{Sym}(P),$$

which is natural in $P \in D^{<-1}_{perf}(\mathrm{Ab})$.

Proof. Theorem 3.2.1 provides the following diagram:



By Theorem 5.1.3 we have a natural equivalence $\mathrm{Sym}|_{\mathrm{Ab}'} \simeq \mathbb{Z}\langle - \rangle \circ I'$ of functors. By Theorem 3.2.1 the diagram commutes up to a canonical isomorphism. □

The proof shows that the Dold-Puppe isomorphisms, which are by the construction multiplicative, are in fact *natural* in $P \in D^{<-1}_{perf}(\mathrm{Ab})$. As a direct corollary we have

Corollary 5.3.2. *The Dold-Puppe isomorphisms of monoids in* $D(\mathrm{Ab})$

$$C_*(K(P)) \simeq \mathbb{L}\,\mathrm{Sym}(P),$$

and

$$C^*(K(P)) \simeq \mathbb{L}\Gamma(\mathcal{RH}om(P,\mathbb{Z})) \simeq \mathbb{L}\,\mathrm{Bin}(\mathcal{RH}om(P,\mathbb{Z})),$$

are natural in $P \in D^{<-1}_{perf}(\mathrm{Ab})$. *In particular the LHS admits a natural split filtration.*

We note that the case $P = A[-n]$ and $A \in \mathrm{Ab}^{free,fg}$ was covered by A. Touzé in [9].

Remark 5.3.3. The isomorphism $\mathbb{L}\Gamma(P^\vee) \simeq \mathbb{L}\,\mathrm{Bin}(P^\vee)$ follows directly by Theorem 3.3.1 by noting that $\mathrm{Bin}'' \simeq \Gamma''$.

6 Applications

6.1 Cohomology of iterated classfying stacks

Given an affine commutative group scheme G one can define the so-called iterated classifying stacks $B^nG = B(B^{n-1}G)$. Its cohomology with coefficient in the structure sheaf can be computed by the formula

$$H^*(B^nG, \mathcal{O}_{B^nG}) := \mathbb{L}\mathcal{O}_G^{\otimes -}(\mathbb{Z}[-n]).$$

Theorem 3.3.1 asserts in particular that for any formal group law G on $\Bbbk[t_1, \dots, t_d]]$ the cohomology of $B^nG^\vee$ are given by $\mathbb{L}\Gamma(\mathbb{Z}^d[-n])$.

Now consider the case $G = \mathbb{G}_a$. Let us recall the décalage isomorphism (mixed cosimplicial/simplicial version is proved in [10] 4.3.2.1):

Theorem 6.1.1. *There is a chain of canonical multiplicative isomorphisms:*

$$\mathbb{L}^{i-2d}\operatorname{Sym}^d(A[-n]) \simeq \mathbb{L}^{i-d}\Lambda^d(A[-n-1]) \simeq \mathbb{L}^i\Gamma^d(A[-n-2])$$

□

Thus

$$\mathbb{L}^i\Gamma(A[-n-2]) \simeq \bigoplus_d \mathbb{L}^{i-2d}\operatorname{Sym}^d(A[-n])$$

By our results

$$H^i(K(\mathbb{Z}, n+2); \mathbb{Z}) \simeq \mathbb{L}^i\operatorname{Bin}(\mathbb{Z}[-n-2]) \simeq \mathbb{L}^i\Gamma(\mathbb{Z}[-n]),$$

and we obtain:

Corollary 6.1.2. *There is a chain of natural multiplicative isomorphisms:*

$$H^i(K(\mathbb{Z}, n+2); \mathbb{Z}) \simeq \bigoplus_d \mathbb{L}^{i-2d}\operatorname{Sym}^d(\mathbb{Z}[-n]) \simeq \bigoplus_{d'} H^{d'}(B^{(n)}\mathbb{G}_a, \mathcal{O}_{B^{(n)}\mathbb{G}_a})_{i-d'}$$

6.2 A topological application

6.2.1 The problem

It is natural to ask if the algebra $H^*(\operatorname{SP}^n X; \mathbb{Z})$ can be recovered by the algebra $H^*(X; \mathbb{Z})$. This question was rised, e.g. in [5].

In general the answer is *no*. To construct a counter-example, let us mention that [7] constructed spaces X_1, X_2 such that $H^*(X_1; \mathbb{Z}) \simeq H^*(X_2; \mathbb{Z})$, but $H^*(X_1 \times X_1; \mathbb{Z}_{(5)}) \not\simeq H^*(X_2 \times X_2; \mathbb{Z}_{(5)})$. Specificall, the example is provided by 3-dimensional lens spaces $X_i = L(i; 5)$. The groups $H^p(X_i; \mathbb{Z})$ are $\mathbb{Z}, 0, \mathbb{Z}/5, \mathbb{Z}$ for $p = 0, \dots 3$ respectively. Inspecting the spectral sequence $E_2^{pq} = H^p(\mathbb{Z}/2; H^q(X_i \times X_i; \mathbb{Z})) \Rightarrow H^{p+q}(\operatorname{SP}^2 X_i; \mathbb{Z})$ implies that the natural morphism $H^*(\operatorname{SP}^2 X_i; \mathbb{Z}_{(5)}) \to H^*(X_i \times X_i; \mathbb{Z}_{(5)})$ is an isomorphism in all degrees except 6. In particular, an isomorphism $H^*(\operatorname{SP}^2 X_1; \mathbb{Z}) \simeq H^*(\operatorname{SP}^2 X_2; \mathbb{Z})$ would imply an isomorphism $H^*(X_1 \times X_1; \mathbb{Z}_{(5)}) \simeq H^*(X_2 \times X_2)$.

On the other hand [8] had shown that it is true if the data $H^*(X; \mathbb{Z})$ is replaced by *cohomological spectrum* of X introduced by Whitehead [11]. Roughly speaking the cohomological spectrum of a space Z is a diagram with vertices $H^*(Z; \mathbb{Z}/m), m \leq \infty$ with morphisms given by the restrictions and bocksteins.

Theorem 6.2.1. *Cohomological spectrum* $\operatorname{SP}^n X$ *is determined by cohomological spectrum of* X*. In particular the algebra* $H^*(\operatorname{SP}^n X; \mathbb{Z})$ *functorially depends on cohomological spectrum of* X*.*

Remark 6.2.2. In the above claim $\operatorname{SP}^n X$ can be replaced by other functors.

Quite remarkabely, D. Gugnin showed the following result:

Theorem 6.2.3 ([5]). *The algebra* $H^*(\operatorname{SP}^n X; \mathbb{Z})/\operatorname{Tors}$ *is functorially depends on the algebra* $H^*(X; \mathbb{Z})/Tors$*.*

6.2.2 Suspension spaces case

Assume $(X, \mathrm{pt}) \in \mathrm{sSet}^{finite,pt}$ is a based space. Recall that $H^*(\mathrm{SP}^\infty X; \mathbb{Z}) \simeq \mathbb{L}\,\mathrm{Bin}''(\mathbb{Z}\langle X/\mathrm{pt}\rangle^\vee)$ admits a canonical increasing filtration which additively splits. So the natural map $H^*(\mathrm{SP}^\infty X, \mathrm{SP}^n X; \mathbb{Z}) \to H^*(\mathrm{SP}^\infty X; \mathbb{Z})$ is an inclusion and we have an isomorphism

$$\mathbb{L}^*\,\mathrm{IBin}''_{>n}(\mathbb{Z}\langle X/\mathrm{pt}\rangle^\vee) \simeq H^*(\mathrm{SP}^\infty X, \mathrm{SP}^n X; \mathbb{Z})$$

. In general this decreasing filtration is not multiplicative.

The discussion above yields

Theorem 6.2.4. *If $X = \Sigma Y$ is a suspension space, then $H^*(\mathrm{SP}^n X; \mathbb{Z})$ is a functor of $\bar{H}^*(X; \mathbb{Z})$ considered as a graded $\mathbb{Z}$-module.*

Proof. We have a morphism $\Delta\colon X \to X \vee X$ such that the induced morphism $\bar{C}_*(X) \to \bar{C}_*(X) \oplus \bar{C}_*(X)$ is given by the diagonal in $D(\mathrm{Ab})$. Then $\Delta \in \mathrm{s\,Ab}'$ induces a morphism $\Delta^\vee\colon \mathbb{Z}\langle X/\mathrm{pt}\rangle^{\oplus 2} \to \mathbb{Z}\langle X/\mathrm{pt}\rangle \in \mathrm{Ho}(\mathrm{cs\,Ab}'')$ which is homotopy equivalent to the usual addition. In particular the additive equivalence

$$\mathbb{L}\,\mathrm{Bin}''(\mathbb{Z}\langle X/\mathrm{pt}\rangle^\vee) \simeq \mathbb{L}\Gamma''(\mathbb{Z}\langle X/\mathrm{pt}\rangle^\vee)$$

is in fact multiplicative in $D(\mathrm{Ab})$. Then the natural isomorphism $H^*(\mathrm{SP}^n X; \mathbb{Z}) \simeq \mathbb{L}^*\,\mathrm{Bin}''/\,\mathrm{IBin}''_{>n}(\mathbb{Z}\langle X/\mathrm{pt}\rangle^\vee)$ provides a natural isomorphism

$$H^*(\mathrm{SP}^n X; \mathbb{Z}) \simeq \mathbb{L}\Gamma''/\,\mathrm{I}\Gamma''_{>n}(\mathbb{Z}\langle X/\mathrm{pt}\rangle^\vee).$$

While $\mathrm{IBin}''_{>n}$ do not extend to functors from Ab, the RHS is given by $\mathbb{L}\Gamma/\,\mathrm{I}\Gamma_{>n}(\mathbb{Z}\langle X/\mathrm{pt}\rangle) \simeq \mathbb{L}\Gamma/\,\mathrm{I}\Gamma_{>n}(\bar{H}^*(X; \mathbb{Z}))$. So, for the suspension space X we have a natural identification:

$$H^*(\mathrm{SP}^n X; \mathbb{Z}) \simeq \mathbb{L}^*\Gamma/\,\mathrm{I}\Gamma_{>n}(\bar{H}^*(X; \mathbb{Z})).$$

This is a functor of $\bar{H}^*(X; \mathbb{Z})$ and this completes the proof. □

As a corollary of the above proof we have

Corollary 6.2.5. *For the suspension space $X = \Sigma Y$, the multiplication*

$$\cup\colon H^*(\mathrm{SP}^\infty X, \mathrm{SP}^{n-1} X; \mathbb{Z}) \otimes H^*(\mathrm{SP}^\infty X, \mathrm{SP}^{m-1} X; \mathbb{Z}) \to H^*(\mathrm{SP}^\infty X, \mathrm{SP}^{\max(n,m)-1}; \mathbb{Z}),$$

naturally passes through $H^*(\mathrm{SP}^\infty X, \mathrm{SP}^{n+m-1} X; \mathbb{Z})$.

References

[1] Albrecht Dold, *Homology of Symmetric Products and Other Functors of Complexes*

[2] Torsten Ekedahl, *On minimal models in integral homotopy theory*

[3] Lawrence Breen, Roman Mikhailov and Antoine Touzé, *Derived Functors of The Divided Power Functors*

[4] R. Mikhailov, *ON THE SPLITTING OF POLYNOMIAL FUNCTORS*

[5] Dmitry V. Gugnin, *On Integral Cohomology Ring of Symmetric Products*

[6] D. Kubrak, G. Shuklin, A. Zakharov, *Derived binomial rings I: integral Betti cohomology of log schemes*

[7] F.P. Palermo, *The Cohomology Ring Of Product Of Complexes*

[8] V. Puppe, *Über Cohomologie von symmetrischen Produkten*

[9] A. Touzé, *Bar complexes and extensions of classical exponential functors*

[10] Luc Illusie, *Complexe Cotangent et Déformations*

[11] J.H.C. Whitehead, *On simply connected, 4-dimensional polyhedra*