GENERIC NON-TRIVIAL RESONANCES FOR ANOSOV DIFFEOMORPHISMS

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Abstract. We study real analytic perturbations of hyperbolic linear automorphisms on the 2-torus. The Koopman and the transfer operator are nuclear of order 0 when acting on a suitable Hilbert space. We show the generic existence of non-trivial Ruelle resonances for both operators. We prove that some of the perturbations preserve the volume and some of them do not.

1. Introduction

Let $T\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a real analytic Anosov diffeomorphism. We define the Ruelle resonances of T to be the zeroes of the (holomorphically continued in $z \in \mathbb{C}$) dynamical determinant

$$d_T(z) := \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n(x)=x} |\det(\mathbb{1} - \mathrm{D}_x T^n)|^{-1} . \tag{1}$$

It is well-known (e.g. combining (1) and Lemma A.1) that 1 is the only resonance if T is a hyperbolic linear toral automorphism M. A subset of the Banach space of $\mathbb{T}^2$-preserving maps, holomorphic and uniformly bounded on some annulus, is called generic if it is open and dense. We show in Theorem 4.3, using an idea of Naud [13], that there is such a set $\mathcal{G}$ so that for all $\psi \in \mathcal{G}$, appropriately scaled, the Anosov diffeomorphism $M + \psi$ admits non-trivial Ruelle resonances. For this, we construct a Hilbert space of anisotropic generalized functions on which the transfer operator $\mathcal{L}_T f := (f/|\det \mathrm{D}T|) \circ T^{-1}$ is nuclear with its Fredholm determinant equal to d_T. Moreover, we prove that some of those generic perturbations preserve the volume while some do not.

The expanding case is easier and was initially studied by Ruelle [14]. More recently, Bandtlow et. al [2], [20] calculated the resonances of real analytic expanding maps $T\colon S \to S$ on the unit circle S explicitly for Blaschke products. Their transfer operator acts on the Hardy space of holomorphic functions on the annulus. (See also Keller and Rugh [11] in the differentiable category.)

In the hyperbolic setting, Rugh proved the holomorphy of the dynamical determinant of real analytic Anosov diffeomorphisms on surfaces [15], [16]. The idea was generalized by Fried to hyperbolic flows in all dimensions [5]. The detailed study of

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anisotropic Banach spaces in the hyperbolic case started with the pioneering work of [3] (in the differentiable setting) and is now a well established tool, see e.g. [1] and [7].
Faure and Roy [4] later addressed real analytic perturbations of hyperbolic linear toral automorphisms on the two-dimensional torus, considering an anisotropic complex Hilbert space, which had already been briefly discussed by Fried [5, Sect 8, I]. Our approach is based on this construction and strongly relies on an idea suggested by Naud [13]. We put the transfer operator central in our analysis. We introduce an anisotropic Hilbert space (Definition 2.4) in Section 2.
In Section 3, we rephrase a result from Faure and Roy [4, Theorem 6] to show that the Koopman operator $\mathcal{K}_T f \coloneqq f \circ T$ is nuclear of order 0 when acting on our anisotropic Hilbert space.
In Section 4, we use this result and an idea of Naud [13] to show that the Koopman operator admits non-trivial Ruelle resonances under a small generic perturbation of the dynamics.
In Section 5, we consider the adjoint of the Koopman operator, which is just the transfer operator, acting on the dual Hilbert space and obtain our final results.
In the Appendix, we recall two needed basic properties of integer matrices (seen as linear maps on the torus) and provide a sufficient condition for determinant preserving perturbations of differentiable real maps.

In principal the analogous problem on any higher dimensional torus can be treated with the presented method. However, one has to modify slightly the used space from Section 2 if the linear toral automorphism has non-trivial Jordan blocks.

Blaschke products were recently generalized to the hyperbolic setting by Slipantschuk et al. [19] who calculate the entire spectrum of these real analytic Anosov volume preserving diffeomorphisms explicitly.

2. An anisotropic Hilbert space

We denote the flat 2-torus by $\mathbb{T}^2 \coloneqq \mathbb{R}^2/\mathbb{Z}^2$. We embed $\mathbb{T}^2$ into the standard polyannulus in $\mathbb{C}^2$ and set for each $r > 0$

$$\mathbb{A}_r \coloneqq \mathbb{T}^2 + \mathrm{i}\,(-r, r)^2.$$

We see $\mathbb{A}_r$ as a submanifold of $\mathbb{C}^2$. The Hilbert space $L_2\left(\mathbb{T}^2\right)$ is equipped with the canonical Lebesgue measure on $\mathbb{T}^2$. This space admits an orthonormal Fourier basis given by

$$\varphi_n \colon \mathbb{T}^2 \to \mathbb{C} \colon x \mapsto \exp\left(\mathrm{i}\, 2\pi n^* x\right), \quad n \in \mathbb{Z}^2, \tag{2}$$

where n^* is the canonical dual of n. We recall a construction from Faure and Roy [4] for a complex Hilbert space $\mathcal{H}_{A_{M,c}}$. This space also has been described briefly by Fried as an "ad hoc example" [5, Sect. 8, I.] of a generalized function space. The construction will be based on:

Definition 2.1 (Hardy space $H_2\left(\mathbb{A}_r\right)$). *For each $r > 0$ and each holomorphic function $f \colon \mathbb{A}_r \to \mathbb{C}$, we define the norm*

$$\|f\|_{H_2(\mathbb{A}_r)} \coloneqq \sup_{y \in (-r,r)^2} \left(\int_{\mathbb{T}^2} \left| f\left(x + \mathrm{i}\, y\right) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Then we set

$$H_2(\mathbb{A}_r) := \left\{ f\colon \mathbb{A}_r \to \mathbb{C} \;\middle|\; f \text{ holomorphic}, \; \|f\|_{H_2(\mathbb{A}_r)} < \infty \right\}.$$

The space $H_2(\mathbb{A}_r)$ is the 2-dimensional analogue of the Hardy space studied in [18, p. 4]. It admits a Fourier basis given by

$$\vartheta_n^r\colon \mathbb{A}_r \to \mathbb{C}\colon x \mapsto \exp(-2\pi r\|n\|)\,\varphi_n, \quad n \in \mathbb{Z}^2,$$

where $\|z\| := |z_1| + |z_2|$ for all $(z_1, z_2) =: z \in \mathbb{C}^2$ and $z \in \mathbb{T}^2$. With this choice of norm, the Fourier basis is orthonormal. Under the canonical isomorphism $L_2(\mathbb{T}^2) \cong L_2(\mathbb{T}^2)^*$, we have the isomorphism

$$(\vartheta_n^r)^* \cong \vartheta_n^{-r}. \tag{3}$$

A matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ is called hyperbolic if its eigenvalues do not lie on the unit circle. We denote by E_M^+ the eigenspace for the eigenvalue of modulus $\lambda_M > 1$ and by E_M^- the eigenspace of the eigenvalue of modulus λ_M^{-1}. We decompose $y \in \mathbb{R}^2$ uniquely as

$$y = y_M^+ + y_M^- \quad \text{with} \quad y_M^+ \in E_{M^*}^+ \quad \text{and} \quad y_M^- \in E_{M^*}^-. \tag{4}$$

We have

$$\|M^* y_M^+\| = \lambda_M \|y_M^+\| \quad \text{and} \quad \|M^* y_M^-\| = \lambda_M^{-1} \|y_M^-\|. \tag{5}$$

Definition 2.2 (Scaling map $A_{M,c}$)**.** *Let $c > 0$, and $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. For every $n \in \mathbb{Z}^2$, we set, recalling (2),*

$$A_{M,c}\varphi_n := \exp\left(-2\pi c\left(\|n_M^+\| - \|n_M^-\|\right)\right)\varphi_n.$$

Lemma 2.3 (Continuous embedding of $H_2(\mathbb{A}_r)$)**.** *Let $c > 0$ and let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. Then the map $A_{M,c}$ can be extended by continuity to an injective linear map*

$$A_{M,c}\colon H_2(\mathbb{A}_c) \to L_2(\mathbb{T}^2),$$

bounded in operator norm by 1.

Proof. By Definition 2.2, for each $f \in H_2(\mathbb{A}_c)$ we have

$$\begin{aligned}\|A_{M,c}f\|^2_{L_2(\mathbb{T}^2)} &= \sum_{n\in\mathbb{Z}^2} |\varphi_n^* A_{M,c} f|^2 = \sum_{n\in\mathbb{Z}^2} \exp\left(-4\pi c\left(\|n_M^+\| - \|n_M^-\|\right)\right) |\varphi_n^* f|^2 \\ &= \sum_{n\in\mathbb{Z}^2} \exp\left(-4\pi c\left(\|n_M^+\| - \|n_M^-\| + \|n\|\right)\right) |\vartheta_n^{c\,*} f|^2,\end{aligned}$$

where we used (3) in the last step. Using the triangle inequality, we find

$$\|n_M^+\| - \|n_M^-\| + \|n\| \geq 0.$$

Hence, it holds

$$\sum_{n\in\mathbb{Z}^2} \exp\left(-4\pi c\left(\|n_M^+\| - \|n_M^-\| + \|n\|\right)\right) |\vartheta_n^{c\,*} f|^2 \leq \|f\|^2_{H_2(\mathbb{A}_c)}.$$

Injectivity follows since $A_{M,c}$ is invertible on the Fourier basis of $L_2(\mathbb{T}^2)$. □

The image of $H_2(\mathbb{A}_c)$ under $A_{M,c}$ is dense in $L_2(\mathbb{T}^2)$ since it contains all Fourier polynomials.

Definition 2.4 (Hilbert space $\mathcal{H}_{A_{M,c}}$)**.** *Let $c>0$ and let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. Let $A_{M,c}$ be the map given by Definition 2.2. Then we set*

$$\mathcal{H}_{A_{M,c}} := \textit{closure of } H_2(\mathbb{A}_c) \textit{ with respect to the norm } \|A_{M,c}\cdot\|_{L_2(\mathbb{T}^2)},$$

and extend $A_{M,c}$ by continuity to a linear map

$$A_{M,c}\colon \mathcal{H}_{A_{M,c}} \to L_2\left(\mathbb{T}^2\right).$$

As a direct consequence of this construction, the scalar product on $\mathcal{H}_{A_{M,c}}$ satisfies

$$\langle\cdot,\cdot\rangle_{\mathcal{H}_{A_{M,c}}} : \mathcal{H}_{A_{M,c}} \times \mathcal{H}_{A_{M,c}} \to \mathbb{C}\colon\ (f,g) \mapsto \langle A_{M,c}f, A_{M,c}g\rangle_{L_2(\mathbb{T}^2)}.$$

An orthonormal Fourier basis of $\mathcal{H}_{A_{M,c}}$ is given by

$$\varrho_n := A_{M,c}^{-1}\varphi_n, \quad n \in \mathbb{Z}^2. \tag{6}$$

Lemma 2.5 (Dual space of $\mathcal{H}_{A_{M,c}}$)**.** *Under the canonical isomorphism $L_2\left(\mathbb{T}^2\right) \cong L_2\left(\mathbb{T}^2\right)^*$, the dual space $\mathcal{H}_{A_{M,c}}^*$ is isomorphic to $A_{M,c}^2\mathcal{H}_{A_{M,c}}$.*

Proof. Under the canonical isomorphism $L_2\left(\mathbb{T}^2\right) \cong L_2\left(\mathbb{T}^2\right)^*$, we have for each n_1, $n_2 \in \mathbb{Z}^2$, using (6),

$$\varphi_{n_1}^*\left(\varphi_{n_2}\right) = \varphi_{n_1}^*\left(A_{M,c}\varrho_{n_2}\right) = \left(A_{M,c}\varphi_{n_1}\right)^*\left(\varrho_{n_2}\right) = \left(A_{M,c}^2\varrho_{n_1}\right)^*\left(\varrho_{n_2}\right).$$

□

Remark 2.6. *By Lemma 2.5, we associate to every linear functional $f^* \in \mathcal{H}_{A_{M,c}}^*$ a unique vector $f \in A_{M,c}^2\mathcal{H}_{A_{M,c}}$. Then, for every $g \in \mathcal{H}_{A_{M,c}}$, the product fg is absolutely integrable with respect to the Lebesgue measure on $\mathbb{T}^2$.*

The decomposition in (4) defines two cones

$$C_M^+ := \left\{y \in \mathbb{R}^2 \;\middle|\; \left\|y_M^+\right\| \geq \left\|y_M^-\right\|\right\} \quad \text{and} \quad C_M^- := \left\{y \in \mathbb{R}^2 \;\middle|\; \left\|y_M^+\right\| \leq \left\|y_M^-\right\|\right\}.$$

Example 2.7. *We let $M = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, then $\lambda_M = 2+\sqrt{3}$. An eigenvector for λ_M for M^* is $\left(1+\sqrt{3},1\right)$ and an eigenvector for λ_M^{-1} is $\left(1-\sqrt{3},1\right)$. The two subspaces $E_{M^*}^+$ and $E_{M^*}^-$ and the two cones C_M^+ and C_M^- are shown in Figure 1.*

We set

$$\mathcal{H}_{A_{M,c}}^+ := \left\{ \sum_{n \in C_M^+ \cap \mathbb{Z}^2} \langle \varrho_n, f\rangle_{\mathcal{H}_{A_{M,c}}} \varrho_n \;\middle|\; f \in \mathcal{H}_{A_{M,c}} \right\} \text{ and}$$

$$\mathcal{H}_{A_{M,c}}^- := \left\{ \sum_{n \in C_M^- \cap \mathbb{Z}^2} \langle \varrho_n, f\rangle_{\mathcal{H}_{A_{M,c}}} \varrho_n \;\middle|\; f \in \mathcal{H}_{A_{M,c}} \right\}.$$

Hence, we have $\mathcal{H}_{A_{M,c}} = \mathcal{H}_{A_{M,c}}^+ + \mathcal{H}_{A_{M,c}}^-$. Comparing for each $n \in C_M^-$ the Fourier basis ϱ_n with φ_n, it follows immediately that $\mathcal{H}_{A_{M,c}}^- \subset L_2\left(\mathbb{T}^2\right)$. For each $n \in C_M^+$, comparing the Fourier basis ϱ_n with $\vartheta_n^{c\,*}$, using (3), shows $\mathcal{H}_{A_{M,c}}^+ \subset H_2\left(\mathbb{A}_c\right)^*$. We conclude therefore that $\mathcal{H}_{A_{M,c}}$ contains linear functionals which do not belong to $L_2\left(\mathbb{T}^2\right)$. By construction, the space $\mathcal{H}_{A_{M,c}}$ is a rigged Hilbert space, i.e.:

$$H_2\left(\mathbb{A}_c\right) \subset \mathcal{H}_{A_{M,c}} \subset H_2\left(\mathbb{A}_c\right)^*. \tag{7}$$

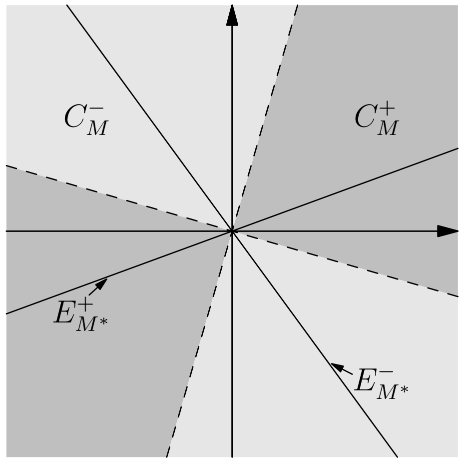

FIGURE 1. The map M is from Example 2.7. The dark gray area is the cone C_M^+ which contains the subspace $E_{M^*}^+$. The light gray area is the cone C_M^- and contains $E_{M^*}^-$. A part $y \in \mathbb{R}^2$ belongs to the dashed lines if and only if $\|y_M^+\| = \|y_M^-\|$.

Remark 2.8. *We note that in the construction of $\mathcal{H}_{A_{M,c}}$, the expanding and contracting directions appear in the dual coordinates $n \in \mathbb{Z}^2$ of the Fourier basis (6). This distinguishes $\mathcal{H}_{A_{M,c}}$ from the space of Rugh [15] where expanding and contracting coordinates are spatial. We observe*

$$n^* x = \left(n_M^+ + n_M^-\right)^* \left(x_{M^*}^+ + n_{M^*}^-\right) = \left(n_M^+\right)^* x_{M^*}^+ + \left(n_M^-\right)^* x_{M^*}^-.$$

Hence, we can rewrite (6) as

$$\begin{aligned} \varrho_n(x) &= \exp\left(2\pi c \left(\|n_M^+\| - \|n_M^-\|\right)\right) \exp\left(\mathrm{i}\, 2\pi n^* x\right) \\ &= \exp\left(2\pi c \|n_M^+\|\right) \exp\left(\mathrm{i}\, 2\pi \left(n_M^+\right)^* x_{M^*}^+\right) \\ &\quad \times \exp\left(-2\pi c \|n_M^-\|\right) \exp\left(\mathrm{i}\, 2\pi \left(n_M^-\right)^* x_{M^*}^-\right). \end{aligned} \tag{8}$$

It is tempting to think of the ϱ_n as basis elements for a tensor product space of a Hardy space on an annulus, with the dual of such a Hardy space. However, we cannot use ϱ_n as such a basis since n_M^+ and n_M^- are not independent of each other. Nevertheless, we can decompose $\mathcal{H}_{A_{M,c}}$ into two generalized Hardy spaces as follows. We define four norms

$$\mu_j(f) \coloneqq \sup_{y \in A_j} \left(\int_{\mathbb{T}^2} |f(x + \mathrm{i}\, y)|^2 \, \mathrm{d}x\right)^{\frac{1}{2}}, \; f \in L_2\left(\mathbb{T}^2\right), \; j \in \{1,2,3,4\}, \; \textit{where}$$

$$A_1 \coloneqq \left\{ y \in \mathbb{R}^2 \;\middle|\; y_{M^*}^- \in (-c,c)^2,\, y_{M^*}^+ \in (c,\infty)^2 \right\},$$

$$A_2 \coloneqq \left\{ y \in \mathbb{R}^2 \;\middle|\; y_{M^*}^- \in (-c,c)^2,\, y_{M^*}^+ \in (-\infty,-c)^2 \right\},$$

$$A_3 \coloneqq \left\{ y \in \mathbb{R}^2 \;\middle|\; y_{M^*}^- \in (-c,c)^2,\, y_{M^*}^+ \in (c,\infty) \times (-\infty,-c) \right\},$$

$$A_4 \coloneqq \left\{ y \in \mathbb{R}^2 \;\middle|\; y_{M^*}^- \in (-c,c)^2,\, y_{M^*}^+ \in (-\infty,-c) \times (c,\infty) \right\}.$$

For all $f \in L_2(\mathbb{T}^2)$ the norms $\mu_j(f)$ cannot be finite but they are so at least for some Fourier polynomials. The spaces H_j, $j \in \{1,2,3,4\}$, are the completions with respect to the norms μ_j above. E.g. using μ_1, it holds for all $f \in H_1$

$$\begin{aligned}\mu_1(f)^2 &= \sup_{y\in A_1}\left(\int_{\mathbb{T}^2} |f(x+\mathrm{i}\,y)|^2\,\mathrm{d}x\right) = \sup_{y\in A_1}\sum_{n\in\mathbb{Z}^2}\exp\left(-4\pi n^* y\right)|\varphi_n^* f|^2\\ &= \sup_{y\in A_1}\sum_{n\in\mathbb{Z}^2}\exp\left(-4\pi\left(n_M^-\right)^* y_{M^*}^- - 4\pi\left(n_M^+\right)^* y_{M^*}^+\right)|\varphi_n^* f|^2\\ &= \sup_{y_{M^*}^+\in(c,\infty)^2}\sum_{n\in\mathbb{Z}^2}\exp\left(4\pi c\left\|n_M^-\right\| - 4\pi\left(n_M^+\right)^* y_{M^*}^+\right)|\varphi_n^* f|^2\\ &= \sum_{\substack{n\in\mathbb{Z}^2\\ n_M^+\in[0,\infty)^2}}\exp\left(4\pi c\left\|n_M^-\right\| - 4\pi c\left\|n_M^+\right\|\right)|\varphi_n^* f|^2 = \sum_{\substack{n\in\mathbb{Z}^2\\ n_M^+\in[0,\infty)^2}}|\varphi_n^* A_{M,c} f|^2.\end{aligned}$$

Similar calculations for the other three norms show then that the spaces H_j, $j \in \{1,2,3,4\}$ disjointly partition the space $\mathcal{H}_{A_{M,c}}$ with respect to the dual coordinate up to $n = 0$. Since E_M^+ is a one dimensional subspace of $\mathbb{R}^2$, always two of the spaces contain only the constant functions (note that $n_M^+ = 0$ implies $n = 0$), say, H_3 and H_4. Then all vectors in the spaces H_1 and H_2 are holomorphic functions on $\mathbb{T}^2 + \mathrm{i}\,A_1$ and on $\mathbb{T}^2 + \mathrm{i}\,A_2$, respectively.

3. The Koopman operator is nuclear

We set for each $r > 0$

$$\mathcal{T}_r := \left\{T\colon \mathbb{T}^2 \to \mathbb{T}^2 \;\middle|\; T \text{ extends holomorphically and boundedly on } \mathbb{A}_r\right\}.$$

For every $T \in \mathcal{T}_r$ the Koopman operator

$$\mathcal{K}_T\colon L_2\left(\mathbb{T}^2\right) \to L_2\left(\mathbb{T}^2\right) : f \mapsto f \circ T$$

is well-defined by differentiability of T. It is well-known that the operator $\mathcal{K}_T$ acting on $L_2(\mathbb{T}^2)$ is not compact. We say that two maps f, $g \in \mathcal{T}_r$ are C^1-close if the distance

$$d(f,g) := \sup_{z\in\mathbb{A}_r}\|f(z) - g(z)\| + \sup_{z\in\mathbb{A}_r}\|\mathrm{D}_z f - \mathrm{D}_z g\|$$

is small. In this section we revisit the proof of Faure and Roy [4]. They showed that $\mathcal{K}_T$, acting on the Hilbert space $\mathcal{H}_{A_{M,c}}$, (see Definition 2.4), is nuclear of order 0 if T is sufficiently C^1-close to a hyperbolic matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ for some $c > 0$. We recall that a linear operator $\mathcal{L}\colon \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|_{\mathcal{H}}$ is called nuclear of order 0 if it can be written as a sum $\mathcal{L} = \sum_{n\in\mathbb{N}} d_n \psi_{1,n}\psi_{2,n}^*$ with $\inf\left\{p > 0 \;\middle|\; \sum_{n\in\mathbb{N}}|d_n|^p < \infty\right\} = 0$ and $\psi_{1,n}, \psi_{2,n} \in \mathcal{H}$, $\|\psi_{1,n}\|_{\mathcal{H}}, \|\psi_{2,n}\|_{\mathcal{H}} \le 1$, $d_n \in \mathbb{C}$, $n \in \mathbb{N}$ [8, II, §1, n°1, p.4]. In particular, such an operator is trace class, hence bounded and admits a trace $\operatorname{Tr}\mathcal{L} := \sum_{n\in\mathbb{N}} e_n^*\mathcal{L}e_n$, invariant for any choice of orthonormal basis e_n, $n \in \mathbb{N}$ of $\mathcal{H}$. Moreover, one can show that $\operatorname{Tr}\mathcal{L}$ equals the sum, including multiplicity (dimension of corresponding generalized eigenspace), over the spectrum $\operatorname{sp}(\mathcal{L})$ of $\mathcal{L}$. The Fredholm determinant, defined for small enough $z \in \mathbb{C}$ by

$$\det(1 - z\mathcal{L}) := \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\operatorname{Tr}\mathcal{L}^n\right), \tag{9}$$

extends to an entire function in z, having zeroes at $z = \lambda^{-1}$, $\lambda \in \mathrm{sp}(\mathcal{L}) \setminus \{0\}$ of same order as the multiplicity of λ.

Theorem 3.1 (Nuclearity of $\mathcal{K}_T$)**.** *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and let $r > 0$. Then there exist constants $\delta_M > 0$ and $0 < c_1 < r$ such that for each $T \in \mathcal{T}_r$ with $d(T, M) \leq \delta_M$ the map*

$$\mathcal{K}_T \colon \mathcal{H}_{A_{M,c_1}} \to \mathcal{H}_{A_{M,c_1}} : f \mapsto f \circ T$$

defines a nuclear operator of order 0. In particular, there exists $c_2 > 0$ depending only on c_1, M, and $\|\cdot\|$ so that for each n_1, $n_2 \in \mathbb{Z}^2$

$$\left| \langle \varrho_{n_1}, \mathcal{K}_T \varrho_{n_2} \rangle_{\mathcal{H}_{A_{M,c_1}}} \right| \leq \exp\left(-2\pi c_2 \left(\|n_1\| + \|n_2\|\right)\right).$$

For every n_1, $n_2 \in \mathbb{Z}^2$, we set

$$I_{n_1,n_2}(T) \coloneqq \langle \varphi_{n_1}, \mathcal{K}_T \varphi_{n_2} \rangle_{L_2(\mathbb{T}^2)}. \tag{10}$$

Estimating this "oscillatory integral" is central for Theorem 3.1. In the case $T = M$, we have simply

$$I_{n_1,n_2}(M) = \begin{cases} 1 & \text{if } M^* n_2 = n_1 \\ 0 & \text{if } M^* n_2 \neq n_1 \end{cases}. \tag{11}$$

The strategy of the proof is as follows. We get an upper bound for $|I_{n_1,n_2}(T)|$ in Lemma 3.2, taking advantage of the holomorphicity of T. In Lemma 3.3, we compare the contribution of n_1 and n_2 in the expanding and contracting directions, using here essentially the hyperbolicity of M. Combining both results, we obtain a weaker bound on $|I_{n_1,n_2}(T)|$ in Proposition 3.4, which finally allows for the proof of Theorem 3.1.
For every $n \in \mathbb{Z}^2$ and $y \in \mathbb{R}^2$ any solution $x \in \mathbb{T}^2$ so that

$$\exp\left(-2\pi \left(n^* \mathrm{D}_x T y\right)\right) = \int_{\mathbb{T}^2} \exp\left(-2\pi \left(n^* \mathrm{D}_z T y\right)\right) \mathrm{d}z \tag{12}$$

is denoted by $x_n(y)$. Since the integrand is continuous in y such a solution exists by the Mean Value Theorem.

Lemma 3.2 (Upper bound on $|I_{n_1,n_2}(T)|$ (I))**.** *Let $r > 0$. Then, there exists $C \geq 0$ so that for each $T \in \mathcal{T}_r$ and n_1, $n_2 \in \mathbb{Z}^2$ and $y \in (-r, r)^2$, recalling (10), we have*

$$|I_{n_1,n_2}(T)| \leq \exp\left(2\pi \left(-n_2^* \mathrm{D}_{x_{n_2}(y)} T y + n_1^* y + C d(T, 0) \|y\|^3 \|n_2\|\right)\right).$$

Proof. By definition

$$I_{n_1,n_2}(T) = \langle \varphi_{n_1}, \mathcal{K}_T \varphi_{n_2} \rangle_{L_2(\mathbb{T}^2)} = \int_{\mathbb{T}^2} \exp\left(\mathrm{i}2\pi \left(n_2^* T(x) - n_1^* x\right)\right) \mathrm{d}x.$$

Since $T \in \mathcal{T}_r$, the $\mathbb{Z}^2$-invariance of the integrand follows. By holomorphicity of T on $\mathbb{A}_r$, we can change the path of integration to $x \mapsto x + \mathrm{i}y$ for every $y \in (-r, r)^2$. Therefore for any $y \in (-r, r)^2$

$$|I_{n_1,n_2}(T)| \leq \int_{\mathbb{T}^2} \exp\left(2\pi \left(n_1^* y - \Im\left(n_2^* T(x + \mathrm{i}\, y)\right)\right)\right) \mathrm{d}x,$$

where $\Im$ is the imaginary part. We expand T (or rather its lift to $\mathbb{R}^2$) at $x \in \mathbb{T}^2$ in a Taylor series to the second order. This yields

$$T(x + \mathrm{i}\, y) = T(x) + \mathrm{i}\, \mathrm{D}_x T y + P(x + \mathrm{i}\, y) + R_2(x + \mathrm{i}\, y).$$

Here, $P(x+\mathrm{i}\,y)$ is the second order term of the expansion which is $\mathbb{R}^2$-valued, and R_2 is the remainder of the series expansion. We find therefore

$$\Im T(x+\mathrm{i}y) = \mathrm{D}_x T y + \Im R_2\,(x+\mathrm{i}\,y)\,.$$

Since T is holomorphic we find a constant $C>0$ independent of T such that

$$|n_2^* R_2\,(x+\mathrm{i}\,y)| \le C d(T,0)\,\|n_2\|\,\|y\|^3.$$

We are left with the evaluation of

$$\int_{\mathbb{T}^2} \exp\left(-2\pi\,(n^*\mathrm{D}_z T y)\right) \mathrm{d}z.$$

Using (12) yields the result. □

The following abbreviation is used in the remaining section. We set for each $y\in\mathbb{R}^2$

$$|y|_M := \left\|y_M^+\right\| - \left\|y_M^-\right\|. \tag{13}$$

Lemma 3.3 (Directional inequality)**.** *Let $M\in\mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. Let $\epsilon>0$ and $\kappa\ge 0$ and let $R\colon\mathbb{R}^2\to\mathbb{R}_{\ge 0}$ be a map such that for all $z\in\mathbb{R}^2$ with $\|z\|<\epsilon$ it holds*

$$R(z)\le\kappa\,\|z\|\,.$$

Then there exists $c_M>0$ such that if $\kappa<c_M$ there exist $0<c_2<c_1<\epsilon$ such that for all $n_1,\ n_2\in\mathbb{Z}^2$ there exists $y_{n_1,n_2}\in\mathbb{R}^2$ independent of R with $\|y_{n_1,n_2}\|<\epsilon$ such that it holds

$$-c_1\left(|n_1|_M - |n_2|_M\right) - \left(n_2^* M - n_1^*\right) y_{n_1,n_2} + \|n_2\|\,R\left(y_{n_1,n_2}\right) \le -c_2\left(\|n_1\|+\|n_2\|\right).$$

Proof. We assume $0<c_2\le c_1$. For $n_1=n_2=0$ there is nothing to prove. For every $(y_1,y_2)\in\mathbb{R}^2$ we set $\|(y_1,y_2)\|_2 := \sqrt{y_1^2+y_2^2}$. We let $0<\tilde{c}_1\le 1\le\tilde{c}_2$ such that

$$\tilde{c}_2^{-1}\left(\left\|y_M^+\right\|+\left\|y_M^-\right\|\right) \le \|y\| \le \tilde{c}_1^{-1}\,\|y\|_2\,,\quad \text{for all } y\in\mathbb{R}^2. \tag{14}$$

Whenever $n_2\ne 0$ we find a linear map M_a such that $M_a n_2 = M^* n_2 - n_1$ and whenever $n_1\ne 0$ we find a linear map M_b such that $M_b n_1 = M^* n_2 - n_1$. For now we let $\tilde{\kappa}>0$ be a variable which will be fixed later on, independently of n_1 and n_2. We consider the following four cases

(a) $\|n_2\|>0$ and $\|n_2\|\ge\|n_1\|$
   (i) $\|M_a n_2\|\ge\tilde{\kappa}\,\|n_2\|$,
   (ii) $\|M_a n_2\|<\tilde{\kappa}\,\|n_2\|$,

(b) $\|n_1\|>0$ and $\|n_1\|\ge\|n_2\|$
   (i) $\|M_b n_1\|\ge\tilde{\kappa}\,\|n_1\|$,
   (ii) $\|M_b n_1\|<\tilde{\kappa}\,\|n_1\|$.

We assume Case (a)(i). For every $\delta>0$ we let

$$y = \delta M_a \frac{n_2}{\|n_2\|}.$$

It follows, using (14), that

$$-\left(n_2^* M - n_1^*\right) y = -n_2^* M_a^* y \le -\tilde{c}_1^2\,\|M_a n_2\|\,\|y\|\,. \tag{15}$$

We recall $|\cdot|_M$ from (13). Using that $c_1+c_2>0$ and that (a) holds, we estimate

$$\begin{aligned}-c_1\left(|n_1|_M - |n_2|_M\right) &\le c_1\left(\|n_1\|+\|n_2\|\right)\\ &= -c_2\left(\|n_1\|+\|n_2\|\right) + (c_1+c_2)\left(\|n_1\|+\|n_2\|\right)\\ &\le -c_2\left(\|n_1\|+\|n_2\|\right) + 2\,(c_1+c_2)\,\|n_2\|\,.\end{aligned}$$

Using (a)(i) and the assumed bound on R for $\|y\| < \epsilon$, we have

$$-c_1\left(|n_1|_M - |n_2|_M\right) - \|n_2\|\left(\tilde{c}_1^2\left\|M_a\frac{n_2}{\|n_2\|}\right\|\|y\| - R(y)\right) \leq$$

$$- c_2\left(\|n_1\| + \|n_2\|\right) + \left(2(c_1+c_2) - \left(\tilde{c}_1^2\tilde{\kappa} - \kappa\right)\|y\|\right)\|n_2\|. \tag{16}$$

We put $c_M := \tilde{c}_1^2\tilde{\kappa}$. Any value $\|y\| \in (0,\epsilon)$ can be attained by controlling δ. Assuming that $c_M > \kappa$, it follows from (15) and (16) that

$$0 < c_1 + c_2 < \frac{c_M - \kappa}{2}\epsilon. \tag{17}$$

The reasoning in Case (b)(i) is completely analogous and yields the same bounds on $c_1 + c_2$.

In Case (a)(ii) and (b)(ii), we take $y = 0$, where $R(0) = 0$ by assumption on R. We assume now Case (a)(ii). We find, using (14),

$$\left\|(M_a n_2)_M^+\right\| + \left\|(M_a n_2)_M^-\right\| \leq \tilde{c}_2\|M_a n_2\|$$

$$< \tilde{c}_2\tilde{\kappa}\|n_2\| \leq \tilde{c}_2\tilde{\kappa}\left(\left\|n_{2,M}^+\right\| + \left\|n_{2,M}^-\right\|\right). \tag{18}$$

We have

$$\left\|(M_a n_2)_M^+\right\| = \left\|M^* n_{2,M}^+ - n_{1,M}^+\right\| \quad \text{and} \quad \left\|(M_a n_2)_M^-\right\| = \left\|M^* n_{2,M}^- - n_{1,M}^-\right\|.$$

Recalling (5), this allows the estimate

$$\left\|(M_a n_2)_M^+\right\| + \left\|(M_a n_2)_M^-\right\| \geq \left\|M^* n_{2,M}^+\right\| - \left\|n_{1,M}^+\right\| - \left\|M^* n_{2,M}^-\right\| + \left\|n_{1,M}^-\right\|$$

$$\geq \lambda_M\left\|n_{2,M}^+\right\| - \lambda_M^{-1}\left\|n_{2,M}^-\right\| - \left\|n_{1,M}^+\right\| + \left\|n_{1,M}^-\right\|.$$

Together with (18) we find therefore

$$-|n_1|_M = -\left\|n_{1,M}^+\right\| + \left\|n_{1,M}^-\right\| < -\left(\lambda_M - \tilde{\kappa}\tilde{c}_2\right)\left\|n_{2,M}^+\right\| + \left(\lambda_M^{-1} + \tilde{\kappa}\tilde{c}_2\right)\left\|n_{2,M}^-\right\|.$$

We set

$$\kappa_+ := \lambda_M - \tilde{\kappa}\tilde{c}_2 - 1 \quad \text{and} \quad \kappa_- := 1 - \lambda_M^{-1} - \tilde{\kappa}\tilde{c}_2.$$

We finally estimate

$$-c_1\left(|n_1|_M - |n_2|_M\right) < -c_1\kappa_+\left\|n_{2,M}^+\right\| - c_1\kappa_-\left\|n_{2,M}^-\right\|.$$

Note that we have $\kappa_+ > \kappa_-$ because $\lambda_M > 1$. Assuming that $c_1\kappa_- \geq 2c_2$, we find

$$-c_1\kappa_-\left\|n_{2,M}^+\right\| - c_1\kappa_-\left\|n_{2,M}^-\right\| < -c_1\kappa_-\|n_2\| \leq -2c_2\|n_2\| \leq -c_2\left(\|n_1\| + \|n_2\|\right).$$

In Case (b)(ii) we consider the bounds

$$|n_2|_M + \lambda_M\left\|n_{1,M}^-\right\| - \lambda_M^{-1}\left\|n_{1,M}^+\right\| \leq \left\|\left((M^*)^{-1}M_b n_1\right)_M^+\right\| + \left\|\left((M^*)^{-1}M_b n_1\right)_M^-\right\|$$

$$\leq \tilde{c}_2\left\|(M^*)^{-1}M_b n_1\right\| < \tilde{\kappa}\tilde{c}_2\left\|(M^*)^{-1}\right\|\|n_1\|.$$

Therefore κ_- is replaced by $1-\lambda_M^{-1}-\left\|(M^*)^{-1}\right\|\tilde{\kappa}\tilde{c}_2$ which we require to be positive. Since $\left\|(M^*)^{-1}\right\|>1$, this yields the stronger conditions

$$0<\tilde{\kappa}<\frac{1-\lambda_M^{-1}}{\left\|(M^*)^{-1}\right\|\tilde{c}_2} \quad \text{and} \quad c_2 \le \frac{1-\lambda_M^{-1}-\left\|(M^*)^{-1}\right\|\tilde{\kappa}\tilde{c}_2}{2}c_1. \tag{19}$$

Any such choice for $\tilde{\kappa}$ is independent of n_1 and n_2 and fixes c_M. Using (19) for an upper bound on c_2 and (17), we find the stronger condition

$$0<c_1<\frac{c_M-\kappa}{3-\lambda_M^{-1}}\epsilon.$$

Therefore the choices of c_1 and c_2 are valid if $\kappa<c_M$. They depend only on ϵ, M and $\|\cdot\|$ and not on n_1 or n_2. □

Proposition 3.4 (Upper bound on $|I_{n_1,n_2}(T)|$ (II))**.** *Let $M\in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and let $r>0$. Then there exist constants $0<\delta_M$ and $0<c_2<c_1<r$ such that for each n_1, $n_2\in\mathbb{Z}^2$ and each $T\in\mathcal{T}_r$ with $d(T,M)\le\delta_M$ it holds that*

$$\exp\left(-2\pi c_1\left(|n_1|_M-|n_2|_M\right)\right)|I_{n_1,n_2}(T)|\le\exp\left(-2\pi c_2\left(\|n_1\|+\|n_2\|\right)\right).$$

Proof. By Lemma 3.2 there is a constant $C>0$ independent of T such that for each $y\in(-r,r)^2$ and n_1, $n_2\in\mathbb{Z}^2$ it holds that

$$|I_{n_1,n_2}(T)|\le\exp\left(2\pi\left(-n_2^*\mathrm{D}_{x_{n_2}(y)}Ty+n_1^*y+Cd(T,0)\|y\|^3\|n_2\|\right)\right). \tag{20}$$

We rewrite

$$n_2^*\mathrm{D}_{x_{n_2}(y)}Ty=n_2^*My+n_2^*\mathrm{D}_{x_{n_2}(y)}\left(T-M\right)y,$$

and set

$$R(y):=\begin{cases}\frac{n_2^*}{\|n_2\|}\mathrm{D}_{x_{n_2}(y)}\left(M-T\right)y+Cd(T,0)\|y\|^3 & \text{if } n_2\neq 0\\ 0 & \text{if } n_2=0\end{cases}.$$

Let $\delta_M>0$ and assume that $d(T,M)\le\delta_M$. We choose $0<\epsilon\le r$ sufficiently small such that for all $y\in\mathbb{R}^2$ with $\|y\|<\epsilon$ there is $\kappa>0$ such that

$$|R(y)|\le\kappa\delta_M.$$

Since $d(T,0)\le d(T,M)+d(M,0)\le\delta_M+d(M,0)$ this choice of ϵ is independent of T. Lemma 3.3 applied to M and $|R|$ gives c_1, c_2 and $y_{n_1,n_2}\in\mathbb{R}^2$ for which the right-hand side of (20) fulfills the desired inequality. □

Proof of Theorem 3.1. Proposition 3.4 yields $0<\delta_M$ and $0<c_2<c_1<r$ such that if $d(T,M)\le\delta_M$ it holds

$$C_{n_1}C_{n_2}^{-1}|I_{n_1,n_2}(T)|\le\exp\left(-2\pi c_2\left(\|n_1\|+\|n_2\|\right)\right), \tag{21}$$

where

$$C_n:=\exp\left(-2\pi c_1\left(\left\|n_M^+\right\|-\left\|n_M^-\right\|\right)\right),\quad n\in\mathbb{Z}^2.$$

We put $c:=c_1$ and M in Definitions 2.2 and 2.4, giving a linear map A_{M,c_1} and a Hilbert space $\mathcal{H}_{A_{M,c_1}}$. Recalling (6), and assuming that $\mathcal{K}_T\colon\mathcal{H}_{A_{M,c_1}}\to\mathcal{H}_{A_{M,c_1}}$ is well-defined, we have

$$\begin{aligned}\left|\langle \varrho_{n_1}, \mathcal{K}_T \varrho_{n_2} \rangle_{\mathcal{H}_{A_{M,c_1}}}\right| &= \left|\left\langle \varphi_{n_1}, A_{M,c_1} \mathcal{K}_T A_{M,c_1}^{-1} \varphi_{n_2} \right\rangle_{L_2(\mathbb{T}^2)}\right| \\ &= C_{n_1} C_{n_2}^{-1} \left| I_{n_1,n_2}(T) \right|. \end{aligned} \tag{22}$$

Using (21) to estimate the right-hand side, the bound in Theorem 3.1 follows. We next obtain well-definedness and nuclearity of order 0 of $\mathcal{K}_T$. Let $f \in \mathcal{H}_{A_{M,c_1}}$ and put $g \coloneqq A_{M,c_1} f$. We have then

$$\begin{aligned} \mathcal{K}_T f \in \mathcal{H}_{A_{M,c_1}} \Leftrightarrow A_{M,c_1}\mathcal{K}_T f \in L_2\left(\mathbb{T}^2\right) &\Leftrightarrow \sum_{n\in\mathbb{Z}^2} \left|\varphi_n^* A_{M,c_1} \mathcal{K}_T f\right|^2 < \infty \\ &\Leftrightarrow \sum_{n_1\in\mathbb{Z}^2} \left| \sum_{n_2\in\mathbb{Z}^2} \varphi_{n_1}^* A_{M,c_1} \mathcal{K}_T A_{M,c_1}^{-1} \varphi_{n_2} \varphi_{n_2}^* g \right|^2 < \infty \\ &\Leftrightarrow \sum_{n_1\in\mathbb{Z}^2} \left| \sum_{n_2\in\mathbb{Z}^2} C_{n_1} C_{n_2}^{-1} I_{n_1,n_2}(T) \varphi_{n_2}^* g \right|^2 < \infty. \end{aligned}$$

Using (21) and the Cauchy-Schwartz inequality, it follows that

$$\sum_{n_1\in\mathbb{Z}^2} \left| \sum_{n_2\in\mathbb{Z}^2} C_{n_1} C_{n_2}^{-1} I_{n_1,n_2}(T) \varphi_{n_2}^* g \right|^2 \le \left(\sum_{n\in\mathbb{Z}^2} e^{-4\pi c_2 \|n\|} \right)^2 \|g\|_{L_2(\mathbb{T}^2)}^2 < \infty.$$

This gives the well-definedness of $\mathcal{K}_T$. Now, using the Cauchy-Schwartz inequality, we have

$$\left|\langle \varrho_n, \mathcal{K}_T f \rangle_{\mathcal{H}_{A_{M,c_1}}}\right|^2 \le \sum_{m\in\mathbb{Z}^2} \left|\langle \varrho_n, \mathcal{K}_T \varrho_m \rangle_{\mathcal{H}_{A_{M,c_1}}}\right|^2 \|f\|_{\mathcal{H}_{A_{M,c_1}}}^2 .$$

Using (22) and (21) to bound $\left|\langle \varrho_n, \mathcal{K}_T \varrho_m \rangle_{\mathcal{H}_{A_{M,c_1}}}\right|$, we find a constant $C > 0$ such that

$$\left|\langle C \exp\left(2\pi c_2 \|n\|\right) \varrho_n, \mathcal{K}_T f \rangle_{\mathcal{H}_{A_{M,c_1}}}\right| \le \|f\|_{\mathcal{H}_{A_{M,c_1}}} .$$

This allows the representation of $\mathcal{K}_T$ as

$$\mathcal{K}_T f = \sum_{n\in\mathbb{Z}^2} C^{-1} \exp\left(-2\pi c_2 \|n\|\right) \langle C \exp\left(2\pi c_2 \|n\|\right) \varrho_n, \mathcal{K}_T f \rangle_{\mathcal{H}_{A_{M,c_1}}} \varrho_n,$$

from which nuclearity of order 0 follows. Finally, a brief inspection of the proofs for Lemma 3.3 and Proposition 3.4 gives the statement about the constants. □

4. Non-trivial resonances for the Koopman operator

Given any hyperbolic matrix $M \in \mathrm{SL}_2(\mathbb{Z})$, we find by Theorem 3.1 constants $0 < \delta_M$ and $c > 0$ such that for each map $T \in \mathcal{T}_r$, satisfying $d(T, M) \le \delta_M$, the operator $\mathcal{K}_T$ acting on the Hilbert space $\mathcal{H}_{A_{M,c}}$ is nuclear of order 0. Therefore it has a well-defined trace

$$\operatorname{Tr} \mathcal{K}_T \coloneqq \sum_{n\in\mathbb{Z}^2} \langle \varrho_n, \mathcal{K}_T \varrho_n \rangle_{\mathcal{H}_{A_{M,c}}} . \tag{23}$$

The map T is an Anosov diffeomorphism (for all small enough δ_M), by structural stability [9, Theorem 9.5.8]. Then the map T has the same number $N_M =$

$|\det(\mathbb{1} - M)|$ of fixed points as the matrix M. We recall a well-known result [4, Proposition 9].

Lemma 4.1 (Trace formula for $\mathcal{K}_T$)**.** *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and let $r > 0$. Then there exist constants $\delta_M > 0$ and $c > 0$ such that for each $T \in \mathcal{T}_r$ with $d(T, M) \leq \delta_M$, letting $\mathcal{K}_T$ act on $\mathcal{H}_{A_{M,c}}$, it holds*

$$\operatorname{Tr} \mathcal{K}_T = \sum_{T(x)=x} |\det(\mathbb{1} - \mathrm{D}_x T)|^{-1}.$$

For the convenience of the reader, we give a proof:

Proof. Using Theorem 3.1 gives constants $c > 0$ and $\delta_M > 0$ and well-definedness of $\mathcal{K}_T$. For small enough $\delta_M > 0$, by structural stability and Lemma A.1 (ii), the map $\mathbb{1} - T$ can be partitioned into N_M surjective submaps. In particular, there are diffeomorphisms $y_j \colon D_j \to \mathbb{T}^2$, $D_j \subseteq \mathbb{T}^2$, $1 \leq j \leq N_M$ such that $\mathbb{1} - T = \bigsqcup_{j=1}^{N_M} y_j$. Then, using (6), we have for each $n \in \mathbb{Z}^2$

$$\begin{aligned}
\langle \varrho_n, \mathcal{K}_T \varrho_n \rangle_{\mathcal{H}_{A_{M,c}}} &= \left\langle \varphi_n, A_{M,c} \mathcal{K}_T A_{M,c}^{-1} \varphi_n \right\rangle_{L_2(\mathbb{T}^2)} = \int_{\mathbb{T}^2} \exp\left(\mathrm{i}\, 2\pi n^* (T - \mathbb{1})(x)\right) \mathrm{d}x \\
&= \sum_{j=1}^{N_M} \int_{y_j^{-1}(\mathbb{T}^2)} \exp\left(\mathrm{i}\, 2\pi n^* y_j(x)\right) \mathrm{d}x \\
&= \sum_{j=1}^{N_M} \int_{\mathbb{T}^2} \frac{\exp(\mathrm{i}\, 2\pi n^* z)}{\left|\det\left(\mathbb{1} - \mathrm{D}_{y_j^{-1}(z)} T\right)\right|} \mathrm{d}z.
\end{aligned}$$

For $N \in \mathbb{N}$ and $z \in \mathbb{T}^2$ the following sum

$$D_N(z) := \sum_{\substack{n \in \mathbb{Z}^2 \\ \|z\| \leq N}} \exp(\mathrm{i}\, 2\pi n^* z)$$

is the 2-dimensional analogue of the Dirichlet kernel [10, p.13]. Together with (23), this yields immediately

$$\operatorname{Tr} \mathcal{K}_T = \lim_{N \to \infty} \sum_{\substack{n \in \mathbb{Z}^2 \\ \|n\| \leq N}} \langle \varrho_n, \mathcal{K}_T \varrho_n \rangle_{\mathcal{H}_{A_{M,c}}} = \sum_{T(x)=x} |\det(\mathbb{1} - \mathrm{D}_x T)|^{-1}.$$

□

Using Lemma 4.1, and the definitions (1) and (9) for the dynamical determinant and Fredholm determinant, respectively, we see directly that

$$\det(1 - z\mathcal{K}_T) = d_T(z). \tag{24}$$

The Ruelle resonances correspond to the zeroes of the Fredholm determinant, hence to the inverses of the non-zero eigenvalues of $\mathcal{K}_T$.

Remark 4.2. *In view of Equation 24 and the relation of the Ruelle resonances of T to the eigenvalues of $\mathcal{K}_T$, one may ask how the spectrum of $\mathcal{K}_T$ would be affected if we let $\mathcal{K}_T$ act on a different Banach space. The following relates a part of the eigenvalues of two linear operators sharing a common dense subspace and is due to a proof of Baladi and Tsujii [1, Appendix A]. Consider two separable Banach spaces $(\mathcal{B}_1, \|\cdot\|_1)$ and $(\mathcal{B}_2, \|\cdot\|_2)$. This induces two other Banach spaces*

$$\left(\mathcal{B}_1 + \mathcal{B}_2, \|\cdot\|_+\right) \text{ and } \left(\mathcal{B}_1 \cap \mathcal{B}_2, \|\cdot\|_\cap\right), \text{ where}$$

$$\|f\|_+ \coloneqq \inf\left\{\|f_1\|_1 + \|f_2\|_2 \;\middle|\; f_1 \in \mathcal{B}_1,\ f_2 \in \mathcal{B}_2,\ f = f_1 + f_2\right\} \text{ and}$$
$$\|f\|_\cap \coloneqq \max\left\{\|f\|_1, \|f\|_2\right\}.$$

Suppose that $\mathcal{B}_\cap$ is dense in $\mathcal{B}_1$ and $\mathcal{B}_2$. Let $\mathcal{L}\colon \mathcal{B}_+ \to \mathcal{B}_+$ be a linear map which preserves the spaces $\mathcal{B}_\cap$, $\mathcal{B}_1$ and $\mathcal{B}_2$ and is a bounded linear operator on the restrictions $\mathcal{L}_{|\mathcal{B}_1}$ and $\mathcal{L}_{|\mathcal{B}_2}$. Then the part of the eigenvalues of $\mathcal{L}_{|\mathcal{B}_1}$ and $\mathcal{L}_{|\mathcal{B}_2}$ coincide which lies outside the closed disc with radius larger to both essential spectral radii. Moreover, the corresponding generalized eigenspaces of $\mathcal{L}_{|\mathcal{B}_1}$ and $\mathcal{L}_{|\mathcal{B}_2}$ coincide and are contained in $\mathcal{B}_\cap$.
For the applications that we have in mind, the map $\mathcal{L}$ is just the Koopman or transfer operator, defined on $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, extended to the space $\mathcal{B}_+$.

The spectrum $\mathrm{sp}(\mathcal{K}_T)$ of $\mathcal{K}_T$ on $\mathcal{H}_{A_{M,c}}$ is invariant under complex conjugation since T is real. The constant functions on $\mathbb{T}^2$ are all fixed by $\mathcal{K}_T$. Therefore we have $1 \in \mathrm{sp}(\mathcal{K}_T)$. If we take $T = M^k$, $k \in \mathbb{N}$ in Lemma 4.1, it follows that $\mathrm{Tr}\,\mathcal{K}_T = 1$. Hence, the dynamical determinant is just $d_T(z) = 1 - z$, also noted in [16, p.3]. We find immediately that 1 is the only Ruelle resonance. We show now that this finding is non-generic in the following sense. The rest of this section is devoted to an idea of Naud [13]. We put for every $r > 0$

$$\mathcal{B}_r \coloneqq \left\{T \in \mathcal{T}_r \;\middle|\; \text{The lift of } T \text{ to } \mathbb{R}^2 \text{ is } \mathbb{Z}^2\text{-periodic}\right\}. \tag{25}$$

Endowed with the uniform norm this is a Banach space.

Theorem 4.3 (Non-trivial Ruelle resonances (I)). *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. For each $r > 0$ there exists an open and dense set $\mathcal{G} \subset \mathcal{B}_r$ such that the linear functional*

$$B_M\colon \mathcal{B}_r \to \mathbb{R}\colon \psi \mapsto {N_M}^{-1} \sum_{Mx=x} \mathrm{Tr}\left((\mathbb{1} - M)^{-1}\,\mathrm{D}_x\psi\right)$$

never vanishes on $\mathcal{G}$. For all $\psi \in \mathcal{G}$ there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$

$$\mathrm{Tr}\,\mathcal{K}_{M+\epsilon\psi} = 1 + \epsilon B_M(\psi) + O\left(\epsilon^2\right).$$

In particular, for all sufficiently small $\epsilon > 0$ it holds

$$\mathrm{sp}\left(\mathcal{K}_{M+\epsilon\psi}\right) \setminus \{0, 1\} \neq \emptyset.$$

Lemma 4.4 (Real analyticity of fixed points). *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and $r > 0$. Then for all $\psi \in \mathcal{B}_r$ the fixed points of the map*

$$M + \delta\psi$$

are real analytic functions of δ where δ lies in a real neighborhood of 0.

Proof. We set for $\delta \in \mathbb{R}$

$$F(\delta, x) \coloneqq Mx + \delta\psi(x) - x.$$

We fix a point $y_j \coloneqq (0, x_j)$ where x_j, $1 \le j \le N_M$, is a fixed point of M. By construction, the map F has a holomorphic extension to $\mathbb{C} \times \mathbb{A}_r$. Since M is hyperbolic, we have $\det \mathrm{D}_{x_j}(F(0, \cdot)) \neq 0$. We apply the Holomorphic Implicit Function Theorem [12, Theorem 1.4.11] on F with $F(y_j) = 0$. This yields a holomorphic function $x_j(\delta)$ such that $x_j(0) = x_j$ and which is obviously real analytic for $\delta \in \mathbb{R}$ in a neighborhood of 0. ∎

Proof of Theorem 4.3. Let $\delta \in \mathbb{R}$ and $\psi \in \mathcal{B}_r$ and set $\tilde{M} \coloneqq M + \delta\psi$. We choose δ small in Lemma 4.4 which gives for each fixed point x of M a real analytic function $\tilde{x}$ with $\tilde{x}(0) = x$. Using a Taylor expansion on $\tilde{x}$ at 0, we have

$$\tilde{x}(\delta) = x + O(\delta).$$

Using real analyticity of the derivative $\mathrm{D}_x\psi$, we have

$$\mathrm{D}_x\psi - \mathrm{D}_{\tilde{x}(\delta)}\psi = O(\delta).$$

We write now for each fixed point x of M

$$\begin{aligned}
\left|\det\left(\mathbb{1} - \mathrm{D}_{\tilde{x}(\delta)}\tilde{M}\right)\right| &= \left|\det\left(\mathbb{1} - M - \delta\mathrm{D}_x\psi + \delta\left(\mathrm{D}_x\psi - \mathrm{D}_{\tilde{x}(\delta)}\psi\right)\right)\right| \\
&= N_M\left|\det\left(\mathbb{1} - (\mathbb{1} - M)^{-1}\left(\delta\mathrm{D}_x\psi + \left(\delta\mathrm{D}_x\psi - \delta\mathrm{D}_{\tilde{x}(\delta)}\psi\right)\right)\right)\right| \\
&= N_M\left|\det\left(\mathbb{1} - \delta(\mathbb{1} - M)^{-1}\mathrm{D}_x\psi + O\left(\delta^2\right)\right)\right| \\
&= N_M\left(1 - \delta\operatorname{Tr}\left((\mathbb{1} - M)^{-1}\mathrm{D}_x\psi\right) + O\left(\delta^2\right)\right).
\end{aligned}$$

We have by Lemma 4.1 for δ small enough

$$\operatorname{Tr}\mathcal{K}_{\tilde{M}} = 1 + \frac{\delta}{N_M}\sum_{Mx=x}\operatorname{Tr}\left((\mathbb{1} - M)^{-1}\mathrm{D}_x\psi\right) + O\left(\delta^2\right).$$

Now we set

$$B_M\colon \mathcal{B}_r \to \mathbb{R}\colon \psi \mapsto {N_M}^{-1}\sum_{Mx=x}\operatorname{Tr}\left((\mathbb{1} - M)^{-1}\mathrm{D}_x\psi\right).$$

We next check that this is a non-trivial linear functional. Note that formally $B_M(\mathbb{1} - M) = 2$. However, no non-zero linear map is in the space of additive perturbations $\mathcal{B}_r$. We denote by v_j, $j \in \{1, 2\}$ the j-th column of the matrix $\left((\mathbb{1} - M)^*\right)^{-1}$ and we fix now j. Let $\psi_0\colon \mathbb{T} + \mathrm{i}(-r, r) \to \mathbb{C}$ be holomorphic and bounded. For every $(x_1, x_2) \eqqcolon x \in \mathbb{T}^2$ we put

$$\psi(x) \coloneqq \psi_0(x_j)\, v_j.$$

By construction, we have $\psi \in \mathcal{B}_r$ and we evaluate

$$B_M(\psi) = \frac{v_j^* v_j}{N_M}\sum_{Mx=x}\psi_0^{(1)}(x_j).$$

The right-hand side is a finite sum and by taking for ψ_0 a suitable Fourier polynomial (e.g. a shifted sine with sufficiently high frequency), we can establish $B_M(\psi) \neq 0$. We set $\mathcal{G} \coloneqq B_M^{-1}(\mathbb{R} \setminus \{0\})$. By continuity of B_M, the set $\mathcal{G}$ is open and dense in $\mathcal{B}_r$. □

5. Non-trivial resonances for the transfer operator

As before, we consider maps $T \in \mathcal{T}_r$, $r > 0$ which are sufficiently C^1-close to a hyperbolic linear map $M \in \mathrm{SL}_2(\mathbb{R})$. We turn to the adjoint of $\mathcal{K}_T$, acting on the dual space $\mathcal{H}^*_{A_{M,c}}$, which we denote by $\mathcal{L}_T$.

Lemma 5.1 (Transfer operator)**.** *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and let $r > 0$. Then there exist constants $0 < \delta_M$ and $c > 0$ such that for each $T \in \mathcal{T}_r$ with $d(T, M) \leq \delta_M$ the map*

$$\mathcal{L}_T \colon \mathcal{H}^*_{A_{M,c}} \to \mathcal{H}^*_{A_{M,c}} \colon f \mapsto \frac{f}{|\det \mathrm{D}T|} \circ T^{-1}$$

defines a nuclear operator of order 0, conjugate to $\mathcal{K}_T$. In particular,

$$\mathrm{sp}\left(\mathcal{L}_T\right) = \mathrm{sp}\left(\mathcal{K}_T\right).$$

Proof. By Theorem 3.1 there is $0 < \delta_M$, $c > 0$ and $\mathcal{H}_{A_{M,c}}$ such that $\mathcal{K}_T$ acting on $\mathcal{H}_{A_{M,c}}$ is nuclear of order 0 if $d(T, M) \leq \delta_M$. The same can be said about its adjoint, acting on $\mathcal{H}^*_{A_{M,c}}$ (e.g. see [17, p. 77]). The trace of $\mathcal{K}_T$ and $\mathcal{L}_T$ coincide, so does their Fredholm determinant, and hence their resonances. By definition of the adjoint, $\forall f^* \in \mathcal{H}^*_{A_{M,c}}$, $\forall g \in \mathcal{H}_{A_{M,c}} \colon \left(\mathcal{L}_T f\right)^*(g) = f^*\left(\mathcal{K}_T g\right)$. Using Lemma 2.5, it holds

$$\begin{aligned} f^*\left(\mathcal{K}_T g\right) = \left\langle A^{-2}_{M,c} f, \mathcal{K}_T g\right\rangle_{\mathcal{H}_{A_{M,c}}} &= \int_{\mathbb{T}^2} \left(A^{-1}_{M,c}\bar{f}\right)(x)\left(A_{M,c}\mathcal{K}_T g\right)(x)\,\mathrm{d}x \\ &= \int_{\mathbb{T}^2} \bar{f}(x)\left(\mathcal{K}_T g\right)(x)\,\mathrm{d}x = \int_{\mathbb{T}^2}\left(\frac{\bar{f}}{|\det \mathrm{D}T|} \circ T^{-1}\right)(x)\, g(x)\,\mathrm{d}x \\ &= \left\langle A^{-2}_{M,c}\left(\frac{f}{|\det \mathrm{D}T|} \circ T^{-1}\right), g\right\rangle_{\mathcal{H}_{A_{M,c}}} = \left(\frac{f}{|\det \mathrm{D}T|} \circ T^{-1}\right)^*(g). \end{aligned}$$

□

By Lemma 5.1, recalling (6), and Lemma 4.1 it holds

$$\mathrm{Tr}\,\mathcal{L}_T = \sum_{n \in \mathbb{Z}^2} \mathcal{L}_T \varrho^*_n\left(\varrho_n\right) = \sum_{T(x) = x} \left|\det\left(\mathbb{1} - \mathrm{D}_x T\right)\right|^{-1}.$$

We have the equality

$$d_T(z) = \det\left(1 - z\mathcal{K}_T\right) = \det\left(1 - z\mathcal{L}_T\right).$$

We give now analogously to Theorem 4.3 a spectral result for the transfer operator (recall $\mathcal{B}_r$ from (25)).

Lemma 5.2 (Non-trivial Ruelle resonances (II))**.** *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. For each $r > 0$ there exists an open and dense set $\mathcal{G} \subset \mathcal{B}_r$ such that for all $\psi \in \mathcal{G}$ there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$*

$$\mathrm{sp}\left(\mathcal{L}_{M+\epsilon\psi}\right) \setminus \{0, 1\} \neq \emptyset.$$

Proof. By Theorem 4.3 we know that under every perturbation $\psi \in \mathcal{G}$ there is $\epsilon_0 > 0$ such that we find for all $0 < \epsilon \leq \epsilon_0$ non-trivial Ruelle resonances. Using Lemma 5.1 for well-definedness of $\mathcal{L}_{M+\epsilon\psi}$ and for the relation $\mathrm{sp}\left(\mathcal{L}_T\right) = \mathrm{sp}\left(\mathcal{K}_T\right)$, the result follows. □

Clearly, the Lebesgue measure (by Remark 2.6, the constant density 1) is fixed by $\mathcal{L}_M$. This does not persist under a generic perturbation of M. However, the spectral relation in Lemma 5.1 implies that $\mathcal{L}_T$ fixes some functionals in $\mathcal{H}^*_{A_{M,c}}$. In particular, using Remark 4.2, we can apply [3, Theorem 3] to our transfer operators

$\mathcal{L}_M$ and $\mathcal{L}_T$. Hence, the eigenvalue 1 of $\mathcal{L}_T$ is simple and the projector Π_1^* onto the corresponding eigenspace of $\mathcal{L}_T$ gives us the SRB measure

$$\mu_{\mathrm{SRB}} \coloneqq \Pi_1^* 1^*,$$

in the usual sense. (It is absolutely continuous with respect to Lebesgue measure in the unstable direction.)
We finish this section by showing the existence of non-zero perturbations $\psi \in \mathcal{B}_r$ which allow the determinant $\det(M + \epsilon \mathrm{D}_x \psi)$ to remain constant or to vary for $x \in \mathbb{T}^2$.

Lemma 5.3 (Volume under perturbations)**.** *Let $r > 0$ and let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. Then there exist non-zero maps $\psi \in \mathcal{B}_r$ in each of the following cases:*

(i) *For all $\epsilon > 0$ and all $x \in \mathbb{T}^2$ it holds $\det(M + \epsilon \mathrm{D}_x \psi) = 1$.*
(ii) *For all $\epsilon > 0$ and Lebesgue almost all $x \in \mathbb{T}^2$ it holds $|\det(M + \epsilon \mathrm{D}_x \psi)| \neq 1$.*

In particular, the map ψ can be chosen such that for all small $\epsilon > 0$ the corresponding transfer operator

$$\mathcal{L}_{M+\epsilon\psi}$$

admits non-trivial Ruelle resonances.

Proof. We prove first Claim (i), including the statement about the non-trivial Ruelle resonances. We will apply Lemma A.2 (i). We choose $j \in \{1, 2\}$, $r > 0$ and let $\phi\colon \mathbb{T} + \mathrm{i}(-r, r) \to \mathbb{C}$ be a holomorphic and bounded map. For $\alpha \in \mathbb{R}^2$ we set for every $(x_1, x_2) \eqqcolon x \in \mathbb{T}^2$

$$\psi_{\phi,\alpha}(x) \coloneqq (\alpha_1 \phi(x_j), \alpha_2 \phi(x_j)).$$

We put $d \coloneqq 2$, j, $T \coloneqq M$, ϕ and $T_\phi \coloneqq \psi_{\phi,\alpha}$ (e.g. as lift to $\mathbb{R}^2$) in Lemma A.2. Since M is a constant matrix, say, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for suitable $a, b, c, d \in \mathbb{Z}$, we can write Condition A.2 (i) as

$$\alpha_1 d = \alpha_2 b \quad \text{if } j = 1 \quad \text{or} \quad \alpha_1 c = \alpha_2 a \quad \text{if } j = 2. \tag{26}$$

Hence, we have non-zero solutions in α independent of x. We choose such a solution α and take $\psi = \psi_{\phi,\alpha}$. Then $\psi \in \mathcal{B}_r$ which yields $\det(M + \epsilon \mathrm{D}_x \psi) = 1$ for every $\epsilon > 0$. We are free to choose any suitable ϕ. In particular, Theorem 4.3 yields a linear functional B_M and a dense subset $\mathcal{G} \subset \mathcal{B}_r$ on which B_M is non-zero. We have to make sure that $\psi \in \mathcal{G}$. Then for ϵ small $\mathcal{L}_{M+\epsilon\psi}$ admits non-trivial Ruelle resonances by Lemma 5.2. To this end, we evaluate B_M at ψ which yields

$$B_M(\psi) = B_M(\psi_{\phi,\alpha}) = {N_M}^{-1} \sum_{Mx=x} \mathrm{Tr}\left((\mathbb{1} - M)^{-1} \mathrm{D}_x \psi_{\phi,\alpha} \right) = \frac{v_j^* \alpha}{N_M} \sum_{Mx=x} \phi^{(1)}(x_j),$$

where v_j^* is the j-th row of $(\mathbb{1} - M)^{-1}$. The sum over the fixed points of M can be made non-zero by a suitable Fourier polynomial. Now we have

$$v_1^* \alpha = \frac{(1-d)\alpha_1 + c\alpha_2}{\det(\mathbb{1} - M)} \quad \text{or} \quad v_2^* \alpha = \frac{b\alpha_1 + (1-a)\alpha_2}{\det(\mathbb{1} - M)}.$$

Using (26), we find

$$v_1^* \alpha = \frac{\left(c - b + \frac{b}{d}\right)\alpha_2}{\det(\mathbb{1} - M)} \quad \text{or} \quad v_2^* \alpha = \frac{\left(b - c + \frac{c}{a}\right)\alpha_1}{\det(\mathbb{1} - M)}.$$

Both equations can never be zero since M is not diagonal. We prove now Claim (ii) by modifying the map ψ. For $\delta \in \mathbb{R} \setminus \{0\}$ we set $\tilde{\alpha} := \alpha + \delta w_j$, where w_j is the j-th column of M and put $\widetilde{\psi} := \psi_{\phi,\tilde{\alpha}}$. We have

$$\det\left(M + \epsilon \mathrm{D}_x \widetilde{\psi}\right) = \det\left(M + \epsilon \mathrm{D}_x \psi + \epsilon \mathrm{D}_x \left(\widetilde{\psi} - \psi\right)\right) = 1 + \delta\epsilon\phi^{(1)}\left(x_j\right).$$

Since ϕ is not constant, the right-hand side differs from 1 (and -1) for Lebesgue almost all x. Since $v_j^* \tilde{\alpha} = v_j^* \alpha + \delta v_j^* w_j \neq 0$ for the right choice of the sign of δ, we have $B_M\left(\widetilde{\psi}\right) \neq 0$.

□

Appendix

For the readers convenience we give a proof of a well-known result:

Lemma A.1 (Fixed points)**.** *Let M be 2×2 integer matrix acting on $\mathbb{T}^2$. Assume that* $\det(\mathbb{1} - M) \neq 0$. *Then the following holds:*

(i) *The number N_M of fixed points of M is given by $N_M = |\det(\mathbb{1} - M)|$.*
(ii) *There exists a disjoint partition $D_j \subseteq \mathbb{T}^2$, $1 \leq j \leq N_M$ of $\mathbb{T}^2$ such that the maps $y_j : D_j \to \mathbb{T}^2 : x \mapsto (\mathbb{1} - M)\, x$ are bijections.*

Proof. We let $\mathbb{1} - M$ act on the cover $\mathbb{R}^2$. The linear map $\mathbb{1} - M$ sends a fundamental region of $\mathbb{T}^2$, e.g. $[0,1)^2$, to a convex polytope having a non-zero volume given by $|\det(\mathbb{1} - M)|$. Each fixed point of M on $\mathbb{T}^2$ is mapped by $\mathbb{1} - M$ to an element of $\mathbb{Z}^2$, and the number of integer points contained in the polytope is just given by its volume. Claim (i) follows.
Let $v_1,\ v_2 \in \mathbb{Z}^2$ be two different such integer points in the polytope. Now assume that there are $f_1,\ f_2 \in [0,1)^2$ such that

$$(\mathbb{1} - M)^{-1}(f_1 - f_2) \equiv (\mathbb{1} - M)^{-1}(v_1 - v_2) \quad \left(\bmod\ [0,1)^2\right).$$

The right-hand side is mapped to a fixed point of M on $\mathbb{T}^2$, implying that $f_1 - f_2$ is an integer point, which is only possible if $f_1 = f_2$. Therefore, $v_1 = v_2$, which contradicts the assumption, and Claim (ii) follows. □

For $d \in \mathbb{N}$ and every real $d \times d$ matrix M we denote by $\Box_{i,j}(M)$, $1 \leq i,\ j \leq d$ the submatrix arising by removing the i-th row and j-th column from M.

Lemma A.2 (Determinant preserving transformation)**.** *Let $d \in \mathbb{N}$, and let $T : \mathbb{R}^d \to \mathbb{R}^d$ and $\phi : \mathbb{R} \to \mathbb{R}$ be differentiable maps. Fix $1 \leq j \leq d$ and $\alpha \in \mathbb{R}^d$ and set*

$$T_\phi : \mathbb{R}^d \to \mathbb{R}^d : x \mapsto \left(\alpha_i \phi\left(x_j\right) \ \middle|\ 1 \leq i \leq d\right).$$

Then for $x \in \mathbb{R}^d$ it holds

$$\det \mathrm{D}_x\left(T + T_\phi\right) - \det \mathrm{D}_x\left(T\right) = 0$$

if and only if at least one of the conditions holds:

(i) $\sum_{i=1}^{d} (-1)^i \alpha_i \det \Box_{i,j}\left(\mathrm{D}_x T\right) = 0$ *or*
(ii) $\phi^{(1)}\left(x_j\right) = 0$.

Proof. We develop the determinant of $\mathrm{D}_x\left(T+T_\phi\right)$ with respect to the j-th column. Since T_ϕ depends only on x_j this gives

$$\det \mathrm{D}_x\left(T+T_\phi\right)=(-1)^j \sum_{i=1}^d(-1)^i \partial_j\left(T+T_\phi\right)_i(x) \det \square_{i,j}\left(\mathrm{D}_x T\right).$$

Hence, it holds

$$\begin{aligned}
\det \mathrm{D}_x\left(T+T_\phi\right)-\det\left(\mathrm{D}_x T\right) &= (-1)^j \sum_{i=1}^d(-1)^i \det \square_{i,j}\left(\mathrm{D}_x T\right) \partial_j\left(T_\phi\right)_i(x) \\
&= (-1)^j \phi^{(1)}\left(x_j\right) \sum_{i=1}^d(-1)^i \alpha_i \det \square_{i,j}\left(\mathrm{D}_x T\right).
\end{aligned}$$

One deduces Claim (i) and (ii) directly from the right-hand side. ☐

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