On the Cohomology of Eilenberg-MacLane Spaces

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Abstract

The goal of the present work is to describe the integral cohomology ring of Eilenberg-MacLane spaces $K(\pi, n)$ localized at a prime p by means of an explicit small DGA model.

Introduction

Our starting point is volume 7 of H. Cartan's seminare notes [1]. The general framework contained there already allows to compute the Pontryagin ring $H_*(K(\mathbb{Z}/p^k, n), \mathbb{Z})$. On the other hand, using the description of $H^*(K(\mathbb{Z}/p^k, n), \mathbb{F}_p)$ in terms of stable cohomological operations P. May[5] computed the corresponding Bockstein spectral sequence, obtaining in particular the additive structure of $H^*(K(\mathbb{Z}/p^k, n), \mathbb{Z}_{(p)})$.

Despite the lack of novelty in the following, there are at least two reasons to write an old fashioned text about such classical subject. First is related to the fact that an explicit expression for *the ring* $H^*(K(\pi, n), \mathbb{Z})$ is still unavailable in the literature. Second concerns the question of functoriality of a description. Namely, according to [4] there is no simple and functorial in π expression for $H_*(K(\pi, n), \mathbb{Z})$ even for free abelian π. This is why we prefer to give an answer by means of a small DGA[1].

The main statements require notation developed through the exposition. As an illustration we sketch the main result in the following form:

Theorem. *Let $p > 2$ be a prime and V is a free abelian group. Then there is a natural quasi-isomorphism*

$$\phi : C^*_{sing}(K(V, n), \mathbb{Z}_{(p)}) \to S^\bullet(V^\vee[-n]) \otimes \Omega(W(V^\vee))$$

of the singular cochain complex of $K(V, n)$ with the tensor product of symmetric super-algebra with trivial differential and the DeRham algebra over a super-affine space $W(V^\vee)$ with differential $p \cdot d_{DeRham}$, i.e.:

$$\mathbb{Z}/p \otimes S^\bullet(V^\vee[-n]) \otimes W(V^\vee) \simeq H^*(K(V, n), \mathbb{F}_p)$$

Here $W(V^\vee)$ functorially depends on $V^\vee = Hom_{\mathbb{Z}}(V, \mathbb{Z})$. In fact K is a cocomutative Hopf-CDGA with the tensor product differential, the map ϕ preserves a natural comultiplication and multiplication up to homotopy.

The detailed form of the theorem and related statements can be found at the page 15.

The main idea is the following. The Eilenberg-MacLane space forms an infinite loop space and its delooping is determined uniquely up to homotopy equivalence. Using the iterated geometric bar construction as a delooping machine one can realize EM-space $K(\pi, n)$ as a strictly abelian monoid with CW-structure. The corresponding chain complex $C^{CW}_*(K(\pi, n))$ is a CDGA equal to the iterated bar construction $\overline{Bar}^{(n)}(\mathbb{Z}[\pi])$ applied to the group algebra of π. Using Cartan's notion of the multiplicative construction one can construct, after fixing a cyclic decomposition of π, a smaller CDGA model for $\overline{Bar}^{(n)}(\mathbb{Z}[\pi]) \otimes \mathbb{Z}_{(p)}$ as a tensor product of elementary algebras. This allows, though non functorialy in π from the very begining, to describe the Pontryagin ring $H_*(K(\pi, n), \mathbb{Z}_{(p)})$. Then we study a comultiplicative structure of elementary algebras and check its compatibility with comultiplication on $\overline{Bar}^{(n)}(\mathbb{Z}[\pi]) \otimes \mathbb{Z}_{(p)}$ to describe the cohomology ring $H^*(K(\pi, n), \mathbb{Z}_{(p)})$. It turns out that this construction is functorial if π is free abelian and $p > 2$. In this case the cohomology ring of $K(\mathbb{Z}, n)$ is closely related to the cohomology of affine de Rham complex over integers.

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[1]In the case $p > 2$ word "small" means it is free and its differential is divisible by p

In the following we accept the following notation. BSS stands for the Bockstein spectral sequence, LSSS for the Leray-Serre spectral sequence of a fibration.

Letter p denotes a prime number, while $q = p^k$ for some k. Our base ring $\Bbbk$ is fixed and is equal to $\mathbb{Z}_{(p)}$. All DG-algebras are connected, i.e. concentrated in non-negative degrees and the degree-zero component is equal to $\Bbbk$, this provides an augmentation with the kernel denoted by the bar symbol.

The shift of a graded module A_* is defined by $(A[n])_k = A_{n+k}$, i.e., a negative n corresponds to a shift to the *right*.

We use the symbol $\boxtimes$ for the usual tensor product and $\boxtimes_\tau$ for a twisted tensor product of DG-modules i.e. a perturbation of the tensor product differential. The details should be clear from the context.

Bar construction

Reminder 1 (the normalized bar construction reminder)**.** For a CDGA A with an augmentation $e : A \to \Bbbk \simeq A/\bar{A}$ the normalized bar construction $B := \overline{Bar}(A)$ is defined to be the tensor algebra $T(\bar{A}[-1])$ with the differential

$$d_{Bar}[a_1, a_2, \dots, a_n] = [a_1 \cdot a_2, a_3, \dots, a_n] \pm \dots \pm [a_1, a_2, \dots, a_{n-1} \cdot a_n]$$

where $[a_1, a_2, \dots, a_n] := \bar{a}_1 \otimes \dots \otimes \bar{a}_n \in \bar{A}^{\otimes n}$. Recall that $|\bar{a}| = |a| + 1$.

There is an associative comultiplication $\Delta : B \to B \boxtimes B$ and an associative commutative multiplication $\mu : B \boxtimes B \to B$ where Δ is given by the usual diagonal map:

$$\Delta[a_1, \dots, a_n] = 1 \boxtimes [a_1, \dots, a_n] + a_1 \boxtimes [a_2, \dots, a_n] + \dots + [a_1, \dots, a_n] \boxtimes 1$$

and μ by the shuffle-product:

$$\mu([a_1, \dots, a_n], [a_{n+1}, \dots, a_{n+m}]) = \sum_{\sigma \in Sh(n,m)} \pm [a_{\sigma(1)}, \dots, a_{\sigma(n+m)}]$$

.

In the last formula the summation goes over all (n, m)-shuffles with the standard sign convention; recall that $|\bar{a}| = |a| + 1$.

Consider the tensor product $A \boxtimes_\tau \overline{Bar}(A)$ of algebras with the differential d_τ defined by:

$$d_\tau(a \boxtimes_\tau [a_1, \dots, a_n]) = da \boxtimes_\tau [a_1, \dots, a_n] + a \cdot a_1 \boxtimes_\tau [a_2, \dots, a_n] + (-1)^a 1 \boxtimes_\tau d_{Bar}$$

i.e. d_τ is a twisted tensor product differential with the twist given by the second term. Then $A \boxtimes_\tau \overline{Bar}(A)$ is acyclic with a contracting homotopy s given by:

$$s(a \boxtimes_\tau [a_1, \dots, a_n]) = \pm(1 - e(a)) \boxtimes_\tau [a, a_1, \dots a_n] \tag{1}$$

, i.e. s is a homotopy between the identity and the natural augmentation on $A \boxtimes_\tau B$.

There is a natural DGA map $A \boxtimes_\tau \overline{Bar}(A) \to \overline{Bar}(A)$ given by reduction modulo $\bar{A}$.

□ (see [1],[3])

The simplicial origin of the bar construction manifests itself in the following:

Reminder 2 (geometric bar construction)**.** Let X be a commutative topological monoid which acts on a space A on the right and on B on the left.

There is a simplicial topological space $Bar(A, X, B) : \Delta^{op} \to Top$. Its value $Bar(A, X, B)[n]$ on n-simplex $[0, \dots, n] \in \Delta^{op}$ is equal to $A \times X^n \times B$. The face and degeneracy maps d_i, s_j are given by

$$\begin{aligned}
&d_i(a \times [x_1, \dots, x_n] \times b) = a \times [x_1, \dots, x_i \cdot x_{i+1}, \dots, x_n] \times b, && i \neq 0, n-1 \\
&d_0(a \times [x_1, \dots, x_n] \times b) = a \cdot x_1 \times [x_2, \dots, x_n] \times b \\
&d_{n-1}(a \times [x_1, \dots, x_n] \times b) = a \times [x_1, \dots, x_{n-1}] \times x_n \cdot b \\
&s_i(a \times [x_1, \dots, x_n] \times b) = a \times [\dots, x_{i-1}, 1, x_i, \dots] \times b, && 1 \in X
\end{aligned}$$

The corresponding geometric realization $|Bar(A,X,B)|$ is equal to $\bigsqcup_n |\Delta^n| \times A \times X^n \times B/\sim := \bigsqcup_n |\Delta^n| \times A \times X^n \times B/(s_i|\Delta^n| \times Bar(A,X,B)[n-1] \sim |\Delta^n| \times s_i(Bar(A,X,B)[n-1]), d_j|\Delta^n| \times Bar(A,X,B)[n+1] \sim |\Delta^n| \times d_j(Bar(A,X,B)[n+1]))$. Identifying $|\Delta^n|$ with a space of configurations of n points in the unit interval $I = [0,1]$ we can view $d_j|\Delta^n| \to |\Delta^{n+1}|$ as doubling of j-th point and $s_i|\Delta^n| \to |Delta^{n-1}|$ as forgetting i-th point. Then a point in $|\Delta^n| \times Bar(A,X,B)[n]$ identifies with a tuple consisting of an A-value at 0, a configuration of n ordered points in $[0,1]$ with values in X, and a B-value at 1. Then the face map d_i corresponds to a merger of i- and $(i+1)$-th point in the tuple. Our equivalence relation identifies a merger of i- and $(i+1)$-th point in the tuple with the multiplication of adjacent values (including the action on A and B). As a set $|Bar(A,X,B)|$ is a disjoint union of tuples as above where all points in the interval are distinct. If A,B,X are CW-complexes this provides a natural CW-structure on $|Bar(A,X,B)|$.

One can verify the following:

1. A contraction of I onto its right end point induces a contraction of $|Bar(X,X,*)|$.
2. There is a fibration sequence $X \to |Bar(X,X,*)| \to |Bar(*,X,*)|$, which implies a natural homotopy equivalence $\Omega|B(*,X,*)| \sim X$.

□

Both constructions are compatible by the following:

Proposition 1 (Milgram [2]). *Assume that X is a commutative monoid with a compatible CW-structure, i.e. $A := C_*^{CW}(X)$ is an augmented DG-algebra. Then $|Bar(*,X,*)|$ has a natural cell decomposition s.t. there is a natural CDGA-isomorphism:*

$$C_*^{CW}(|Bar(*,X,*)|) \simeq \overline{Bar}(C_*^{CW}(X))$$

compatible with a natural diagonal approximation on $|Bar(*,X,*)|$.

Moreover $|Bar(X,X,*)|$ is also has a natural cell decomposition s.t. the induced map

$$|Bar(X,X,*)| \to |Bar(*,X,*)|$$

is cellular and the induced map of cellular chain complexes is reduction $\mod \bar{A}$:

$$A \boxtimes_\tau \overline{Bar}(A) \to \overline{Bar}(A)$$

□

Remark 1. The above statements are easily checked by considering the geometric realization $|Bar(*,X,*)|$ as the configuration space of X-valued points in the unit interval. The only non-evident part is an explicit expression for a diagonal approximation of $|Bar(*,X,*)|$, which directly follows from a diagonal approximation for simplices $\Delta^n \to \Delta^n \times \Delta^n$. For details see [2].

Corollary 1. *Let X and A be as in Proposition1. Then the filtration on the bicomplex $A \boxtimes_\tau \overline{Bar}(A)$ with respect to the degree filtration on $\overline{Bar}(A)$ is consistent with the Leray filtration of the corresponding contractible fibration $pt \to X$.*

In particular, the spectral sequence associated to the bicomplex $A \boxtimes_\tau \overline{Bar}(A)$ is naturally isomorphic to the LSSS associated with the path-loop fibration $pt \to X$.

The previous construction can be iterated, in turn clarifying an algebraic origin of EM-spaces

Theorem 1 (Eilenberg-MacLane [6]). *Let π be an abelian group. Then there is a natural, up to homotopy, quasi-isomorphism*

$$\overline{Bar}^{(n)}(\mathbb{Z}[\pi]) \to C_*^{sing}(K(\pi,n))$$

compatible with the comultiplication and multiplication of both sides. In particular, there is a natural isomorphism of Hopf-CDGA:

$$H^*(K(\pi,n),\mathbb{Z}) \simeq H^*(\overline{Bar}^{(n)}\mathbb{Z}[\pi],\mathbb{Z})$$

□

For the sake of completness let us mention the following

Theorem 2 (Milgram([3])). *Let $S^\infty(Y, pt)$ be the infinite symmetric power of a based space (Y, pt). For a connected space X there is a natural isomorphism of monoids*

$$S^\infty(X \wedge S^1) \simeq |Bar(*, S^\infty(X, pt), *)|$$

Proof. One can identify a point in $S^\infty(\Sigma X)$ with a configuration of ΣX-valued points on $S^1 = [0, 1]/0 \sim 1$ s.t. values at the ends of I are $1 \in \Sigma X$. This is a homeomorphism compatible with multiplication. □

Corollary 2 (Dold-Thom([7])). *There is a natural homotopy equivalence*

$$S^\infty M(\pi, n) \sim K(\pi, n)$$

where $M(\pi, n)$ is a Moore space, i.e. $\tilde{H}_*(M(\pi, n)) = \pi[-n]$.

Corollary 3. *For a strictly abelian monoid X there is a natural homomorphism inducing a weak homotopy equivalence:*

$$\prod K(\pi_n(X), n) \xrightarrow{\sim} X$$

Proof. The map is provided by a natural homorphism $S^\infty(\vee_n M(\pi_n, n)) \to X$. (see [9] on Dold-Thom theorem for details) □

Cartan's construction

To work with small models for delooping recall Cartan's notion of construction.

Definition 1. ([1, exposé 4]) Let A be a commutative DG-algebra. Cartan's multiplicative construction is a triple $(A, A \boxtimes_\tau B, B)$, where B is a CDGA, $A \boxtimes_\tau B$ has the usual algebra structure with a twisted differenatial, i.e., it is a DG-module over A and the reduction $A \boxtimes_\tau B/(\bar{A} \boxtimes_\tau B) \to B$ is an isomorphism of DG-algebras. We require that $A \boxtimes_\tau B$ be acyclic.

The normalized bar construction is the universal Cartan's construction in the following sense:

Proposition 2. *([1]) For a multiplicative construction $(A, A\boxtimes_\tau B, B)$ there is a natural DGA-quasi-isomorphism $g : A \boxtimes_\tau B \to A \boxtimes_\tau \overline{Bar}(A)$ over A s.t. $g \mod \bar{A}$ induces a quasi-isomorphism $g : B \to \overline{Bar}(A)$.*

Proof. (see Cartan[1], exposé 4 Théorém 5)

The module $A \boxtimes_\tau B$ is naturally bigraded. We will construct g by induction on the second degree corresponding to B. Let $g : A \boxtimes_\tau B \supset A \to A \subset A \boxtimes_\tau \overline{Bar}(A)$ be equal to the identity.

Assume that g has being defined on a submodule $M_{<n} := A \boxtimes_\tau B_{<n}$. From the fact that the total differential d_a restricted onto $1 \boxtimes_\tau \overline{Bar}(A)$ equal to $\pm d_{Bar}$ and is an inclusion, remembering that $A \boxtimes_\tau \overline{Bar}(A)$ is acyclic via the contraction s (equation 1) we need to have $g(b) = s(g(db))$ to ensure that $g(db) = dg(b)$. Thus, to perform an inductive step, we put $g(b) := s(g(db))$ for $b \in B_n$ and extend g on $M_{\leq n}$ as a map of modules over A.

Because the constructed g is the identity on A and both $A \boxtimes_\tau B$ and $A \boxtimes_\tau \overline{Bar}(A)$ are acyclic we deduce, using a spectral sequences argument, that $g : B \to \overline{Bar}(A)$ is a quasi-isomorphism. Thus we get A-modules morphism $g : A \boxtimes_\tau B \to A \boxtimes_\tau \overline{Bar}(A)$. One can verify, using induction, that in fact $g(1 \boxtimes_\tau a \cdot 1 \boxtimes_\tau b) = g(1 \boxtimes_\tau ab)$. Hence g actually is a DGA-morphism. □

Proposition 3. *Let $G, G' : A \boxtimes_\tau B \to A' \boxtimes_\tau \overline{Bar}(A')$ be maps of a multiplicative constructions s.t. images of $1 \boxtimes_\tau B$ are in $1 \boxtimes_\tau \overline{Bar}(A')$.*

Assume that $G|_A - G'|_A = [d, h]$ are homotopic via $h : A \to A'[-1]$, then there is an extension $\tilde{h}$ of h s.t. $G - G' = [d_\boxtimes, \tilde{h}]$.

Proof. Similarly to the previous proposition we will define an extension of h by induction on the degree corresponding to B. Let $\tilde{h} = h$ on $A \boxtimes_\tau 1$. Assume that $\tilde{h}$ has being defined on a submodule $M_{<n} = A \boxtimes_\tau B_{<n}$ s.t. $[\tilde{h}, d] = G - G'$ on $M_{<n}$. Take $b \in B$ with $|b| = n$. The expression $\tilde{h}(db) - s(d\tilde{h}(db))$ is closed in $A' \boxtimes_\tau \overline{Bar}(A')$, hence it is exact and we may define $\tilde{h}(b)$ using equation:

$$d\tilde{h}(b) = \tilde{h}(db) - s(d\tilde{h}(db))$$

To extend $\tilde{h}$ on $M_{\le n}$ we put:

$$\tilde{h}(a \boxtimes_\tau b) := \tilde{h}(a) \boxtimes_\tau G(b) + G'(a) \boxtimes_\tau \tilde{h}(b)$$

One can check that $[\tilde{h}, d_\boxtimes] = G - G'$. □

In light of the above, in order to understand the chain algebras of EM-spaces, it is sufficient to start with the group ring of an abelian group viewed as $C_*(K(\pi, 0))$, then iteratively choose a Cartan's construction of previous algebra that would be a small as possible, and proceed. It turns out that DG-algebras that will appear in this way are limited to the following.

Elementary algebras

Let $q = p^k$ or $q = 0, k = -\infty$.

Definition 2 (elementary algebras).

- Exterior algebra $[\nu] := E(\nu)$ on an odd variable ν
- Divided powers algebra $[x] := D(x)$ on an even variable x
- DG algebra $[x, \nu_{x-1}]_q := D_q(x, \nu_{x-1})$ equal to $D(x) \otimes E(\nu_{x-1})$ as an algebra with differential $dx = q\nu_{x-1}$
- DG algebra $[\mu, x_{\mu-1}]_q := E_q(\mu, x_{\mu-1})$ equal to $E(\mu) \otimes D(x_{\mu-1})$ with differential $d\mu = q \cdot x_\mu$

Remark 2. Square brackets corresponds to the divided powers structure.

Remark 3 (notation). All algebras as above allows divided powers, i.e. for any $x \in A$ and a natural n element x^n is divisible by $n!$ in A. Denote by $[N]$ a maximal p^k dividing N.
One subscript under a variable, e.g. x_{p^i}, means that $x_{p^i} = \frac{x^{p^i}}{[p^i!]}$. Two subscripts under a variable, e.g. $x_{\mu-1,p^i}$ indicates that $|x| = |\mu - 1|$ and $x_{\mu-1,p^i} = \frac{{x_{\mu-1}}^{p^i}}{[p^i!]}$. Differentiation naturally extends applying the Leibniz's rule.

For example we can write $x_{p^i}^p = px_{p^{i+1}}$, $dx_{p^i} = x_{p^{i-1}}dx$, $x_{\mu-1,p} = \frac{x_{|\mu|-1}}{p}$ etc.

Theorem 3 (B-construction, on elementary delooping). *Let A be an elementary algebra as above.*

Then there is an explicit construction $(A, *, B(A))$ where $B(A)$ as DGA is a tensor product of elementary algebras and is given by:

(i) $B(E(\nu)) := D(y_{\nu+1})$

(ii) $B(D(x)) := E(\nu_{x+1}) \otimes \bigotimes_{i>0} D_p(z_{p^ix+2}, \nu_{p^ix+1})$ *with differential* d_B:

$$d_B z_{p^ix+2} = -p \cdot \nu_{p^ix+1}$$

(iii) $B(D_q(x, \mu_{x-1})) := E_q(\nu_{x+1}, y_{\mu+1}) \otimes \bigotimes_{i>0} D_p(z_{p^ix+2}, \nu_{p^ix+1})$, *with differential* d_B:

$$d_B\nu_{x+1} = q \cdot y_x$$
$$d_B z_{p^ix+2} = -p \cdot \nu_{p^ix+1}$$

(iv) $B(E_q(\mu, x_{\mu-1})) := D_q(y_{\mu+1}, \nu_{x+1}) \otimes \bigotimes_{i>0} D_p(z_{p^ix+2}, \nu_{p^ix+1})$, *with differential* d_B:

$$d_B y_{\mu+1} = \nu_{x+1}$$
$$d_B z_{p^ix+2} = -p\nu_{p^ix+1}$$

Proof

(i). $E(\nu) \boxtimes_\tau D(y_{\nu+1})$
Let $d_\tau y = \nu$, this naturally extends as a derivation on all divided powers and gives an acyclic complex. $\square$

(ii). $D(x) \boxtimes_\tau E(\nu_{x+1}) \otimes \bigotimes_{i>0} D_p(z_{p^i x+2}, \nu_{p^i x+1})$
Let

$$d_\tau \nu_{p^i x+1} = x_{p^i}$$

$$d_\tau z_{p^i x+2} = -p\nu_{p^i x+1} + x_{p^{i-1}}^{p-1} \nu_{p^{i-1}x+1}$$

We have $d_\tau^2 = 0$ and because the reduction $\mod p$ of $D(x)$ is a product of algebras of the form $\mathbb{F}_p[x_{p^i}]/x_{p^i}{}^p$ it gives an acyclic complex (similarly to the simple part of the Kudo's transgression theorem), thus it is automatically acyclic over $\mathbb{Z}_{(p)}$ $\square$

(iii). We want to construct $B(D_q(x, \mu_{x-1}))$ as a deformation of $B(D(x)) \otimes B(E(\mu_{x-1}))$. Namely, in

$$D(x) \otimes E(\mu_{x-1}) \boxtimes_\tau D(y_\mu) \otimes E(\nu_{x+1}) \otimes \bigotimes_{i>0} D_p(z_{p^i x+2}, \nu_{p^i x+1})$$

one can adjust bare differential d_τ of the tensor product as below:

$$d_\tau \nu_{p^i x+1} = (x - q y_{\mu+1})_{p^i}$$

$$d_\tau y_{\mu+1} = \mu$$

$$d_\tau z_{p^i x+2} = -p\nu_{p^i x+1} + \left((x - q y_{\mu+1})_{p^{i-1}}\right)^{p-1} \nu_{p^{i-1}x+2}$$

Note that all mentioned algebras allow divided powers and hence all coefficients in formulas above are in fact integer numbers. It is straightforward to check that the introduced formula obeys $d^2 = 0$. Since our differential is a deformation of the bare differential along q, the resulting complex is acyclic.

Let d_B be the reduction of the differential d_τ on $B(D_q(x, \mu_{x-1}))$. It is evaluated by augmenting x, μ terms:

$$d_B \nu_{p^i x+1} = x_{p^i} - q^{p^i} y_{\mu+1,p^i}$$

$$d_B y_{\mu+1} = 0$$

$$d_B z_{p^i x+2} = -p\nu_{p^i x+1} + (-q^{p^{i-1}} y_{\mu+1,p^{i-1}})^{p-1} \nu_{p^{i-1}x+1}$$

Put

$$\widetilde{\nu_{p^i x+1}} := \nu_{p^i x+1} - \frac{(-q)^{p^{i-1}(p-1)}}{p} y_{\mu+1,p^{i-1}}^{p-1} \nu_{p^{i-1}x+1}$$

for $i > 0$. Direct calculation shows that:

$$d_B \nu_{x+1} = -q y_{\mu+1}$$

$$d_B \widetilde{\nu_{p^i x+1}} = 0$$

$$d_B z_{p^i x+2} = -p\widetilde{\nu_{p^i x+1}}$$

Thus $B(D_q(x, \mu_{x-1}))$ is indeed the tensor product of an elementary algebras as was stated. $\square$

(iv). We specify differential on $E_q(\mu, x_{\mu-1}) \boxtimes_\tau D_q(y_{\mu+1}, \nu_{x+1}) \otimes \bigotimes_{i>0} D_p(z_{p^i x+2}, \nu_{p^i x+1})$ explicitly:

$$d_\tau y_{\mu+1} = \mu - q\nu_{x+1}$$

$$d_\tau \nu_{p^i x+1} = x_{\mu-1,p^i}$$

$$d_\tau z_{p^i x+2} = -p\nu_{p^i x+1} + x_{p^{i-1}}^{p-1} \nu_{p^{i-1}x+1}$$

We have $d_\tau^2 = 0$, hence it is acyclic and its reduction modulo $\mu, x_{\mu-1}$ gives differential d_B as needed. □

■

In addition let us state

Proposition 4. *Let $\pi = \mathbb{Z}/q$ be a cyclic group with a chosen generator η and an augmentation defined by $\Bbbk[\pi] \to \Bbbk$ by $\eta \to 1$.*
Put $B(\Bbbk[\pi]) := D_q(x_2, \nu_1)$. Then there is a construction $(\Bbbk[\pi], *, D_q(x_2, \nu_1))$

Proof. An acyclic complex $\Bbbk[\pi] \boxtimes_\tau D_q(x_2, \nu_1)$ is given by the standard cyclic group-homology resolution with $d(x_2) = N\nu_1$ and $d\nu_1 = 1 - \eta$, where $N = 1 + \eta + .. + \eta^{q-1}$. Reduction modulo the augmentation ideal gives $D_q(x_2, \nu_1)$. □

Remark 4. The complex $D_q(x_2, \nu_1)$ appears to be the standard cellular chain complex of the lens space $K(\mathbb{Z}/q, 1)$. Then Proposition 1 [2] provides an explicit morphism $g : D_q(x_2, \nu_1) \to \overline{Bar}(\Bbbk[\pi])$, which can be used to compute a diagonal approximation and multiplication maps on $C_*^{CW}(K(\mathbb{Z}/q, 1))$ rather than in ad hoc way given in (Steenrod-Epstein[8], on a cyclic group).

Let A be an elementary algebra. Here we define a comultiplication for the B-construction.

Definition 3. The B-construction $B(A)$ introduced in 3 admits a bialgebra structure given by a comultiplication $\Delta_B : B(A) \to B(A) \boxtimes B(A)$ by declaring d.p.-generators to be primitive.

Recall that P, E denotes the usual polynomial and exterior algebras respectively. The end of this section concerns only the case $p = 2$. The following is necessary refinement of the B-construction needed for a description of right coalgebra structure compatible with the diagonal map of the Bar-construction.

Definition 4 (p=2)**.** The $\widetilde{B}$-construction $\widetilde{B}(A)$ is equal to $B(A)$ when $p > 2$ or A is not of the form $D_2(x, \mu_{x-1})$. When $A = D_2(x, \mu_{x-1})$ we define $\widetilde{B}(A) = B(A) \otimes E(\Phi_{2x+1}) \otimes P(\tau_{\Phi-1})$ with the differential $d_{\widetilde{B}}$ extended from d_B that of $B(A)$ onto the tensor product by

$$d_{\widetilde{B}}\Phi = y_x^2 - \tau \qquad\qquad d_{\widetilde{B}}\tau = 0$$

where $y_x \in B(A)$ is the variable from the theorem above.

Proposition 5. *The $\widetilde{B}$-construction provides a multiplicative Cartan's construction $(A, *, \widetilde{B}(A))$.*

Proof. We have to check the case $A = D_2(x, \mu_{x-1})$. Note that a natural inclusion $B(A) \to \widetilde{B}(A)$ is a quasi-isomorphism. It is enough to extend the construction $A \boxtimes_\tau B(A)$ with acyclic differential d to $A \boxtimes_\tau \widetilde{B}(A)$ by:

$$d_\tau \Phi = (\nu_{x+1}\mu + y^2) - \tau$$
$$d_\tau = x\mu$$

One can check that the natural quasi-isomorphism $g : \widetilde{B}(A) \to \overline{Bar}(A)$ extends by:

$$g(\tau) = \overline{x\mu}$$
$$g(\Phi) = \overline{\mu}|\overline{x}$$

□

Definition 5 (p=2)**.** Assume $p = 2$. We define a diagonal map $\Delta_{\widetilde{B}} : \widetilde{B}(A) \to \widetilde{B}(A) \boxtimes \widetilde{B}(A)$ according to the following:

(i) If A is not of the form $D_2(x, \mu)$, then we declare variables y and $\nu_{2^i x+1}, i \geq 0$ to be primitive and put:

$$\Delta_{\widetilde{B}} z_{2^i x+2} = z_{2^i x+2} \boxtimes 1 + 1 \boxtimes z_{2^i x+2} + \nu_{2^{i-1}x+1} \boxtimes \nu_{2^{i-1}x+1}, i > 0$$

(ii) If $A = D_2(x, \mu_{x-1})$ we declare variable y, τ and $\nu_{2^i x+1}, i \neq 1, 2$ to be primitive. Let

$$\begin{aligned}
&\Delta_{\widetilde{B}} \nu_{2x+1} = \nu_{2x+1} \boxtimes 1 + 1 \boxtimes \nu_{2x+1} + y \boxtimes \nu_{x+1} - \nu_{x+1} \boxtimes y \\
&\Delta_{\widetilde{B}} \nu_{2^2 x+1} = \nu_{2^2 x+1} \boxtimes 1 + 1 \boxtimes \nu_{2^2 x+1} + \tau \boxtimes \nu'_{2x+1} - \nu'_{2x+1} \boxtimes \tau \\
&\Delta_{\widetilde{B}} z_{2^i x+2} := z_{2^i x+1} \boxtimes 1 + 1 \boxtimes z_{2^i x+1} + \nu_{2^{i-1} x+1} \boxtimes \nu_{2^{i-1} x+1} \qquad i \neq 2 \\
&\Delta_{\widetilde{B}} z_{2^2 x+2} := z_{2^2 x+2} \boxtimes 1 + 1 \boxtimes z_{2^2 x+2} + \nu'_{2x+1} \boxtimes \nu'_{2x+1} \\
&\Delta_{\widetilde{B}} \Phi = \Phi \boxtimes 1 + 1 \boxtimes \Phi + \nu_{x+1} \boxtimes y
\end{aligned}$$

where

$$\begin{aligned}
&\nu'_{2x+1} = \nu_{2x+1} - y\nu_{x+1} + 2\Phi \\
&\Delta_B \nu'_{2x+1} = 1 \boxtimes \nu'_{2x+1} + \nu'_{2x+1} \boxtimes 1
\end{aligned}$$

Remark 5. Algebra $\widetilde{B}(A)$ contains a polynomial factor, hence it does not allows divided powers, nonetheless, since the right hand side of the above expressions contains only odd variables of the new factor, the homomorphism $\Delta_{\widetilde{B}}$ extends correctly.

The following assertion is crucial for our computation of the cohomology ring of EM-spaces for $p = 2$:

Proposition 6. *The diagonal map $\Delta_{\widetilde{B}}$ introduced above for any p is coassosiative and commutes with the $\widetilde{B}$-construction differential d_B. This naturally defines a bialgebra structure on $\widetilde{B}$.*

Proof. A straightforward computation. □

Dual elementary algebras

Definition 6. For an algebra with divided powers A we say that A is d.p.-generated by a subset $S \subset A$, if A is equal to the minimal subalgebra with divided powers containing S.

Note that all elementary algebras and its duals can be defined without specifying d.p.-generators variables, but as functors from the category of graded free $\Bbbk$-modules to the category of CDGA-algebras in obvious way.

If V is free $\Bbbk$-module and an elementary algebra $A = [V, V[1]]_q$ or $A = [V]$ we say that V *corresponds* to A.

Definition 7. In such case we will refer to V and $V[1]$ as the d.p.-generators of A.

Let us introduce another three algebras:

Definition 8 (elementary duals)**.**

- $(x) := P(x)$ is the polynomial algebra on x
- $(x, \nu_{x+1})_q := \Omega_q(x, \nu_{x+1})$ is equal to $P(x) \otimes E(\nu_{x+1})$ with the differential determined by $dx = q\nu_{x+1}$. It is the DeRham complex with q-multiply of the exterior derivative.
- $(\nu, x_{\nu+1})_q := K_q(\nu, x_{\nu+1})$ is equal to $E(\nu) \otimes P(x_{\nu+1})$ with differential determined by $d\nu = qx_{\nu+1}$. It is the Kozsul complex.

Proposition 7 (p>2)**.** *Let V be a free graded $\Bbbk$-module. There is a natural isomorphism of Hopf-CDGA $[V]^\vee \simeq (V^\vee)$ and $[V, V[-1]]^\vee \simeq (V^\vee[1], V^\vee)$, i.e.:*

$$E(V)^\vee \simeq E(V^\vee)$$

$$D(V)^\vee \simeq P(V^\vee)$$
$$D_q(V, V[1])^\vee \simeq K_q(V^\vee[1], V^\vee)$$
$$E_q(V, V[1])^\vee \simeq \Omega_q(V^\vee[1], V^\vee)$$

Proof. The proof follows from the given definitions. Note that under this dualization d.p.-generators of an elementary algebra are naturally duals to the non-decomposable elements in the dual algebra. □

By the previous results $\widetilde{B}(A)^\vee$ admits an algebra structure. For $p > 2$ it is naive dual version of the B-construction:

Proposition 8. $p > 2$

(i) $A = E(\nu)$
$\widetilde{B}(E(\check{\nu}))^\vee = P(\check{y}_{\nu+1})$

(ii) $A = D(x)$
$\widetilde{B}(D(x))^\vee = E(\check{\nu}_{x+1}) \otimes \bigotimes_{i>0} (\check{\nu}_{p^i x+1}, \check{z}_{p^i x+2})_p$ *with differential* d*:*

$$d\check{\nu}_{p^i x+1} = p\check{z}_{p^i x+1}$$

(iii) $A = D_q(x, \mu_{x-1})$
$\widetilde{B}(D_q(x, \mu_{x-1}))^\vee = (\check{y}_{\mu+1}, \check{\nu}_{x+1})_q \otimes \bigotimes_{i>0} (\check{\nu}_{p^i x+1}, \check{z}_{p^i x+2})_p$*, with differential* d*:*

$$d\check{y} = -q \cdot \check{\nu}_{x+1}$$
$$d\check{\nu}_{p^i x+1} = p \cdot \check{z}_{p^i x+2}$$

(iii) $A = E_q(\mu, x_{\mu-1})$
$\widetilde{B}(E_q(\mu, x_{\mu-1}))^\vee = (\check{\nu}_{x+1}, \check{y}_{\mu+1})_q \otimes \bigotimes_{i>0} (\check{\nu}_{p^i x+1}, \check{z}_{p^i x+2})_p$*, with differential* d_B*:*

$$d_B\check{y}_{\mu+1} = \mu$$
$$d_B\check{z}_{p^i x+2} = -p\check{\nu}_{p^i x+1}$$

Proposition 9. $p = 2$

(i) $A = E(\nu)$
$\widetilde{B}(E(\check{\nu}))^\vee = P(\check{y}_{\nu+1})$

(ii) $A = D(x)$
$\widetilde{B}(D(x))^\vee = P(\check{\nu}_{x+1}) \otimes \bigotimes_{i>0} P(\check{\nu}_{2^i x+1})$ *with graded differential* d *determined by Leibniz rule and:*

$$d\check{\nu}_{2^i x+1} = p \cdot \check{\nu}^2_{2^{i-1} x+1}$$

(iii) $A = D_q(x, \mu_{x-1}), q \neq 2$
$\widetilde{B}(D_q(x, \mu_{x-1}))^\vee = (\check{y}_{\mu+1}, \check{\nu}_{x+1})_q \otimes \bigotimes_{i>0} P(\check{\nu}_{2^i x+1})$*, with graded differential* d *determined by:*

$$d\check{y} = -q \cdot \check{\nu}_{x+1}$$
$$d\check{\nu}_{2^i x+1} = p \cdot \check{\nu}^2_{2^{i-1} x+2}$$

(iii) $A = D_2(x, \mu_{x-1}), p = 2$
As a complex $\widetilde{B}(D_2(x, \mu_{x-1})) = (\check{y}_{\mu+1}, \check{\nu}_{x+1})_q \otimes [\check{\tau}, \check{\Phi}]_1 \bigotimes_{i>0} (\check{\nu}_{2^i x+1})$ *with a differential* d *determined by:*

$$d\check{y} = 2 \cdot \check{\nu}_{x+1}$$
$$d\check{\tau} = -\check{\Phi}$$

$d\check{\nu}_{2^i x+1} = 2\cdot\check{\nu}^2_{2^{i-1}x+1}$

and a multiplication determined by:

$$[\check{y},\check{\nu}_{x+1}] = 2\check{\nu}_{2x+1} - \check{\Phi}$$
$$(\check{\Phi}_{2x+1})^2 = 4\check{\nu}^2_{2x+1}$$
$$[\check{\tau}_{2x},\check{\nu}_{2x+1}] = 2\check{\nu}_{4x+1}$$
$$[\check{\tau}_{2x},\check{\Phi}] = 4\check{\nu}_{4x+1}$$
$$[\check{\nu}_{2x+1},\check{\Phi}] = -4\check{\nu}^2_{2x+1}$$

and leaves other quadratic relations to be super-commutative. Here brackets denote a super-commutator.

(iv) $A = E_q(\mu, x_{\mu-1})$
$\widetilde{B}(E_q(\mu,x_{\mu-1}))^\vee = (\check{\nu}_{x+1},\check{y}_{\mu+1})_q \otimes \bigotimes_{i>0} P(\check{\nu}_{2^i x+1})$, *with differential* d*:*

$d\check{y}_{\mu+1} = \mu$
$d\check{\nu}_{2^i x+1} = \check{\nu}^2_{2^{i-1}x+1}$

Proof. A straightforward computation. In all cases for $p = 2$ we have $\check{\nu}^2_{2^{i-1}x+1} = \check{z}_{2^i+2}$, this way we eliminated $\check{z}$ variables. Note that $\check{\Phi}$ is the only variable with divided powers. □

Pontryagin ring of Eilenberg-MacLane spaces

Here we give a non-invariant description of a small-model of cellular chains algebra $C^{CW}_*(K(\pi,n))$ for $\pi = \mathbb{Z}_{(p)}/q$, where $q = p^k$ for some k or $q = 0$.

Iterating the B-construction we obtain the following: (3) we obtain:

Theorem 4. *Let* $\pi = \mathbb{Z}/q$ *be a cyclic group with a chosen generator. There is a CDGA quasi-isomorphism*

$$C^{CW}_*(K(\pi,n)) \simeq \overline{Bar}^{(n)}(\Bbbk[\pi]) \sim B^{(n)}(\Bbbk[\pi]) \simeq K$$

with a CDGA K *equal to:*

(i) $q = 0$

$$K := [\eta_n] \otimes \bigotimes_{I>0} [P_I\eta_n, \beta P_I \eta_n]_p$$

where formal symbols $P_I\eta_n$ *and* $\beta P_I\eta_n$ *runs over all Steenrod's admissible sequences ([8],[5]) with excess* $e(I) \leq n-1$ *which are not begins or ends on Bockstein, we put* $|P_I\eta_n| = n + |I|, |\beta P_I\eta_n| = n + |I| + 1$.

(ii) q *is finite*

$$K := [\eta_n, \beta\eta_n]_q \otimes \bigotimes_{I>0} [P_I\eta_n, \beta P_I\eta_n]_p$$

where I *runs over all admissible sequences with excess* $e(I) \leq n-1$ *which are not ends (on the left) on Bockstein.*

In other words, modulo p d.p.-generators of K *naturally correspond to the generators of* $H^*(K(\pi,n),\Bbbk/p)$ *in terms of Steenrod powers.*

Proof. (cf. 10) Acording to the description of the B-construction (3) we already know that $B^{(n)}(\Bbbk[\pi])$ is a tensor product of elementary algebras and by (2,1) we have $B^{(n)}(\Bbbk[\pi]) \sim \overline{Bar}(\Bbbk[\pi]) \sim C^{CW}_*(K(\pi,n)) \otimes \Bbbk$. It remains to explicitly enumerate d.p.-generators (6). Iterating of B-construction naturally leads to the notion of Cartan's admissible words (see [1], exposé 9) on the letters σ, γ_p, ψ_p. In our terms, this letters allows to pass from a d.p.-generator of an elementary algebra A to a d.p.-generator of $B(A)$, using notation of Proposition (3), as follows:

$$\sigma x_{p^i} = \nu_{p^i x+1}, \sigma\mu = y$$

$$\gamma_p x_{p^i} = x_{p^{i+1}}$$

$$\psi_p x_{p^i} = z_{p^{i+1}x+2}$$

assuming that in all non-covered cases this operations act by zero. Thus, to pass from a d.p.-generator of $B^{(n-1)}(\Bbbk[\pi])$ to a d.p-generator of $B^{(n)}(\Bbbk[\pi])$ on can apply the word $\sigma\gamma_p{}^i$ or ψ_p. Proper definition due to Cartan implies that the d.p.-generators of $B^{(n)}(\Bbbk[\pi])$ are in 1-1 correspondence with all Cartan's admissible words. On the other hand there is 1-1 correspondence between Cartan's admissible words and Steenrod's admissible sequences. For a combinatorial proof of this fact see ([1], exposé 9).

For completness, assuming the description of $\mathbb{F}_p$-cohomology of EM-space to be known, let us relate enumeration of d.p.-generators with the Steenrod powers directly. By Theorem (1) and Corollary (1) the iterated geometric bar construction gives a cellular decomposition of $K(\pi, n)$ and the spectral sequence $E^{\geq 2}_{**}$ associated to the acyclic bicomplex $\overline{Bar}^{(n-1)}(\Bbbk[\pi]) \boxtimes_\tau \overline{Bar}^{(n)}(\Bbbk[\pi])$ of the corresponding construction is naturally isomorphic to the LSSS of the corresponding path-loop fibration $K(\pi, n-1) \to pt \to K(\pi, n)$. According to Proposition 1 (2) this also applies to the iterated B-construction we are working with.

For $p > 2$, dualizing everything and passing to the reduction modulo p we observe that the differentials in the bicomplex of B-construction are exactly correspond to the Kudo's transgression patterns. Namely, note that $B^{(n)}(\Bbbk[\pi])/p \simeq H_*(B^{(n)}(\Bbbk[\pi]), \mathbb{F}_p)$, hence d.p.-generators of $B^{(n)}(\Bbbk[\pi])$ naturally correspond to the non-decomposable elements in $H^*(B^{(n)}(\Bbbk[\pi]), \mathbb{F}_p)$ i.e. its free generators. Then, abusing notation and denoting the generators of the dual algebra $A^\vee/p$ by the same letters (see 7), we can write transgression patterns in the following manner:

$$\tau x^{p^i} = \nu_{p^i x+1} := P^{p^{i-1}x/2} \ldots P^{x/2}\tau x, \tau\mu = y$$

$$\pm\tau((x^{p^{i-1}})^{p-1}\nu_{p^{i-1}x+1}) = z_{p^{i+1}x+2} := \beta P^{p^{i-1}x/2}\nu_{p^{i-1}x+1} = \beta P^{p^{i-1}x/2} \ldots P^{x/2}\tau x$$

where τ is the transgression map in the spectral sequence associated with the bicomplex $B^{(n-1)}(\Bbbk[\pi]) \boxtimes_\tau B^{(n)}(\Bbbk[\pi]) \mod p$. We can consider this as a way to pass from generators of $A^\vee/p$ to generators of $B(A)^\vee/p$. In particular, we have established a formal bijection between Steenrod's admissible sequences with Cartan's admissible words. This way we denote by $P_I\eta_n$ a d.p.-generator in $B^{(n)}(\Bbbk[\pi])$ corresponding to the Steenrod admissible sequence I. Notation is justified due to the fact that modulo p element $P_I\eta_n \in B^{(n)}(\Bbbk[\pi])/p \simeq H_*(B^{(n)}(\Bbbk[\pi]), \mathbb{F}_p)$ is dual to $P^I\eta_n \in H^*(K(\pi, n), \mathbb{F}_p) \simeq H^*(B^{(n)}(\Bbbk[\pi]), \mathbb{F}_p)$ viewed as an application of Steenrod power P^I to the fundametal class $\eta_n \in H^n(K(\pi, n), \mathbb{F}_p)$.

The case $p = 2$ is literally the same. Recall that we can write $Sq^{2k} = P^k$ and $Sq^{2k+1} = \beta P^k$. Thus we have an uniform description for our model in all cases. ☐

Remark 6. Expressions $\beta P^I\eta_n$ correspond to the stable integral operations and exhaust all of them. By contrast, their "potentials" $P^I\eta_n$ are formal and are not related to some integral operations in any obvous way. Both results can be verified independently using the Adams spectral sequence to show, that all positive-degree integral stable operations have a form $x \to \beta' \circ P^I \circ red \circ x, x \in H^*(X, \mathbb{Z}_{(p)})$, where red is the reduction modulo p, P^I is a stable operation over $\mathbb{F}_p$ not ending on the Bockstein and β' is a Bockstein associated with coefficients $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_p \to 0$.

Remark 7. Note that for $p = 2$ the standard description ([8],[5]) of $H^*(K(\pi, n), \Bbbk/p)$ does not allows $e(\beta P^I) = n$ and our does.

The crucial difference here is that in $\mathbb{F}_2$-cohomology the dual to $\beta Sq_{2k}Sq_I\eta_n$ with $1 + 2k = |I| + n$ or equivalently $e(I) = 2k - |I| = n - 1$ is $\beta Sq^{2k}Sq^I\eta_n = (Sq^I\eta_n)^2$, hence it is not a generator.

There is no contradiction here, because modulo 2 cohomology ring is polynomial algebra with no skew-symmetric parts. For example algebras $(\nu_{x+1}) \otimes \bigotimes_{i>0}(\nu_{2^i x+1}, z_{2^i x+2})$ and the free polynomial algebra on $\nu_{2^i x+1}, i \geq 0$ are of the same size.

Diagonal map

The following result indentifies, up to homotopy, the formal comultiplication on $B(A)$ (7) with that is induced from the comultiplication in $\overline{Bar}(A)$. Recall that $\widetilde{B}(A) = B(A)$ if $p > 2$. For any prime p the following holds

Theorem 5 (on comultiplication)**.** *Let A be a product of elementary algebras and $g : \widetilde{B}(A) \to \overline{Bar}(A)$ is the natural CDGA-quasi isomorphism (2). Let Δ_{Bar} be the comultiplication in $\overline{Bar}(A)$ and $\Delta_{\widetilde{B}} : \widetilde{B}(A) \to \widetilde{B}(A) \boxtimes \widetilde{B}(A)$ be a comultiplication introduced in 3.*

Then the following square is homotopy-commutative:

$$\begin{array}{ccc} \widetilde{B}(A) & \xrightarrow{\Delta_{\widetilde{B}}} & \widetilde{B}(A) \boxtimes \widetilde{B}(A) \\ \downarrow g & & \downarrow g \boxtimes g \\ \overline{Bar}(A) & \xrightarrow{\Delta_{Bar}} & \overline{Bar}(A) \boxtimes \overline{Bar}(A) \end{array}$$

Proof. Let $C := \overline{Bar}(A) \boxtimes \overline{Bar}(A)$. We have two CDGA-maps $\nabla^1, \nabla^0 : \widetilde{B}(A) \to C$, where ∇^* are two ways for passing through the diagram. We need to construct connecting homotopy between them.

First we prove that if A is an elementary tensor component of $\widetilde{B}(A)$, then the restrictions of ∇^* to $A' \to C$ are homotopical (lemma1 and proposition10 below). One can verify that over a tensor component of the form $E(\Phi) \otimes P(\tau)$, by the very definition of comultiplication 5, equality $\nabla^1 = \nabla^0$ automatically holds. Then we show that if ∇^* are products of such restrictions, which are homotopy equivalent on each elementary part A', then they are homotopy equivalent on the whole tensor product $B(A)$ (lemma 2). ☐

Proofs of the following statements are given in Routine section17.

Definition 9 (first order homotopy)**.** Let A be a tensor product of elementary algebras with two CDGA-maps $\phi^0, \phi^1 : A \to C$. Denote by $G(A)$ the subspace of d.p.-generators, it is a subcomplex in A. We say that ϕ^0, ϕ^1 are first order homotopic if there is they are homotopic in restriction onto $G(A)$.

Lemma 1 (on existence)**.** *Let A be an elementary algebra with the natural CDGA-quasi-isomorphism $g : \widetilde{B}(A) \to \overline{Bar}(A)$. Let $\nabla^1 = \Delta_{Bar} \circ g$ and $\nabla^0 = g \boxtimes g \circ \Delta_{\widetilde{B}}$.*

Then ∇^1 and ∇^0 are first order homotopic on each tensor factor of $\widetilde{B}(A)$.

Proposition 10 (on extension of the first order homotopy)**.** *Let A be an elementary algebra and C is some CDGA allowing divided powers. Let $\nabla^i : A \to C$ be two CDGA-morphisms which are homotopical in the first order via $h : G(A) \to C[1]$, then there is an extension of h to $\tilde{h} : A \to C[1]$ s.t. $[d, \tilde{h}] = \nabla^1 - \nabla^0$.*

The following holds for any algebras:

Lemma 2 (on a tensor product homotopy)**.** *Let $\phi_A, \phi'_A : A \to C$ and $\phi_B, \phi'_B \to C$ be two pairs of homotopical CDGA-maps i.e. $\phi_A - \phi'_A = [d, h_A]$ and $\phi_B - \phi'_B = [d, h_B]$.*

Consider $A \otimes B \to C$, then $\phi_A \otimes \phi_B$ and $\phi'_A \otimes \phi'_B$ are homotopical by
$h_{A\otimes B} := \phi_A \otimes h_B + h_A \otimes \phi'_B$

Invariance of B-construction

Now, for $p > 2$ it is possible to reformulate elementary delooping theorem(3) in a functorial way. Let V be a free $\Bbbk$-module, x be an even integer and ν is odd. Consider an elementary algebra A corresponding to V. We can define a construction $B[A]$ according to 3 by:

1. $B[(V_x)] = [V_{x+1}] \otimes \bigotimes_{i>0} [V_{p^i x+2}, V_{p^i x+1}]_p$
2. $B[(V_x, V_{x-1})_q] = [V_{x+1}, V_x]_q \otimes \bigotimes_{i>0} [V_{p^i x+2}, V_{p^i x+1}]_p$
3. $B[(V_\nu)] = [V_{\nu+1}]$
4. $B[(V_\nu, V_{\nu-1})_q] = [V_{\nu+1}, V_\nu]_q \otimes \bigotimes_{i>0} [V_{p^i(\nu-1)+2}, V_{p^i(\nu-1)+1}]_p$

The definition of $B[A]$ is deceptively functorial on V: it defines a multiplicative construction $(A, A \boxtimes_\tau B[A], B[A])$ only after picking a basis in the corresponding V. We have to investigate covariance of the corresponding Cartan's construction w.r.t basis transformation in V.

Proposition 11 (p>2). *Let $V = \langle e_i \rangle$ and $W = \langle f_j \rangle$ be a pair of free $\Bbbk$-modules with a chosen generators.*

Let $A = [V_x, V_{x-1}]_q$ and $A' = [W_x, W_{x-1}]_q$ and x is even.

Assume we have a CDGA-morphism $f : A \to A'$ induced by a linear map $f : V \to W$ viewed as a matrix. There is a CDGA-map $F : A \boxtimes_\tau B[A] \to A' \boxtimes_\tau B[A']$ s.t. $F|_A = F$ and its reduction $F^{[p]} := F_{B[A]} : B[A] \to B[A']$ induces CDGA-maps of tensor factors s.t.:

1. *$F^{[p]} : [V_{x+1}, V_x]_q \to [W_{x+1}, W_x]_q$ and its restriction is naturally induced by f*
2. *$F^{[p]} : [V_{p^i x+2}, V_{p^i x+1}]_p \to [W_{p^i x+2}, W_{p^i x+1}]_p$ and its restriction is naturally induced by $f^{(p^i)}$*

here $f^{(p^i)}$ denotes a matrix obtained from f by exponentiation all its elements in power p^i.

Similarly, if $A = [V_y, V_{y-1}]_q$ and $A' = [W_y, W_{y-1}]_q$ and y is odd then the same holds:

1. *$F^{[p]} : [V_{y+1}, V_y]_q \to [W_{y+1}, W_y]_q$ and its restriction is naturally induced by f*
2. *$F^{[p]} : [V_{p^i(y-1)+2}, V_{p^i(y-1)+1}]_p \to [W_{p^i(y-1)+2}, W_{p^i(y-1)+1}]_p$ and its restriction is naturally induced by $f^{(p^i)}$*

Proof. It is enough to prove the statement in the case V is one-dimensional and W is two-dimensional. Indeed, then the result for any W with $dimW = n$ follows from factorization through $(n-1)$-dimensional module, for example:

$$\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & a_4 \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ 1 & \\ 1 & \end{bmatrix}$$

this decomposition respects element-wise exponentiation and we may proceed by induction on n. It is clear that adding dimension to V corresponds to introducing new independend variable(s) and all matrices are obtained by adding new column, thus if it holds for any W and one-dimensional V the same tautologically true for all V.

Now, assume $V = \langle u \rangle$ and $W = \langle e, f \rangle$. Lets extends scalars to $\Bbbk[\alpha, \beta]$ and consider the universal map $f : A \to A'$ induced by $f : u \to \alpha e + \beta f$. We have

$$A = [u, \mu_{u-1}]_q$$

$$A' = [e, \mu_{e-1}]_q \otimes [f, \mu_{f-1}]_q$$

(i) The case $A = [V_x, V_{x-1}]_q$ and x is even

Then $|u| = |e| = |f| = x$ is even and:

$$B[A] = E_q(\nu_{u+1}, y_u) \otimes \bigotimes D_p(z_{p^i u+2}, \nu_{p^i u+1})$$

and similarly for $B[A']$.

First, we have to prove the existense of an extension $F : A \boxtimes_\tau B[A] \to A' \boxtimes_\tau B[A']$. We will proceed by induction on the degree corresponding to $B[A]$. Let $F(\nu_{u+1}) = \alpha\nu_{e+1} + \beta\nu_{f+1}$ and $F(y_u) = \alpha y_e + \beta y_f$. Consider i-th tensor component of $B[A]$ equal to $D_p(z_{p^i u+2}, \nu_{p^i u+1})$ and assume that F has being defined below i. Consider an expansion

$$F(d\nu_{p^i u+1}) = F((u - qy_u)_{p^i}) = d(\alpha^{p^i}\nu_{p^i e+1} + \beta^{p^i}\nu_{p^i f+1}) + C_i(\alpha(e - qy_e), \beta(f - qy_f))$$

where $C_i(\alpha(e - qy_e), \beta(f - qy_f))$ is a cycle divisible by $\alpha\beta$, thus we may write $d\Phi_i = C_i$ for some Φ_i, assuming that Φ_i does not contain α^{p^i} or β^{p^i} also. It follows that putting

$$F(\nu_{p^i u+1}) := \alpha^{p^i}\nu_{p^i e+1} + \beta^{p^i}\nu_{p^i f+1} + \Phi_i$$

gives a well-defined extension. It remains to prove that $F(\nu_{p^i u+1})$ modulo $\bar{A}'$ is equal to $\alpha^{p^i}\nu_{p^i e+1} + \beta^{p^i}\nu_{p^i f+1}$. Though we do not control Φ_i it is possible to verify the statement using dimension counting. Namely, by the construction an expansion of Φ_i does not contains $\nu_{p^i e+1}$ or $\nu_{p^i f+1}$. Moreover, an expression for Φ_i may depend only on the parity of x, not the degree itself.

Assume there are exists terms in Φ_i without variables $e, \mu_{e-1}, f, \mu_{f-1}$. Then there is a partition of $|\Phi_i| = p^i x + 1$ into $|\nu_{p^{i-t}X+1}|$,$|z_{p^{i-t}X+1}|$,y_X for $X = e, f$ i.e. there are non-negative integers a_t, b_t, c s.t.

$$a_1(p^{i-1}x + 2) + b_1(p^{i-1}x + 1) + ... + a_{i-1}(px + 2) + b_{i-1}(px + 1) + b_i(x + 1) + c = p^i x + 1$$

holds for all x of parity $|u|$, thus they obey:

$$(a_1 + b_1)p^{i-1} + ... + (a_{i-1} + b_{i-1})p + b_i + c = p^i$$
$$2a_1 + b_1 + ... + 2a_{p-1} + b_{p-1} + b_p = 1$$

One may check that for $p > 2$ this pair of equations have no common solution, hence

$$F(\nu_{p^i u+1}) = \alpha^{p^i} \nu_{p^i e+1} + \beta^{p^i} \nu_{p^i f+1} \mod \bar{A}'$$

(For $p = 2$ a solution exists and our proposition is false).

The same argument applies to $z_{p^i u+2}$ and we can continue by induction.

(ii) The case of $A = [V_y, V_{y-1}]_q$ and odd y is similar.

□

Remark 8. One can reduce the case $A = [V_x]$ to $[V_x, V_{x-1}]_q$ for $q = 0$.

Remark 9. According to (3) the composition

$$g \circ F : A \boxtimes_\tau B[A] \to A' \boxtimes_\tau \overline{Bar}(A')$$

is unique up to homotopy.

Let us denote by $F^{[p]}$ the reduction on $B[A] \to B[A']$ of the map constructed in the previous proposition.

Lemma 3. *Let $f, f' : A \to A'$ be a pair of homotopical CDGA-morphisms $f \sim f'$ induced by linear maps $f, f' : V \to V'$.*

Consider an extensions constructed above $F^{[p]}, F'^{[p]} : B[A] \to B[A']$ and the functorial extensions $F, F' : B[A] \to B[A']$.

Then

$$F^{[p]} \sim F'^{[p]} \sim F \sim F'$$

Proof. The case $A = [V_x]$ reduces to $[V_x, V_{x-1}]_{q=0}$, so we may assume that $A = [V_x, V_{x-1}]_q$ and $A' = [V_x, V_{x-1}]_q$ for some x.

Because $f, f' : V \to V'$ induce homotopical maps their difference $f - f'$ is divisible by q. Conversly, if $f - f'$ is divisible by q, then, according to the first order homotopy lemma 10 the induced maps $f, f' : A \to A'$ are homotopical. Similarly, map F is the tensor product of natural maps induced on each component of $B[A]$ with $q' = q$ or $q' = p$. Then $F - F'$ is divisible by q on component with $q' = q$ and divisible by p on component with $q' = p$, thus $F \sim F'$. The map $F^{[p]}$ is obtained by exponentiation matrix elements of F in some power of p for components with $q' = q$ and equal to F for $q' = p$. Hence, by the same argument, $F^{[p]} \sim F$. □

Using the lemma we can safely iterate the construction:

Corollary 4 (p>2). *For a free $\Bbbk$-module V there is a natural, up to homotopy, quasi-isomorphism*

$$B^{(n)}[\Bbbk[V]] \xrightarrow{\sim} \overline{Bar}^{(n)}[\Bbbk[V]]$$

which commutes with the diagonal maps of both sides up to homotopy.

By previous we can restate a small CDGA model construction (4) for EM-spaces in the functorial manner:

Corollary 5 (p>2). *For a free $\Bbbk$-module V there is a natural CDGA-quasi-isomorphism of*

$$C_*^{CW}(K(V,n)) \simeq \overline{Bar}^{(n)}(\Bbbk[V])$$

with:

$$[V_n] \otimes \bigotimes_{I>0} [P_I V_n, \beta P_I V_n]_p$$

where I runs over all admissible sequences as in 4 and $P_I V_n$ (resp. $\beta P_I V_n$) denotes $\mathbb{Z}_{(p)}$-module naturally isomorphic to V concentrated in degree $n + |I|$ (resp. $1 + n + |I|$).

□

Cohomology model of Eilenberg-MacLane spaces

Finally we can state the main result. There is a crucial difference in $p > 2$ and $p = 2$ cases, let us treat them separately.

Theorem 6. *Let π be a cyclic group with a chosen generator η. Consider $C^*_{sing}(K(\pi,n), \mathbb{Z}_{(p)})$ as an associative DG-algebra. There is a quasi-isomorphism of complexes*

$$\phi : C^*_{sing}(K(\pi,n), \mathbb{Z}_{(p)}) \xrightarrow{\sim} K$$

such that ϕ preserves the multiplication up to homotopy, where

(i) *$p > 2$ and $q = 0$*

$$K = (\eta_n) \otimes \bigotimes_{I>0} (P^I \eta_n, \beta P^I \eta_n)_p$$

(ii) *$p > 2$ and q is finite*

$$K = (\eta_n, \beta\eta_n)_q \otimes \bigotimes_{I>0} (P^I \eta_n, \beta P^I \eta_n)_p$$

*In above cases formal variables $P_I\eta_n$ and $\beta P_I \eta_n$ modulo p corresponds to generators in the standard description of the $\mathbb{F}_p$cohomology, i.e., I runs over all admissible sequences with excess $e(I) \leq n-1$ which are not ends (on the left) on Bockstein (**i,ii**), not begins on Bockstein (**i**). We put $|P_I\eta_n| = n+|I|$, $|\beta P_I \eta_n| = n+|I|+1$. Similarly for $Sq^{2t} = P^t$ in the case $p = 2$.*

In fact the monoidal structure on $K(\pi,n)$ provides a comultiplication on $C^*_{sing}(K(\pi,n), \mathbb{Z}_{(p)})$ which is associative up to homotopy. The map ϕ also preserves the comultiplication map up to homotopy (7).

Theorem. *There is a quasi-isomorphism of complexes*

$$\phi : C^*_{sing}(K(\pi,n), \mathbb{Z}_{(2)}) \xrightarrow{\sim} K$$

such that ϕ preserves the multiplication up to homotopy, where

(i) *$p = 2$ and $q = 0$*
(K,d) is a DG algebra generated by symbols $\eta_n, Sq^I\eta_n, \Phi_{J,n}, \tau_{J,n}$ of degrees $|Sq^I\eta_n| = |I|+n$, $|\Phi_{J,n}| = 2(|J|+n)+1$, $|\tau_{J,n}| = 2(|J|+n)$ respectively, allowing divided powers on $\tau_{J,n}$, modulo quadratic relations $\mathcal{R}$ defined

below. Here I, J run over all non-empty admissible sequences with $e(I) \leq n-1$,$e(J) \leq n-2$, J_1 and $|J|+n$ even, that do not starts (on the right) on the Bockstein. The ideal $\mathcal{R}$ is generated by (cf. 10):

$$
\begin{aligned}
&[Sq^J\eta_n, \beta Sq^J\eta_n] = 2Sq^{|J|+n}\beta Sq^J\eta_n - \Phi_{J,n} \\
&\Phi_{J,n}^2 = 4(Sq^{|J|+n}\beta Sq^J\eta_n)^2 \\
&[\tau_{J,n}, Sq^{|J|+n}\beta Sq^J\eta_n] = 2Sq^{2(|J|+n)}Sq^{|J|+n}\beta Sq^J\eta_n \\
&[\tau_{J,n}, \Phi_{J,n}] = 4Sq^{2(|J|+n)}Sq^{|J|+n}\beta Sq^J\eta_n \\
&[Sq^{|J|+n}\beta Sq^J\eta_n, \Phi_{J,n}] = -4(Sq^{|J|+n}\beta Sq^J\eta_n)^2
\end{aligned}
$$

for each I with I_1, $|I|+n$ even and $e(I) \leq n-2$ and super-commutativity relations for all other pairs of variables.

The differential d extends by the Leibniz rule according to:

$$
\begin{aligned}
&dSq^I\eta_n = 2\cdot\beta Sq^I\eta_n \\
&d\eta_n = 0 \\
&d\tau_{I,n} = -\Phi_{I,n}
\end{aligned}
$$

***(ii)** $p = 2$, q is finite and $q > 2$*
K is defined as above with I allowing empty sequences, the differential is redefined by $d\eta_n = q\cdot\beta\eta_n$.

***(iii)** $p = 2$, q is finite and $q = 2$*
K is defined as above with I and J allowing empty sequences, the differential is redefined by $d\eta_n = q\cdot\beta\eta_n$.

Proof. The description of the small chain model in 4 provides an explicit correspondence between Cartan's admissible words enumerating y, ν, z variables in $B^{(n)}$ and Steenrod admissible sequences. Applying the theorem on comultiplication(5) and then carefully following the dualization procedure of 9 we can rewrite all relations of $B'(B^{n-1})^\vee$ in terms of symbols in the statement. □

Corollary 6 ($p > 2$)**.** *Let V be a free $\mathbb{Z}$-module dual to $V^\vee$. There is a natural, up to homotopy, CDGA-quasi-isomorphism:*

$$C^*_{sing}(K(V,n))\otimes\mathbb{Z}_{(p)} \sim (V_n)\otimes\bigotimes_{I>0}(P^IV_n, \beta P^IV_n)_p$$

where $|P_I| = |I|, |\beta P_I| = |I|+1$ and I runs over all admissible sequences with excess $e(I) \leq n-1$ which are not begins or ends on Bockstein β, P_IV_n (resp. βP_IV_n) denotes $\mathbb{Z}_{(p)}$-module naturally isomorphic to V concentrated in degree $n+|I|$ (resp. $1+n+|I|$).

In particular, the cohomology of EM-space $H^*(K(V,n),\mathbb{Z}_{(p)})$ are naturally isomorphic to the cohomology of a super-affine DeRham algebra tensored by the free-commutative algebra (V):

$$(V)\otimes H^*(\Omega^\bullet(\bigoplus_I P^IV_n), p\cdot d)$$

Remark 10 (on Bockstein spectral sequence of EM-space)**.** We can interpret the above result in terms of computation done by P. May ([5]). Recall that the BSS sheet $E_{k+1} = H^*_{\beta_k}(E_k)$ is equal to the cohomology of previous sheet E_k with respect to the well-defined higher Bokstein derivative $\beta_k^2 = 0$. Here $\beta_1 = \beta$ and E_1 is equal to $\mathbb{F}_p$-cohomology i.e. $H^*(K(\pi,n),\mathbb{F}_p)$.

Assume $p > 2$. It is well-known that $H^*(K(n,\mathbb{Z}),\mathbb{F}_p)$ is the free-commutative algebra isomorphic to the tensor product of duals of an elementary algebras (7) $(P^I\nu_n, \beta P^I\nu_n)_p$, (ν_n) modulo p. In order to compute the BSS P. May made a simple observation that if β_k is well-defined on an element $x \in H^*(X,\mathbb{F}_p)$ for some X, then $\beta_{k+1}(x^p) = x^{p-1}\beta_k(x)$. Thus the corresponding BSS-sheets E_k and E_{k+1} of each elementary part are related by the following pattern

$$(x, \beta_k x) \rightsquigarrow (x^p, \beta_{k+1}x^p = x^{p-1}\beta_{k+1}x)$$

hence the BSS of $H^*(K(n,\mathbb{Z}_{(p)}),\mathbb{Z})$ is obtained by tensoring BSS of all elementary tensor components together. One can see that exactly the same pattern appears for the elementary algebras of the form $(a,b)_p$. Using Corollary 1(1) on the LSSS we coclude that our answer is a natural lifting over $\mathbb{Z}_{(p)}$ of the BSS described by May.

The case $p = 2$ is more delicate. First of all, the above formula for Bockstein does holds only for $\beta_k, k \neq 2$. For $k = 2$ we have $\beta_2(x^2) = \beta_1 x{\cdot}x + Sq^{|x|}\beta_1 x$. This means that there is no CDGA which provides a model for the cohomology. One can guess a DGA with the same BSS patterns. This an algebra obtained by forgetting all $\Phi_{I,n}, \tau_{I,n}$ terms and all corresponding quadratic relations in the description the main theorem 6. Note that this procedure gives a new algebra, not a quotient of the former. This naive answer is a deformation given by the Weyl algebra, i.e., corresponding to the only relation $[Sq^I\eta_n, \beta Sq^I\eta_n] = 2Sq^{|I|+n}\beta Sq^I\eta_n$. In dual terms, we would have a natural morphism $g : B(A) \to \overline{Bar}(A)$ for $A = D_2(x,\mu)$ compatible with a comultiplication constrained to $\Delta_B\nu_{2x+1} = \nu_{2x+1}\boxtimes 1 + 1\boxtimes\nu_{2x+1} + y\boxtimes\nu_{x+1} - \nu_{x+1}\boxtimes y$. It turns out that there is no such morphism and that is the main reason for introducing the $\widetilde{B}$-construction.

We finish this section with a remark about the cohomology $H^*([V,V[-1]]_1)$ of the de Rham algebra $[V,V[-1]]_1$ for even-graded V. Pick a basis $x_i \in V$ and $dx_i \in V$. Though there is no simple and functorial description of the cohomology groups (see [4]), it is not difficult to give an answer in terms of x_i [10]. For simplicity let $|x_i| = 0$.

There is a decomposition of a complex

$$\mathbb{Z}_{(p)}[x_1,\dots,x_n,dx_1,\dots,dx_n] = \bigoplus_{\substack{S\subset\{1,\dots,n\}\\ \text{a tuple } e_s>0 \text{ for } s\in S}} K^{S,e_{s\in S}}$$

On each S and tuple (e_s) the subcomplex $K^{S,e_{s\in S}}$ is defined by:

$$K^{S,e_{s\in S}} = \bigoplus_{\substack{\varepsilon_s\in\{0,1\}\\ s\in S}} \prod_{s\in S} {x_s}^{e_s-\varepsilon_s} {dx_s}^{\varepsilon_s}$$

Let $\nu(e_{s\in S}) = ord_p(e_n)$ for some $n \in S$ be equal to the smallest p-primary part of numbers $e_{s\in S}$. Then

$$H^d(K^{S,e_{s\in S}}) \simeq (\mathbb{Z}/p^{\nu(e)})^{\oplus\binom{|S|-1}{d-1}}$$

and representatives of the cohomology group can be given by:

$$\omega = (-1)^{|g|}\frac{dg}{p^{\nu(e)}}\cdot {x_n}^{e_n} + g{x_n}^{e_n-1}dx_n$$

where g is a differential form on variables $x_s, s \neq n \in S$, which defines an cohomology class of order $p^{\nu(e)}$.

Routine

Definition 10 (first order homotopy)**.** Let A be a tensor product of elementary algebras with two CDGA-maps $\phi^0,\phi^1 : A \to C$. Denote by $G(A)$ the subspace of generators, it is a subcomplex in A. We say that ϕ^0,ϕ^1 are first order homotopic if there is they are homotopic in restriction to $G(A)$.

Lemma 4 (existence)**.** *Let A be an elementary algebra with CDGA-quasi-isomorphism $g : \widetilde{B}(A) \to \overline{Bar}(A)$.*

Let $\nabla_1 = \Delta_{Bar}\circ g$ and $\nabla_0 = g\boxtimes g\circ\Delta_{\widetilde{B}}$.

Then ∇_1 and ∇_0 are first order homotopic on each factor of $\widetilde{B}(A)$. If $p = 2$, then at $(\Phi,\tau)_1$ factor of $\widetilde{B}$ they are equal.

Proof

It is possible to express g iteratively by $g(a) = s(g(da))$ (Proposition 1), where d is acyclic differential of the bar-resolution, but I will need only a few such expressions. Recall that except the special case $A = D_2(x,\nu)$ we defined $\widetilde{B}(A) = B(A)$.

(i) $A = E(\nu)$. Then $\widetilde{B}(A) = D(x_{\nu+1})$ and $g(x) = \bar{\nu}$, hence $g(x)$ is already primitive and $h = 0$. □

(ii) $A = D(x)$. Then $\widetilde{B}(D(x)) = E(\nu_{x+1}) \otimes \bigotimes D_p(z_{p^i x+2}, \nu_{p^i x+1})$ and we have(2):

$$g(\nu_{p^i x+1}) = \overline{x_{p^i}} \qquad\qquad g(z_{p^i x+2}) = \overline{x_{p^{i-1}}^{p-1}}|\overline{x_{p^{i-1}}}$$

We see that generators ν already primitive. If $p > 2$, then

$$(\nabla_1 - \nabla_0)(z_{p^i x+2}) = \overline{x_{p^{i-1}}^{p-1}} \boxtimes \overline{x_{p^{i-1}}}$$

which is obviously a boundary for $p > 2$, thus h exists. if $p = 2$, then by the definition of ∇_1 we have $(\nabla_1 - \nabla_0)(z_{p^i x+2}) = 0$.

□

(iii) $A = D_q(x, \mu_{x-1}), q \neq 2$. Then $\widetilde{B}(A) = E_q(\nu_{x+1}, y_{\mu+1}) \otimes \bigotimes D_p(z_{p^i x+2}, \nu_{p^i x+1})$ and:

$$g(y_{\mu+1}) = \overline{\mu_{x-1}}, \qquad\qquad g(\nu_{x+1}) = \overline{x}$$

are already primitive. It remains to define a first order homotopy on $\nu_{p^i x+1}, z_{p^i x+2}$, $i \geq 1$. However expressions for g-values are complex. Actually it is possible to construct a homotopy by showing that $(\nabla_1 - \nabla_0)z_{p^i x+2}$ is exact modulo p.

Let us for a moment go back to notations from the proof in (3,**(iii)**). Then, in fact, we work with variables

$$\widetilde{\nu_{p^i x+1}} := \nu_{p^i x+1} - \frac{(-q)^{(p-1)p^{i-1}}}{p} y_{p^{i-1}}^{p-1} \nu_{p^{i-1}x+1}$$
$$d\nu_{p^i x+1} = (x - qy)_{p^i}$$
$$dz_{p^i x+2} = -p\nu_{p^i x+1} + (x - qy)_{p^{i-1}+1}^{p-1} \nu_{p^{i-1}x+1}$$

d is the acyclic differential of the $\widetilde{B}$-construction, we have

$$d_B z_{p^i x+2} = -p\widetilde{\nu_{p^i x+1}}$$

where d_B is reduced d, i.e., the $\widetilde{B}$-construction differential.

Using the definition of comultiplication 3 it is straightforward to check that modulo p (for all primes):

$$(\nabla_1 - \nabla_0)z_{p^i x+2} = -p \cdot u + d_{Bar \boxtimes Bar} v$$

for some $u, v \in \overline{Bar}(A) \boxtimes \overline{Bar}(A)$. For instance, if $p > 2$:

$$(\nabla_1 - \nabla_0)z_{p^i x+2} = \overline{x_{p^{i-1}}^{p-1}} \boxtimes \overline{x_{p^{i-1}}} \mod p$$

is a boundary, if $p = 2$ the right hand side vanishes by definition of ∇_1 using introduced comultiplication.

Then, putting $h(\widetilde{\nu_{p^i x+1}}) := u$ and $h(z_{p^i x+2}) := v$ we obtain $(\nabla_1 - \nabla_0)z_{p^i x+2} = [d, h](z_{p^i x+2})$ and automatically $(\nabla_1 - \nabla_0)\widetilde{\nu_{p^i x+2}} = [d, h](\widetilde{\nu_{p^i x+1}})$. This will define a first order homotopy on $\nu_{p^i x+1}, z_{p^i x+2}$ for $i \geq 1$. □

(iv) $A = E_q(\mu, x_{\mu-1})$. Then $B(A) = D_q(y_{\mu+1}, \nu_{x+1}) \otimes \bigotimes_{i>0} D_p(z_{p^i x+2}, \nu_{p^i x+1})$. One can apply the same argument as in **(ii)**. □

(v) $A = D_2(x, \nu), q = 2$.

Then $\widetilde{B}(A) = B(A) \otimes E(\Phi) \otimes P(\tau)$. It is straightforward to see that

$$
\begin{aligned}
&g(\nu_{x+1}) = \overline{x} \\
&g(\nu_{2x+1}) = \overline{x_2} + \overline{x}|\overline{\mu} - \overline{\mu}|\overline{x} \\
&g(\nu_{2^i x+1}) = \overline{x_{2^i}} \mod 2 && i > 0 \\
&g(z_{2^i x+1}) = \overline{x_{2^{i-1}}}|\overline{x_{2^{i-1}}} && i > 0 \\
&g(\tau) = \overline{x\mu} \\
&g(\Phi) = \overline{\mu}|\overline{x}
\end{aligned}
$$

Comparing with the definition of comultiplication 5 gives that $\nabla_1 - \nabla_0$ is zero on $\tau, \phi, \nu_{x+1}, \nu_{2x+1}$ and is exact modulo p for $z_{2^i x+2}$. Now one can proceed as in **(iii)**. □

■

Proposition 12 (extension of the first order homotopy)**.** *Let A be an elementary algebra and C is some CDGA with divided powers. Let $\nabla^i : A \to C$ two CDGA-morphisms which are homotopical in first order by $h : G(A) \to C[1]$, then there is an extension of such homotopy to $h : A \to C[1]$ s.t. $[d, h] = \nabla^1 - \nabla^0$.*

proof (see [1], expose 11, proposition 3) In the following subscripts, as usual, indicate divided power, while superscript play the role of index. Fix an acyclic resoulution $L \to C$ with divided powers (e.g. $L = C \boxtimes \overline{Bar}(C)$). One can check two simple facts:

- $(dv)_k = dw^k$ for some $w^k \in C$. Indeed, pick a lifting $\tilde{v} \in L$, then $(d_L\tilde{v})_k$ is closed, thus $(d_L\tilde{v})_k = d_L\tilde{w}^k$, for some $\tilde{w}^k \in L$, hence $(d_Cv)_k = d_Cw^k$ after the reduction $L \to C$.
- $v(dv)_k - (k+1)w_{k+1} = dt^k$ for some $t^k \in C$. Similarly, the expression has a closed lifting and hence is exact in C.

(i) $A = E(\nu)$.
nothing to prove □

(ii) $A = D(x)$.
We have $\nabla^1(x) = \nabla^0(x) + dv$, then

$$\nabla^1(x_k) = \nabla^0(x_k) + [\nabla^0(x_{k-1})dv + \nabla^0(x_{k-2})(dv)_2 + ... + (dv)_k]$$

and by above facts expression in brackets is exact, thus we can find $h(x_k)$ s.t. $\nabla^1 - \nabla^0 = [d, h]$. □

(iii) $A = D_q(x, \nu_{x-1})$.
For some u, v we have

$$\nabla^1(x) = \nabla^0(x) + qv + du$$
$$\nabla^1(\nu) = \nabla^0(\nu) + dv$$

First we reduce to a particular case. Let $\nabla'(x) = \nabla^0(x) + du, \nabla'(\nu) = \nabla^0(\nu)$ and extend it to a CDGA-map. We have

$$\nabla'(x_k) = \nabla^0(x_k) + [\nabla^0(x_{k-1})dv + \nabla^0(x_{k-2})(dv)_2 + .. + (dv)_k]$$

Take v^k as above s.t. $dv^k = (dv)_k$. Let

$$h(x_k) = \nabla^0(x_{k-1})v^1 + \nabla^0(x_{k-2})v^2 + .. + v^k$$
$$h(x_{k-1}\nu) = \nabla^0(x_{k-2}\nu)v^1 + .. + \nabla^0(\nu)v_{k-1}$$

It is easy to check that:

$$\nabla'(x_k) = \nabla^0(x_k) + dh(x_k) + qh(x_{k-1}\nu)$$
$$\nabla(x_{k-1}\nu) = \nabla^0(x_{k-1}\nu) + dh(x_{k-1}\nu)$$

Thus $\nabla' = \nabla^0 + [d, h]$ and replacing ∇^0 with ∇' we may assume from now that:

$$\nabla^1(x) = \nabla^0(x) + qv$$
$$\nabla^1(\nu) = \nabla^0(\nu) + dv$$

Lets put $h(x_k) = 0$, $h(\nu) = v$. Define $h(x_{k-1}\nu)$ requiring:

$$\nabla^1(x_k) - \nabla^0(x_k) = h(dx_k) = qh(x_{k-1}\nu)$$

It is clear that taking

$$h(x_{k-1}\nu) := \frac{1}{q}(\nabla^1(x_k) - \nabla^0(x_k))$$

we obtain an integer expression. We want to satisfy:

$$\nabla^1(x_{k-1}\nu) = \nabla^0(x_{k-1}\nu) + dh(x_{k-1}\nu)$$

But $dh(x_{k-1}\nu) = \frac{1}{q}d[\nabla^1(x_k) - \nabla^0(x_k)] = \nabla^1(x_{k-1}\nu) - \nabla^0(x_{k-1}\nu)$. Thus we get a desired homotopy. □

(iv) $A = E_q(\nu, x_{\nu-1})$.

We have $d\nu = qx, d(x_k\nu) = q(k+1)x_{k+1}$. Start with:

$$\nabla^1(\nu) = \nabla^0(\nu) + qv + du$$
$$\nabla^1(x) = \nabla^0(x) + dv$$

As above, let

$$\nabla'(\nu) = \nabla^0(\nu) + du$$
$$\nabla'(x) = \nabla^0(x)$$

Then $\nabla'(x_k\nu) = \nabla^0(x_k\nu) + d(\nabla^0(x_k) \cdot u)$ We can put

$$h(x_k) = 0$$
$$h(x_k\nu) = \nabla^0(x_k)u$$
$$h(\nu) = u$$

to obtain homotopy $h : \nabla' \sim \nabla^0$. Now, replacing ∇' with ∇^0 we can assume that:

$$\nabla^1(\nu) = \nabla^0(\nu) + qv$$
$$\nabla^1(x) = \nabla^0(x) + dv$$

Then

$$\nabla^1(x_k) = \nabla^0(x_k) + \nabla^0(x_{k-1})dv + .. + (dv)_k$$

Let v^k be s.t. $dv^k = (dv)_k$ and $v^1 = v$. Put $h(x_k) = \nabla^0(x_{k-1})v^1 + .. + v^k$ and $h(\nu) = 0$ Then

$$\nabla^1(x_k) = \nabla^0(x_k) + dh(x_k)$$

We want to satisfy:

$$\nabla^1(x_k\nu) = \nabla^0(x_k\nu) + q(k+1) \cdot h(x_{k+1}) + dh(x_k\nu)$$

On the other hand we have:

$$\nabla^1(x_k\nu) = \nabla^0(x_k\nu) + dh(x_k) \cdot \nabla^0(\nu) + \nabla^1(x_k) \cdot qv + dh(x_k) \cdot qv$$

Rewrite remaining parts as:

- $dh(x_k) \cdot \nabla^0(\nu) = d[h(x_k) \cdot \nabla^0(\nu)] + [\nabla^0(x_{k-1})v^1 + .. + v^k]\nabla^0(qx) =$
$$dA_k + q \cdot \nabla^0(k \cdot x_k)v^1 q[\nabla^0((k-1) \cdot x_{k-1})v^2 + .. + \nabla^0(x) \cdot v^k]$$
- $$dh(x_k) \cdot qv = q[\nabla^0(x_{k-1})vdv + .. + v(dv)_k)]$$
- $$\nabla^1(x_k) \cdot qv = q \cdot \nabla^1(x_k)v^1$$

Then:

$$\nabla^1(x_k\nu) - \nabla^0(x_k\nu) =$$
$$dA_k + q\{\nabla^0(x_{k-1})((k-1)v^2 + vdv) + \nabla^0(x_{k-2})((k-2)v^3 + v(dv)_2) + .. + \nabla^0(x_1)(v^k + v(dv)_k)\} + qv \cdot \nabla^1((k+1)x_k)$$

After subtracting

$$q(k+1) \cdot h(x_{k+1}) = q(k+1)[\nabla^0(x_k)v^1 + \nabla^0(x_{k-1})v^2 .. + v^{k+1}]$$

We are left with:

$$\nabla^1(x_k\nu) - \nabla^0(x_k\nu) - q(k+1) \cdot h(x_{k+1}) =$$
$$dA_k + \{\nabla^0(x_{k-1})(-2v^2 + vdv) + \nabla^0(x_{k-2})(-3v^3 + v(dv)_2) + .. + (-(k+1)v^{k+1} + v(dv)_k)\}$$

There are exists w^i s.t. $dw^i = -iv^i + v(dv)_{i-1}$, we fix them once and for all i.
Then the right hand side is exact, thus we may extend h with $\nabla^1 - \nabla^0 = [d, h]$

□

■

Lemma 5. *homotopy on tensor product*

Let $\phi_A, \phi'_A : A \to C$ *and* $\phi_B, \phi'_B \to C$ *be two pairs of homotopical CDGA-maps i.e.* $\phi_A - \phi'_A = [d, h_A]$ *and* $\phi_B - \phi'_B = [d, h_B]$.

Consider $A \otimes B \to C$*, then* $\phi_A \otimes \phi_B$ *and* $\phi'_A \otimes \phi'_B$ *are homotopical by*
$h_{A\otimes B} := \phi_A \otimes h_B + h_A \otimes \phi'_B$

Proof. $[d, \phi_A \otimes h_B + h_A \otimes \phi'_B] = \phi_A \otimes [d, h_B] + [d, h_A] \otimes \phi'_B = \phi_A \otimes (\phi_B - \phi'_B) + (\phi_A - \phi'_A) \otimes \phi'_B = \phi_A \otimes \phi_B - \phi'_A \otimes \phi'_B$

□

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