A NOTE ON THE CONSTRUCTION OF THE
ŚRĪ YANTRA

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ABSTRACT. The Śrī Yantra is an ancient geometric diagram used for meditation in various schools of tantrism whose study stimulated a vast effort of specialists from different fields: mathematics, history, psychology, ethnography, philosophy etc. Its construction sets an elementary and nontrivial problem. In this note, we work out a straight-edge and compass method for constructing concurrent models of Śrī Yantras. The problem proves to be equivalent to Apollonius circle-line-point problem.

1. INTRODUCTION

The study of the Śrī Yantra from a purely mathematical point of view is natural and saw some developments in the last decades [11, 16, 17, 20, 21].

The Śrī Yantra\(^1\) belongs to a variety of objects (yantras) used in schools of tantrism for meditation. It can be described as a polygon bearing a central point (the so called bindu) inscribed within a circular motive of 8 lotus petals, surrounded in turn by a 16-petalled lotus and by a square, the so-called square of defence. The polygon and the surrounding circle passing through at least four of its vertices is the subject of this note and is reproduced in the handwritten version of Fig. 1.

\[\text{Figure 1. A Śrī Yantra diagram.}\]

\(^1\)Sometimes written as Shri Yantra or Śrī Yantra and also referred to as Shri Chakra or Nava Chakra.
The origins of the Śrī Yantra are unknown. In 2002, Huet [8] points out that the actual divinities and mantras associated with the Śrī Yantra depend on sectarian traditions (see [15] for a complete account by Padoux). Michaël [12] describes a particular tradition of ritual associated with the Śrī Yantra taken from the Saundarya Laharī, the hymn attributed to the philosopher of the eighth century Adi Shankara. Back in 1984, Kulaichev [9] mentions that a realisation of the Śrī Yantra may be found in the religious institution of Sringeri Matha established by Adi Shankara. This is one of the earliest known examples that can be seen nowadays. Kulaichev alludes also to a hymn from Atharva Veda (c. 12th century BC) dedicated to a Śrī Yantra-like figure consisting of nine triangles. However, as Huet puts it, “although it is hinted in several sources that this symbol is very old” we do not know of “any published representation anterior to the 17th century, leaving open its date of creation.” In [14], Mookerjee and Khanna, go as far as commenting that the “Śrī Yantra, in its formal content, is a visual masterpiece of abstraction, and must have been created through revelation rather than by human ingenuity and craft.”

Indeed, the constructibility of the polygon and that of the circumscribed circle is the subject of several writings. Various methods of approximation of numerical solution of algebraic equations are discussed in [4, 8, 9, 10, 17, 16, 18, 19, 20]. In [8], Huet provides an explicit computer programme for representing the diagram based on a Newton approximation of the coordinates of its vertices. A natural question is whether the Śrī Yantra is constructible with compass and straight-edge.

Constructibility in this sense is one of the most classical subjects in geometry and arithmetic and it was completely elucidated by Wantzel’s theorem in 1837 based on the theory of Gaussian periods. For instance, by this method, we can see that a regular $n$-sided polygon (an $n$-gon) can be constructed by means of straight-edge and compass if and only if $n$ is of the form $2^h p_1 \ldots p_k$ where $h$ and $k$ are nonnegative integers and, when $k > 0$, the numbers $p_1, \ldots, p_k$ are distinct Fermat primes.\footnote{These are prime numbers of the form $2^{2^m} + 1$. The only known examples are 3, 5, 17, 257, 65537. Thus, a regular $n$-gon is constructible if and only if $n = 3, 4, 5, 6, 8, 10, 12, 15, 16, 17 \ldots$ and any of the numbers of the sequence \url{https://oeis.org/A003401}.}

In fact, the Śrī Yantra should be regarded as a family of diagrams (see Fig. 4) whose shapes are determined uniquely by four real parameters (see §3 and [8], §1.2, Thm.]). To the best of my knowledge there is no method to construct the diagram for any choice of these parameters by straight-edge and compass. We provide such a construction for every

\footnote{We owe this remarkable quote to Huet again, see [8], §2.6.}
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choice of such parameters. By the Mohr–Mascheroni theorem, this means that the vertices of the Śrī Yantra can be placed only by means of a compass without any use of a straight-edge.

As Kulaichev [9] among others shows, there is a “traditional method of copying”, see [1] which divides the vertical diameter of the circle in ten parts of prescribed length (we refer to it in the beginning of §4). This heuristic method does not ensure the required three-line concurrency properties of Fig. 1 (see §2;1–3 for a formal statement). Actually, the lack of concurrency is visible to the eye.

Further methods of construction appeared in the latest years. See for instance [3] for a straight-edge and compass method, as well as the collection of methods of [19]. The inaccuracies are not visible to the eye, but they have been shown not to yield the desired concurrency properties. See [19] and the list of methods discussed there.

The purpose of this note is to state the problem in simple terms and derive an exact construction method from a known classical problem. It turns out that drawing a Śrī Yantra ultimately depends on solving a problem of Euclidian geometry dating back to the third century BC: the Apollonius problem of tracing a circle tangent to a given circle, a given line, and passing through a given point.

The solution to this problem, provided by Apollonius of Perga himself in the second half of the third century BC was lost and later reported in the fourth century AD. It is of course a classical and widely treated example of straight-edge and compass construction, see for instance [6, 7].

After formally describing the diagram in §2 (and deducing a nice feature, see https://www.geogebra.org/m/cmny6ypg), we illustrate why exactly four parameters determine a unique shape in §3. Then, we show the equivalence with Apollonius problem in §4. Finally, in §5, we go back to deduce an explicit construction of it; we provide an animated version of it here https://www.geogebra.org/m/zdvxtdvv.

Beside the beauty of the construction and the independent interest of anything relating to the Apollonius contact problem especially from the algebro-geometrical point of view (see for instance [3 §4] and [2 §2.3]), it seems important for many other reasons to know whether other exact straight-edge and compass constructions existed. This may prove useful to the historians aiming to understand the alternative methods of construction adopted in the course of history.

Furthermore, the work by Kulaichev and Ramendic suggests that unravelling the quite intricate mutual dependences of choices at work in
the Śrī Yantra may also shed light on its psychophysiological aspects; see [11].

Relying on an exact, relatively simple, plane construction may also allow further progress. The following four extra requirements are often made. It is required (i) that the circle passes through the extremes of the lowest horizontal triangle base in the diagram, (ii) that the circle passes also through the extremes of the highest horizontal base, (iii) that the middle black triangle in Fig. 1 is equilateral, and (iv) that the center of such equilateral triangle and that of the circle coincide (on the first two extra conditions see [9, 10] suggest an algorithm).

From an historical point of view another interesting subject is the pursuit of a spherical version. Indeed, the Śrī Yantra also exists in three-dimensional versions since it provides a model in architecture and for other three-dimensional objects, [9]. Constructions via spherical geometry have been explored and the hypothesis of a spherical origin has been formulated and mathematically studied, see Kulaichev [9, 10] and Rao [17]. This inspiring idea, which we do not study here, consists in replacing lines in the diagram by portions of great circles or simply cross sections on a sphere.

The three-dimensional origin would have been reinforced by the statement that the Śrī Yantra could not be constructed on a plane by compass and straight-edge but only arise as the projection to the plane of a construction made in three-dimensions. Instead, the constructibility on a plane shown here implies that the coordinates of the diagram fit the field of constructible numbers: the smallest field extension of the rationals that includes the square roots of all of its positive numbers. This may also shed light on the attempt by [9, §Analysis], [16] and [17] of writing explicit equations for it; the theorem of Wantzel says that the minimal polynomial’s degree is a power of two.

1.1. Presentation of this text. I made my best effort to write this text as simply as possible; it should be understandable to a reader who is not a professional mathematician. As made clear in this introduction, I have recently discovered the problem, and my presentation of the state of the art may not be precise; I would be very glad to correct it and complete it. Each argument (especially in §4), is supported by geogebra animations each one available via an explicit url. Section 2 describes the diagram formally. Section 3 recalls why the diagram can be deformed along four real parameters. Section 4 draws the connection to Apollonius problem and illustrates a method of construction. Section 5 goes through the construction in five figures that can be found in the complete construction provided here

https://www.geogebra.org/m/zedxtdv4
2. Description of the diagram

We focus on the diagram lying at the center of the Śrī Yantra. It consists of nine plane triangles sharing a common axis of symmetry Ω. We draw Ω vertically, oriented from bottom to top, and we parametrize it by a real parameter $t$.

Since the symmetry with respect to Ω transforms each triangle onto itself, the nine triangles are isosceles and they are entirely determined by three real coordinates:

(a) the position on the axis Ω of the apex $a$, the only vertex on Ω;

(b) the position on Ω of the midpoint $b$ of the base, the edge opposite to $a$;

(c) the distance from Ω of (either of) the remaining vertices, the base vertices $c$ and $c'$.

We refer to the two edges joining the apex to any of the base vertices as the legs. Triangles as the one drawn above satisfy $t(a) > t(b)$ (resp. $t(a) < t(b)$) and will be referred to as pointing upward (resp. downward).

A triangle in the diagram is maximal if it is not contained in any other triangle within the diagram. There are exactly nine maximal triangles: five point downward and four upward. We label the five downward triangles by $d_1, d_2, d_3, d_4,$ and $d_5$ from the lowest to the highest apex (see Fig. 3). The remaining four triangles $u_1, u_2, u_3,$ and $u_4$ point upward and we label them from the highest to the lowest apex (see again Fig. 3).

Within these two sets, $d_1$ and $u_1$ share the same circumscribed circle $\Pi$: their apices are antipodal on it.

We observe that two legs of two distinct upward triangles never meet; the same holds for the legs of two distinct downward triangles.

The apex and the base midpoint of all remaining triangles are all contained between the apices of $u_1$ and $d_1$, i.e. they all lie in the diameter of $\Pi$.

We can now introduce the Śrī Yantra by drawing its diagram and by
listing its concurrency properties corresponding to \(\text{\textdegree}\), \(\text{\textbullet}\), and \(\text{\textbullet}\) in Fig. 2 (see [8, §1.2] for an equivalent list).

Concurrency properties:

1. The circle \(\Pi\) passes through the vertices of \(u_1\) and of \(d_1\);

2. The apex of \(T_1\) is the base point of \(T_2\) for \((T_1, T_2) = (u_2, d_5), (u_3, d_2), (u_4, d_1), (d_5, u_3), (d_4, u_1), (d_3, u_2), (d_2, u_4)\);

3. The downward triangle \(T_1\), the upward triangle \(T_3\), and the base of the triangle \(T_2\) intersect at exactly two points for \((T_1, T_2, T_3) = (d_5, d_2, u_1), (d_2, d_1, u_1), (d_5, d_1, u_2), (d_5, d_3, u_3), (d_2, d_4, u_4), (d_3, u_3, u_4), (u_4, u_1, d_2), (u_2, u_1, d_1), (d_1, u_2, u_4), (d_3, d_3, u_2), (d_4, d_3, u_3), (d_2, u_3, u_3)\).

We will refer to specific conditions within this list by \((1; T_1), (2; T_1, T_2), (3; T_1, T_2, T_3)\). Notice that the in last three triples at point (3), namely \((3; d_3, d_3, u_2), (3; d_4, d_4, u_3), (3; d_2, u_3, u_3)\), the second term equals the first or the third: this means that the base vertices of \(T_2 = T_i\) lie on the legs of \(T_j\) with \(i = 1\) and \(j = 3\) or vice versa.
Remark 2.1. We conclude the section by illustrating a feature of the diagram following the arrows in Fig. 2 based on properties (1) and (2) in the above list. This is a curiosity, not mentioned in the literature to the best of my knowledge. The reader can refer to its formal statement, or simply look here [https://www.geogebra.org/m/cmny6ypg](https://www.geogebra.org/m/cmny6ypg) or skip to the next section.

The diagram of the Śrī Yantra, i.e. the nine triangles and the circle, can be drawn by a single continuous path, with no overlaps apart from isolated points. The path respects the following rule: *it switches from a figure to another (circle or triangle) if and only if it reaches an apex of a triangle.*

The rule indicates the path without any ambiguity at all concurrency points. We illustrate explicitly what this means. Namely, the apex of $d_4$ is the base point of $u_1$; the apex of $u_1$ is joined by a semicircle within $\Pi$ to the apex of $d_1$; the base point of $d_1$ is the apex of $u_4$; the base point of $u_4$ is the apex of $d_2$; the base point of $d_2$ is the apex of $u_3$; the base point of $u_3$ is the apex $d_5$; the base point of $d_5$ is the apex $u_2$; the base point of $u_2$ is the apex $d_3$. We could summarise diagrammatically as follows:

\[
\begin{array}{c}
4 \searrow 1 \circ 1 \nearrow 4 \swarrow 2 \nearrow 3 \swarrow 5 \nearrow 2 \swarrow 3.
\end{array}
\]

3. Deformations of the Śrī Yantra: Four Moduli

The Śrī Yantra depends on four parameters (or moduli). We make this statement precise and recall the argument in two different ways. An equivalent discussion can be found in [8, §1.2]; we felt that this section serves as a preparation to the following one, but the reader can skip to §4.
Diagrams satisfying the concurrency properties (1,2,3) listed above are not unique. Depending on the choice of four real parameters we can construct a unique diagram. The parameters may vary continuously each one within a limited interval yielding a family of diagrams all satisfying properties (1,2,3). In Fig. 4 we have illustrated that there can be families of diagram possibly constant in sizes but with different shapes (in the far away diagram the circle touches two more vertices).

Figure 4. A picture to illustrate that there can be at least a one-parameter family of different Śrī Yantras. While we vary one parameter on the lower line the circle acquires two more tangency points. In fact one can deform a diagram in four independent directions.

In this section we show why fixing four parameters suffice in order to determine a unique shape of Śrī Yantra (up to translation on the plane or rescaling). The subsection 3.1 does it via a well known method. The subsection 3.2 also shows the claim, but placing it in a different perspective. In §4, this leads us to provide a simple straight-edge and compass construction based on the choice of four parameters.

3.1. The inside-outside construction. There is a tentative method of construction which is recalled in several places in the literature, see for instance [9, Fig. 6] and [23]. It is referred to as the inside-outside constrution attributed to Bhāskara’s Nityāśodśikārnaga; see [20]. We illustrate it here with Fig. 5 and Fig. 6. One starts by fixing six points $P_1, P_2, P_3, P_4, P_5, P_6$ along $Ω$: the apices of the triangles $u_4, d_3, d_4, d_5$ and the base points of $d_4$ and $d_3$. 
Two more parameters are fixed: the measures of the apex angles of $d_4$ and $d_5$. This is done by choosing two points $P_7$ and $P_8$. The first point, $P_7$, is the base vertex of $d_4$ pictured in blue on the right hand side of Fig. 5. The last point, $P_8$, will tell us where the leg of $d_5$ crosses the base of $d_4$. These eight choices allow us to complete all triangles satisfying the concurrency properties (2) and (3) listed above. One can proceed from the darker to the lightest region; we illustrate the construction in Fig. 6 and here https://www.geogebra.org/m/skz7hgfr.

Fig. 5 shows that the resulting diagram can fail to satisfy condition (1): $u_1$ and $d_1$ have two different circumscribed circles. In Fig. 6 instead, the two bigger triangles share the same circumscribed circle — for this to happen we have drawn the triangles in a special way. In the first three steps we have respected the construction presented in this text.

The eight choices we started from amount to eight real parameters: the first six apices are determined by their position on $\Omega$ and the two remaining points are determined by their distance from $\Omega$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5}
\caption{Four parameters: eight points determine a diagram satisfying the concurrency properties (2) and (3). Here, and in general, condition (1) cannot be satisfied since the circumscribed circle of $u_1$ differs from that of $d_1$.}
\end{figure}
Figure 6. The inside-outside method.

Figure 7. The inside-outside method: in order to draw the first figure one needs three of the eight points (namely $P_4, P_5, P_8$). Four points are needed for the second figure (namely $P_1, P_3, P_6, P_7$). An eighth point, $P_2$, is used for the third figure. The next figure (number five and six) are entirely determined from the previous choices.

However, as shown in Fig. 5, in the resulting diagram, the triangles $u_1$ and $d_1$ do not necessarily share a circumscribed circle. In order for this to happen we need to impose that the two circles circumscribing $u_1$ and $d_1$ have the same center and radius (see Fig. 5). Ultimately, this amounts to imposing two conditions on the eight parameters, one for the common center and one for the equal radii.

The diagram in Fig. 6 satisfies such conditions. In practice, six points have been chosen freely and the remaining two have been chosen conveniently in order to satisfy the concurrency condition (1).

Finally it should be noticed that rescaling or translating all choices $P_1, \ldots, P_8$ on the line $\Omega$ yields rescaled or translated diagrams. In this way the shape of the Śrī Yantra only depends on four real parameters.

3.2. A second method. An alternative method yielding the same conclusion consists in fixing the apices of seven triangles: $u_1, u_2, u_4, d_1, d_2, d_4, d_5$. For simplicity, we perform a right-angle rotation in the anti-clockwise direction: the right hand side of the diagram becomes the upper half plane and $\Omega$ becomes the horizontal real axis oriented from right to left (in practice downward triangles point toward the right and vice versa). It is natural to draw only the upper side of the diagram, the other being symmetrical (in practice we only retain one
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half of the isosceles triangles: we are left with right triangles having a cathetus on $\Omega$.

Since we are interested in the shape of the diagram and not on its position along $\Omega$ nor on its size, we can translate and rescale the positions of the seven points above. This reduces again the degrees of freedom by two. For instance, we can normalise the position of the apices of $u_1$ and $d_1$ and fix them at two values $t_\star$ and $t_\bullet > 0$ on the real line $\Omega$. We then choose the remaining apices of $u_2, u_4, d_2, d_4, d_5 \in [t_\star, t_\bullet]$. By applying the concurrency conditions, these data determine the entire triangles $u_1, u_2, u_3, u_4, d_1, d_2$, and $d_5$. As Fig. 8 illustrates, the apex and the base point of $d_3$ is also determined. However, Fig. 8 also illustrates that the initial five parameters should be conveniently chosen in order to allow us to place the entire triangle $d_3$. This happens because of the following crucial remark (a well known problem for which the reader can consult [9, Fig. 5] and numerous other references, see for instance [19]).

Remark 3.1. The apex of the triangle $d_3$ is determined. In order to completely draw the rest of it, we need to place the base vertices of the triangle $d_3$ in agreement with the concurrency conditions $(3; d_5, d_3, u_4)$ and $(3; d_3, d_3, u_2)$. Indeed, these two conditions refer to the base of such triangle and, if read explicitly, state the following.

- The base of $d_3$, intersects $d_4$ and $u_4$ at two internal (symmetrical) points (this is $(3; d_5, d_3, u_4)$);

- The extremes of the base of $d_3$ lie on the legs of $u_2$ (this is $(3; d_3, d_3, u_2)$).

Fig. 8 illustrates how one can respect the first requirement above (i.e. $(3; d_5, d_3, u_4)$) and fail to satisfy the second one: $(3; d_3, d_3, u_2)$. Geogebra applet are a useful tool to see this at work [https://www.geogebra.org/m/h247ufsa.]

Once conditions $(3; d_5, d_3, u_4)$ and $(3; d_3, d_3, u_2)$ are satisfied the rest of the figure can be completed guided by the remaining concurrency conditions.

Again we find that the shape of the Śrī Yantra depends on four parameters. We started from seven parameters, but we reduced to five by fixing the two extremes of the diameter. Imposing the concurrency conditions means that one of the five parameters should be chosen compatibly with the other four choices. The construction presented in this text shows how the position of the apex of $u_2$ is determined by the position of the apices of $u_4, d_2, d_4$, and $d_5$. 
Figure 8. The failure to satisfy condition $(3; d_5, d_3, u_2)$ (recall we have performed a right-angle rotation in anti-clockwise direction).

4. FROM THE ŚRĪ YANTRA TO APOLLONIUS

Once the apices of $u_4, d_2, d_4,$ and $d_5$ are fixed, it remains to place the apex of $u_2$. Before treating this choice, let us point out that the four above mentioned apices of $u_4, d_2, d_4,$ and $d_5$ should have coordinates $t_1, t_2, t_3, t_4$ on $\Omega$ placing them within the interval set by the apex of $u_1$ and the apex of $d_1$ (i.e. within the diameter) and satisfying

$$t_1 > t_2 > t_3 > t_4.$$ 

Furthermore, some choices lead to degenerate diagrams. Degeneracy means that the diagram still satisfies the concurrency conditions at the expense of adapting the definition of triangle, allowing for instance for the basis to extend beyond the vertices. We do not treat this problem. The reader can work with the apex of $d_1$ at the origin of $\Omega$, the apex of $u_1$ at $t = 1$ on $\Omega$ and the apices of $u_4, d_2, d_4,$ and $d_5$ fixed at $t_1 = \frac{2}{7} > t_2 = \frac{4}{5} > t_3 = \frac{2}{5} > t_4 = \frac{1}{5}$. We should mention that [9, 10] discuss this issue at lengths referring to a method starting from prescribed values for all apices and base points. His choice are equivalent to setting $(t_1, t_2, t_3, t_4) = \left(\frac{6}{48}, \frac{18}{48}, \frac{21}{48}, \frac{14}{48}\right)$. Huet goes further to study the æsthetics of the Śrī Yantra. His choices, stated in [8, §1.2, Defn.], are equivalent to $(t_1, t_2, t_3, t_4) = (0.668, 0.463, 0.398, 0.165)$ (i.e. the values of $Y_F, Y_P, Y_J, Y_L$ in his notation).

4.1. The problem in broad terms. As illustrated in the previous section, the problem boils down to fixing the slope of the straight line originating from the apex of $d_5$ in a such a way that the concurrency conditions $(3; d_5, d_3, u_4)$ and $(3; d_3, d_3, u_2)$ are satisfied.

In Fig. 9 we redraw Fig. 8 and highlight the relevant data in black. In these terms we can phrase the concurrency conditions more explicitly.

4.2. The precise statement of the problem. Fig. 10 illustrates the problem a little further. In black we retain only the relevant data.
Figure 9. Retaining only the relevant data (black lines and black dots).

The dotted ray is the straight line whose slope should be determined. The dashed arrows describe a geometric construction identifying two circled points depending on the dotted line. Unravelling conditions $(3; d_5, d_3, u_4)$ and $(3; d_3, d_3, u_2)$ leads us to conclude that the concurrency conditions are satisfied if and only if the straight line through the circled points is orthogonal to $\Omega$ (in order to see this in concrete terms we provide an applet here: https://www.geogebra.org/m/b76w889d).

Figure 10. The two circled dots should be aligned on a vertical line.

4.3. An equivalent elementary problem. In this section we state a simple problem, which is entirely equivalent to ours. We refer to it as the triangle-and-rays problem.

We fix a triangle as in Fig. II and an edge within it, which we refer to as the base. Then we fix two points $P_1$ and $P_2$ each one lying within the interior of the exterior angle adjacent to the base. As the caption of Fig. II reads, the problem consists in finding a point $T$ on the base in such a way that the two points $P_1'$ and $P_2'$ intercepted on the edges by the segments joining $T$ to $P_1$ and $P_2$ have the same distance from the line passing through the base. In other words, the segment joining $P_1'$ to $P_2'$ should be parallel to the base.
The triangle-and-rays problem. Move the point on the base so that the two points intercepted on the edges have the same distance from the base. See https://www.geogebra.org/m/fc9yb kpz.

The problem always has a solution: if we place the point \( T \) on the left base vertex the segment joining the two points intercepted on the edges is certainly not horizontal; its left extreme is on the base, so its slope is positive. Similarly if we place the white dot on the right base vertex the resulting segment has negative slope. The slope varies continuously and, somewhere between the right and the left base vertex, it vanishes.

We show the equivalence between the problem in Fig. 10 and the problem in Fig. 11 in four simple but not straightforward remarks.

Remark 4.1. In Fig. 10 the slope of the required dotted ray is determined by the position of the point marked with a cross on the height of the triangle.
Remark 4.2. In Fig. [10] when this point varies along the height, the lower circled point moves along a ray stemming from the left base vertex of the initial triangle.

Remark 4.3. In Fig. [10] such ray is uniquely identified by the following property. It passes through the point where the diagonals of the right trapezoid in the figure meet.

Figure 13. We represent Remarks 4.2 and 4.3. While we move along the height the intersection point traces a ray; see https://www.geogebra.org/m/uby77vvc

Remark 4.4. We conclude that the circled dots range on the edges of a triangle shaded in Fig. [14]

The equivalence between the two problems is now immediate by turning Fig. [14] by a right angle anticlockwise.

Figure 14. A right angle rotation now yields the problem illustrated in Fig. [11]. We provide an applet here https://www.geogebra.org/m/b76w889d.
4.4. **Two parabolæ.** We now focus on the triangle-and-rays problem of Fig. 11. The data are as follows. The triangle has an horizontal base lying on the line $\ell$. The two points $P_1$ and $P_2$ lie in the interior of the two exterior angles adjacent to the base. We will refer to the two remaining edges as the right hand side edge and the left hand side edge.

Fig. 15 illustrates that the solution is a point determined by projecting on $\ell$ the intersection of two parabolæ (see the arrow pointing upwards there). We explain why it works in three steps: the setup, the formalisation of the condition imposed to the point $T$, its algebraic formulation, the conclusion.

4.4.1. **Setup.** Project the right (resp. left) hand side external point on the line $\ell$ passing through the base by drawing through it a straight line parallel to the right (resp. left) hand side edge. In this way we get two new points on $\ell$. These, alongside with the extremes of the base, complete a set of four points on the line $\ell$. We denote by $v_i < l_1 < l_2 < v_2$ their real coordinates from left to right. The initial data boil down to these four real numbers alongside with $t_1$ and $t_2$, the distances of the two external points from $\ell$. In other words, we set $t_i = d(P_i, \ell)$.

We now turn to the points $P'_1$ and $P'_2$ intercepted on the edges by the ray stemming from $T$ and joining $P_i$. We refer to their distance from $\ell$ as $y'_i$. In other words, we set $y_i = d(P'_i, \ell)$.

4.4.2. **The required condition** $P'_1P'_2 \parallel \ell$. With all this data in place, we impose the required condition: the distance of $P'_1$ from $\ell$ equals the distance $P'_2$ from $\ell$:

$$y_1 = y_2;$$

or, in other words, the line through $P'_1$ and $P'_2$ and $\ell$ are parallel.

4.4.3. **Imposing** $y_1 = y_2$ **to** $T$. There is a convenient way to express $y_1$ and $y_2$ in terms of the position of the point $T$ on the line $\ell$. Let us use the parameter $x$ to express the position of $T$. Notice that $T$ ranges between the extremes of the base; hence $x$ ranges between $l_1$ and $l_2$

$$x \in [l_1, l_2].$$

Fig. 15 represents our initial triangle $\Theta$ and two triangles $R$ and $L$ on the right and the left of it obtained by tracing parallels to the right and left edges as described in the Setup. The right hand side triangle $R$ overlaps $\Theta$. The intersection is a little triangle $\Theta \cap R$, which is similar to the right hand side triangle. The same applies on the left hand side. This means that the ratio height/base for the right triangle $R$ and for $\Theta \cap R$ coincide. In other words we have $y_i/(x - l_i) = t_i/(x - v_i)$; which we can rewrite as

$$y_i = t_i(x - l_i)/(x - v_i).$$
Conclusion. In order to solve the triangle-and-rays problem we need to find a solution within $|l_1, l_2|$ of the equation

$$t_1 \frac{x - l_1}{x - v_1} = t_2 \frac{x - l_2}{x - v_2}.$$ 

In other words, the required identity $y_1 = y_2$ boils down to

$$(1) \quad t_1(x - l_1)(x - v_2) = t_2(x - l_2)(x - v_1).$$

This explains that the problem is finally solved by finding the abscissa $x$ of the intersection of two parabolae expressed in terms of degree-two polynomials: $t_1(x - l_1)(x - v_2)$ and $t_2(x - l_2)(x - v_1)$.

A parabola is determined by a point $F$, the focus, and a line $d$, the directrix. The points $P$ of the parabola are characterised by $d(P, F) = d(P, d)$. We can entirely determine the foci of the two parabolae above.

First, since the parabolae are the graph of two polynomials, their directrix is horizontal. Second, the leading terms of the two polynomials are $t_1x^2$ and $t_2x^2$ and this means that, for the two parabolae, the distance between the focus and the directrix is $t_1/2$ and $t_2/2$. It is now possible to find the foci and the directrices, the geometric construction is in Fig. 15.

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**Figure 15.** The distances of the external points are halved and projected on the axes of symmetry of the two parabolae. We join the point obtained in this way to the root $v_i$ by a segment whose orthogonal line intercepts the focus and determines the directrix.
4.5. **Apollonius.** We are considering two parabolæ $p$ and $p'$ with parallel directrices $d$ and $d'$. In our situation, the foci $F$ and $F'$ corresponding to $p$ and $p'$ lie in one of the two half-planes bounded by one of the directrices, which we refer to as $d$.

It is not difficult to see that the intersection $p \cap p'$ is the center of the circle tangent to the directrix $d$, passing through the corresponding focus $F$, and tangent to a circle centred in the other focus $F'$ whose ray is the distance between $d$ and $d'$ (see Fig. 16).

![Figure 16](image)

**Figure 16.** Intersecting these two parabolæ amounts to finding the circle tangent to the upper directrix, passing through the corresponding focus, and tangent to the dashed circle. In other words this is Apollonius Circle-Line-Point (CLP). The solution is the dotted circle; its center is the intersection point.

In Euclidean plane geometry, Apollonius’s problem is about constructing circles that are tangent to three given circles. Apollonius of Perga (born in the second half of the third century BC) posed and solved this problem. His work has been lost, but his results are reported by Pappus of Alexandria in the fourth century AD. We can approach this problem via a series of reductions to simpler problems, among which the Apollonius CLP problem that we are considering.

Let $c$ be a circle, let $l$ be a line, let $P$ be a point. The CLP problem of Apollonius is to determine a circle tangent to $c$, to $l$, and passing through $P$. The problem admits four solutions in general: two of them contain the given circle $c$ and two lie outside of it.

The solution needed here is one of this last type. It results from identifying two more points through which the circle passes. In this way, we reduce Apollonius CLP to CPP: the problem of identifying a circle passing through two given points and tangent to a given circle. This problem easily reduces to the much easier problem of passing a circle through three points. See [6, 7].
Figure 17. Apollonius CLP, reduced to CPP and then to PPP. The initial data are drawn with a continuous line, the desired circle is dotted, the intermediate steps are dashed.

In Fig. [17] we find a second point through which the desired (dotted) circle passes. This can be obtained as the point $P' = c' \cap l'$. The straight line $l'$ joins the given point $P$ to the top of the circle initially given (the point of maximal distance from $l$). The circle $c'$ passes through the given point $P$, through the point of least distance between $c$ and $l$ and through its projection on the line.

In Fig. [17] we also find a third point $P''$ through which the desired (dotted) circle passes. This is actually the point of tangency between the circle we are looking for and the circle $c$. It is identified in two steps. First, pass a straight line $l''$ through the two points of the intersection $c' \cap c$. The point $P''$ is the point of tangency between a line through the point $l' \cap l''$ and the circle $c$ (see [6, 7]).

5. The construction

We recapitulate the entire construction in five figures. A complete geogebra version of the construction is available here: https://www.geogebra.org/m/zdvxtdvv
Figure 18. Determining the apex of $d_3$ as discussed in Rem. 3.1.

Figure 19. We recognise the triangle of Fig. 11 (upside down). The problem: find the position for the big black dot so that the segment joining it to the big grey intersect two points lying on a horizontal line.
Figure 20. Identifying the two directrices (horizontal dashed lines) and the two foci. We use the method of Fig. 15.

Figure 21. By applying the method of Fig. 17 we solve Apollonius CLP problem. The center of the circle projects on the big black point solving problem of Fig. 11. We have found an answer to the question posed at Fig. 19.
Figure 22. The ray through the point found in the previous figure allows us to complete the diagram.

Figure 23. The entire construction with the bindu, the center of the circle inscribed within the innermost triangle (the intersection between $\Omega$ and the bisector of any of the base angles).
A NOTE ON THE CONSTRUCTION OF THE ŚRĪ YANTRA

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