RESONANCES FOR ANOSOV DIFFEOMORPHISMS

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Supplement

An illustrative process of measure theoretical mixing is the stirring of two initially separated liquids in a glass. As the stirring continuous we may observe that the two liquids uniformly distributes within the glass. We suppose that every repetition of the experiment (possibly with different initial conditions) will lead to the same observation. Let X be the (locally compact Hausdorff) space within the glass. Let $T\colon X \to X$ be the stirring map. Then it is reasonable to ask if for all $A, B \subseteq X$

$$\lim_{n\to\infty} \frac{\mu(T^n(A)\cap B)}{\mu(B)} = \mu(A), \tag{1}$$

where μ is a (complete) T-invariant probability measure (defined on the σ-algebra of Borel sets over X). Later on, we may not be too much concerned about the probability interpretation of the measures except that they are finite, e.g. $\mu(X) = 1$. The relation (1) is called *strong* 2*-mixing*. (In this context there is *Rokhlin's problem* [5, p.50], e.g. whether strong 2-mixing implies always strong 3-mixing, which is in general an open question.) Clearly, this can only happen if T is somehow expansive. An example of such a process is the expansive map $F\colon S^1 \to S^1\colon z \mapsto z^2$ for the normalized Lebesgue measure of the angle [3, p.77, Proposition 4.4.2]. If we want to reverse the stirring we have to require T to be invertible. Since we assumed T to have some sort of expansion we now also have to require some sort of contraction. The C^1-class of Anosov diffeomorphisms provides maps which have this property in the following sense. Let X be a compact Riemannian manifold. A diffeomorphism $T\colon X \to X$ is *Anosov* [10, Section I.3.] if it admits a splitting (continuous on the base) of the tangent space $TX = E_- \oplus E_+$ such that it exist $C \geq 1$, $0 < \gamma < 1$ such that for all $n \in \mathbb{N}$

$$\begin{array}{ll} \|\mathrm{D}\, T^n v\| \leq C\gamma^n \|v\|, & \text{for all } v \in E_- \\ \|\mathrm{D}\, T^{-n} v\| \leq C\gamma^n \|v\|, & \text{for all } v \in E_+ \end{array}. \tag{2}$$

Such a splitting is called *uniform hyperbolic* because the constants C, γ are independent of the base points. A famous example is the linear map $M \coloneqq \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, also called Arnold's cat map, which preserves the Lebesgue measure on $\mathbb{T}^2$. This map induces a linear automorphism on the flat 2-torus $\mathbb{T}^2 \coloneqq \mathbb{R}^2/\mathbb{Z}^2$ because it preserves $\mathbb{Z}^2$. The eigenvalues of M are $\lambda_+ = \frac{1}{2}(3+\sqrt{5})$ and $\lambda_- = \frac{1}{2}(3-\sqrt{5}) = \lambda_+^{-1}$. The tangent space $T\mathbb{T}^2 \cong \mathbb{R}^2$ admits a uniform hyperbolic splitting (2) into the eigenspaces of M. Back to the problem of mixing, using indicator functions 1_A for

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sets $A \subseteq X$, we can write $\mu(A) = \int_X 1_A \,\mathrm{d}\,\mu$. Together with T-invariance of μ this gives

$$\begin{aligned}\mu(T^n(A) \cap B) - \mu(A)\mu(B) &= \int_X 1_{T^n(A)} \cdot 1_B \,\mathrm{d}\,\mu - \int_X 1_A \,\mathrm{d}\,\mu \int_X 1_B \,\mathrm{d}\,\mu \\ &= \int_X 1_A \cdot 1_B \circ T^n \,\mathrm{d}\,\mu - \int_X 1_B \,\mathrm{d}\,\mu \int_X 1_A \,\mathrm{d}\,\mu.\end{aligned}$$

This difference approximates the limit in (1). Now consider a Hilbert space $\mathcal{H}(\mu)$ consisting of (hyper)function vectors such that for all $g \in \mathcal{H}(\mu)$, $f \in \mathcal{H}(\mu)^*$ we have $\int_X fg \,\mathrm{d}\,\mu < \infty$ (we identify an element of the dual space $\mathcal{H}(\mu)^*$ with its defining generalized density). Replacing 1_A and 1_B with f and g, respectively, this leads to the correlation function

$$C_{f,g}(n, T, \mu) \coloneqq \int_X f \cdot g \circ T^n \,\mathrm{d}\,\mu - \int_X f \,\mathrm{d}\,\mu \int_X g \,\mathrm{d}\,\mu_1,$$

where $\mathrm{d}\,\mu_1 = \rho \,\mathrm{d}\,\mu$ such that $\rho \in \mathcal{H}(\mu)^*$ and $T^*\mu_1 = \mu_1$ is fixed (by the push-forward of T). In our applications μ_1 is the unique SRB-measure. The correlation captures the asymptotic independence of the *observables* f and g under the evolution of T. In the considered examples of the maps K and M either $X = S^1$ or $X = \mathbb{T}^2$, respectively, $\mathcal{H}(\mu) = L_2(\mathbb{T}^2, \lambda)$ is the space of square-integrable functions on $\mathbb{T}^2$ and $\mu = \mu_1 = \lambda$, the Lebesgue measure. Using Fourier series to represent f, g, one finds $C_{f,g}(n, K, \lambda) = o_{f,g}(1)$ and $C_{f,g}(n, M, \lambda) = o_{f,g}(1)$ as $n \to \infty$. The correlation decays at least arbitrarily slow to 0. (There are however certain observables for which the decay is exponential, e.g. for simple or smooth functions.) We conclude that T and M are strongly 2-mixing with respect to Lebesgue measure λ.
From now on we assume $X = \mathbb{T}^2$. The subsets $H_r \subset L_2(\mathbb{T}^2, \lambda)$, $r > 0$, of square-integrable functions, holomorphic on an annulus $\mathbb{T}^2 + \mathrm{i}\,(-r, r)^2$, are again Hilbert spaces, so-called Hardy-Hilbert spaces. Those spaces contain no non-trivial simple function. For every $f \in H_r$ let $c_m(f)$, $m \in \mathbb{Z}^2$, denote its Fourier coefficients. They fulfill the summability condition $\sum_{m \in \mathbb{Z}} |c_m(f)|^2 \exp(4\pi r\,|m|) < \infty$, where $|m| = |m_1| + |m_2|$. An analogous use of Fourier representation as before allows us now to conclude that $C_{f,g}(n, M, \lambda) = O_{f,g}(\exp(-r\lambda_+^n))$ as $n \to \infty$ for all $f, g \in H_r$. Hence for such regular observables the decay of correlation is super-exponential (in fact "2-exponential") [2, Section 4.1.1].
What happens now to the decay rate if we consider (small) real-analytic perturbations T of M (not necessarily preserving λ)? The composition operator $\mathcal{K}_T g \coloneqq g \circ T$, acting on a suitable Hilbert space $\mathcal{H}(\lambda) = \mathcal{H}_{r,M}$, allows us to study the correlation from a functional analytic point of view. This separable Hilbert space contains H_r as a dense subspace. The operator $\mathcal{K}_T \colon \mathcal{H}_{r,M} \to \mathcal{H}_{r,M}$ is of trace class. In particular, the spectrum $\sigma(\mathcal{K}_T)$ is countable and contains at most one accumulation point at 0. The trace $\operatorname{tr} \mathcal{K}_T = \sum_{\lambda \in \sigma(\mathcal{K}_T)} \operatorname{ord}(\lambda)\lambda$ is well-defined, where $\operatorname{ord}(\lambda)$ is the dimension of the generalized eigenspace of $\lambda \in \sigma(\mathcal{K}_T)$ [4, §2, Exercise 20], [7, II, §1, n°1, p.4]. The Fredholm determinant satisfies $\det(1 - z\mathcal{K}_T) = \prod_{\lambda \in \sigma(\mathcal{K}_T)} (1 - z\lambda)^{\operatorname{ord}(\lambda)}$. It equals the analytically continued dynamical determinant of T

$$d_T(z) \coloneqq \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n(x) = x} |\det(1 - \mathrm{D}_x T^n)|^{-1}, \quad z \in \mathbb{C},$$

connected through the trace formula

$$\operatorname{tr}(\mathcal{K}_T^n) = \sum_{T^n(x)=x} \left|\det\left(1 - \mathrm{D}_x T^n\right)\right|^{-1}.$$

The right-hand side is invariant under C^1-conjugation of T. It admits a (Dolbeault) cohomological interpretation given by the holomorphic Lefschetz fixed-point formula [6, p.422]. Moreover, the non-zero eigenvalues of $\mathcal{K}_T$ are the reciprocals of the poles ϱ - the Ruelle resonances of T [8] - of the meromorphic continuation of the Z-transform of the correlation function. We assume that 1 is the spectral radius of $\mathcal{K}_T$. Then it holds for all $|z| < 1$ and all $f, g \in H_r$ such that $\int_{\mathbb{T}^2} g \,\mathrm{d}\,\mu_1 = 0$ where $\mathcal{K}_T^* \mu_1 = \mu_1$ is a fixed dual vector (in other words g is orthogonal with respect to the generalized density ρ of μ_1)

$$\sum_{n=0}^{\infty} z^n C_{f,g}(n, T, \lambda) = \int_{\mathbb{T}^2} f \cdot \sum_{n=0}^{\infty} z^n \mathcal{K}_T^n g \,\mathrm{d}\,\lambda = \int_{\mathbb{T}^2} f \cdot \frac{1}{1 - z\mathcal{K}_T} g \,\mathrm{d}\,\lambda.$$

Note that $C_{f,g}(n, T, \lambda) = O(\|f\|_{\mathcal{H}_{r,M}^*} \|g\|_{\mathcal{H}_{r,M}})$ by boundedness of $\mathcal{K}_T$ (bounded in operator norm by 1). The right-hand side is analytic in $z \in \mathbb{C}$ if $\frac{1}{z} \notin \sigma(\mathcal{K}_T)$ and has possible poles at $z = \frac{1}{\lambda}$ of order at most $\operatorname{ord}(\lambda)$ for all $\lambda \in \sigma(\mathcal{K}_T)$. In the case $T = M$ there are no resonances. From the existence of a resonance ϱ such that $|\varrho| > 1$ and that 1 is the unique and simple peripheral eigenvalue one deduces that the decay of correlation is exponential (e.g. $C_{f,g}(n, T, \lambda) = O(|\varrho|^{-n})$, using the inverse Z-transform). Note that $1 \cdot \mu_1$ is just the projector on the eigenspace of the eigenvalue 1 and μ_1 is the SRB-measure. It appears in a contracted form as the term after the minus-sign in the definition of the correlation function if we drop the orthogonality condition on g. For the considered perturbations [1], [9] at least one non-trivial resonance of T appears and 1 is a simple peripheral eigenvalue and it is the only one on the unit circle.

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