Research Statement

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1 Introduction

My research primarily focuses on problems arising in algebraic topology, homotopical algebra, complex and algebraic geometry. The common tools in my practice usualy come from homological algebra, Hodge theory, operads and differential geometry. In the next section I will summarize the content of my work in [Zak24],[KSZ23] and then turn to the future directions.

2 Previous work

2.1 Rational Homotopy Type of Complements of Submanifold Arrangements

In [Zak24] I study the topology of complements $U := X - \cup_i Z_i$ in terms of the given space X, the arrangement $\{Z_i\}$ and its various intersections $Z_I = \cap_{i\in I} Z_i$. The main results assume that X is proper algebraic over $\mathbb{C}$ and the reduced schemes ${Z_I}^{red}$ are smooth. In this case the work provides an explicit model for the rational homotopy type of U in the sense of Sullivan, by means of a natural commutative dg-algebra formed by terms $H^*(X, X - Z_I)$ together with natural morphisms induced by inclusions $Z_I \to Z_J, I \supset J$. The default coefficients ring is $\mathbb{Q}$.

2.2 Context

One of the earliest general results in the subject is the identification of the cohomology of the complements of complex hyperplane arrangements $U = \mathbb{C}^n - \cup_i L_i$ with the Orlik-Solomon algebra [OS80]. For example in the case of the configuration space $U := F(\mathbb{C}, n) = \mathbb{C}^n - \cup_{ij}\Delta_{ij}$ of points in $\mathbb{C}$, it is given by the dg-algebra generated by the forms $d\ln(z_i - z_j)$ which satisfy Arnold's relations. In general we get an explicit quasi-isomorphism of the Orlik-Solomon algebra with the de Rham complex of the complement U.

Deligne [Del71], using the notion of mixed Hodge complexes, showed that if X is a proper algebraic vareity over $\mathbb{C}$ and $Z_i \subset X$ form a divisor with normal crossings, then the Leray spectral sequence of the inclusion $X - \cup_i Z_i \hookrightarrow X$ degenerates after the E_2 term. In particular the cohomology of U coincides with the cohomology of the complex (E_2^{**}, d_2). The latter in turn has a very simple expression in terms of $H^*(Z_I)$ for the various intersections $Z_I := \cap_{i\in I} Z_i$ and Gysin's maps $H^*(Z_I) \to H^*(Z_J)$ for $I \supset J$.

Soon after, Morgan [Mor78] developed a multiplicative analog of mixed Hodge complexes, called mixed Hodge diagrams. He showed that in fact (E_2^{**}, d_2) is a commutative dg-algebra which models U in the sense of Sullivan's rational homotopy theory. In the simplest case, taking the empty arrangement, this recovers the more classical result stating that a smooth projective variety X is Kähler and hence formal, i.e. $H^*(X; \mathbb{C})$ is quasi-isomorphic to the de Rham algebra of X.

The primary obstacle to applying Morgan's result in practice is the assumption that $\{Z_i\}$ should be a divisor with normal crossings. Of course any complement U can be realized in this way via Hironaka's resolution of singularities. In general, however, such a resolution is rarely accessible.

There are important cases in which a nice compactification of U is available. For instance, consider the configuration spaces $U = F(Y, n)$ of n-tuples of points in Y. Kriz [Kri94] used the Fulton-MacPherson compactification of $F(Y, n)$ and by applying Morgan's theory expressed the rational homotopy type of $F(Y, n)$ in terms of $H^*(Y)$. The result, called the Kriz-Totaro[Kri94][Tot96] model, is given by the Leray spectral sequence of the inclusion $F(Y, n) \hookrightarrow Y^n$.

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In general the relation between a nice compactification of $U = X - \cup_i Z_i$ and the given one $U \hookrightarrow X$ is usually obscure. Ideally we still want to express everything in terms of the given arrangement $\{Z_i\}$. If Z_i are hypersurfaces with hyperplane-like intersection this was done by Dupont [Dup15], who extended the notion of the log-de Rham complex to this case. This immediately extends Morgan's result to this case.

To the best of my knowledge, no general results in case $\operatorname{codim} Z_i/X \geq 2$ case were known.

2.3 Results

There is a Mayer-Vietoris spectral sequence with the page E_1 given by $H^*_{Z_I}(X) := H^*(X, X - Z_I)$, and the differential d_1 induced by the inclusions of the various intersections Z_I. In case $\{Z_i\}$ is a divisor with normal crossings, the décalage of this spectral sequence recovers the corresponding Leray spectral sequence.

Assuming algebraicity and set-theoretical smoothness of the intersections Z_I, the Mayer-Vietoris spectral sequence degenerates by considering the weights, and thus one can expect an isomorphism $H^*(E_1^{**}, d_1) \simeq H^*(U)$. Note that we omit the requirement on Z_i to form a divisor.

Observe that the spectral sequence is multiplicative. It is natural to ask if the commutative dg-algebra (E_1^{**}, d_1) models the rational homotopy type of U in general.

Assume $\{Z_i \subset X\}$ is a proper algebraic arrangement such that the set-theoretical intersections $Z_I := \cap_{i \in I} Z_i$ for all multi-indices I are smooth. Let $U = X - \cup_i Z_i$. The main result is the following.

Theorem 2.3.1 ([Zak24]). *There is a multiplicative spectral sequence* $E_1^{pq} = \bigoplus_{|I|=-p} H^q_{Z_I}(X)$ *such that*

1. *We have* $E_2^{pq} = E_\infty^{pq}$*;*
2. *There is a zig-zag of quasi-isomorphisms of cdga's* $(E_1^{**}, d_1) \overset{\sim}{\leftrightsquigarrow} A^*_{PL}(U)$*, i.e.* (E_1^{**}, d_1) *models the rational homotopy type of* U *in the sense of Sullivan.*

To my knowledge, previous results of this type usually require studying a nice compactification of U, e.g. the Fulton-MacPherson compactification for configuration spaces in [Kri94], and hyperplane-like intersections in [Dup15],[Mor78].

Let me mention two additional advantages of the above construction. First, it is possible to omit the assumption that the intersection be space-like. For example the arrangement consisting of a conic and a tangent line in $\mathbb{P}^2$ satisfies the hypotheses of the theorem.

Second, by using Cirici-Horel's [CH20] functorial approach to the E_1-formality of mixed Hodge diagrams over $\mathbb{Q}$, and the naturality of the above construction, we obtain a functorial zig-zag of quasi-isomorphisms $E_1 \sim A^*_{PL}(U)$. In particular, if the arrangement admits an action of a group G, then the above zig-zag of quasi-isomorphisms is G-equivariant.

In the course of the work I developed necessary tools to handle homological constructions with respect to the combinatorics of the intersections. For example when the intersection poset Z_I is a geometric lattice, we get a much smaller model quasi-isomorphic to the one stated in the main theorem. It relies on the Orlik-Solomon algebra of the lattice. In the case of configuration spaces it recovers the Kriz-Totaro model. Below we explicitly describe the result for chromatic configuration spaces.

2.4 Applications

Let me mention two applications of the main result.

2.4.1 Chromatic configuration spaces

For any finite unoriented graph G one can consider the *chromatic configuration space* $F(M, G)$ of tuples of points $x_i \in M$ labeled by the vertices $i \in V(G)$ with the condition $x_i \neq x_j$ for each edge $\{ij\} \in E(G)$. Let $\Delta_{ab} \subset M^n$ be the diagonal $x_a = x_b$ and $[\Delta_{ab}] \in H^{2 \dim M}(M^n)$ the corresponding cohomology class.

Remark 2.4.1. The word "chromatic" is justified by the fact that $\chi(F(M, G)) = p_G(\chi(M))$ where p_G is the chromatic polynomial of G and χ is any motivic measure on the Grothendieck ring of varieties.

Assume M is a smooth proper algebraic variety over $\mathbb{C}$. The following theorem is a generalization of the Kriz-Totaro model:

Theorem 2.4.2 ([Zak24]). *The rational homotopy type of $F(M,G)$ has a model equal to the cdga over $H^*(M^{V(G)})$ given by the quotient*

$$H^*(M^{V(G)}) \otimes \Lambda^*\langle\tilde{\Delta}_{ab}\rangle/\mathcal{J}$$

where $\langle\tilde{\Delta}_{ab}\rangle$ is the $\mathbb{Q}$-vector space spanned by the generators $\tilde{\Delta}_{ab}$ of degree $2\dim_{\mathbb{C}} M - 1$ for all ordered *pairs $(a,b) \in E(G)$, and $\mathcal{J}$ is the ideal generated by the following relations*

1. $\tilde{\Delta}_{ab} = \tilde{\Delta}_{ba}$;

2. $\partial(\tilde{\Delta}_{i_1 i_2} \wedge \ldots \wedge \tilde{\Delta}_{i_{k-1} i_k}) = 0$ *for each simple cycle* $(i_1,i_2),\ldots,(i_{k-1},i_k),(i_k,i_1) \in E(G)$ *in* G;

3. $p_a^*(\gamma) \cdot \tilde{\Delta}_{ab} = p_b^*(\gamma) \cdot \tilde{\Delta}_{ab}$ *for all* $\gamma \in H^*(M)$ *and* $(a,b) \in E(G)$.

The differential d is zero on $H^*(M^{V(G)})$, and we have $d\tilde{\Delta}_{ab} = [\Delta_{ab}] \in H^{2\dim M}(M^{V(G)})$ where $[\Delta_{ab}]$ is the Poincaré dual class of the diagonal $[\Delta_{ab}]$.

We define an action of $Aut(G)$ on our model by setting $\sigma.\tilde{\Delta}_{ab} = \tilde{\Delta}_{\sigma(a)\sigma(b)}, \sigma \in Aut(G)$ and using the permutation action on $M^{V(G)}$. With this definition, the model is $Aut(G)$-equivariant in the sense that there is a zig-zag of $Aut(G)$-equivariant quasi-isomorphisms connecting the model and $A_{PL}(F(M,G))$.

2.4.2 Formality of locally geometric arrangements of affine subspaces

The next application is about an arrangement $L_x \subset \mathbb{C}^n$ of affine subspaces. Let $(L,\leq)$ be the corresponding intersection poset and $U = \mathbb{C}^n - \bigcup_{x>0} L_x$ be the complement. Assume that for all $p \in \mathbb{C}^n$ the lattice $\{x \in L \mid L_x \ni p\} \subset L$ is geometric. We have

Theorem 2.4.3 ([Zak24]). *The complement U is formal.*

Remark 2.4.4. In the case of a central arrangement, i.e. when L is a geometric lattice, the theorem was proved by Feichtner-Yuzvinsky in [FY07]. Note that an arrangement given by affine hyperplanes is always locally geometric; this case was covered by Dupont [Dup16].

2.5 Derived binomial rings I: integral Betti cohomology of log schemes

In this joint work with Dmitry Kubrak and Georgii Shuklin we study derived binomial algebras. Classical binomial algebras are commutative algebras over $\mathbb{Z}$ equipped with operations $\binom{-}{n}$ which satisfy the usual binomial identities. The free binomial algebra $\mathrm{Bin}(V)$ on a free finitely generated $\mathbb{Z}$-module V is the subset of maps in $\mathrm{Maps}_{\mathrm{Sets}}(V^\vee,\mathbb{Z})$ which are given by polynomials with rational coefficients.

In particular any algebra of the form $\mathrm{Maps}_{\mathrm{Sets}}(S,\mathbb{Z})$ is binomial. More generally for any topological space X, the integer cochains $C^*_{sing}(X;\mathbb{Z})$ can be viewed as a cosimplicial binomial ring. In fact, the cochains of an Eilenberg-MacLane space $C^*_{sing}(K(A,n);\mathbb{Z})$ are equivalent to $\mathrm{Bin}(A[n]^\vee)$, which is the free binomial algebra generated by the complex $A[n]^\vee = \mathcal{H}om(A[n],\mathbb{Z})$ in the derived sense by means of a projective resolution of $A[n]$ followed by the Dold-Kan correspondence and term-wise application of $\mathrm{Bin}(-)$.

In our work we defined derived binomial algebras for all complexes, not neccessary coconnective. Then we identified the cohomology of all free derived binomial algebras. We introduced a relative version of binomial algebras, and as an application we proved an integer version of Steenbrink's formula for the Betti cohomology of a sufficiently nice log-scheme.

2.6 Context

The relation of binomial algebras to topology goes back to the observation that the inclusion of the "algebraic" cochains in all cochains $\mathrm{Bin}(\mathbb{Z}[-n]) \hookrightarrow \mathrm{Maps}(\mathbb{Z}[n],\mathbb{Z})$ is a quasi-isomorphism. Here $\mathbb{Z}[n] = \mathbb{Z}[-n]^\vee$, concentrated in homological degree n, is considered as a simplicial abelian group via the Dold-Kan correspondence. In particular the geometric realization $|\mathbb{Z}[n]|$ is homotopy equivalent to $K(\mathbb{Z},n)$. This observation lead Ekedahl [Eke02] to the discovery of minimal models of p-local homotopy types. Namely, the Postnikov tower $X \to X_n$ of a simply-connected space X corresponds to a cotower of binomial algebras B_n such that $B_{n+1} = B_n \otimes_\tau \mathrm{Bin}(\pi_{n+1}(X)[n+1]^\vee)$. Here $- \otimes_\tau -$ denotes a twisted tensor product in the sense of May. In more detail, B_{n+1} is the homotopy colimit of the diagram of cosimplicial binomial algebras

$$B_n \leftarrow \mathrm{Bin}(\pi_{n+1}(X)[n+2]^\vee) \to \mathbb{Z}[0].$$

It turns out that working over $\mathbb{Z}_{(p)}$ and representing $\pi_{n+1}(X)[n+1]^\vee$ by a minimal free resolution we get a binomial algebra $B_\infty = \varinjlim_n B_n$ which is minimal in the sense of [Eke02]. Note that any quasi-isomorphism of any two simply-connected Ekedahl minimal binomial algebras is in fact an isomorphism (ibid.).

Let me note that one can think of the binomial monad $\mathrm{Bin}(-)$ as a "continous" dual to the comonad $A \rightsquigarrow \mathbb{Z}\langle |A| \rangle$ appearing in the comonadicity of sets over abelian groups:

$$\mathbb{Z}\langle - \rangle \colon \mathrm{Sets} \rightleftarrows \mathbb{Z}\text{-}\mathrm{Mod} \colon |-|.$$

Blomquist and Harper[BH19] showed that passing to simplicial objects and then to the corresponding homotopy categories leads to a fully faithful embedding of simply connected spaces in derived coalgebras over the comonad. Moreover the unit of the adjunction yields the Bousfield-Kan $H\mathbb{Z}$-completion. These constructions could be though as an integer version of $H\mathbb{F}_p$-models by Kriz and Goerss established in [Kri93] and [Goe95].

Latter Horel [Hor24], using a less model dependent approach, showed that homotopically, simply connected spaces form a full subcategory of coconnective binomial algebras. Also, $C^*(-)\colon \mathrm{Top} \to \mathrm{Bin}\text{-}\mathrm{Alg}$ admits a right adjoint functor $|-|$, the realization functor. Again, it turns out that the unit $X \to |C^*(X)|$ is related to the $H\mathbb{Z}$-completion functor of Bousfield-Kan.

In a sense this gives a more straightforward way of modeling homotopy types than the well-known Mandell's approach[Man06] via $\mathbb{E}_\infty$-algebras.

2.7 Results

In this work following Raksit [Rak20] we define the derived binomial monad on the whole category $D(\mathbb{Z})$. One can consider binomial algebras appearing in the works cited above as coconnected derived binomial algebras lying in coconnective part of $\mathrm{Bin}\text{-}\mathrm{Alg}^{\geq 1}$.

First we identify the cohomology of all free binomial algebras including non coconnected ones. One has

Proposition 2.7.1 ([KSZ23])**.** *If $M \in D(\mathbb{Z}\text{-}\mathrm{Mod})^{\leq -1}$, then $\mathrm{Bin}(M) \to \mathrm{Sym}(M \otimes \mathbb{Q})$ is an equivalence.*

Thus the free binomial algebra on a connective complex is very simple and can be computed by décalage. On the other hand the coconnective case is covered by

Theorem 2.7.2. *If $M \in D(\mathbb{Z}\text{-}\mathrm{Mod})^{\leq -1}$, then there is a natural equivalence*

$$\mathrm{Bin}(M^\vee) \to C^*(K(M); \mathbb{Z}),$$

where $K(M) = |M| \in \mathrm{sSet}$ is a generalized Eilenberg-MacLane space.

In the remaining case we have

Theorem 2.7.3 ([KSZ23])**.** *For a finite abelian group A, we have a natural isomorphism:*

$$H^1(\mathrm{Bin}(A[-1])) \simeq \varinjlim_t (J\mathbb{Z}[A^D]/J^{t+1})^D,$$

where $(-)^D = Hom(-, \mathbb{Q}/\mathbb{Z})$ is the Pontryagin dual and J is the augmentation ideal of the group algebra $\mathbb{Z}[A^D]$. Moreover $H^q(\mathrm{Bin}(A[-1])) = 0$ for $q > 1$ and $H^0(\mathrm{Bin}(A[-1])) = \mathbb{Z}$.

The following corollary has a particularly nice form:

Corollary 2.7.4 ([KSZ23])**.** *For a prime p, there is an identification*

$$H^1(\mathrm{Bin}(\mathbb{Z}/p[-1])) \simeq K/\mathcal{O}_K,$$

where $K = \mathbb{Q}_p(\zeta_p)$ is the totally ramified extension of degree $p-1$ of $\mathbb{Q}_p$ by a primitive p-th root of unity ζ_p.

Also we show that

Theorem 2.7.5 ([KSZ23]). *The forgetful functor* $\mathrm{Bin}\text{-}\mathrm{Alg} \to \mathbb{E}_\infty\text{-}\mathrm{Alg}$ *commutes with small limits and colimits.*

This illustrates the fact that the binomial monad $\mathrm{Bin}(A)$ for a free abelian group A appears naturally as the following functor

$$A \rightsquigarrow \mathbb{Z} \sqcup_{C^*_{sing}(K(A^\vee,1))} \mathbb{Z} \simeq \mathrm{Bin}(A),$$

where the pushout is taken in the category of $\mathbb{E}_\infty$-algebras with respect to a chosen augmentation.

We are also able to translate the following result by Ekedahl [Eke02] without appealing to cosimplicial structures:

Theorem 2.7.6 ([KSZ23]). *The singular cochain functor*

$$C^*(-;\mathbb{Z})\colon\ \mathrm{Top}^{1-ct,ft} \to \mathrm{Bin}\text{-}\mathrm{Alg}^{\mathrm{op}}$$

induces a fully faithful embedding from the category of based simply-connected finite type spaces.

Further one can introduce a relative version of binomial algebras by means of a sheaf of binomial algebras over a space X. As an application, let $(X, \alpha\colon \mathcal{M} \to (\mathcal{O}_X, \times))$ be a nice log-scheme. In other words, $\mathcal{M}$ is a sheaf of monoids, and so is $\mathcal{O}_X$ with the multiplication operation, and α is a morphism of such sheaves satisfying certain conditions. Let $p\colon X^{KN} \to X(\mathbb{C})$ be the the associated Kato-Nakayama space. For example if $D \subset X$ is a divisor with normal crossings and $\mathcal{M}$ is locally generated by the functions that do not have zeros outside D, then $X^{KN} \to X(\mathbb{C})$ is a proper continous map and it is homotopy equivalent to the inclusion $X - D \hookrightarrow X$. In general $H^*(X^{KN};\mathbb{Z})$ is called the Betti cohomology of the log-scheme $(X, \mathcal{M})$.

One has a natural structure of a coaugmented binomial ring on $Rp_*\underline{\mathbb{Z}}$. The composition

$$\mathcal{O}_X \xrightarrow{\exp(-)} \mathcal{O}_X^\times \hookrightarrow \mathcal{M}^{gr}$$

from $\mathcal{O}_X$ to the group completion $\mathcal{M}^{gr}$, defines a cone $\exp(\alpha) = \mathrm{Cone}(\mathcal{O}_X \to \mathcal{M}^{gr})[-1]$, where $\mathcal{O}_X$ sits in cohomological degree 0. The natural map $\underline{\mathbb{Z}} \to \mathcal{O}_X$ induces $\mathrm{Bin}(\underline{\mathbb{Z}}) \to \mathrm{Bin}(\exp(\alpha))$. We have

Theorem 2.7.7 ([KSZ23]). *There is an equivalence*

$$\mathrm{Bin}(\exp(\alpha)) \sqcup_{\mathrm{Bin}(\underline{\mathbb{Z}})} \underline{\mathbb{Z}} \simeq Rp_*\underline{\mathbb{Z}}$$

of coaugmented derived binomial algebras on X*.*

This extends Steenbrink's formula [Ste95] to the integers.

3 Further directions

3.1 Complements in families

Assume we have a smooth family of arrangements over a base B, i.e. each $b \in B$ gives a complement $U|_b = X|_b - \cup_i Z_i|_b \subset X|_b$. The vector spaces $H^*(X|_b, X|_b - Z_I|_b)$ admit the Gauss-Manin connection with respect to the maps induced by $Z_I \subset Z_J, I \supset J$. Since the model $E_1|_b$ depends functorially on $H^*(Z_I|_b)$, we obtain a natural connection ∇' on $H^*(U|_b) \simeq E_2|_b$. It turns out that ∇' does not necessary coincide with the natural Gauss-Manin connection ∇^{GM} on $H^*(U|_b)$. This make sense, because monodromy of ∇^{GM} is induced by a non holomorphic endomorphism of X_b, while the established identification $H^*(U_b) \simeq E_2|_b$ is functorial only with respect to algebraic maps.

An example of this sort was observed by Looijenga [Loo23]. He showed that Torelli group of a topological surface C with $g(C) > 2$ acts non trivially on $H^*(Conf(C,3))$. Geometrically this means that there is a family of configuration spaces $Conf(C_b, 3)$ induced by a family of curves C_b over B such that the monodromy of ∇' is trivial, while monodromy of ∇^{GM} is not. More general results on the action of Torelli group on $H^*(Conf(C,k))$ were obtained in [BMW22] by combinatorial considerations.

On the other hand we see that the monodromy action comes from the difference $\nabla^{GM} - \nabla'$, which defines in a natural way a certain smooth 1-form on B. The local properties of this object deserve further study.

3.2 Binomial algebras and non-abelian derived functors

This project is a continuation of studies started in [KSZ23] focusing on applications to some classical problems. Below I will mention results which are proven, but have not yet been written up. Given a complex K, the object $\mathrm{Bin}(K)$ admits a filtration $\mathrm{Bin}^{\leq n}(K)$ such that $\mathrm{Gr}^n \mathrm{Bin}^{\leq *}(K) \simeq \Gamma^n(K)$, where Γ is a functor of divided powers. Thus the functor $\mathrm{Bin}(-)$ is a deformation of $\Gamma(-)$. For $M \in D_{perf}(\mathbb{Z}\text{-}\mathrm{Mod})$ let $M^\vee = \mathcal{H}om(M, \mathbb{Z})$. The objects $M^\vee$ for $M \in D_{perf}(\mathbb{Z}\text{-}\mathrm{Mod})^{<-1}$ form a full subcategory denoted by $(D_{perf}(\mathbb{Z}\text{-}\mathrm{Mod})^{<-1})^\vee$.

I proved the following.

Theorem 3.2.1. *For any $M \in (D_{perf}(\mathbb{Z}\text{-}\mathrm{Mod})^{<-1})^\vee$ there is a functorial isomorphism of $\mathbb{E}_1$-algebras $\Gamma(M) \simeq \mathrm{Bin}(M)$.*

In particular the cohomology of the Eilenberg-Maclane spaces admit a natural multiplicative filtration which canonically splits. Note that without the coconectedness assumption the claim is clearly false.

Corollary 3.2.2. *There is a natural isomorphism of algebras*

$$H^i(K(\mathbb{Z}, n+2); \mathbb{Z}) \simeq \oplus_d H^{i-2d} \operatorname{Sym}^d(\mathbb{Z}[-n]) \simeq \oplus_{d'} H^{d'}(B^{(n)}\mathbb{G}_a, \mathcal{O}_{B^{(n)}\mathbb{G}_a})_{i-d'}.$$

Here $B^{(n)}\mathbb{G}_a$ is the n-th iterated classifying stack for the group scheme $\mathbb{G}_a$ over $\mathbb{Z}$, while the lower index denotes the $i-d'$-graded part corresponding to the action $\mathbb{G}_m$ on $\mathbb{G}_a$ with character $t \mapsto t^2$. This statement for $n = 1$ was shown by Kubrak-Prikhodko in [KP22] by means of an explicit computation. This observation served as an original motivation for the theorem above. In a current work in progress with Dmitry Kubrak and Georgii Shuklin we are going to generalize this statement to other group schemes.

Let me mention a surprising result which is a byproduct of the technique developed in order to prove the above statements. Recall the Dold-Puppe isomorphism between the values of the left derived functors of Sym and the homology of the Eilenberg-Maclane spaces:

$$DP\colon H_*(\operatorname{Sym} A[n]) \simeq H_*(K(A, n); \mathbb{Z}).$$

Mikhailov [Mik12] showed that for $n = 1$ and A being 2-torsion both parts are non-isomorphic functors in A. On the other hand Touzé [Tou14] proved that this isomorphism is nevertheless functorial after restriction to free abelian groups. In the work by Breen-Mikhailov-Touzé [BMT16] it was stated that there is no reason to expect functoriality of the Dold-Puppe isomorphism in general. However by using the techniques mentioned above it is possible to prove that

Theorem 3.2.3. *The Dold-Puppe isomorphism is functorial for all $n > 1$ and all finitely generated abelian groups A.*

3.3 Binomial algebras and applications to Log geometry

Assume $p\colon T \to X$ is a sufficiently nice proper continous map with toric fibers. An example is provided by the Kato-Nakayama map $p\colon X^{KN} \to X_{an}$ as in Theorem 2.7.7. The distinguished triangle of abelian groups

$$\mathbb{Z} \to \tau^{\leq 1} Rp_*\underline{\mathbb{Z}} \to R^1 p_*\underline{\mathbb{Z}}[-1] \tag{3.3.1}$$

induces (Theorem 2.7.7) a natural quasi-isomorphism

$$\mathrm{Bin}(\tau^{\leq 1} Rp_*\underline{\mathbb{Z}}) \sqcup_{\mathrm{Bin}(\underline{\mathbb{Z}})} \underline{\mathbb{Z}} \xrightarrow{\sim} Rp_*\underline{\mathbb{Z}}. \tag{3.3.2}$$

In the work-in-progress with Dmitry Kubrak and Georgii Shuklin we express the Leray spectral sequence E_1 for $Rp_*\underline{\mathbb{Z}}$, corresponding to the canonical filtration, in terms of the degree filtration on the left hand side of (3.3.2) induced by filtration $\mathrm{Bin}^{\leq n}(-)$ on the binomial monad. Namely we have the second quadrant (i.e. $p \leq 0, q \geq 0$) spectral sequence

$$E_1^{pq} = H^{2p+q}(X; \Lambda^{-p} R^1 p_*\underline{\mathbb{Z}}) \implies H^{p+q}(T; \mathbb{Z}).$$

Moreover we are able to express the differential d_1 in terms of the derivation induced by the morphism

$$R^1 p_*\underline{\mathbb{Z}} \to \mathbb{Z}[2]$$

from the triangle (3.3.1) and the natural morphisms

$$\Lambda^{-p}(-)[p] \to \Lambda^{-p-1}(-)[p+2],$$

induced by the extension

$$0 \to \Gamma^{n-1}(-) \to \mathrm{Bin}^{\leq n} / \mathrm{Bin}^{\leq n-1}(-) \to \Gamma^n(-) \to 0,$$

followed by classical décalagé. This refines a result proven by Achinger and Ogus [AO20]. Namely, Theorem 4.2.2 [AO20] identifies $d_1^{pq}\colon E_1^{pq} \to E_1^{p+1,q}$ only up to multiplication by $(-p)!$, whereas our result gives an explicit formula for d_1^{pq}. In particular we are able to show that the Achinger-Ogus formula holds after multiplication by 2.

3.4 Kaledin classes

Below we are working over a base commutative ring k which contains $\mathbb{Q}$. Assume (L, δ) is a dg-algebra corresponding to a formal deformation stack $\mathcal{M}$ in the sense of Hinich [Hin01]. For example one can fix a non-unital associative algebra A and take $L = \mathrm{Coder}(\bar{T}A[1])$ to be the shifted Hochshild cochain complex of A equipped with the Gerstenhaber bracket, so $H^i(L) = HH^{i+1}(A)$. For a formal scheme $Z = \operatorname{Spec} R$ with an Artian R and the maximal ideal $m \subset R$ we have $\mathrm{Hom}(Z, \mathcal{M}) = \mathrm{MC}(m \otimes L)$ is the set of Maurer-Cartan elements. More generally $\mathrm{Hom}(Z, \mathcal{M})_\bullet = \mathrm{MC}(m \otimes \Omega^*(\Delta^\bullet) \otimes L)$ forms a simplicial set, so it make sence to speak about homotopical maps and so on. We think of $\mathrm{Hom}(Z, \mathcal{M})$ as flat deformations over Z of the given object A. Recall that $\mu \in \mathrm{MC}(m \otimes L)$ is equivalent to saying that $\mu \in m \otimes L^1$ and the derivation $\delta_\mu := \delta + [\mu, -]$ squares to zero.

Following Kaledin's [Kal07] work, Lunts [Lun10] showed that for a given family $\mu \in \mathrm{MC}(m \otimes L)$ for $R = k[t]/t^p$ one can define a class $\Theta_\mu \in H^1(k[t]/t^{p-1} \otimes L, \delta_\mu)$, such that $\Theta_\mu = 0$ iff the family is equivalent to the trivial one, i.e. μ is gauge equivalent to zero. A remarkable feature of Kaledin's class is that it provides a necessary and sufficient conditions for triviality of a deformation over $R = k[t]/t^p$ in terms of an obstruction Θ which took values in a simple homotopicaly invariant object. If this object satisfy base change properties with respect to R, this could be used for example to check deformation triviality after a base change.

In the current project with Enrico Lampetti we attempt to generalize the Kaledin class beyond the case of 1-dimensional formal disk $R = k[t]/t^p$ via derived algebraic geometry point of view. Namely, given $f\colon Z = \operatorname{Spec} R \to \mathcal{M}$ corresponding to $\mu \in \mathrm{MC}(m \otimes L)$ induces a morphism $df\colon \mathbb{T}_Z \to f^*\mathbb{T}_\mathcal{M}$ from tangent complex of Z to the pullback of tangent complex of $\mathbb{T}_\mathcal{M}$. In case $R = k[[t]] := \varprojlim_p k[t]/t^p$ the (homotopy class of) morphism df is equivalent to the class Θ_μ. In this setting the classical Kaledin's assertion simply says that if the differential df of a map f from a formal scheme $\operatorname{Spec} R$ is trivial, then f is constant. It turns out that in derived setting this is false for general R. In the simplest n-dimensional case $R = k[[t_i]], i \leq n$, assume $f\colon \operatorname{Spec} R \to \mathcal{M}$ is equivalent to 0 via a homotopy $df \sim_h 0$. One can show that the original Kaledin's proof is differential geometrical in nature and boils down to the integrability of a certain connection on a principal $\exp(m \otimes L^0)$-bundle over $\operatorname{Spec} R$ determined by h. Note that in 1-dimensional case $R = k[[t]]$ the flatness of the connection is automatic, while in case of general R it provides an extra obstruction for triviality of the deformation over R.

The differential geometric approach mentioned above can be relaxed in the following way. Consider the sequence of derived square-zero extensions

$$Z \to Z^{dR,1} \to \ldots \to Z^{dR,\infty} = Z^{dR},$$

where $Z^{dR,k} = \mathrm{Spec}(\Omega_Z^{\leq k}, d_dR)$. Note that the de Rham stack of a formal scheme Z^{dR} is equivalent to $\operatorname{Spec} k$. An equivalence of $df\colon \mathbb{T}_Z \to f^*\mathbb{T}_\mathcal{M}$ and 0 amount to the extension of $f\colon Z \to \mathcal{M}$ to $f^{(1)}\colon Z^{dR,1} \to \mathcal{M}$. An extension of $f^{(1)}$ to $f^{(2)}\colon Z^{dR,2} \to \mathcal{M}$ is equivalent to the vanishing of a cohomological class represented by the curvature. And so on. This produce a sequence of obstructions for triviality of f.

The main issue with this approach is familar from algebraic topology: it is possible to start with a map homotopical to the constant map, but chosing wrong homotopies on the first few extensions may result in a non trivial obstruction for the following extension.

An alternative perspective was initiated by Emprin in [Emp24], where she introduced a notion of *prismatic decomposition* for the dg-Lie algebra $\mathfrak{g} := m \otimes L$ in terms of $\delta-$grading $\mathfrak{g} = \prod_{i\geq 1} \mathfrak{g}^{(i)}$ such that $d\colon \mathfrak{g}^{(i)} \to \mathfrak{g}^{(i+\delta)}$ and a certain map $\mathcal{D}\colon \mathfrak{g}^1 \oplus \mathfrak{g}^0 \to \mathfrak{g}[h]]$. In particular $\mathcal{D}$ preserves Mauerer-Cartan elements. Using $\mathcal{D}$ one can define an obstruction

$$\Theta'_\mu := \Theta_{\mathcal{D}(\mu)} \in H^1(\mathfrak{g}[h]], \delta_{\mathcal{D}(\mu)}).$$

In [Emp24] it was shown that $\Theta'_\mu = 0$ iff $\mu = \mu^\delta + \mu^{\delta+1} + \dots$ is gauge equivalent to μ^δ *in* $\mathfrak{g}$.

One of the goals of this project is to generalize this statement and interpret it in geometrical terms.

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