Kaledin's class computations

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1 Intro

Let A be a flat associative dg-algebra over $k[h]$, where $k \supset \mathbb{Q}$ is a ring. In particular $h\colon A \hookrightarrow A$ is an inclusion of an ideal. We consider A as a deformation of $A^\circ = A/hA$ over $\operatorname{Spec} k[h]$.

Our first goal is to construct the Kaledin class in $\Theta \in HH^2(A)$ and to show that $\Theta = 0 \in HH^2(A \otimes_{k[h]} k[h]/h^p)$ iff A is quasi-isomorphic to the trivial deformation $A^0[h]$ over the formal disk $\operatorname{Spec} k[h]/h^p$.

2 Zero-square extensions

Fix $\bar{A}$ an associative dg-algebra over a base ring R (e.g. $R = k[h]$). All tensor products below are over R. Suppose we have a square-zero extension of algebras

$$0 \to M \to A \to \bar{A} \to 0,$$

where $M^2 = 0$ in A. We have

$$\begin{tikzcd}
I_{\bar{A}} \arrow[r] & \bar{A} \otimes \bar{A} \arrow[r] & \bar{A} \\
I_A \arrow[u] \arrow[r] & A \otimes \bar{A} \arrow[u] \arrow[r] & \bar{A} \arrow[u] \\
I_M \arrow[r] & M \otimes \bar{A} \arrow[u] \arrow[r] & M
\end{tikzcd} \tag{2.0.1}$$

Here all rows and the middle column are exact. In particular

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, phantom, "\wr"] & K/I_M \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$=:$}"] & I_{\bar{A}} \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\
0 \arrow[r] & M \otimes \bar{A}/I_M \arrow[r] & \ker[A \otimes \bar{A} \to \bar{A}]/I_M \arrow[r] & I_{\bar{A}} \arrow[r] & 0
\end{tikzcd} \tag{2.0.2}$$

is exact. Note that the middle term is in fact an $\bar{A} \otimes \bar{A}^{\mathrm{op}}$-module:

$$\begin{tikzcd}
M \otimes K \arrow[r, hook] \arrow[drr, dashed] & M \otimes A \otimes \bar{A} \arrow[r, "m\otimes \mathrm{id}"] & M \otimes \bar{A} \arrow[r, "m"] & M \\
 & & I_M \arrow[u, hook] &
\end{tikzcd} \tag{2.0.3}$$

because the top composition vanishes the dashed arrow exists and hence the ideal M acts on K/I_M from the left trivially.

The extension (2.0.2) defines a class $\Theta_{A/\bar{A}} \in \operatorname{Ext}^1_{\bar{A}\otimes\bar{A}^{\mathrm{op}}}(I_{\bar{A}}, M)$. It remains to note that $\Theta_{A/\bar{A}}$ also lives in:

$$HH^2(\tilde{A}, M) = \operatorname{Ext}^2_{\bar{A}\otimes\bar{A}^{\mathrm{op}}}(\bar{A}, M) \simeq \operatorname{Ext}^1_{\bar{A}\otimes\bar{A}^{\mathrm{op}}}(I_{\bar{A}}, M). \tag{2.0.4}$$

In a commutative setting of algebraic geometry we expect that $\Theta_{A/\bar{A}}$ should lift to $\operatorname{Ext}^1_{\bar{A}}(I_{\bar{A}}/I^2_{\bar{A}}, M)$ or rather $\operatorname{Ext}^1_{\bar{A}}(\Omega^1_{\bar{A}}, M)$, where $\Omega^1_{\bar{A}}$ is the cotangent complex. The last ext should genuinely classify square-zero extensions and this also should make sense globally for sheaves.

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3 Formula for the cocyle

First recall the following construction. Here all module are over associative dg-algebra R. Start with a class $\Theta \in \operatorname{Ext}^1_R(C, A)$ given by a short exact sequence of left dg-modules over R:

$$0 \to A \xrightarrow{j} B \xrightarrow{\pi} C \to 0.$$

Suppose $P_* := [\ldots \to P_2 \to P_1 \to P_0] \xrightarrow{\sim} C$ is a free resolution. We want to describe a representative of Θ in $\operatorname{Hom}(P_*, A[1])$.

$$\begin{tikzcd}
& P_1/d(P_2) \arrow[r, hook, "d"] \arrow[dl, dashed, "s'"'] & P_0 \arrow[d] \arrow[dll, dotted, "v"' near start] \arrow[dl, dashed, "s"] \\
A \arrow[r, "j"'] & B \arrow[r, two heads, "\pi"'] & C
\end{tikzcd} \tag{3.0.1}$$

By projectivity of P_* we first pick a lifting s which defines a morphism s'. Define $v \colon P_0 \to A[1]$ as a unique map s.t. $jv = [s, d] \in B$.

Proposition 3.0.2. *The map $P_1[1] \oplus P_0 \xrightarrow{s'[1]+v} A[1]$ defines a morphism of complexes of R-dg-modules $P_* \to A[1]$ which represents the class $\Theta \in \operatorname{Ext}^1_R(C, A)$ of the extension $0 \to A \to B \to C \to 0$.*

Proof. Straightforward. □

4 Remainder on Hochshild cohomology

Assume A is a dg-algebra. Here the base ring is a dg-algebra R. The Hochshild complex

$$C^*(A) := [\operatorname{Hom}_R(R, A) \to \operatorname{Hom}_R(A, A) \to \operatorname{Hom}_R(A \otimes A, A) \to \ldots]$$

admits a mutiplication of degree zero and a shifted Poisson bracket of degree -1 called Gerstenhaber bracket. The latter can be described by the identification

$$C^*(A) = \operatorname{Coder}_R(\bar{T}(A[1]))[-1]. \tag{4.0.1}$$

Here

$$\bar{T}(A[1]) = A[1] \oplus A[1]^{\otimes 2} \oplus \ldots$$

is the reduced cofree coalgebra on $A[1]$ aka bar construction. Coderivations form a dg-Lie algebra with a bracket $[-,-]$ given by the commutator. For the cofree coalgebra the corresponding complex is given by:

$$\operatorname{Coder}_R(\bar{T}A[1]) \simeq \prod_n \operatorname{Hom}_R(A[1]^{\otimes n}, A[1]) \simeq \prod_n \operatorname{Hom}_R(A^{\otimes n}, A)[1-n].$$

The isomorphism of complexes in (4.0.1) is with respect to the differential on the RHS

$$d_A + [\delta_m, -],$$

where d_A is the differential on A and δ_m is the coderivation of degree 1 corresponding to the multiplication $m \in \operatorname{Hom}^1_R(A[1]^{\otimes 2}, A[1])$ on A.

5 The construction of Θ

Now, let $\bar{A}$ be a flat dg-algebra over $k[h]$, where k is a field. Let $A = \bar{A}[\varepsilon]/\varepsilon^2$, but with a deformed $k[h]$-structure given by $h' := h + \varepsilon$. The square-zero extension

$$0 \to \varepsilon \bar{A} \to A \to \bar{A} \to 0$$

gives a class $\Theta \in \operatorname{Ext}^1_{\bar{A}}(I_{\bar{A}}, \varepsilon \bar{A})$ by (2.0.4).

5.1 Formula

First we give a formula for Θ using Proposition3.0.2 in terms of the given splitting $A^0 \to A$, i.e. an identification of $k[h]$-modules $A \simeq A^0[h]$. Let $a(h) = \sum_{i\geq 0} a_i h^i, a_i \in A^0$, the multiplication $- \star -$ and differential d in A can be written as:

$$\sum_{i\geq 0} a_i h^i \star \sum_{j\geq 0} b_j h^j := \sum_{i,j,k\geq 0} m_k(a_i, a_j) h^k h^{i+j}$$

and

$$d(a(h)) = \sum_{i,k\geq 0} d_k(a_i) h^k \cdot h^i,$$

where $d_k(-), m_k(-,-)$ are tensors on A^0. Note that d_0, m_0 encodes dg-algebra A^0. We have $M = \varepsilon\,\bar{A}$ and

$$\begin{tikzcd} 0 \arrow[r] & I_M \arrow[r] \arrow[d, equal] & M \otimes \bar{A} \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \varepsilon\, I_{\bar{A}} \arrow[r] & \varepsilon\,\bar{A} \otimes \bar{A} \arrow[r] & \varepsilon\,\bar{A} \arrow[r] & 0. \end{tikzcd} \tag{5.1.1}$$

$$I_{\bar{A}} \quad = \quad \begin{tikzcd} {[\dots} \arrow[r] & \bar{A}^{\otimes 5} \arrow[r] \arrow[d, "0"] & \bar{A}^{\otimes 4} \arrow[r] \arrow[dl, "s'"'] & \bar{A}^{\otimes 3}] \arrow[d] \arrow[dll, "v"] \arrow[dl, dashed, "s"] \\ & M = \varepsilon\,\bar{A} \arrow[r] & \ker[A \otimes \bar{A} \to \bar{A}]/I_M \arrow[r] & I_{\bar{A}} \end{tikzcd}$$

Proposition 5.1.2. *The maps v and s' are morphisms of $\bar{A} \otimes \bar{A}^{\mathrm{op}}$-modules and in terms of the identification we have:*

$$v(1|a(h)|1) = \varepsilon \sum_k k h^{k-1} d_k . a(h), \tag{5.1.3}$$

$$s'(1|a(h)|b(h)|1) = \varepsilon \sum_k k h^{k-1} m_k(-,-).(a(h), b(h)). \tag{5.1.4}$$

Thus

$$\Theta = \sum_k k h^{k-1} \cdot (d_k(-) + m_k(-,-)) \in HH^2(\bar{A}) \simeq \mathrm{Ext}^1_{\bar{A}\otimes\bar{A}^{\mathrm{op}}}(I_{\bar{A}}, \varepsilon\,\bar{A}).$$

Proof. Since s is a morphism of $\bar{A}\otimes\bar{A}^{\mathrm{op}}$-modules over $k[h]$, it is enough to fix a k-linear lifting $1|A^0|1 \to A\otimes\bar{A}$ given by $s(1|a|1) = a \otimes 1 - 1 \otimes a$. Thus

$$s(1|a(h)|1) = a_i \otimes h^i - 1 \otimes a_i h^i = a(h+\varepsilon) \otimes 1 - 1 \otimes a(h).$$

1. $v = [d, s]$ is evaluated by

$$-sd.(1|a(h)|1) = d(a_i)(h+\varepsilon) \cdot (h+\varepsilon)^i \otimes 1 - 1 \otimes d(a_i)(h) \cdot h^i$$

and

$$ds.(1|a(h)|1) = d(a(h+\varepsilon) \otimes 1 - 1 \otimes a(h)) = d(a_i)(h) \cdot (h+\varepsilon)^i \otimes 1 - 1 \otimes d(a_i)(h) \cdot h^i.$$

Thus

$$[s,d](1|a(h)|1) = \varepsilon \frac{d(a_i)(h)}{dh} \cdot h^i = \varepsilon \frac{d}{dh}(d_k(a_i)h^k)h^i = \varepsilon \sum_k k h^{k-1} d_k(-).a(h).$$

2. Similarly it is straightforward to check that

$$s'(1|a(h)|b(h)|1) = \varepsilon(a' \star b + a \star b' - (a \star b)')(h),$$

which implies

$$s'(1|a(h)|b(h)|1) = \varepsilon \sum_k k h^{k-1} m_k(-,-).(a(h), b(h)).$$

$\square$

6 A_∞ setting

Proposition 5.1.2 suggest the following generalization for the definition of the Kaledin class Θ.

First recall

Definition 6.0.1. An A_∞-algebra A is a coderivation δ in the *bar complex* $\mathrm{Coder}^1(\bar{T}(A[-1]))$ of cohomological degree 1 such that $\delta^2 = 0$.

A morphism between A_∞-algebras is a morphism of the corresponding bar complexes viewed as dg-coalgebras.

Any dg-algebra A with the differential d and multiplication m defines A_∞-structure by setting

$$\delta = d + m,$$

whereas the associativity together with the Leybniz identity ensures that $\delta^2 = 0$.

Definition 6.0.2. The Hochshild cohomology $HH^{*+1}(A)$ of A_∞-algebra A is the cohomology of the complex $(\mathrm{Coder}(\bar{T}(A[-1])), [\delta, -])$.

Assume now $\bar{A}$ is a flat A_∞ algebra over $k[h]$. As before fix a splitting $A^0 \to \bar{A}$, i.e. an isomorphism of $k[h]$-modules $A^0[h] \simeq \bar{A}$. This gives an expansion $\delta = \sum_{k\geq 0} \delta_k h^k$. We will use the similar expansions for $\bar{T}_{k[h]}(\bar{A}[1]) \simeq \bar{T}_k(A^0[1])[h]$. In particular δ_0 is an A_∞-structure on A^0.

By super-commutativity $0 = \partial_h[\delta, \delta] = 2[\delta, \partial_h \delta]$, thus

$$[\delta, \partial_h \delta] = 0.$$

Definition 6.0.3 (Lunts)**.** The Kaledin class is given by $\Theta_{\bar{A}/A^0} := [\partial_h \delta] \in HH^2(\bar{A})$.

The fact that the definition is independent on the splitting $A^0 \to \bar{A}$ follows from

Lemma 6.0.4. *Assume ι is an automorphism of the plain coalgebra $\bar{T}(\bar{A}[1])$. For a coderivation $\delta \in \bar{T}(\bar{A}[1])$ let $\iota_*(\delta) := \iota \circ \delta \circ \iota^{-1}$. Then*

$$\iota_*(\partial_h \delta) - \partial_h(\iota_*(\delta)) = [\iota_*(\delta), X],$$

for some coderivation X.

Proof. Here we dont ask $\delta^2 = 0$. Note that we can always write $\partial_h(\iota) = \iota \circ Q$ for some coderivation Q. As usual we have

$$\partial_h(\iota^{-1}) = -\iota^{-1} \circ \partial_h(\iota) \circ \iota^{-1} = -Q \circ \iota^{-1}.$$

Then

$$\iota_*^{-1}(\partial_h(\iota_* \delta)) = \iota^{-1} \circ (\partial_h(\iota) \circ \delta \circ \iota^{-1} + \iota \circ \partial_h \delta \circ \iota^{-1} + \iota \circ \delta \circ \partial_h(\iota^{-1})) \circ \iota =$$
$$Q \circ \delta + \partial_h(\delta) - \delta \circ Q,$$

thus

$$\partial_h(\delta) - \iota_*^{-1}(\partial_h(\iota_* \delta)) = [\delta, Q].$$

Setting $X := \iota_*(Q)$ solves the problem. □

Theorem 6.0.5 (Kaledin)**.** *Assume $\Theta \mod h^{p-1}$ is trivial, then there is A_∞-isomorphism*

$$\iota_p \colon \bar{A} \xrightarrow{\sim} A^0[h] \mod h^p,$$

which is identity $\mod h$.

Proof. For $p = 1$ we take $\iota_1 := \mathrm{id}$. Assume the statement holds for some $p \geq 1$, i.e.

$$\iota_p \colon \bar{T}(A^0[1])[h] \to \bar{T}(A^0[1])[h]$$

is an automorphism of the plain coalgebra such that $(\iota_p)_* \delta = \delta_0 + u h^p \mod h^{p+1}$ for some coderivation u.

Suppose $\Theta = \partial_h(\delta) = [\delta, T] \mod h^p$. The step of induction is to find $\iota_{p+1} = \iota_p \circ \exp(vh^p)$ for some coderivation v, such that $(\iota_{p+1})_*\delta = \delta_0 \mod h^{p+1}$. Modulo h^{p+1} we have

$$\iota_p(\exp(vh^p)(\delta)) \equiv \iota_p(\delta + h^p[v,\delta]) \mod h^{p+1} \equiv \delta_0 + uh^p + [v,\delta]h^p \mod h^{p+1}.$$

It is enough to show that u is $[\delta_0, -]$-exact modulo h.

The cohomological vanishing $\Theta \equiv 0 \mod h^p$ gives

$$(\iota_p)_*\partial_h\delta = [(\iota_p)_*\delta, (\iota_p)_*T] \equiv [\delta_0 + uh^p, (\iota_p)_*T] = [\delta_0, (\iota_p)_*T] \mod h^p$$

for some coderivation T. On the other hand the lemma says that

$$(\iota_p)_*\partial_h\delta = \partial_h((\iota_p)_*\delta) + [(\iota_p)_*\delta, X] \equiv \partial_h(\delta_0 + uh^p) + [(\iota_p)_*\delta, X] \mod h^p$$

for some coderivation X. Thus

$$u \cdot ph^{p-1} = [\delta_0, (\iota_p)_*T - X] \mod h^p,$$

which clearly implies that u is $[\delta_0, -]$-exact $\mod h$. □

Remark 6.0.6. In order to establish an equivivalence $\bar{A} \to A^0[h]$ over $\operatorname{Spec} k[h]/h^p$ starting from $\Theta = 0 \mod h^{p-1}$ it is enough to $(p-1)! \in k$ be invertible.

7 Classical Kaledin's class revisted

Here we present an easy and a natural way to show basic main properties of the Kaledin's class. The only thing we will need is the fact that flat deformations of an object A^0 over Artinian rings are determined by dg-Lie algebra $(\mathcal{L}, \delta)$. For example in case of associative algebras one can take $\mathcal{L} = \operatorname{Coder}(\bar{T}(A^0[1]))$.

Assume $(\mathcal{L}, \delta)$ is a dg-Lie algebra over k. Given $\mu \in \operatorname{MC}(\mathcal{L} \otimes_k R) \subset \mathcal{L}^1 \otimes R$ by definition determines new dg-Lie algebra $(\mathcal{L} \otimes_k R, \delta + [\mu, -])$. We require μ to be zero modulo the maximal ideal of R. Let $R = k[t]/t^p$ and put $\delta_t := \delta^\mu$ and $\delta_0 := \delta$. The equality $\partial_t[\delta_t, \delta_t] = 2[\partial_t\delta_t, \delta_t]$ justifies

Definition 7.0.1. The class

$$[\partial_t\delta_t] \in H^1(\mathcal{L} \otimes k[t]/t^{p-1}, \delta_t)$$

is called Kaledin's class.

The rest of the section is devoted to a proof of one of the main results concerning Kaledin's classes.

Theorem 7.0.2. *If $[\partial_t\delta_t] = 0$, then there is an isomorphism of dg-Lie algebras*

$$(\mathcal{L} \otimes k[t]/t^p, \delta_t) \simeq (\mathcal{L} \otimes k[t]/t^p, \delta_0)$$

over $k[t]/t^p$ which is identity modulo t.

For arbitrary nilpotent Lie algebra L over R with $char = 0$, denote by $\exp(L)$ the corresponding group of exponents. Recall that $\exp(L)$ is an algebraic group over R with a product defined by the BHC formula and as a scheme is canonically identified with L via the exponent.

Suppose L is a Lie algebra over k. The following is an algebraic version of an obvious statement in differential gemetry.

Lemma 7.0.3. *For any given path $v \in L \otimes k[t]/t^{p-1}$ there is $g \in \exp(L \otimes tk[t]/t^p)$ such that $g = \mathrm{id} \mod t$ and $g^{-1}\partial_t g = v \in L \otimes k[t]/t^{p-1}$.*

Proof. Recall that $\exp(-x)\frac{d}{dt}\exp(x) = \operatorname{Td}^{-1}(\operatorname{ad}_x).\partial_t x$, where $\operatorname{Td}(-)$ is the Todd class. Under the substitution $g = \exp(x)$ for some $x \in L \otimes tk[t]/t^p$, the equation $g^{-1}\partial_t g = v$ is equivalent to a solution

$$\operatorname{Td}(\operatorname{ad}_x).\partial_t x = v \in L \otimes k[t]/t^{p-1} \tag{7.0.4}$$

for the given v.

Now, sending x to $\mathrm{Td}(\mathrm{ad}_x).\partial_t x$ is a map

$$\Phi \colon L \otimes tk[t]/t^p \to L \otimes k[t]/t^{p-1}$$

of sets. Since $\mathrm{Td}(t) = 1 \mod t$, Φ maps $L \otimes t^a k[t]/t^p$ to $L \otimes t^{a-1} k[t]/t^{p-1}$. Moreover the induced map on the associated quotients of sets

$$\mathrm{Gr}_a \Phi \colon L \otimes t^a k[t]/(t^p, t^{a+1}) \to L \otimes t^{a-1} k[t]/(t^{p-1}, t^a)$$

is the multiplication by at^{-1} in the obvious sense. It follows that Φ is a bijection and 7.0.4 admits a unique solution. □

Remark 7.0.5. It is interesting to have an explicit solution to the previous equation on x in terms of iterated integrals with $v, \partial_t v, \partial_t^2 v$,etc. We were unable to find such.

proof of Theorem 7.0.2. A trivialization of the family is equivalent to an existence of

$$g \in \mathrm{End}_{k[t]/t^p}(\mathcal{L} \otimes k[t]/t^p)$$

which (1) respects the Lie bracket,(2) $g \circ \delta_t = \delta_0 \circ g \mod t^p$ and (3) $g = \mathrm{id} \mod t$. If the latter holds, then by taking derivative, the former is equivalent to

$$\partial_t g \circ \delta_t + g \circ \partial_t \delta_t = \delta_0 \circ \partial_t g \mod t^{p-1},$$

or

$$\partial_t \delta_t = [\delta_t, g^{-1} \circ \partial_t g] \tag{7.0.6}$$

in $\mathrm{End}_k(\mathcal{L}) \otimes k[t]/t^{p-1}$.

Now, by our assumption $\partial_t \delta_t = \delta_t(v)$ for some $v \in \mathcal{L}^0 \otimes k[t]/t^{p-1}$. By the lemma above there is unique $g \in \exp(\mathcal{L}^0 \otimes tk[t]/t^p)$ such that $g = \mathrm{id} \mod t$ and $g^{-1}\partial_t g = v \in \mathcal{L}^0 \otimes k[t]/t^{p-1}$. It remains to note, that $g = \exp(x), x \in \mathcal{L}^0 \otimes tk[t]/t^p$ acts on $\mathcal{L} \otimes k[t]/t^p$ by $\exp(\mathrm{ad}_x)$, hence the image of g in $\mathrm{Aut}_{k[t]/t^p}(\mathcal{L} \otimes k[t]/t^p)$ acts with respect to the Lie bracket. This finishes the proof. □

7.1 Arbitrary base

Note that a generalization of Lemma 7.0.3 to an arbitrary base other then $S = \operatorname{Spec} k[t]/t^p$ usually doesn't work. Namely let G be a Lie group and $\mathfrak{g}$ is the Lie algebra. Given $v \in \Omega^1_S \otimes \mathfrak{g}$ we want to solve equation $g^{-1} dg = v$ for some $g \colon S \to G$. More precisely consider the Mauer-Cartan form $\nu \in \Omega^1_G \otimes \mathfrak{g}$ defined by $\nu_h(h.a) = a, a \in \mathfrak{g}$ at any given point $h \in G$. In our notation $g^{-1} dg = g^*(\nu)$, so the required equation is $g^*(\nu) = v$.

Recall that ν is left invariant and $d\nu(g.v, g.w) = [v, w]$.

Proposition 7.1.1. *Given $v \in \Omega^1_S \otimes \mathfrak{g}$, the equation $g^*(\nu) = v$ admits a solution in $g \colon S \to G$ iff $dv = \frac{1}{2}[v, v]$.*

Proof. The integrability condition comes from the following picture. Let $p \colon S \times G \to S$ be the trivial G-bundle. Consider ν as a vertical 1-form, then $\eta := \nu - p^* v \in \Omega^1_S \otimes \mathfrak{g}$ is a G-equivariant connection. Any $g \colon S \to G$ defines a section s of p, we have $s^*\eta = s^*(\nu - p^*v) = g^*(\nu) - v$. In other words $g^*(\nu) = v$ is equivalent to flatness of s. It is well-known that principal G-bundle admits flat section iff it is flat, i.e. $\omega = d\eta + \frac{1}{2}[\eta, \eta]$ vanishes. Since ω is a pullback from S it can be computed in terms of restriction to the constant section $S \times e$, we have $\omega|_e = -dv + \frac{1}{2}[v, v]$. □

For example if $S = \operatorname{Spec} k[t_1, t_2]]$ is the 2-dimensional formal disk, then by trying to prove Theorem 7.0.2 amounts to solution of equations $g^{-1}\partial_i g = v_i$ for some v_i satisfying $\partial_i \mu = \delta_\mu(v_i)$. Then setting $v := v_1 dt_1 + v_2 dt_2$ the integrability condition from Proposition 7.1.1 takes a form

$$\partial_1 v_2 - \partial_2 v_1 = [v_1, v_2].$$

7.2 De Rham's stack

A homotopical approximation to the above is given by the following approximation of $\operatorname{Spec} k$. Let Z be a formal scheme over k. Define $Z^{dR,k} := \operatorname{Spec}(\Omega_Z^{\leq k}, d_{dR})$. So $Z^{dR,0} = Z$ and $Z^{dR,\infty} = Z^{dR}$. Moreover, since Z is formal we have $Z^{dR} \sim \operatorname{Spec} k$!

Lemma 7.2.1. *Given a morphism $f\colon Z \to \mathcal{M}$:*

1. *An extension of f to $f^1\colon Z^{dR,1} \to \mathcal{M}$ is controlled by $\mu_1 \in \Omega_Z^1 \otimes \mathcal{L}^0$ such that*

$$d_{dR}\mu_f + \delta_f(\mu_1).$$

2. *An extension of f^1 to $f^2\colon Z^{dR,2} \to \mathcal{M}$ is controlled by $\mu_2 \in \Omega_Z^2 \otimes \mathcal{L}^{-1}$ such that*

$$\delta_f(\mu_2) + d\mu_1 + \frac{1}{2}[\mu_1, \mu_1] = 0.$$

3. *etc.*

Proof. □

8 Cotangent complex in $char = 0$

Let k be a commutative ring containing $\mathbb{Q}$. Assume $B \to C$ is a morphism of cdga over k. We will treat everything in derived setting. For example $\operatorname{Spec} C \xrightarrow{f} \operatorname{Spec} B$ is the corresponding morphism of stacks/derived schemes etc.

8.1 Basic properties of $\mathbb{L}$ and $\mathbb{T}$

Let us postulate that in general there is a natural (in derived sense) $\mathcal{O}_X$-module $\mathbb{L}(X \xrightarrow{f} Y)$ called *the cotangent complex of f*. By the definition *the tangent complex* $\mathbb{T}(f) = \mathbb{L}(f)^\vee$ is its $\mathcal{O}_X$-dual.

Below we list some basic properties:

1. Base change:

   The pullback square:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd} \tag{8.1.1}$$

   gives a natural equivalence:

$$p_2^*\mathbb{L}(g) \simeq \mathbb{L}(p_1).$$

2. Smooth morphism $f\colon X \twoheadrightarrow Y$:

$$\mathbb{L}(f) \simeq \Omega^1_{X/Y}.$$

3. Locally complete intersection $f\colon X \hookrightarrow Y$:

$$\mathbb{L}(X/Y) \simeq I_{X/Y}/I^2_{X/Y}[1],$$

   where $I_{X/Y}$ is the sheaf of ideals defining $X \subset Y$.

4. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ ($X/Z \xrightarrow{f} Y/Z$ in short) we have a triangles of $\mathcal{O}_X$-modules:

$$\mathbb{T}(X/Y) \to \mathbb{T}(X/Z) \xrightarrow{df} f^*\mathbb{T}(Y/Z)$$

   and

$$f^*\mathbb{L}(Y/Z) \xrightarrow{df} \mathbb{L}(X/Z) \to \mathbb{L}(X/Y).$$

We put $\mathbb{T}_X := \mathbb{T}(X \to \operatorname{Spec} k)$.

The last property applied to $X \xrightarrow{f} Y \to \operatorname{Spec} k$ gives a triangle

$$\mathbb{T}(X/Y) \to \mathbb{T}_X \xrightarrow{df} f^*\mathbb{T}_Y \tag{8.1.2}$$

in particular it relates the relative tangent complex $\mathbb{T}(X/Y)$ to absolute ones $\mathbb{T}_X$ and $f^*\mathbb{T}_Y$.

8.2 $\mathbb{L}_{\operatorname{Sym} V,\delta}$

Here we indicate an operational receipt to compute $\mathbb{L}(k \to B)$ for a cdga B. Take a cdga B/k and a B-module M.

Definition 8.2.1. The B-module of derivatives $\operatorname{Der}^i_k(B, M)$ of degree i is given by maps $\partial\colon B \to M[i]$ such that $\partial(b_1b_2) = \partial(b_1)b_2 + (-1)^{i\cdot|b_1|}b_1\partial(b_2)$.

Definition 8.2.2. The module of Kähler differentials Ω^1_B is a B-module characterized by the following natural equivalence of B-modules:

$$\operatorname{Hom}_B(\Omega^1_B, M) = \operatorname{Der}^*_k(B, M),$$

for all B-modules M.

As usual one can identify Ω^1_B with B-module I_Δ/I^2_Δ, where $I_\Delta \subset B\otimes B$ is the ideal of the diagonal. The natural map $B \xrightarrow{d_{dR}} \Omega^1_B$ commutes with the differential on B and is given by $d_{dR}(b) = b\otimes 1 - 1\otimes b \in I_\Delta/I^2_\Delta$.

Now assume B is semi-free, i.e. $B = (\operatorname{Sym}(V), \delta)$, where V is some graded free module over k. Let $e_i \in V$ be a basis.

Proposition 8.2.3. *We have*

$$\Omega^1_{(\operatorname{Sym}(V),\delta)} \simeq I_\Delta/I^2_\Delta \simeq (\operatorname{Sym}(V) \otimes V, \delta + \delta'),$$

where $I_\Delta \subset \operatorname{Sym}(V) \otimes \operatorname{Sym}(V)$ *is the ideal of the diagonal, and*

$$\delta'(1 \otimes v) = d_{dR}(\delta(v)) := \sum_i \partial_{e_i}(\delta(v)) \otimes e_i, v \in V.$$

Proof. The first isomorphism is classical and works for all cdga's. The second one

$$\operatorname{Sym}(V) \otimes V \to I_\Delta/I^2_\Delta$$

is given by sending $f d_{dR}v := f \otimes v$ to $f(v \otimes 1 - 1 \otimes v)$. The rest is straightforward. □

Assume $\tilde{B} \xrightarrow{\sim} B$ is a cofibrant replacement, i.e. a semi-free model $\tilde{B} = (\operatorname{Sym} V, \delta)$ of B.

Corollary 8.2.4. *Given* $f\colon \operatorname{Spec}(R, \delta_R) \to \operatorname{Spec} B \simeq \operatorname{Spec}(\operatorname{Sym} V, \delta)$, *one has a natural equivalence:*

$$f^*\mathbb{L}_{\operatorname{Spec} B} \simeq (R \otimes V, \delta_R + \delta_f),$$

where

$$\delta_f(1 \otimes v) = f^*(\partial_{e_i}\delta(v)) \otimes e_i.$$

The natural morphism $df\colon f^*\mathbb{L}_{\operatorname{Sym} V,\delta} \to \mathbb{L}_{R,\delta_R}$ *is determined by:*

$$df(1 \otimes V) = d_{dR}(f(v)).$$

8.3 $D_p := \operatorname{Spec} k[t]/t^p$

A cofibrant resolution of the truncated formal disk $D_p = \operatorname{Spec} k[t]/t^p$ has a form $\operatorname{Spec}(k[t, \tau], \delta)$, where $\deg(\tau) = -1$ and $\delta(\tau) = t^p$ is the Koszul differential. From this we have $\mathbb{L}_{D_p} \simeq (k[t, \tau] \otimes \langle d_{dR}t, d_{dR}\tau\rangle, \delta)$, where δ is such that $\delta d_{dR}\tau = -d_{dR}\delta(\tau) = -pt^{p-1}d_{dR}t$. The following quasi-isomorphic model is also useful

$$\mathbb{L}_{D_p} \sim \left[k[t]/t^p d_{dR}\tau \xrightarrow{d_{dR}\tau \to -pt^{p-1}d_{dR}t} k[t]/t^p dt\right],$$

where $\deg(d_{dR}t) = 0$ and $\deg(d_{dR}\tau) = -1$.

8.4 Application to formal moduli stacks

As the main application we formulate the dual version of the above statement. Assume $\mathcal{L}$ is an L_∞-algebra and $\operatorname{Sym}\mathcal{L}^\vee[-1]$ is the corresponding Chevalley-Eilenberg complex. Thus

$$\mathcal{M}^{\mathcal{L}} \simeq \operatorname{Spec}\mathcal{L}^\vee[-1]$$

is the formal moduli stack associated with the Lie algebra $\mathcal{L}$. Recall that

$$\{f\colon \operatorname{Spec}(R,\delta_R)\to\mathcal{M}^{\mathcal{L}}\} = \mathrm{MC}(R\otimes\mathcal{L},\delta_R+\delta_{\mathcal{L}}).$$

We denote the Mauer-Cartan element corresponding to f by $\mu_f\in\mathrm{MC}(R\otimes\mathcal{L})$.

Theorem 8.4.1. *Given $f\colon \operatorname{Spec}(R,\delta_R)\to\mathcal{M}^{\mathcal{L}}$ one has a natural equivalence:*

$$f^*\mathbb{T}_{\mathcal{M}^{\mathcal{L}}} \simeq (R\otimes\mathcal{L}[1],\delta_f[1]),$$

where δ_f is the differential $\delta_R+\delta_{\mathcal{L}}$ twisted by $\mu_f\in\mathrm{MC}(R\otimes\mathcal{L})$:

$$\delta_f = \delta_R+\delta_{\mathcal{L}}+[\mu_f,-].$$

Proof. Dualize Corollary 8.2.4 for $V=\mathcal{L}^\vee[-1]$. ◻

From $\operatorname{Spec}R\xrightarrow{f}\mathcal{M}^{\mathcal{L}}\to\operatorname{Spec}k$ we obtain a triangle

$$\mathbb{T}(f)\to\mathbb{T}_{\operatorname{Spec}R,\delta_R}\to(R\otimes\mathcal{L},\delta_f)[1]=f^*\mathbb{T}_{\mathcal{M}^{\mathcal{L}}}.$$

Precomposition with the natural morphism $T_{\operatorname{Spec}R,\delta_R}\to\mathbb{T}_{\operatorname{Spec}R,\delta_R}$ gives $T_{\operatorname{Spec}R,\delta_R}\to(R\otimes\mathcal{L},\delta_f)[1]$.

Proposition 8.4.2. *The map $F\colon T_{\operatorname{Spec}R,\delta_R}\to(R\otimes\mathcal{L},\delta_f)[1]$ is given by the formula:*

$$\partial\to\partial(\mu_f),$$

where $\mu_f\in R\otimes\mathcal{L}[1]$ and $\partial\in\operatorname{Der}_k((R,\delta_R),(R,\delta_R))=T_{\operatorname{Spec}R}$ acts by the derivation on the first term.

Proof. ◻

8.5 Case $D_p\to\mathcal{M}$

Recall $D_p=\operatorname{Spec}k[t]/t^p$. The morphism $f\colon D_p\to\mathcal{M}$ amounts to a Mauerer-Cartan element $\mu\in\mathrm{MC}(k[t]/t^p\otimes\mathcal{L},\delta_{\mathcal{L}})$. The equivalence $D_p\xrightarrow{\sim}\operatorname{Spec}(k[t,\tau],\delta_p)$ sugests an extension $\operatorname{Spec}(k[t,\tau],\delta_p)\to\mathcal{M}$. One can describe it as a lifting of μ to an element $\tilde\mu-\tau\cdot\nu\in(k[t,\tau]\otimes\mathcal{L})^1$. Here $\tilde\mu$ is an arbitrary extension of μ to an element in $k[t]\otimes\mathcal{L}^1$, while $\nu\in\tau\cdot\mathcal{L}^2$ is the unique element such that

$$t^p\nu=\delta_{\mathcal{L}}\tilde\mu+\frac{1}{2}[\tilde\mu,\tilde\mu].$$

Remark 8.5.1. Note that $\nu \mod t^p\in(k[t]/t^p\otimes L[2],\delta_\mu)$ is the obstruction for an extension of μ to $k[t]/t^{2p}$. Equivalently, it is an obstruction for the zero-extension $D_p\to\mathcal{M}$ to $D_{2p}\to\mathcal{M}$.

Lemma 8.5.2. *One has*

$$\tilde\mu-\tau\cdot\nu\in\mathrm{MC}(k[t,\tau]\otimes\mathcal{L},\delta_{\mathcal{L}}+\delta_p).$$

Proof. For the twisted differential $\delta_{\tilde\mu}:=\delta_{\mathcal{L}}+[\tilde\mu,-]$, notice that $[\delta_{\tilde\mu},\delta_{\tilde\mu}]$ is divided by t^p and then use the fact that $[\delta_{\tilde\mu},[\delta_{\tilde\mu},\delta_{\tilde\mu}]]=0$. ◻

Recall that $f\colon D_p\to\mathcal{M}$ induces $df\colon\mathbb{T}_{D_p}\to f^*\mathbb{T}_{\mathcal{M}}$.

Corollary 8.5.3. *The morphism*

$$k[t,\tau]\otimes\langle\partial_t,\partial_\tau\rangle\xrightarrow{d\tilde f}(k[t,\tau]\otimes\mathcal{L}[1],\delta_p+\delta_{\mathcal{L}}),$$

is given by

$$d\tilde f(\partial_i)=\partial_i(\tilde\mu-\tau\cdot\nu).$$

Thus $d\tilde f(\partial_\tau)=-\nu$ and $d\tilde f(\partial_t)=\partial_t\tilde\mu+\tau\partial_t\nu$.

In particular the morphism

$$\left[k[t]/t^p\partial_t \xrightarrow{-pt^{p-1}} k[t]/t^p\partial_\tau\right] \xrightarrow{d\tilde{f}} (k[t]/t^p \otimes \mathcal{L}[1], \delta_\mu)$$

is given by

$$d\tilde{f}(\partial_t) = \partial_i\tilde{\mu} \mod t^p \tag{8.5.4}$$

$$d\tilde{f}(\partial_\tau) = -\nu \mod t^p \tag{8.5.5}$$

8.6 Back to the Kaledin class

Let $R = k[t]] = \varinjlim_n \operatorname{Spec} k[t]/t^n$ and $\mathcal{L} = C^*_{HH}(A^0/k)[1]$. The morphism $f\colon \operatorname{Spec} R \to \mathcal{M}^{\mathcal{L}}$ amounts to a flat family $\bar{A}/k[t]$. A splitting $A^0 \to \bar{A}$ gives an isomorphism of $k[t]$-modules $A^0[t] \simeq \bar{A}$ and hence $(k[t]/t^n \otimes \mathcal{L}[1], \delta_f) \simeq f^*\mathbb{T}_{\mathcal{M}^{\mathcal{L}}}$. Recall that

$$\Theta_{\bar{A}/A^0} = \partial_t \mu_f$$

for the constant vector field $\partial_t \in T_{\operatorname{Spec} R}$, and by the previous theorem we get

$$\Theta_{\bar{A}/A^0} = F(\delta_t) \in H^0(R \otimes \mathcal{L}[1], \delta_f) = HH^2(k[t]] \otimes A^0, \mu_f).$$

In truncated case $k[t]/t^p$ the class Θ can recovered as follows. Consider the embedding of truncated disks $i\colon D_{p-1} \to D_p$. Then $i^*(\mathbb{T}_{D_p} \xrightarrow{df} f^*\mathbb{T}_{\mathcal{M}})$ by (8.5.4) takes a form

$$\left[k[t]/t^{p-1}\partial_t \xrightarrow{0} k[t]/t^p\partial_\tau\right] \xrightarrow{i^*(df)} (k[t]/t^{p-1} \otimes \mathcal{L}[1], \delta_{\mu \mod t^{p-1}}),$$

and is given by:

$$i^*(df).\partial_t = \partial_t\mu \tag{8.6.1}$$

$$i^*(df).\partial_\tau = -\nu \mod t^{p-1}. \tag{8.6.2}$$

9 Obstructions for equivalence of two morphisms

Let $Z \to \tilde{Z}$ be a square-zero extension with an ideal I, assume $Z = \operatorname{Spec}(R, \delta)$. Assume we have morphisms $f_i\colon \tilde{Z} \to \mathcal{M} = \operatorname{Spec}(\operatorname{Sym} V, \delta), i = 1, 2$, let $\bar{f}_i\colon Z \to \mathcal{M}$ be the restrictions. Assume $\bar{f}_1 \sim \bar{f}_2$ we want to know obstructions for an existence of equivalence $f_1 \sim f_2$.

Explicitely the equivalence $\bar{f}_1 \sim_{\bar{H}} \bar{f}_2$ amounts to a morphism of cdga's:

$$\bar{H}\colon (\operatorname{Sym} V, \delta) \to (R[t, dt], \delta + d).$$

Theorem 9.0.1. *The full obstruction to a lifting of $\bar{H}$ to H lives in*

$$\operatorname{Der}((\operatorname{Sym} V, \delta), I[t, dt]^\circ[1]) \simeq \mathcal{H}om(f^*\mathbb{L}_\mu, I[t, dt]^\circ[1]) \simeq H^1(I \otimes \mathcal{L}, \delta_f),$$

where $I[t, dt]^\circ = (I[t, dt])^{ev_0 = ev_1} \sim I[-1]$.

Proof. It is straightforward. ☐