RESEARCH STATEMENT

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1. Introduction

My research interest are dynamical systems with chaotic behavior and the spectral properties of associated transfer operators.
It is one of the branches in mathematics which roots directly to physics: to give equilibrium statistical mechanics a solid mathematical basis. This is the thermodynamic formalism [26].
My long-term research will contribute to the understanding in which way spectral problems for the geodesic flow on the unit tangent bundle of hyperbolic Riemann surfaces differ from those of arising by more general chaotic dynamics.
To shed light on the interplay between properties of the dynamics and the transfer operator, I use mainly functional analysis and certain analytic objects.
On the dynamical side one associates dynamical zeta functions and determinants which capture the topological or differentiable behavior of the dynamics ([14], [25]). Among those are e.g. the Artin-Mazur and Selberg zeta function.
On the operator side one associates functional analytic objects, e.g. the flat trace and determinant [4].
After identifying both sides with each other, one passes to the spectral information of the operator. It is this technique which allows e.g. to study or extend the domain of the associated zeta function.
Measure theoretical aspects can be treated by spectral questions as well. For example, exponential decay of correlation (usually for Hoelder observables C^r where $r > 0$ is given) for an invariant measure essentially amounts in showing, that the operator admits a spectral gap. Progress in understanding the spectral structure of transfer operators for expanding and hyperbolic dynamics has been made over the last several years ([3], [4], [6], [7], [10], [16]–[18], [23], [27], [28]).

2. Past research

In my master thesis, supervised by Anke Pohl, I made myself familiar with a discretization algorithm for the geodesic flow on the unit tangent bundle of Riemann surfaces arising from non-uniform lattices. This algorithm was developed by Anke Pohl [24] in order to extend the approach of Dieter Mayer [19], [21] on the Selberg zeta function for the modular group $\mathrm{PSL}_2(\mathbb{Z})$. In particular, I studied the Theta group, a cofinite group with two cusps and one conic point, as the limit group of the cofinite Hecke triangle group family. The parabolic elements in the group force the dynamical system to be non-uniformly expanding. By passing to an accelerated dynamics one establishes:

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Theorem 2.1 (Meromorphic continuation (Adam, 2015))**.** *There exists a Banach space $\mathcal{B}$ and a meromorphic transfer operator family $\{\mathcal{L}_s \colon \mathcal{B} \to \mathcal{B} \mid s \in \mathbb{C}\}$ such that*

$$Z_{Selberg}(s) = \det\left(1 - \mathcal{L}_s\right).$$

Here Z_{Selberg} denotes the Selberg zeta function for the Theta group and det the Fredholm determinant whenever $\mathcal{L}_s$ (which is the transfer operator associated to the accelerated dynamical system) is well-defined. At the end of my master study I realized how to formally invert the acceleration procedure on the operator side. Together with Anke Pohl, we put it on a solid base [1] and confirmed a conjecture of her and Martin Möller [20]. This inversion generalizes an approach on $\mathrm{PSL}_2(\mathbb{Z})$ which was used by Roelof Bruggeman, John Lewis and Don Zagier [9].

3. Current research

I began my doctoral research, supervised by Viviane Baladi, investigating a question about non-trivial resonances for real analytic Anosov diffeomorphisms on the two-dimensional torus $\mathbb{T}^2$. A hyperbolic linear toral automorphism A has only trivial resonances $\{0, 1\}$.
Frédéric Faure and Nicolas Roy [11] addressed real analytic perturbations $A + \epsilon\psi$ on $\mathbb{T}^2$, considering an anisotropic complex Hilbert space, such that the composition operator $\mathcal{K}_{A+\epsilon\psi}$ is trace class. My approach was based on this construction and relied on an idea suggested by Frédéric Naud [22].

Theorem 3.1 (Non-trivial Ruelle resonances [2])**.** *Let $A \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. There exists a Banach space $\mathcal{B}$ and dense set $\mathcal{G} \subset \mathcal{B}$ such that for all $\psi \in \mathcal{G}$ there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$*

$$\sigma\left(\mathcal{K}_{A+\epsilon\psi}\right) \setminus \{0, 1\} \neq \varnothing.$$

Here σ denotes the spectrum. It is then a direct consequence that generic small perturbations of A are not C^1-conjugated to a linear toral automorphism.
I then moved on to study the ergodic average of minimal stable horocycle flows h_ρ, arising from Anosov flows g_α on compact Riemannian manifolds M of dimension 3 without boundary. Both flows are connected by

$$g_\alpha \circ h_\rho = h_{\tau(\rho,\alpha,\cdot)} \circ g_\alpha, \quad \alpha, \rho \in \mathbb{R},$$

where $\tau(\rho, \alpha, \cdot)$ is what I call the pointwise renormalization time. In the case that g_α is the geodesic flow of a surface of constant negative sectional curvature one has $\tau(\rho, \alpha, \cdot) = \rho \exp(-\alpha)$ [13].
I should emphasize that I accomplish a generalization of this well-studied case to the more general situation of $C^{3+\epsilon}$ contact Anosov flows whenever $\dim M = 3$.
By passing to a suitable semigroup of weighted transfer operators, acting on a certain Banach space $\mathcal{W}$ (I constructed the flow-analogue of the space used in [5].), one is led to study the spectral properties of its generator X with domain $D(X)$. In particular, I can identify a part of the spectrum $S \subseteq \sigma(X)$ such that for the real parts it holds

$$\sup \Re\left(\sigma(X) \setminus S\right) \leqslant \min \Re S,$$

such that the following holds:

Theorem 3.2 (Expansion of horocycle average (Adam, 2017))**.** *Let $r > 3$ and assume that $\partial_\alpha \partial_\rho \tau(0, 0, \cdot) \in C^1(M)$ exists. For all $x \in M$, all $T \geqslant 0$ and all $v \in S$ there exist functionals*

$$\mathcal{O}_v \in D(X)', \quad R_{T,x} \colon C^r(M) \to \mathbb{C},$$

and complex numbers $c_v(T, x) \in \mathbb{C}$ such that for all $\varphi \in C^r(M)$ it holds

$$\int_0^T \varphi \circ h_\rho(x) \,\mathrm{d}\,\rho = \sum_{v \in S} c_v(T, x)\, \mathcal{O}_v(\varphi) + R_{T,x}(\varphi).$$

I can give upper bounds (please see my dissertation statement) on the coefficients $c_v(T,x)$ and the remainder $R_{T,x}$ analogous to the work of Livio Flaminio and Giovanni Forni [13] on cocompact subgroups of $\mathrm{PSL}_2(\mathbb{R})$.
If g_α is a C^r-contact Anosov flow on M of dimension 3 the assumptions in the theorem are satisfied whenever $r > 3$ with a non-trivial estimate on the remainder term $R_{T,x}$.
Moreover, Theorem 3.2 confirms the first part of [15, Conjecture 2.14]. However I was not yet able to show that there is a spectral gap for X which would immediately show that the ergodic average has polynomial speed of convergence.

4. Future research

In my current research of studying the horocycle flow on compact manifolds, there are still open questions I would like to answer in the near future:

- Notably, there is the problem concerning the regularity of the strong stable distribution which induces the regularity of the pointwise renormalization time. This has been already addressed by Paolo Giulietti and Carlangelo Liverani [15] in case that g_α is replaced by an Anosov diffeomorphism, using an extension to the Grassmanian. Doing the same for the flow case seems very feasible.
- Also I have not yet shown that the coefficients c_v in Theorem 3.2 are never zero for all choices of x and large T. This could be addressed by a perturbation argument. This and the first point could be established within the first nine months.
- A seemingly more complicated question is to show the existence of a spectral gap for the generator. However we know by the work of Dimitry Dolgopyat [10] that Hölder observables admit exponential decay of correlation (at least if r is large) in the presence of the weight of maximal entropy.
- Also the problem in higher dimensions is not well-posed since a global one-dimensional horocycle flow may not exist. One knows however [8] that an extension of the horocycle foliation of Anosov flows to higher dimension is uniquely ergodic. Any extension of my problem to treat such higher dimensional foliations would certainly be of interest. This or the third point could be established within one year which would leave the other problem for the remaining research time.

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