HOROCYCLE AVERAGES ON CLOSED MANIFOLDS AND TRANSFER OPERATORS

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Abstract. We study semigroups of weighted transfer operators for Anosov flows of regularity C^r, $r > 1$, on compact manifolds without boundary. We construct an anisotropic Banach space on which the resolvent of the generator is quasi-compact and where the upper bound on the essential spectral radius depends continuously on the regularity. We apply this result to the ergodic average of the horocycle flow for C^3 contact Anosov flows in dimension three.

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1. Introduction

Let M be a closed (compact without boundary) orientable Riemannian manifold of arbitrary finite dimension $d \geqslant 3$. On such manifolds Anosov introduced C^2 flows

$$g_{\alpha_1} \circ g_{\alpha_2} = g_{\alpha_1+\alpha_2} = g_\alpha \colon M \to M, \quad \alpha, \alpha_1, \alpha_2 \in \mathbb{R},$$

to study the geodesic flow on the unit tangent bundle of closed Riemannian manifolds with variable negative sectional curvature [2, 1]. As pointed out by Anosov [1], the topological entropy h_{top} of the time-one map g_1 of an Anosov flow is positive. A special class of such *Anosov* flows are those which preserve a *contact* structure. The geodesic flows are well-studied examples of contact Anosov flows. We give the precise definition of a (contact) Anosov flows in Section 2.
Every Anosov flow admits a contracting transversal foliation. The underlying vector bundle E_- is called the strong stable distribution. If the leaves of the contracting foliation are one-dimensional and orientable, one associates with g_α another flow, the *horocycle* flow $h_\rho \colon M \to M$, $\rho \in \mathbb{R}$. (The term *horocycle flow* was used originally only in the case of the geodesic flow, e.g. see [35, p.84] or [27].) For every $x \in M$ the flow trajectory $h_{\mathbb{R}}(x)$ is such a contracting leaf. Statistical properties of contact Anosov flows are nowadays fairly well understood (see [17, 34, 24]). Regarding the horocycle flow one knows by the work of Bowen and Marcus unique ergodicity of and minimality of the horocycle flow (e.g. see [12, 36]). The corresponding invariant probability measure μ will play an important role below. (It is related to but distinct from the measure of maximal entropy of the flow.)
Since the horocycle flow is induced by the Anosov flow the following pointwise equality for all $x \in M$ holds for a suitable function $\tau(\rho, \alpha, x)$:

$$g_\alpha \circ h_\rho(x) = h_{\tau(\rho,\alpha,x)} \circ g_\alpha(x).$$

We call $\tau(\rho, \alpha, x)$ the *renormalization time*.
This kind of renormalization has been used effectively in the work of Flaminio and Forni [21] to give a precise understanding of the horocycle integral

$$\gamma_x(\varphi, T) \coloneqq \int_0^T \varphi \circ h_\rho(x) \, \mathrm{d}\rho, \quad x \in M, \quad T > 0,$$

in the setting of unit speed geodesic flows on hyperbolic compact (more generally finite volume) Riemannian surfaces with constant negative sectional curvature (i.e. Riemann surfaces), for $\varphi : M \to \mathbb{R}$ in Sobolev spaces of positive order. In this case, $h_{\text{top}} = 1$. Flaminio and Forni found that the speed of convergence of $\gamma_x(\varphi, T)/T$ to $\mu(\varphi)$ as $T \to \infty$ is controlled by *invariant distributions* under the push-forward of the horocyclic vector field. These distributions are also eigendistributions under the push-forward of the geodesic vector field and the eigenvalues give the powers of T appearing in the expansion of $T^{-1}\gamma_x(\varphi, T) - \mu(\varphi)$.
Their approach inspired Giulietti and Liverani [22] to study a toy model, replacing the Anosov flow with a hyperbolic diffeomorphism, using the renormalization dynamics as a key to study $\gamma_x(\varphi, T)$. They show analogously (for the corresponding invariant measure μ) that the speed of convergence to zero of $T^{-1}\gamma_x(\varphi, T) - \mu(\varphi)$ is controlled by eigendistributions for a weighted transfer operator of the hyperbolic diffeomorphism.
Giulietti and Liverani conjectured that a similar behavior holds in the setting of more general Anosov flows, e.g. for the geodesic flow on the unit tangent bundle of

a Riemannian manifold with variable negative sectional curvature [22, Conjecture 2.12]. More precisely, we expect for smooth enough observables φ an expansion like

$$\gamma_x(\varphi, T) = T \int \varphi \, \mathrm{d}\, \mu + \sum_{\delta < \Re\lambda < h_{\mathrm{top}}} T^{\frac{\Re\lambda}{h_{\mathrm{top}}}} c(\lambda, T, x) \mathcal{O}_\lambda(\varphi) + \mathcal{E}_{T,x}(\varphi), \tag{1}$$

with $\mathcal{E}_{T,x} = O(T^{\frac{\delta}{h_{\mathrm{top}}}})$, uniformly in x. The $\mathcal{O}_\lambda$ are generalized eigendistributions associated to the eigenvalue λ for the adjoint of the generator $X + V$ of a certain weighted transfer operator $\mathcal{L}_{\alpha,\phi_\alpha}$, acting on an anisotropic Banach space (see below). The real parameter δ is an upper bound on the essential spectral bound of $X + V$. The complex coefficients $c(\lambda, T, x)$ are bounded from above independently of x by $|\log T|^c$ for some $c = c(\lambda) \geqslant 0$ which depends whether $\Re\lambda < 0$, $\Re\lambda = 0$ or $\Re\lambda > 0$ and if there are non-trivial Jordan blocks for λ. This is analogous to the bounds in [21],[22]. However our methods show no substantial improvement of the error term $\mathcal{E}_{T,x}$ if the summation in λ includes some $\Re\lambda < 0$ (this is seen also in [21],[22]). We restrict ourself therefore to $\delta \geqslant 0$ (i.e. always $\Re\lambda > 0$).

The main result of this work, Theorem 5.7, gives conditions under which such an asymptotic expansion indeed holds, for some $\delta > 0$, for codimension one topologically mixing Anosov flows, under an assumption of "spectral gap with (Dolgopyat) bounds" (Condition 4.11 below). In Proposition 5.10 we specialize to C^3 contact Anosov flows in dimension $d = 3$. For compact Riemann surfaces (recall that this is the constant negative curvature case) Randol [38] proved that there exist eigenvalues arbitrarily close to 1 (his result is for the associated Laplacian). This provides examples with a non-trivial expansion.
Analogous to the work of Giulietti and Liverani [22], the key idea to study $\gamma_x(\varphi, T)$ is to introduce a weighted transfer operator family

$$\mathcal{L}_{\alpha,\phi_\alpha} \colon W_p^{s,t,q} \to W_p^{s,t,q}, \quad \mathcal{L}_{\alpha,\phi_\alpha}\varphi = \phi_\alpha \cdot \varphi \circ g_{-\alpha}, \quad \alpha \geqslant 0,$$

where the weight is $\phi_\alpha = \partial_\rho \tau(0, -\alpha, \cdot)$ and where $W_p^{s,t,q}$ is an anisotropic Banach space with certain real regularity parameters s, t, q and p. In the case of the unit speed parametrization of the flow h_ρ, the weight $\partial_\rho \tau(0, -\alpha, \cdot)$ is just the Jacobian along the strong stable distribution evaluated at negative time $-\alpha$.

The paper is organized as follows: After recalling some facts about Anosov flows in Section 2, the transfer operator $\mathcal{L}_{\alpha,\phi_\alpha}$ is defined in Section 3.1 (for more general weights) and the Banach spaces $W_p^{s,t,q}$ are constructed in Section 3.2. These spaces are a flow analogue to the spaces constructed by Baladi and Tsujii [6] to study hyperbolic diffeomorphisms. Anisotropic Banach spaces are now considered a standard tool (yet with still ongoing research) for investigating transfer operators and zeta functions associated to hyperbolic dynamics [9, 5, 3, 4, 7, 17, 23, 24, 34, 37, 41, 42]. Although we do not study here the dynamical zeta function for the transfer operator $\mathcal{L}_{\alpha,\phi_\alpha}$, we believe that this space could be a suitable choice to be dealt with.
In Section 4 we establish properties of the transfer operator, its generator $X + V$ and the resolvent $\mathcal{R}_z$. Most of these results do not require the contact assumption. Among those are norm estimates which yield a Lasota–York inequality for the resolvent. This is Theorem 4.5. Then in Lemma 4.10 one obtains a strip in

the spectrum of the generator, containing at most countable eigenvalues of finite multiplicity. Those are precisely the eigenvalues λ in the summation over λ in (1). Finally, these results are used in Section 5 to give the expansion (1) of $\gamma_x(\varphi, T)$ in terms of eigendistributions and eigenvalues of $X + V$ under a *spectral gap with bounds* condition, see Condition 4.11.

We end this introduction with two remarks about possible further work:

First, the conjecture that the distributions $\mathcal{O}_v$ appearing in the expansion (1) are fixed by the (adjoint) of the horocycle flow remains still open. (In contrast this was the starting point in [21]!) Here, progress has been made by Faure and Guillarmou [18] in dimension 3 for smooth contact Anosov flows.

Second, the renormalization time $\tau(\rho, \alpha, x)$ inherits the regularity properties of the underlying Anosov foliation and horocycle flow, i.e. the regularity in x is expected to be no more than Hölder. To deal with such irregular flows one can lift the dynamics to the Grassmanian. This has been used with success, e.g. in [24, 22] and more recently in [42]. However in this work we wish to avoid such technicalities and we will make additional assumptions ensuring that $\tau(\rho, \alpha, x)$ enjoys sufficient regularity.
In particular, if the Anosov flow is C^r we require $\partial_\rho \tau(0, \alpha, \cdot)$ to be C^{r-1} for all $\alpha \geqslant 0$. This is reasonably only if r is small since the regularity of the stable foliation is usually only Hölder. In the setting of C^3 contact Anosov flows in dimension 3 we can take $r = 2 - \epsilon$ for all $\epsilon > 0$ by a result of [29] (see also Remark 5.8 in Section 5).

The Appendix comprises our computational tools. On the lowest level, we utilize Fourier transform, integration by parts, and Young's inequality [11, Theorem 3.9.4] to estimate convolutions.

2. Geometric setting

Let M be a closed, connected, orientable, smooth Riemannian manifold of dimension $d \geqslant 3$. We let $g_\alpha \colon M \to M$, $\alpha \in \mathbb{R}$, be a C^r Anosov flow on M [1]for $r > 1$. That is, there exists a decomposition of the tangent space TM of M as a direct sum

$$\mathrm{T}\, M = E_- \oplus E_+ \oplus E_0, \tag{2}$$

such that for some constants $C \geqslant 1$, $0 < \theta < 1$ and every $\alpha \geqslant 0$

$$\begin{aligned} \|\mathrm{D}\, g_\alpha v\| &\leqslant C\theta^\alpha \|v\|, \quad \text{for all } v \in E_-, \\ \|\mathrm{D}\, g_{-\alpha} v\| &\leqslant C\theta^\alpha \|v\|, \quad \text{for all } v \in E_+, \end{aligned} \tag{3}$$

and $E_0 = \langle X \rangle$ where X is the generator of the Anosov flow

$$X \coloneqq \partial_\alpha g_{-\alpha|\alpha=0}. \tag{4}$$

Note that the conditions in (3) are closed. Hence by compactness of M the distributions E_- and E_+ are uniformly continuous and so are the weak-stable $E_- \oplus E_0$ and weak-unstable $E_+ \oplus E_0$ distributions. The restriction of the tangent space to a base point $x \in M$ is denoted by

$$\mathrm{T}_x\, M = E_{-,x} \oplus E_{+,x} \oplus E_{0,x}, \tag{5}$$

[1]In this paper, if $r > 0$, is not an integer, C^r means $C^{\lfloor r \rfloor}$ with all partial derivatives of order $\lfloor r \rfloor$ being $(r - \lfloor r \rfloor)$-Hölder continuous.

The dimensions of those vector spaces do not vary with x and we set for some $x \in M$

$$d_- := \dim E_{-,x}. \tag{6}$$

The cotangent space T^*M is the dual space of $\mathrm{T}\,M$ and has the canonical splitting

$$\mathrm{T}^*\,M = E_-^* \oplus E_+^* \oplus E_0^* \quad \text{and} \quad \mathrm{T}_x^*\,M = E_{-,x}^* \oplus E_{+,x}^* \oplus E_{0,x}^*, \quad x \in M, \tag{7}$$

where $E_-^* \cong (E_+ \oplus E_0)^\perp$, $E_+^* \cong (E_- \oplus E_0)^\perp$, $E_0^* \cong (E_- \oplus E_+)^\perp$. This splitting is $(\mathrm{D}\,g_\alpha)^{\mathrm{tr}}$-invariant and satisfies an analogue of (3).

A contact form is a 1-form $\eta \in T^*M$ such that $\eta \wedge \bigwedge_{n=1}^{\frac{d-1}{2}} \mathrm{d}\,\eta$ vanishes nowhere ($\mathrm{d}\,\eta$ is the exterior derivative of η). An Anosov flow is a contact flow if there exists a C^1 contact form η which is preserved by the pullback of g_α. Clearly, a contact form can only exist if d is odd.

We mean by "$\Subset$" for sets $A, B \subseteq T^*M$ (or $\subseteq \mathbb{R}^d$) that

$$A \Subset B \Leftrightarrow \bar{A} \subseteq (\operatorname{int} B \cup \{0\}).$$

Here $\bar{A}$ denotes the closure of A and $\operatorname{int} B$ the interior of B. We say that a cone A is *compactly included* in a cone B if and only if $A \Subset B$. We say that a cone A and a cone B are *transversal* if and only if $A \cap B = \{0\}$.

We introduce two closed convex cone fields on M in the cotangent space:

For every $x \in M$ and for every $v \in \mathrm{T}_x^*\,M$ we have $v = v^- + v^+ + v^0$, where $v^\sigma \in E_{\sigma,x}^*$, $\sigma \in \{-, +, 0\}$. For every $0 < \gamma < 1$ we set

$$\begin{aligned} C_\gamma^-(x) &:= \left\{ v \in \mathrm{T}_x^*\,M \mid \|v^+\| + \left\|v^0\right\| \leqslant \gamma \|v^-\| \right\}, \\ C_\gamma^+(x) &:= \left\{ v \in \mathrm{T}_x^*\,M \mid \|v^-\| + \left\|v^0\right\| \leqslant \gamma \|v^+\| \right\}. \end{aligned} \tag{8}$$

If $\gamma' > \gamma$ then we have the compact inclusions

$$C_\gamma^-(x) \Subset C_{\gamma'}^-(x) \quad \text{and} \quad C_\gamma^+(x) \Subset C_{\gamma'}^+(x).$$

Moreover, this construction implies $E_{-,x}^* \subset C_\gamma^-(x)$, and $E_{+,x}^* \subset C_\gamma^+(x)$ and also transversality $E_{0,x}^* \cap \left(C_\gamma^-(x) \cup C_\gamma^+(x)\right) = \{0\}$ and $C_\gamma^-(x) \cap C_\gamma^+(x) = \{0\}$.

We have (see Lemma A.1) for all $\alpha \geqslant 0$ so that $C^2\theta^\alpha\gamma < \gamma' < 1$ and for all $x \in M$ the compact inclusions

$$(\mathrm{D}\,g_{-\alpha})^{\mathrm{tr}}\, C_\gamma^-(x) \Subset C_{\gamma'}^-(g_\alpha(x)) \quad \text{and} \quad (\mathrm{D}\,g_\alpha)^{\mathrm{tr}}\, C_\gamma^+(x) \Subset C_{\gamma'}^+(g_{-\alpha}(x)). \tag{9}$$

The cones defined in (8) are expanding and contracting, respectively (see Lemma A.2). Note that the cones in (8) have non-empty interior while [32, Proposition 17.4.4] uses "flat" cones included in $E_+^* \oplus E_-^*$.

Let $V_\omega \subseteq \Omega$, $\omega \in \Omega$, be an open cover of M, where Ω is a finite index set. We let $\mathcal{A}$ be an atlas for M, containing diffeomorphic C^r-charts $\kappa_\omega \colon V_\omega \to \mathbb{R}^d$, compatible with the splittings (2) and (7), as we explain now. Fixing coordinates $(x_1, \ldots, x_d) \in \mathbb{R}^d$ and recalling X from (4), we may and do require the flowbox condition

$$\mathrm{D}\,\kappa_\omega X_{|V_\omega} = \partial_{x_d}. \tag{10}$$

Since g_α is C^r the chart maps κ_ω, $\omega \in \Omega$, are also C^r diffeomorphisms. We set

$$C_{\gamma,\omega}^\sigma := \bigcup_{x \in V_\omega} \left(\mathrm{D}\,\kappa_\omega^{-1}\right)^{\mathrm{tr}} C_\gamma^\sigma(x), \quad \sigma \in \{-, +\}, \omega \in \Omega. \tag{11}$$

We require the sets V_ω to be small enough such that for small $0 < \gamma_-, \gamma_+ \leqslant 1$ there exist $0 < \gamma_-^*, \gamma_+^* \leqslant 1$ such that for all $\omega \in \Omega$ and for all $x \in V_\omega$

$$(\mathrm{D}_x\, \kappa_\omega)^{\mathrm{tr}}\, C^-_{\gamma_-,\omega} \subseteq C^-_{\gamma_-^*}(x) \quad \text{and} \quad (\mathrm{D}_x\, \kappa_\omega)^{\mathrm{tr}}\, C^+_{\gamma_+,\omega} \subseteq C^+_{\gamma_+^*}(x)\,. \tag{12}$$

This is possible by uniform continuity of the weak-stable and weak-unstable distributions and the flowbox condition in (10). Note that the cones $C^\sigma_{\gamma,\omega}$ are not necessarily convex. This poses no problem since the differential is linear and hence the convex closure of $C^\sigma_{\gamma_\sigma,\omega}$ is contained in $C^\sigma_{\gamma_\sigma^*}(x)$ (this is already a convex, closed cone) for all $x \in M$. Without loss of generality we identify $C^\sigma_{\gamma_\sigma,\omega}$ with its convex closure.

Definition 2.1 (Cone ensemble)**.** *Let $C^-, C^+ \subset \mathbb{R}^d$, $d \geqslant 3$, be transversal, convex, closed cones with non-empty interiors. Let $\Phi_\sigma\colon \mathbb{R}^d \setminus \{0\} \to [0,1]$ be C^∞ maps, $\sigma \in \{-,+,0\}$, such that*

$$\Phi_{-|\,\mathrm{int}\, C^-} \equiv 1, \qquad \Phi_{+|\,\mathrm{int}\, C^+} \equiv 1, \qquad \Phi_- + \Phi_+ + \Phi_0 \equiv 1 \qquad \textit{and}$$
$$C^- = \mathbb{R}^d \setminus (\mathrm{supp}\,\Phi_+ \cup \mathrm{supp}\,\Phi_0)\,,\ C^+ = \mathbb{R}^d \setminus (\mathrm{supp}\,\Phi_- \cup \mathrm{supp}\,\Phi_0)\,.$$

We call $\Theta := (\Phi_-, \Phi_+, \Phi_0)$ a cone ensemble.[2]

Definition 2.2 (Cone hyperbolicity)**.** *Let $K \subset \mathbb{R}^d$ be open and let $F\colon K \to F(K)$ be a diffeomorphism. Let Θ, Θ° be two cone ensembles. Let*

$$C^- := \mathbb{R}^d \setminus (\mathrm{supp}\,\Phi_+ \cup \mathrm{supp}\,\Phi_0)\,.$$

We say that F is (Θ°, Θ)-cone hyperbolic on K if there exists C^∞ maps

$$\widetilde{\Phi}_+, \widetilde{\Phi}^\circ_\sigma\colon \mathbb{R}^d \setminus \{0\} \to [0,1]$$

such that $\widetilde{\Phi}_{+|\,\mathrm{supp}\,\Phi_+}, \widetilde{\Phi}^\circ_{\sigma|\,\mathrm{supp}\,\Phi^\circ_\sigma} \equiv 1$ for all $\sigma \in \{-,0\}$ such that for all $z \in K$

$$(\mathrm{D}_z\, F)^{\mathrm{tr}}\, \mathrm{supp}\, \widetilde{\Phi}^\circ_- \Subset C^- \ \textit{and}\ (\mathrm{D}_z\, F)^{\mathrm{tr}}\, \mathrm{supp}\, \widetilde{\Phi}^\circ_0 \Subset \mathbb{R}^d \setminus \mathrm{supp}\, \widetilde{\Phi}_+\,. \tag{13}$$

In Section 3.2 an anisotropic Banach space is constructed where the cones C^-, C^+ determine the directions of lowest and highest regularity, respectively. The inclusions (13) ensure that no parts of higher regularity are mapped to parts of lower regularity.

Lemma 2.3 (Existence of admissible cones)**.** *Let $\alpha \in \mathbb{R}$ and let $\omega, \omega' \in \Omega$. Set $V_{\alpha,\omega\omega'} := V_\omega \cap g_\alpha(V_{\omega'})$ and set*

$$F_{-\alpha,\omega\omega'}\colon \kappa_\omega(V_{\alpha,\omega\omega'}) \to \kappa_{\omega'}(V_{-\alpha,\omega'\omega}) : y \mapsto \kappa_{\omega'} \circ g_{-\alpha} \circ \kappa_\omega^{-1}(y)\,.$$

Then there exists $\alpha_0 > 0$ such that for all $\omega, \omega' \in \Omega$ there exist cone ensembles

$$\Theta_\omega = (\Phi_{-,\omega}, \Phi_{+,\omega}, \Phi_{0,\omega}) \quad \textit{and} \quad \Theta^\circ_{\omega'} = \left(\Phi^\circ_{-,\omega'}, \Phi^\circ_{+,\omega'}, \Phi^\circ_{0,\omega'}\right),$$

such that for all $\alpha \geqslant \alpha_0$ the map $F_{-\alpha,\omega\omega'}$ is $(\Theta^\circ_{\omega'}, \Theta_\omega)$-cone hyperbolic. Moreover, for every $\omega \in \Omega$ it holds

$$\mathrm{supp}\,\Phi^\circ_{0,\omega} \Subset \mathrm{supp}\,\Phi_{0,\omega} \cup \mathrm{supp}\,\Phi_{+,\omega} \qquad \textit{and} \qquad \mathrm{supp}\,\Phi^\circ_{+,\omega} \Subset \mathrm{supp}\,\Phi_{+,\omega}\,. \tag{14}$$

[2] By the support of a function $f\colon S \to \mathbb{C}$ we mean $\mathrm{supp} := \{x \in S \mid f(x) \neq 0\}$ which can be an open set in the topology of S.

Proof. We let $\omega, \omega' \in \Omega$. We assume $V_{\alpha,\omega\omega'} \neq \varnothing$ (otherwise we are done). We let $0 < \gamma_-, \gamma_+ \leqslant 1$ be small such that $\gamma_-^*, \gamma_+^* > 0$ are the values attained in (12) for all cones $C^-_{\gamma_-,\omega}$, $C^+_{\gamma_+,\omega}$, $\omega \in \Omega$. These cones are transversal, convex and closed by construction. We repeat the construction, resulting in values $\widetilde{\gamma}_-^* < \gamma_-^*$ and $\widetilde{\gamma}_+^* < \gamma_+^*$, using now values

$$\widetilde{\gamma}_- < \gamma_-, \quad \widetilde{\gamma}_+ < \gamma_+,$$

sufficiently small (possibly by passing to a finer open cover) such that for all $\omega \in \Omega$ and all $x \in V_\omega$

$$\left(\mathrm{D}_{\kappa_\omega(x)}\, \kappa_\omega^{-1}\right)^{\mathrm{tr}} C^-_{\widetilde{\gamma}_-^*}(x) \Subset C^-_{\gamma_-,\omega}, \quad \left(\mathrm{D}_{\kappa_\omega(x)}\, \kappa_\omega^{-1}\right)^{\mathrm{tr}} C^+_{\widetilde{\gamma}_+^*}(x) \Subset C^+_{\gamma_+,\omega}. \tag{15}$$

We note that the map $F_{\alpha,\omega\omega'}$ is a diffeomorphism by construction. We construct further cones as follows: By the construction of local cones in (11) and the compact inclusion given in (12) for some $C^2\beta^\alpha\gamma_+^* \leqslant \gamma'_+ < \widetilde{\gamma}_+^*$ and for all $\alpha \geqslant \alpha_0$ we have for all $x \in V_{\alpha,\omega\omega'}$

$$(\mathrm{D}\, g_\alpha)^{\mathrm{tr}} \left(\mathrm{D}_x\, \kappa_\omega\right)^{\mathrm{tr}} C^+_{\gamma_+,\omega} \subseteq (\mathrm{D}\, g_\alpha)^{\mathrm{tr}}\, C^+_{\gamma_+^*}(x) \Subset C^+_{\gamma'_+}\left(g_{-\alpha}(x)\right) \Subset C^+_{\widetilde{\gamma}_+^*}\left(g_{-\alpha}(x)\right).$$

Comparing with the compact inclusion in (15), there exists a convex, closed cone $\widetilde{C}^+_{\gamma_+,\omega'} \subset \mathbb{R}^d$ such that

$$(\mathrm{D}\, F_{\alpha,\omega'\omega})^{\mathrm{tr}} C^+_{\gamma_+,\omega} \Subset \widetilde{C}^+_{\gamma_+,\omega'} \Subset C^+_{\gamma_+,\omega'}. \tag{16}$$

Analogously we find

$$(\mathrm{D}\, F_{-\alpha,\omega'\omega})^{\mathrm{tr}} C^-_{\gamma_-,\omega} \Subset \widetilde{C}^-_{\gamma_-,\omega'} \Subset C^-_{\gamma_-,\omega'}. \tag{17}$$

Recalling Definition 2.1, we let

$$\Theta_\omega = \left(\Phi_{-,\omega}, \Phi_{+,\omega}, \Phi_{0,\omega}\right) \quad \text{and} \quad \Theta^\circ_{\omega'} = \left(\Phi^\circ_{-,\omega'}, \Phi^\circ_{+,\omega'}, \Phi^\circ_{0,\omega'}\right)$$

be the cone ensembles such that

$$\Phi_{-,\omega|\,\mathrm{int}\,\widetilde{C}^-_{\gamma_-,\omega}} \equiv \Phi_{+,\omega|\,\mathrm{int}\,C^+_{\gamma_+,\omega}} \equiv \Phi^\circ_{-,\omega'|\,\mathrm{int}\,C^-_{\gamma_-,\omega'}} \equiv \Phi^\circ_{+,\omega'|\,\mathrm{int}\,\widetilde{C}^+_{\gamma_+,\omega'}} \equiv 1.$$

The supports of $\Phi_{-,\omega}, \Phi_{+,\omega}$ and $\Phi^\circ_{-,\omega'}, \Phi^\circ_{+,\omega'}$ are taken to be disjoint, respectively, considering slightly larger convex cones. We check $(\Theta^\circ_{\omega'}, \Theta_\omega)$-cone hyperbolicity of $F_{\alpha,\omega\omega'}$, recalling Definition 2.2. The supports of $\widetilde{\Phi}^\circ_{-,\omega'}, \widetilde{\Phi}_{+,\omega'}, \widetilde{\Phi}^\circ_{0,\omega'}$ are chosen analogously on corresponding slightly larger cones. The first compact inclusion in (13) is a direct consequence of the compact inclusion in (17). To see the second compact inclusion in (13) note that

$$\left(\left(\mathrm{D}\, F_{-\alpha,\omega\omega'}\right)^{\mathrm{tr}}\right)^{-1}\left(\mathbb{R}^d \setminus \operatorname{supp}\Phi_{+,\omega}\right) = \mathbb{R}^d \setminus \left(\mathrm{D}\, F_{\alpha,\omega'\omega}\right)^{\mathrm{tr}} \operatorname{supp}\Phi_{+,\omega}.$$

Comparing with the compact inclusion in (16), we conclude. The claim in (14) follows again by comparing with the compact inclusions in (16) and (17). □

3. The transfer operator and the anisotropic Banach space

3.1. **The transfer operator.** We denote by $C^r(M)$ the space of $C^{\lfloor r\rfloor}$ functions whose $\lfloor r\rfloor$-th partial derivatives in charts are $C^{r-\lfloor r\rfloor}$. We let $C^{r-1}_X(M)$[3] be the space of C^{r-1} functions which are C^r in the flow direction X defined by (4). Fixing

[3] If $\varphi \in C^{r-1}(M)$ then $\varphi_c := \frac{1}{c}\int_0^c \varphi \circ g_{-\alpha}\,\mathrm{d}\,\alpha \in C^{r-1}_X(M)$ for all $c > 0$. In the Banach spaces we construct the limit $\lim_{c\to 0}\varphi_c$ exists.

a "potential function" $V \in C^{r-1}(M,\mathbb{R})$, we introduce the ϕ_α-weighted transfer operator family

$$\mathcal{L}_{\alpha,\phi_\alpha} : \varphi \mapsto \phi_\alpha \cdot (\varphi \circ g_{-\alpha}), \quad \alpha \geqslant 0, \tag{18}$$

acting on $\varphi \in C^{r-1}_X(M)$, where

$$\phi_\alpha(x) := \exp\left(\int_0^\alpha V \circ g_{-\alpha'}(x)\,\mathrm{d}\,\alpha'\right).$$

We will construct Banach spaces $W^{s,t,q}_p$ containing $C^{r-1}_X(M)$ as a dense subspace (for suitable choices $p,s,t,q \in \mathbb{R}$) on which the family (18) of operators extends continuously to a strongly continuous semigroup (see Lemma 4.4 below). Note that

$$V = \partial_\alpha \phi_{\alpha|\alpha=0^+}. \tag{19}$$

Our construction will show that for all $\varphi \in C^{r-1}_X(M)$

$$\partial_\alpha \mathcal{L}_{\alpha,\phi_\alpha}\varphi_{|\alpha=0^+} = X\varphi + V\varphi,$$

is well-defined in the sense that $(X+V)\varphi \in W^{s,t,q}_p$ if $\varphi \in C^{r-1}_X(M)$. The operator $X+V$ is the generator of the semigroup $\left\{\mathcal{L}_{\alpha,\phi_\alpha} : W^{s,t,q}_p \to W^{s,t,q}_p \mid \alpha \geqslant 0\right\}$. We denote by

$$\sigma(X+V)|_{W^{s,t,q}_p}$$

the spectrum of $X+V$ to emphasize the dependency of the domain and hence the spectrum of $X+V$ on $W^{s,t,q}_p$. We show in Theorem 4.5 that the resolvent of $X+V$

$$\mathcal{R}_z\varphi := (z - V - X)^{-1}\varphi, \quad z \notin \sigma(X+V)|_{W^{s,t,q}_p}, \quad \varphi \in W^{s,t,q}_p, \tag{20}$$

admits a Lasota–Yorke inequality for large $\Re z > 0$. This allows us to identify a vertical left-open strip in the complex plane in which $\sigma(X+V)|_{W^{s,t,q}_p}$ contains only isolated eigenvalues of finite multiplicity of $X+V$ (see Lemma 4.10).

3.2. **The anisotropic Banach space.** We work locally with the atlas $\mathcal{A}$, introduced in Section 2. We let $\Psi_n : \mathbb{R}^d \to [0,1]$, $n \in \mathbb{Z}_{\geqslant 0}$, be a Paley–Littlewood decomposition as follows:

Let $\chi : \mathbb{R}_{>0} \to [0,1]$ be a C^∞ map so that $\chi_{|(0,1]} \equiv 1$ and $\operatorname{supp}\chi \subseteq [0,2]$. Let $|\cdot| : \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ be a smooth norm on $\mathbb{R}^d\setminus\{0\}$. Define Ψ_n by setting for all $\xi \in \mathbb{R}^d\setminus\{0\}$

$$\Psi_0(\xi) := \chi(|\xi|) \quad \text{and} \quad \Psi_n(\xi) := \chi\left(\left|2^{-n}\xi\right|\right) - \chi\left(\left|2^{1-n}\xi\right|\right),\ n \geqslant 1. \tag{21}$$

This defines a partition of unity on $\mathbb{R}^d\setminus\{0\}$ since we have

$$\sum_{n=0}^\infty \Psi_n(\xi) = \lim_{n\to\infty}\chi\left(\left|2^{-n}\xi\right|\right) = 1.$$

For all $n \geqslant 1$ it holds $\Psi_n(\xi) = \Psi_1(2^{-n+1}\xi)$ from which one finds

$$\operatorname{supp}\Psi_n \subseteq \left\{\xi \in \mathbb{R}^d \mid 2^{n-1} \leqslant |\xi| \leqslant 2^{n+1}\right\}. \tag{22}$$

The inverse Fourier transform is given by

$$\mathbb{F}^{-1}\varphi(x) := (2\pi)^{-d}\int_{\mathbb{R}^d} e^{\mathrm{i}\,\xi x}\varphi(\xi)\,\mathrm{d}\,\xi,$$

where

$$\xi x := \langle \xi, x\rangle$$

is the canonical scalar product on $\mathbb{R}^d$. The convolution of two complex valued functions φ_1, φ_2 on $\mathbb{R}^d$ (and extended to distributions) is given by

$$\varphi_1 * \varphi_2(x) := \int_{\mathbb{R}^d} \varphi_1(x-y)\varphi_2(y)\,\mathrm{d}\,y.$$

We will make frequent use (e.g. in the proof of Lemma 3.1 below and Lemma 4.16 in Section 4.4) of a special case of Young's inequality for convolutions,

$$\|\varphi_1 * \varphi_2\|_{L_p} \leqslant \|\varphi_1\|_{L_1} \|\varphi_2\|_{L_p}, \quad \text{for all } p \in [1, \infty].$$

Given a cone ensemble $\Theta = (\Phi_-, \Phi_+, \Phi_0)$, we set for all $\sigma \in \{-,+,0\}$, $n \in \mathbb{Z}_{\geqslant 0}$,

$$\Psi_{\sigma,n} := \Psi_n \Phi_\sigma \quad \text{and} \quad \Psi^{\mathrm{Op}}_{\sigma,n}\varphi := \left(\mathbb{F}^{-1}\Psi_{\sigma,n}\right) * \varphi. \tag{23}$$

We let $\widetilde{\Psi}_0, \widetilde{\Psi}_1 \in C^\infty$ such that $\widetilde{\Psi}_{0|\operatorname{supp}\Psi_0} \equiv 1$ and $\widetilde{\Psi}_{1|\operatorname{supp}\Psi_1} \equiv 1$. We set for every $n \in \mathbb{N}$

$$\widetilde{\Psi}_n := \widetilde{\Psi}_1 \circ 2^{1-n}.$$

(In principal it is enough to require the condition on the support of $\widetilde{\Psi}_n$ for each n individually. Regarding the bounds in (25) below our choice here is reasonable.) Then we set for every $\sigma \in \{-,+,0\}$ and every $n \in \mathbb{Z}_{\geqslant 0}$

$$\widetilde{\Psi}_{\sigma,n} := \widetilde{\Psi}_n \widetilde{\Phi}_\sigma \quad \text{and} \quad \widetilde{\Psi}^{\mathrm{Op}}_{\sigma,n}\varphi := \left(\mathbb{F}^{-1}\widetilde{\Psi}_{\sigma,n}\right) * \varphi, \tag{24}$$

where $\widetilde{\Phi}_\sigma \in C^\infty$ and $\operatorname{supp}\widetilde{\Phi}_\sigma$ is a closed convex cone such that $\widetilde{\Phi}_{\sigma|\operatorname{supp}\Phi_\sigma} \equiv 1$ and $\Phi_{\sigma_1}\Phi_{\sigma_2} \equiv 0 \Rightarrow \widetilde{\Phi}_{\sigma_1}\widetilde{\Phi}_{\sigma_2} \equiv 0$ for all $\sigma_1, \sigma_2 \in \{-,+,0\}$. We have the following estimates for all σ and all $n \in \mathbb{N}$:

$$\left\|\mathbb{F}^{-1}\Psi_n\right\|_{L_1} = \left\|\mathbb{F}^{-1}\Psi_1\right\|_{L_1} < \infty, \quad \left\|\mathbb{F}^{-1}\Psi_{\sigma,n}\right\|_{L_1} = \left\|\mathbb{F}^{-1}\Psi_{\sigma,1}\right\|_{L_1} < \infty. \tag{25}$$

Analogous estimates hold for $\mathbb{F}^{-1}\Psi_{\sigma,0}$ and $\mathbb{F}^{-1}\Psi_0$ and for the $\sim$-versions as well. If Θ° is another cone ensemble we define $\Psi^\circ_{\sigma,n}, \Psi^{\circ\,\mathrm{Op}}_{\sigma,n}$ and $\widetilde{\Psi}^\circ_{\sigma,n}, \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\sigma,n}$ analogously. We set

$$B := \left\{x \in \mathbb{R}^d \mid |x| < 1\right\} \quad \text{and} \quad B^{\mathrm{c}} := \mathbb{R}^d \backslash B. \tag{26}$$

In order to show a continuous embedding of certain spaces we will use very often the following statement about convolution operators (an extension of [39, Theorem 0.3.1] for the case $r = 1$ and $K(x,y) = K(x-y)$ in his notation). In Lemma 3.1 below all the occurring L_p-spaces are understood (as Bochner spaces, cf. [10]) such that if $a \in L_p\left(\mathbb{R}^d, \mathcal{B}\right)$ for some complex Banach space $\mathcal{B}$ then the norm of a is given by

$$\|a\|_{L_p(\mathbb{R}^d,\mathcal{B})} := \left\|\|a\|_{\mathcal{B}}\right\|_{L_p(\mathbb{R}^d,\mathbb{R}_{\geqslant 0})}.$$

The following lemma handles the range $p \in [1, \infty]$. (For parameters $p \in (1, \infty)$ one could apply instead the classical Marcinkiewicz theorem quoted e.g. as [6, Theorem 3.1].)

Lemma 3.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be (complex) Banach spaces, let $d \in \mathbb{N}$ and let $Q \in C^{d+1}\left(\mathbb{R}^d, \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)\right)$ satisfy for its partial derivatives*

$$\left\|\partial^\beta_\xi Q(\xi)\right\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} \leqslant C(\beta)\,|\xi|^{-|\beta|} \quad \textit{as} \quad |\xi| \to \infty,$$

for some constants $C(\beta) > 0$ and all multi-indices $\beta \in \{0, \dots, d+1\}^d$ such that $|\beta| \leqslant d+1$, where $|\beta| := \beta_1 + \dots + \beta_d$. Then for all $p \in [1, \infty]$ the map

$$Q^{\mathrm{Op}} \colon L_p\left(\mathbb{R}^d, \mathcal{B}_1\right) \to L_p\left(\mathbb{R}^d, \mathcal{B}_2\right) : a \mapsto \int_{\mathbb{R}^d} \left(\mathbb{F}^{-1} Q\right)(x-y) a(y)\,\mathrm{d}\,y, \tag{27}$$

defines a bounded linear operator, where for every $b \in \mathcal{B}_1$ and every $x \in \mathbb{R}^d$

$$\mathbb{F}^{-1} Q(x)\, b := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{\mathrm{i}\, x\xi} Q(\xi)\, b\,\mathrm{d}\,\xi.$$

It holds

$$\left\| Q^{\mathrm{Op}} \right\|_{\mathcal{L}(L_p(\mathbb{R}^d, \mathcal{B}_1), L_p(\mathbb{R}^d, \mathcal{B}_2))} \leqslant \left\| \mathbb{F}^{-1} Q \right\|_{L_1(\mathbb{R}^d, \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2))} < \infty.$$

Proof. Linearity of Q^{Op} follows if Q^{Op} is a bounded operator. Suppose first that $\mathbb{F}^{-1} Q \in L_1\left(\mathbb{R}^d, \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)\right)$. We estimate

$$\begin{aligned}
\left\| Q^{\mathrm{Op}} a \right\|_{L_p(\mathbb{R}^d, \mathcal{B}_2)} &= \left\| \left\| \int_{\mathbb{R}^d} \left(\mathbb{F}^{-1} Q\right)(\cdot - y) a(y)\,\mathrm{d}\,y \right\|_{\mathcal{B}_2} \right\|_{L_p} \\
&\leqslant \left\| \int_{\mathbb{R}^d} \left\| \left(\mathbb{F}^{-1} Q\right)(\cdot - y) a(y) \right\|_{\mathcal{B}_2} \mathrm{d}\,y \right\|_{L_p} \\
&\leqslant \left\| \int_{\mathbb{R}^d} \left\| \left(\mathbb{F}^{-1} Q\right)(\cdot - y) \right\|_{\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)} \|a(y)\|_{\mathcal{B}_1} \mathrm{d}\,y \right\|_{L_p} \\
&= \left\| \left\| \mathbb{F}^{-1} Q \right\|_{\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)} * \|a\|_{\mathcal{B}_1} \right\|_{L_p}.
\end{aligned}$$

Using Young's inequality, we estimate and conclude

$$\begin{aligned}
\left\| \left\| \mathbb{F}^{-1} Q \right\|_{\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)} * \|a\|_{\mathcal{B}_1} \right\|_{L_p} &\leqslant \left\| \left\| \mathbb{F}^{-1} Q \right\|_{\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)} \right\|_{L_1} \left\| \|a\|_{\mathcal{B}_1} \right\|_{L_p} \\
&= \left\| \mathbb{F}^{-1} Q \right\|_{L_1(\mathbb{R}^d, \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2))} \|a\|_{L_p(\mathbb{R}^d, \mathcal{B}_1)}.
\end{aligned}$$

We now show $\left\| \mathbb{F}^{-1} Q \right\|_{L_1(\mathbb{R}^d, \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2))} < \infty$. It remains to show an upper bound for

$$I := \int_{\mathbb{R}^d} \left\| \int_{\mathbb{R}^d} e^{\mathrm{i}\, y\xi} Q(\xi)\, a\,\mathrm{d}\,\xi \right\|_{\mathcal{B}_2} \mathrm{d}\,y.$$

Inside I we substitute, whenever $y \neq 0$

$$\xi \mapsto \langle y, y \rangle^{-\frac{1}{2}} \xi$$

which yields

$$I = \int_{\mathbb{R}^d} \langle y, y \rangle^{-\frac{d}{2}} \|J(y)\|_{\mathcal{B}_2} \mathrm{d}\,y = \left(\int_B + \int_{B^c} \right) \langle y, y \rangle^{-\frac{d}{2}} \|J(y)\|_{\mathcal{B}_2} \mathrm{d}\,y,$$

where

$$J := J(y) = \int_{\mathbb{R}^d} e^{\mathrm{i}\, y \langle y, y \rangle^{-\frac{1}{2}} \xi} Q\left(\langle y, y \rangle^{-\frac{1}{2}} \xi \right) a\,\mathrm{d}\,\xi.$$

For every $y \in \mathbb{R}^d \backslash \{0\}$ we set

$$\xi_0 := y \langle y, y \rangle^{-\frac{1}{2}} \pi.$$

Clearly, it holds

$$\langle \xi_0, \xi_0 \rangle = \pi^2 \quad \text{and} \quad y \langle y, y \rangle^{-\frac{1}{2}} (\xi + \xi_0) = y \langle y, y \rangle^{-\frac{1}{2}} \xi + \pi.$$

We now repeat the following substitution

$$\xi \mapsto \xi + \xi_0,$$

inductively $(d+1)$-times in one summand of the following splitting

$$2^{-1}J + 2^{-1}J = J = \int_{\mathbb{R}^d} e^{\mathrm{i}\, y \langle y,y\rangle^{-\frac{1}{2}} \xi} Q_n\left(\xi\right) a \,\mathrm{d}\,\xi,$$

where for all $0 \leqslant n \leqslant d+1$

$$Q_n\left(\xi\right) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \left(-1\right)^k Q\left(\langle y,y\rangle^{-\frac{1}{2}} \left(\xi + k\xi_0\right)\right),$$

which yields

$$J = \int_{\mathbb{R}^d} e^{\mathrm{i}\, y \langle y,y\rangle^{-\frac{1}{2}} \xi} Q_{d+1}\left(\xi\right) a \,\mathrm{d}\,\xi.$$

We let $0 < \epsilon < 1$. We split the part in Q_{d+1} if $y \in B$ for every $0 \leqslant n \leqslant d+1$ as

$$1 \equiv \chi\left(\left(\xi + n\xi_0\right)\langle y,y\rangle^{\frac{\epsilon}{2}}\right) + \left(1-\chi\right)\left(\left(\xi + n\xi_0\right)\langle y,y\rangle^{\frac{\epsilon}{2}}\right),$$

for every corresponding summand in Q_{d+1}, respectively. The part in I which corresponds to $\chi\left(\left(\xi + n\xi_0\right)\langle y,y\rangle^{\frac{\epsilon}{2}}\right)$ is estimated trivially, using boundedness of Q and integrability of $\langle y,y\rangle^{\frac{\epsilon-d}{2}}$ on B. Using the identity

$$\begin{aligned} Q\left(\xi\right) - Q\left(\xi+\xi_0\right) &= \int_0^1 \partial_t Q\left(\left(\xi + \xi_0 - t\xi_0\right)\langle y,y\rangle^{-\frac{\epsilon}{2}}\right) \mathrm{d}\,t \\ &= -\langle y,y\rangle^{-\frac{\epsilon}{2}} \int_0^1 \left(\mathrm{D}\, Q\right)\left(\left(\xi + \xi_0 - t\xi_0\right)\langle y,y\rangle^{-\frac{\epsilon}{2}}\right) \xi_0 \,\mathrm{d}\,t, \end{aligned}$$

we now write the remaining part in Q_{d+1} as

$$\begin{aligned} Q_{d+1}\left(\xi\right) = 2^{-d-1} \int_{[0,1]^{d+1}} \sum_{n=0}^{d+1} \binom{d+1}{n} \langle y,y\rangle^{\frac{n\epsilon}{2}} \langle y,y\rangle^{-\frac{d+1-n}{2}} \\ \times \left(\left(\mathrm{D}^n\left(1-\chi\right)\right)\circ\langle y,y\rangle^{\frac{\epsilon}{2}} \left(\mathrm{D}^{d+1-n} Q\right)\circ\langle y,y\rangle^{-\frac{1}{2}}\right)\left(\xi\left(t\right)\right)\left(-\xi_0\right)^{\otimes(d+1)} \mathrm{d}\,t, \end{aligned} \tag{28}$$

where we put

$$\xi\left(t\right) \coloneqq \xi + \left(d+1\right)\xi_0 - t\xi_0^{\otimes(d+1)} \quad \text{and} \quad \xi_0^{\otimes(d+1)} \coloneqq \underbrace{\left(\xi_0,\dots,\xi_0\right)}_{(d+1)-\text{times}}.$$

We observe that the part where derivatives of $\left(\left(1-\chi\right)\circ\langle y,y\rangle^{\frac{\epsilon}{2}}\right)\left(\xi\right)$ contribute implies

$$\xi \in \left(\left(2B\right)\backslash B\right)\langle y,y\rangle^{-\frac{\epsilon}{2}}.$$

Using the decay condition on all the partial derivatives of Q, recalling that

$$\langle y,y\rangle^{\frac{\epsilon-d}{2}} \log\langle y,y\rangle$$

is integrable on the unit ball, and exchanging the order of integration with respect to t as the outermost (justified by absolute integrability), we find for the corresponding part in I, for some constants $C_1, C_2, C_3 > 0$

$$\begin{aligned} I &\leqslant C_1 \frac{\pi^{d+1}}{2^{d+1}} \|a\|_{\mathcal{B}_1} \sum_{n=0}^{d+1} \binom{d+1}{n} \int_B \int_{((2B)\setminus B)\langle y,y\rangle^{-\frac{\epsilon}{2}}} |\xi|^{-d-1+n} \,\mathrm{d}\,\xi \,\langle y,y\rangle^{\frac{n\epsilon-d}{2}} \,\mathrm{d}\,y \\ &\qquad + C_1 \pi^{d+1} \|a\|_{\mathcal{B}_1} \int_B \int_{2\langle y,y\rangle^{-\frac{\epsilon}{2}} B^{\mathrm{c}}} |\xi|^{-d-1} \,\mathrm{d}\,\xi \,\langle y,y\rangle^{-\frac{d}{2}} \,\mathrm{d}\,y \\ &\leqslant C_2 \pi^{d+1} (\log d) \|a\|_{\mathcal{B}_1} \int_B \left(1 + \frac{\epsilon}{2} |\log \langle y,y\rangle|\right) \langle y,y\rangle^{\frac{\epsilon-d}{2}} \,\mathrm{d}\,y \\ &\leqslant C_3 \|a\|_{\mathcal{B}_1}. \end{aligned}$$

In the case $y \in B^{\mathrm{c}}$ we proceed analogously, using the formular for $Q_{d+1}(\xi)$ given in (28), but without splitting the integral with respect to ξ. We have now

$$Q_{d+1}(\xi) = 2^{-d-1} \int_{[0,1]^{d+1}} \langle y,y\rangle^{-\frac{d+1}{2}} \left(\mathrm{D}^{d+1} Q\right) \left(\langle y,y\rangle^{-\frac{1}{2}} \xi(t)\right) (-\xi_0)^{\otimes(d+1)} \,\mathrm{d}\,t.$$

If $\xi \in B \langle y,y\rangle^{\frac{\epsilon}{2}}$ we bound the corresponding part in I trivially, using boundedness of the $(d+1)$-th partial derivatives of Q and integrability of $\langle y,y\rangle^{\frac{d\epsilon-2d-1}{2}}$ on B^{c}. If $\xi \in B^{\mathrm{c}} \langle y,y\rangle^{\frac{\epsilon}{2}}$ we use instead the decay condition of the $(d+1)$-th partial derivatives of Q and integrability of $\langle y,y\rangle^{-\frac{d+\epsilon}{2}}$ on B^{c}. □

For every open set $K \subseteq \mathbb{R}^d$ with compact closure we let $C_0^{r-1}(K)$ be the space of C^{r-1} functions which vanish at the boundary of K. Since $C_0^{r-1}(K) \subset L_p(K, \mathbb{C})$ for all $p \in [1, \infty]$, the following definition makes sense.

Definition 3.2 (Local norm and local Banach space). *Let $p \in [1, \infty]$ and let $s, t, q < r-1$. Let Θ be a cone ensemble from Definition 2.1 and let $K \subset \mathbb{R}^d$ be an open set with compact closure. For every $\varphi \in C_0^{r-1}(K)$ we set as the local norm*

$$\|\varphi\|_{W^{s,t,q}_{p,\Theta,K}} := \left\| \left(\sum_{n=0}^{\infty} 4^{ns} \left|\Psi^{\mathrm{Op}}_{-,n} \varphi\right|^2 + 4^{nt} \left|\Psi^{\mathrm{Op}}_{+,n} \varphi\right|^2 + 4^{nq} \left|\Psi^{\mathrm{Op}}_{0,n} \varphi\right|^2 \right)^{\frac{1}{2}} \right\|_{L_p(\mathbb{R}^d)}.$$

The completion $W^{s,t,q}_{p,\Theta,K}$ of $C_0^{r-1}(K)$ under $\|\cdot\|_{W^{s,t,q}_{p,\Theta,K}}$ is our local anisotropic Banach space.

This is an anisotropic version of a Triebel–Lizorkin space [40, p.45, Definition 2] with a certain inner l_2-norm and an outer L_p-norm. More precisely, we relate the summation in n and σ which appears in the norm of $W^{s,t,q}_{p,\Theta,K}$ to the norm of a Hilbert space of complex valued sequences defined on $\{-, +, 0\} \times \mathbb{Z}_{\geqslant 0}$. We set

$$c(-) := s, \quad c(+) := t, \quad c(0) := q. \tag{29}$$

Then we denote by ℓ_2^{c} the Hilbert space with norm given for all $a \in \ell_2^{\mathrm{c}}$ by

$$\|a\|_{\ell_2^{\mathrm{c}}} := \left(\sum_{\sigma,n} 4^{c(\sigma)n} |a_{\sigma,n}|^2 \right)^{\frac{1}{2}}. \tag{30}$$

For $s', t', q' \in \mathbb{R}$ we define c' and ℓ_2^{c} analogously.

Lemma 3.3 (Multiplication and composition operator). *Let $p \in [1, \infty]$ and let $s', t', q', s, t, q < r-1$. Let $\tilde{r} > \max\{0, s, t, q\} - \min\{0, s', t', q'\}$ and let $f \in C_0^{\tilde{r}}(K)$*

for some open set $K \in \mathbb{R}^d$ with compact closure and let $F\colon K \to F(K)$ be a $C^{\tilde{r}}$ diffeomorphism. Let Θ and Θ° be two cone ensembles. Then the linear operator

$$\mathcal{M}_{F,f}\colon W^{s',t',q'}_{p,\Theta^\circ,K} \to W^{s,t,q}_{p,\Theta,F(K)}\colon \varphi \mapsto f\cdot(\varphi\circ F)$$

is bounded if $c(\sigma) \leqslant c'(\tau)$ whenever $\bigcup_{x\in K}\operatorname{supp}\Psi_\sigma \cap \mathrm{D}\,F(x)^{\mathrm{tr}}\operatorname{supp}\Psi^\circ_\tau \neq \varnothing$. Moreover, if $F = \mathrm{id}$ and $\Theta = \Theta^\circ$ the linear operator

$$\mathcal{M}_{\mathrm{id},f}\colon W^{s,t,q}_{p,\Theta,K} \to W^{s,t,q}_{p,\Theta,K}$$

is bounded if $s \leqslant q \leqslant t$.

Proof. We exclude first the indices for given $\sigma,\tau \in \{-,+,0\}$ such that

$$\bigcup_{x\in K}\operatorname{supp}\Psi_\sigma \cap \mathrm{D}\,F(x)^{\mathrm{tr}}\operatorname{supp}\Psi^\circ_\tau \neq \varnothing, \tag{31}$$

and given $n,\ell \in \mathbb{Z}_{\geqslant 0}$ such that

$$\left|\sup_{x\in F(K),\xi\in(\operatorname{supp}\Psi_\sigma\cap B)}\left|\mathrm{D}\,F^{-1}(x)^{\mathrm{tr}}\xi\right|\right|^{-1}2^{-4} \leqslant 2^{n-l} \leqslant 2^4 \sup_{x\in K,\eta\in(\operatorname{supp}\Psi^\circ_\tau\cap B)}\left|\mathrm{D}\,F(x)^{\mathrm{tr}}\eta\right|.$$

For all remaining $\sigma,\tau \in \{-,+,0\}$ and $n,\ell \in \mathbb{Z}_{\geqslant 0}$ we bound the local norm for every $\epsilon > 0$ and some constant $C_1 = C_1(\epsilon) > 0$

$$\begin{aligned}\|\mathcal{M}_{F,f}\varphi\|_{W^{s,t,q}_{p,\Theta,K}} &= \left\|\left(\sum_{\sigma,n}4^{-\epsilon n}4^{(c(\sigma)+\epsilon)n}\left|\Psi^{\mathrm{Op}}_{\sigma,n}\mathcal{M}_{F,f}\varphi\right|^2\right)^{\frac{1}{2}}\right\|_{L_p(\mathbb{R}^d)}\\ &\leqslant C_1\sup_{\sigma,n}2^{(c(\sigma)+\epsilon)n}\left\|\Psi^{\mathrm{Op}}_{\sigma,n}\mathcal{M}_{F,f}\varphi\right\|_{L_p(\mathbb{R}^d)}.\end{aligned} \tag{32}$$

On the excluded indices we estimate as in the proof of Lemma 4.13 below, using Lemma 3.1 and Cauchy–Schwarz in ℓ and that $n \sim \ell$ and using $c(\sigma) \leqslant c'(\tau)$. We recall the map $\widetilde{\Psi}^\circ_{\tau,\ell}$ defined in (24). Then we bound for every $n \geqslant 0$ and every $\sigma \in \{-,+,0\}$

$$\begin{aligned}\left\|\Psi^{\mathrm{Op}}_{\sigma,n}\mathcal{M}_{F,f}\varphi\right\|_{L_p(\mathbb{R}^d)} &\leqslant \sum_{\tau,\ell}\left\|\Psi^{\mathrm{Op}}_{\sigma,n}\mathcal{M}_{F,f}\widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell}\Psi^{\circ\,\mathrm{Op}}_{\tau,\ell}\varphi\right\|_{L_p(\mathbb{R}^d)}\\ &= \sum_{\tau,\ell}2^{-c'(\tau)\ell}2^{c'(\tau)\ell}\left\|\Psi^{\mathrm{Op}}_{\sigma,n}\mathcal{M}_{F,f}\widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell}\Psi^{\circ\,\mathrm{Op}}_{\tau,\ell}\varphi\right\|_{L_p(\mathbb{R}^d)}.\end{aligned} \tag{33}$$

Note that if $\operatorname{supp}\Psi_\sigma$ and $\mathrm{D}\,F(x)^{\mathrm{tr}}\operatorname{supp}\Psi^\circ_\tau$ have empty intersection, since the supports are open, we may assume that $\operatorname{supp}\Psi_\sigma$ and $\mathrm{D}\,F(x)^{\mathrm{tr}}\operatorname{supp}\widetilde{\Psi}^\circ_\tau$ have empty intersection as well. Since we excluded the conditions regarding certain σ,τ and n,ℓ given in (31) and below of it then by construction of $\widetilde{\Psi}^\circ_{\tau,\ell}$, for some constant $C_2 > 0$ it holds, in the following assuming $n,\ell > 0$,

$$\begin{aligned}&\inf_{x\in K}\left|\operatorname{supp}\Psi_{\sigma,n} - \mathrm{D}\,F(x)^{\mathrm{tr}}\operatorname{supp}\widetilde{\Psi}^\circ_{\tau,\ell}\right| \geqslant C_2 2^{\max\{n,\ell\}} \text{ or}\\ &\inf_{x\in F(K)}\left|\mathrm{D}\,F^{-1}(x)^{\mathrm{tr}}\operatorname{supp}\Psi_{\sigma,n} - \operatorname{supp}\widetilde{\Psi}^\circ_{\tau,\ell}\right| \geqslant C_2 2^{\max\{n,\ell\}}.\end{aligned} \tag{34}$$

In the following we assume the first inequality in (34). Otherwise the next estimates are done with the substitution $F(y) \mapsto y$. If $n = 0$ or $\ell = 0$ the following estimate is done analogously, using that either ξ or η is bounded. We set

$$\tilde{\xi} := 2^{-n}\xi, \quad \tilde{\eta} := 2^{-\ell}\eta, \quad \text{and} \quad U := \mathbb{R}^d\times\mathbb{R}^d\times K\times\mathbb{R}^d.$$

We write for every $x \in \mathbb{R}$

$$\begin{aligned}I(x) &:= I_{\sigma,n,\tau,\ell}(x) := \frac{(2\pi)^{2d}}{2^{(n+\ell)d}} \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi(x) \\ &= \frac{(2\pi)^{2d}}{2^{(n+\ell)d}} \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi(x) \\ &= \int_U e^{\mathrm{i}\, 2^n \widetilde{\xi}(x-y)} e^{\mathrm{i}\, 2^\ell \widetilde{\eta}(F(y)-z)} \Psi_{\sigma,1}\left(\widetilde{\xi}\right) \widetilde{\Psi}^{\circ}_{\tau,1}(\widetilde{\eta}) f(y) \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi(z) \,\mathrm{d}\, z \,\mathrm{d}\, y \,\mathrm{d}\, \widetilde{\xi} \,\mathrm{d}\, \widetilde{\eta}.\end{aligned}$$

Note that by assumption we have

$$\widetilde{r} > \max\{0, s, t, q\} - \min\{0, s', t', q'\} \geqslant 0. \tag{35}$$

Integrating $\widetilde{r}$-times by parts (see Lemma A.3-Lemma A.5) in y, using the lower bound in (33), we arrive at

$$I(x) = \int_U e^{\mathrm{i}\, 2^n \widetilde{\xi}(x-y)} e^{\mathrm{i}\, 2^\ell \widetilde{\eta}(F(y)-z)} \Psi_{\sigma,1}\left(\widetilde{\xi}\right) \widetilde{\Psi}^{\circ}_{\tau,1}(\widetilde{\eta}) \frac{f_{\widetilde{r}}(y)}{2^{\max\{n,\ell\}\widetilde{r}}} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi(z) \,\mathrm{d}\, z \,\mathrm{d}\, y \,\mathrm{d}\, \widetilde{\xi} \,\mathrm{d}\, \widetilde{\eta},$$

where all derivatives of $f_{\widetilde{r}}(y)$ with respect to $\widetilde{\eta}$ and $\widetilde{\xi}$ are bounded uniformly for all $\left(\widetilde{\xi}, \widetilde{\eta}, y\right) \in \operatorname{supp} \Psi_{\sigma,1} \times \operatorname{supp} \widetilde{\Psi}^{\circ}_{\tau,1} \times K$. We set for every $y \in \mathbb{R}^d$ and for every $n \geqslant 0$

$$u(y) := \begin{cases} 1, & |y| \leqslant 1 \\ |y|^{-d-1}, & \text{otherwise} \end{cases}, \quad u_n := u \circ 2^n.$$

If $|x-y|\, 2^n > 1$ we integrate $(d+1)$-times by parts in $\widetilde{\xi}$ and if $|z - F(y)|\, 2^\ell > 1$ we integrate $(d+1)$-times by parts in $\widetilde{\eta}$. Hence we arrive at

$$I(x) = 2^{-\max\{n,\ell\}\widetilde{r}} \int_U \widetilde{f}_{\widetilde{r}}\left(\widetilde{\xi}, \widetilde{\eta}, y\right) u_n(x-y)\, u_\ell(z - F(y))\, \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi(z) \,\mathrm{d}\, z \,\mathrm{d}\, y \,\mathrm{d}\, \widetilde{\xi} \,\mathrm{d}\, \widetilde{\eta},$$

where $\widetilde{f}_{\widetilde{r}}\left(\widetilde{\xi}, \widetilde{\eta}, y\right)$ is uniformly bounded for all $\left(\widetilde{\xi}, \widetilde{\eta}, y\right) \in \operatorname{supp} \Psi_{\sigma,1} \times \operatorname{supp} \widetilde{\Psi}^{\circ}_{\tau,1} \times K$. Hence we estimate for some constant $C_3 > 0$

$$|I(x)| \leqslant C_3 2^{-\max\{n,\ell\}\widetilde{r}} u_n * (u_\ell \circ F) * \left|\Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi\right|(x). \tag{36}$$

We estimate for every $\sigma, \tau \in \{-, +, 0\}$ and every $n, \ell \geqslant 1$, using the equality in (35) and assuming $\epsilon > 0$ small enough,

$$2^{(c(\sigma)+\epsilon)n - c'(\tau)\ell - \max\{n,\ell\}\widetilde{r}} \leqslant 2^{(\max\{s,t,q\}+\epsilon)n - \min\left\{s',t',q'\right\}\ell - \max\{n,\ell\}\widetilde{r}} \leqslant 2^{-\epsilon\ell}. \tag{37}$$

Hence we bound, using the estimates in (32), (33), (36), two times Young's inequality and the bound in (37), for some constants $C_4, \ldots, C_6 > 0$

$$\begin{aligned}\|\mathcal{M}_{F,f}\varphi\|_{W^{s,t,q}_{p,\Theta,K}} &\leqslant C_1 \sup_{\sigma,n} \sum_{\tau,\ell} 2^{(c(\sigma)+\epsilon)n - c'(\tau)\ell} 2^{c'(\tau)\ell} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right\|_{L_p} \\ &= C_1 (2\pi)^{-2d} 2^{d(n+\ell)} \sup_{\sigma,n} \sum_{\tau,\ell} 2^{(c(\sigma)+\epsilon)n - c'(\tau)\ell} 2^{c'(\tau)\ell} \|I_{\sigma,n,\tau,\ell}\|_{L_p} \\ &\leqslant C_4 \sup_{\sigma,n} \sum_{\tau,\ell} 2^{(c(\sigma)+\epsilon)n - c'(\tau)\ell - \max\{n,\ell\}\widetilde{r}} 2^{(n+\ell)d} 2^{c'(\tau)\ell} \left\| u_n * (u_\ell \circ F) * \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right\|_{L_p} \\ &\leqslant C_5 \sum_{\tau,\ell} 2^{-\ell\epsilon} 2^{c'(\tau)\ell} \left\| \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right\|_{L_p} \leqslant C_6 \sup_{\tau,\ell} 2^{c'(\tau)\ell} \left\| \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right\|_{L_p}.\end{aligned}$$

To see the statement if $F = \mathrm{id}$ we estimate the corresponding cases $c'(\tau) < c(\sigma)$ if $\sigma \neq \tau$ and $n \sim \ell$ and $n, \ell \neq 0$ in a different way. We use

$$\widetilde{I}(x) := \int_U e^{\mathrm{i}\, 2^n \widetilde{\xi}(x-y)} e^{\mathrm{i}\, 2^\ell \widetilde{\eta}(y-z)} \widetilde{\Psi}_{\sigma,1}\left(\widetilde{\xi}\right) \Psi_{\sigma,1}\left(\widetilde{\xi}\right) \widetilde{\Psi}^{\circ}_{\tau,1}\left(\widetilde{\eta}\right) f(y)\, \Psi^{\mathrm{Op}}_{\tau,\ell}\varphi(z)\,\mathrm{d}\,z\,\mathrm{d}\,y\,\mathrm{d}\,\widetilde{\xi}\,\mathrm{d}\,\widetilde{\eta}.$$

We express $\Psi_{\sigma,1}\left(\widetilde{\xi}\right)$, using the identity

$$\Psi_{\sigma,1}\left(\widetilde{\xi}\right) - \Psi_{\sigma,1}\left(\widetilde{\eta}\right) = \int_0^1 \left(\mathrm{D}\,\Psi_{\sigma,1}\right)\left(\widetilde{\xi} + (1-h)\left(\widetilde{\eta} - \widetilde{\xi}\right)\right)\mathrm{d}\,h\left(\widetilde{\xi} - \widetilde{\eta}\right).$$

We repeat this k-times in the right-hand side of this identity, replacing $\widetilde{\xi}$ and yielding in total $k+1$ terms. The first k terms are linear combinations of

$$\Psi_{\sigma,1}\left(j\widetilde{\eta} - (j-1)\,\widetilde{\xi}\right),$$

where $1 \leqslant j \leqslant k+1$. If $j = 1$ then this is just $\Psi_{\sigma,1}(\widetilde{\eta})$. The corresponding part in $\widetilde{I}(x)$ is hence

$$\widetilde{I}_1(x) = \widetilde{\Psi}^{\mathrm{Op}}_{\sigma,n}\left(f \cdot \Psi^{\mathrm{Op}}_{\sigma,n}\Psi^{\mathrm{Op}}_{\tau,\ell}\varphi\right) = \widetilde{\Psi}^{\mathrm{Op}}_{\sigma,n}\left(f \cdot \Psi^{\mathrm{Op}}_{\sigma,n}\left(1 - \sum_{\sigma' \neq \tau} \Psi^{\mathrm{Op}}_{\sigma',\ell}\right)\varphi\right).$$

Note that $\widetilde{\Psi}_{\sigma,n}$ and $\Psi_{\sigma,n}$ satisfy the vanishing conditions in Lemma 3.1 as seen as an operator $\ell_2^{\mathrm{c}} \mapsto \ell_2^{\mathrm{c}}$. Then we bound with some constant $C_5 = C_5(f)$

$$\left\|\widetilde{I}_1\right\|_{L_p} \leqslant C_5 \left\|\left(\sum_{n=0}^{\infty}\left|4^{\sigma n}\Psi^{\mathrm{Op}}_{\sigma,n}\varphi\right|^2\right)^{\frac{1}{2}}\right\|_{L_p},$$

using two times Lemma 3.1 and that $c(-) \leqslant c(0) \leqslant c(+)$ and that $\Psi^{\mathrm{Op}}_{+,\ell}\Psi^{\mathrm{Op}}_{-,n} \equiv 0$. The terms where $1 < j \leqslant k+1$ are dealt with, using first the substitution

$$j\widetilde{\eta} - (j-1)\,\widetilde{\xi} \mapsto \widetilde{\eta},$$

and then $\widetilde{r}$-times integration by parts analogous as before. The $(k+1)$-th term is

$$\widetilde{\Psi}_k\left(\widetilde{\eta}\right) := \int_{[0,1]^k}\left(\mathrm{D}^k\,\Psi_{\sigma,1}\right)\left(\widetilde{\xi} + \sum_{j=1}^{k}(1-t_j)\left(\widetilde{\eta} - \widetilde{\xi}\right)\right)\mathrm{d}\,t\left(\widetilde{\xi} - \widetilde{\eta}\right)^{\otimes k}.$$

We split now according to the size $\left|\widetilde{\eta} - \widetilde{\xi}\right|$. We let $\epsilon > 0$. We note that

$$2^{-n\epsilon((d+1)-k)}\chi\left(\left|\widetilde{\eta} - \widetilde{\xi}\right| 2^{n\epsilon}\right)\widetilde{\Psi}_k\left(\widetilde{\eta}\right)$$

satisfies the vanishing conditions in Lemma 3.1 uniformly in $\widetilde{\xi}$ as seen as an operator $\ell_2^{\mathrm{c}} \mapsto \ell_2^{\mathrm{c}}$ in $\widetilde{\eta}$. We bound the L_p norm of the corresponding part analogous as in the case $\widetilde{I}_1$. This is bounded appropriately with the choice of k below. On the range $\left(1 - \chi\left(\left|\widetilde{\eta} - \widetilde{\xi}\right| 2^{n\epsilon}\right)\right) > 0$ we integrate $\widetilde{r}$-times by parts in y and then $(d+1)$-times in $\widetilde{\xi}$ and $\widetilde{\eta}$ in the corresponding part of $\widetilde{I}(x)$. The terms which depend on χ are treated as in the range $\chi\left(\left|\widetilde{\eta} - \widetilde{\xi}\right| 2^{n\epsilon}\right) > 0$. In the remaining part we gained a factor $\sim 2^{(-n+\epsilon)\widetilde{r} + n\epsilon(d+1)}$. We choose ϵ small compatible with the inequality given in (35) and then k large enough such that

$$2^{(c(\sigma)-c(\tau))n} \leqslant 2^{-n\epsilon((2d+2)-k)}.$$

□

Lemma 3.4 (Continuity and compactness). *Let $p \in [1,\infty]$, let $s' \leqslant s$, $q' \leqslant q$, $t' \leqslant t$, and $s \leqslant q \leqslant t$ and let Θ, Θ° be two cone ensembles, recalling Definition 2.1. Suppose the compact inclusions*

$$\operatorname{supp}\Phi_0^\circ \Subset \operatorname{supp}\Phi_0 \cup \operatorname{supp}\Phi_+ \quad \textit{and} \quad \operatorname{supp}\Phi_+^\circ \Subset \operatorname{supp}\Phi_+. \tag{38}$$

Then the inclusion

$$W^{s,t,q}_{p,\Theta,K} \subseteq W^{s',t',q'}_{p,\Theta^\circ,K}$$

is continuous for every open subset $K \subset \mathbb{R}^d$ with compact closure. Moreover, if $s' < s, t' < t$ and $q' < q$ then the inclusion

$$W^{s,t,q}_{p,\Theta,K} \subseteq W^{s',t',q'}_{p,\Theta,K}$$

is compact.

Proof. We prove first the claim on the continuous inclusion. We set for all $n \in \mathbb{Z}_{\geqslant 0}$

$$\begin{aligned} F_{-,n} &\coloneqq 2^{(s'-s)n}\left(\Psi^\circ_{-,n}\Big/\left(2^{(q-s)n}\Psi_{0,n} + 2^{(t-s)n}\Psi_{+,n} + \Psi_{-,n}\right)\right), \\ F_{+,n} &\coloneqq 2^{(t'-t)n}\left(\Psi^\circ_{+,n}/\Psi_{+,n}\right), \\ F_{0,n} &\coloneqq 2^{(q'-q)n}\left(\Psi^\circ_{0,n}\Big/\left(2^{(t-q)n}\Psi_{+,n} + \Psi_{0,n}\right)\right). \end{aligned}$$

We define a map Q on the Hilbert space ℓ_2^c (with norm as given in (30)) by setting for all $\sigma \in \{-,+,0\}$, $n \in \mathbb{Z}_{\geqslant 0}$ and all $a \in \ell_2^c$

$$(Qa)_{\sigma,n} \coloneqq F_{\sigma,n} a_{\sigma,n}.$$

In Lemma 3.1 we take $\mathcal{B}_1 = \mathcal{B}_2 = \ell_2^c$. It follows from the definition of $\Psi_{\sigma,n}$ in (23), the compact inclusion assumptions in (38) and the assumptions on s,t,q,s',t',q' that Q satisfies the decay conditions on Q in Lemma 3.1. It follows that the corresponding operator Q^{Op} in (27) is bounded. Let $\varphi \in W^{s,t,q}_{p,\Theta,K}$. We set for all $n \in \mathbb{N}_{\geqslant 0}$

$$\begin{aligned} b_{-,n} &\coloneqq 2^{sn}\mathbb{F}^{-1}\left(2^{(q-s)n}\Psi_{0,n} + 2^{(t-s)n}\Psi_{+,n} + \Psi_{-,n}\right) * \varphi, \\ b_{+,n} &\coloneqq 2^{tn}\mathbb{F}^{-1}\Psi_{+,n} * \varphi, \\ b_{0,n} &\coloneqq 2^{qn}\mathbb{F}^{-1}\left(2^{(t-q)n}\Psi_{+,n} + \Psi_{0,n}\right) * \varphi. \end{aligned}$$

Then $(b_{\sigma,n} \mid \sigma \in \{-,+,0\}, n \in \mathbb{N}_{\geqslant 0}) =: b \in L_p\left(\mathbb{R}^d, \ell_2^c\right)$ by assumption on φ and in particular it holds, for some constant $C \geqslant 1$, $\|b\|_{L_p(\mathbb{R}^d,\ell_2^c)} \leqslant C\|\varphi\|_{W^{s,t,q}_{p,\Theta,K}}$. We estimate, using Lemma 3.1, and conclude

$$\|\varphi\|_{W^{s',t',q'}_{p,\Theta^\circ,K}} = \left\|\left\|Q^{\mathrm{Op}}b\right\|_{\ell_2^c}\right\|_{L_p} \leqslant C\|b\|_{L_p(\mathbb{R}^d,\ell_2^c)}.$$

We show the claim on the compact inclusion. We let $U \subset W^{s,t,q}_{p,\Theta,K}$ be a bounded set in $W^{s,t,q}_{p,\Theta,K}$ with bound $R > 0$. We set c' with respect to s',t',q' analogous to c. It is enough to find for each $\epsilon > 0$ an open cover of U in $W^{s',t',q'}_{p,\Theta,K}$ where each open set in the cover has size $\sim \epsilon$. (This yields total boundedness of U in $W^{s',t',q'}_{p,\Theta,K}$ and hence compactness.) Now there is $\delta > 0$ such that for all $\sigma \in \{-,+,0\}$

$$c'(\sigma) + \delta - c(\sigma) < 0. \tag{39}$$

For all $\varphi \in U$ and all $N \in \mathbb{N}$ we bound

$$\begin{aligned}\left\| \sqrt{\sum_{\sigma,n \geqslant N} 4^{-\delta n} \left| 2^{(c'(\sigma)+\delta)n} \Psi^{\mathrm{Op}}_{\sigma,n} \varphi \right|^2} \right\|_{L_p} &\leqslant C \sup_{\sigma,n\geqslant N} 2^{\left(c'(\sigma)+\delta\right)n} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} \varphi \right\|_{L_p} \\ &\leqslant C 2^{\left(c'(\sigma)+\delta-c(\sigma)\right)N} R, \end{aligned} \tag{40}$$

for some $\sigma \in \{-,+,0\}$. Recalling the bound in (39), we make the bound in (40) smaller than ϵ by taking $N = N(\epsilon, R)$ large enough. Suppose now that the embedding is not compact. Then there are infinitely many $\varphi_m \in U$, $m \in \mathbb{N}$, such that for all $m_1 > m_2$ it holds

$$\left\| \varphi_{m_1} - \varphi_{m_2} \right\|_{W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} > \epsilon. \tag{41}$$

Recalling the bound in (40), it holds for some $n < N$ and some $\sigma \in \{-,+,0\}$

$$\left\| \varphi_{m_1} - \varphi_{m_2} \right\|_{W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} \leqslant C 2^{\left(c'(\sigma)+\delta\right)n} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} \left(\varphi_{m_1} - \varphi_{m_2} \right) \right\|_{L_p}. \tag{42}$$

Since $C^{r-1}_0(K)$ is dense in $W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$ we may assume $\varphi_m \in C^{r-1}_0(K)$. We set $S := \cup_{\sigma,n<N} \operatorname{supp} \Psi_{\sigma,n}$. Since all φ_m are uniformly bounded in $W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$-norm and $\operatorname{supp} \varphi_m$ is uniformly bounded in m as well, the Fourier transform of φ_m cannot diverge on a dense subset of S as $m \to \infty$ (this would violate the Paley–Wiener Theorem [28, Theorem 1.7.7]). By passing to a subsequence in m we may split $S = S_1 \sqcup S_2$ such that the family $\left\{ \mathbb{F}\varphi_{m|S_1} \mid m \in \mathbb{N} \right\}$ is uniformly bounded. Then, using again that φ_m has compact support with maximal diameter independent of m, the family $\left\{ \mathbb{F}\varphi_{m|S_1} \mid m \in \mathbb{N} \right\}$ is also uniformly equicontinuous. Hence by the Arzelà–Ascoli Theorem there is a subsequence in m such that $\varphi_{m|S_1}$ is a Cauchy sequence in C^0. Repeating the argument inductively for the part $\mathbb{F}\varphi_{m|S_2}$, then using a diagonal argument, we find a subsequence in m such that $\mathbb{F}\varphi_{m|S}$ is a Cauchy sequence in C^0. Hence the right-hand side in (42) can be made arbitrary small which contradicts the lower bound in (41) and we conclude. □

Lemma 3.5 (Local derivative)**.** *Let $p \in [1,\infty]$, $s,t,q < r-1$ and let $\varphi \in W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$. It holds for some constant $C > 0$, for every $1 \leqslant j \leqslant d$, for every $\sigma \in \{-,+,0\}$ such that*

$$\xi_j \neq 0 \quad \textit{if} \quad (\xi_1,\ldots,\xi_d) = \xi \in \operatorname{supp} \Psi_{1,\sigma},$$

and for every $\tilde{r} \in \mathbb{R}$

$$\left\| \left(\sum_{n=0}^{\infty} 4^{\tilde{r}n} \left| \Psi^{\mathrm{Op}}_{\sigma,n} \varphi \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \leqslant C \left\| \left(\sum_{n=0}^{\infty} 4^{(\tilde{r}-1)n} \left| \Psi^{\mathrm{Op}}_{\sigma,n} \partial_{x_j} \varphi \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p}. \tag{43}$$

Proof. Using the triangle inequality, it is enough to consider only the terms with $n > 0$. For every $\xi \in \mathbb{R}^d \setminus \{0\}$ and $b \in \mathbb{C}$ we put

$$\left(\mathcal{D}(\xi) b \right)_n := \mathrm{i} \frac{\xi_j}{2^n} \Psi_{\sigma,n}(\xi) b, \quad n \in \mathbb{N}.$$

We note

$$\Psi^{\mathrm{Op}}_{\sigma,n} \left(\partial_{x_j} \varphi \right) = \left(\mathbb{F}^{-1} \Psi_{\sigma,n} \right) * \partial_{x_j} \varphi = \left(\partial_{x_j} \mathbb{F}^{-1} \Psi_{\sigma,n} \right) * \varphi = 2^n \left(\mathcal{D}^{\mathrm{Op}} \varphi \right)_n.$$

We let ℓ_2 be the space of complex valued sequences space over $\mathbb{N}$. As norm we set $\|a\|_{\ell_2} := \sqrt{\sum_{n=1}^{\infty} 4^{\tilde{r}n} |a_n|^2}$. For every $\xi \in \mathbb{R}^d \setminus \{0\}$, every $a \in \ell_2$ and every $n \in \mathbb{N}$ we put

$$(Q(\xi)a)_n := -\mathrm{i}\, \frac{2^n}{\xi_j} \widetilde{\Psi}_{\sigma,n}(\xi) a_n,$$

where $\widetilde{\Psi}_{\sigma,n}$ is defined in (24). Note that $\left(Q^{\mathrm{Op}} \mathcal{D}^{\mathrm{Op}} \varphi\right)_n = \Psi^{\mathrm{Op}}_{\sigma,n} \varphi$. Moreover since $\xi_j \neq 0$ by assumption, the map Q satisfies the decay condition on its derivatives as required in Lemma 3.1. Hence, using Lemma 3.1 with $\mathcal{B}_1 = \mathcal{B}_2 = \ell_2$, the map $Q^{\mathrm{Op}} : L_p\left(\mathbb{R}^d, \ell_2\right) \to L_p\left(\mathbb{R}^d, \ell_2\right)$ is a bounded linear operator. We conclude, using the estimate for some constant $C > 0$

$$\left\| \left\| Q^{\mathrm{Op}} \mathcal{D}^{\mathrm{Op}} \varphi \right\|_{\ell_2} \right\|_{L_p} \leqslant C \left\| \left\| \mathcal{D}^{\mathrm{Op}} \varphi \right\|_{\ell_2} \right\|_{L_p}.$$

□

We recall the open cover $V_\omega \subseteq M$ and the chart maps $\kappa_\omega \in \mathcal{A}$, $\omega \in \Omega$, introduced in Section 2. Also we recall the vector space $C^{r-1}_X(M)$ from the beginning of Section 3.2.

Definition 3.6 (Anisotropic Banach space)**.** *Let $\vartheta_\omega : V_\omega \to [0,1]$ be a C^r partition of unity adapted to the chart maps κ_ω and let Θ_ω be hyperbolic cone ensembles, recalling Definition 2.2, where $\omega \in \Omega$. Let $p \in [1,\infty]$, let $s,q,t < r-1$ and let $\alpha_0 > 0$. We put for every $\varphi \in C^{r-1}_X(M)$ and every $p \in [1,\infty]$*

$$\|\varphi\|_{W^{s,t,q}_p} := \left(\sum_{\omega\in\Omega} \int_0^{\alpha_0} \left\| \left(\vartheta_\omega \cdot (\mathcal{L}_{\alpha,\phi_\alpha}\varphi) \circ \kappa_\omega^{-1}\right) \right\|^2_{W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} \,\mathrm{d}\,\alpha \right)^{\frac{1}{2}}, \tag{44}$$

We denote by $W^{s,t,q}_p$ the completion of $C^{r-1}_X(M)$ under this norm.

Remark 3.7. *Note that $W^{s,t,q}_p$ depends on the dynamics, α_0, the atlas $\mathcal{A}$ and the cone ensembles Θ_ω, $\omega \in \Omega$. We understand each $\vartheta_\omega \circ \kappa_\omega^{-1}$ in (44) as extended to $\mathbb{R}^d$ by zero. By Lemma 3.3 a C^r change of the atlas and hence a change of the cone ensemble yields an equivalent norm if $s \leqslant q \leqslant t < r-1$. The integration with respect to α is a way to "project out" the small times where the flow is not sufficiently hyperbolic. This is similar to [20, Definition 8.1] and also Baladi–Liverani [5, p.705, (3.2)] with the supremum replaced by an integral in the latter case. In turn, for $p = 2$ the space $W^{s,t,q}_2$ is a Hilbert space because the parallelogram law*

$$\|\varphi_1 + \varphi_2\|^2_{W^{s,t,q}_p} + \|\varphi_1 - \varphi_2\|^2_{W^{s,t,q}_p} = 2\|\varphi_1\|^2_{W^{s,t,q}_p} + 2\|\varphi_2\|^2_{W^{s,t,q}_p}$$

holds [8, Proposition 15.2].

The compact inclusion of the local Banach space in Lemma 3.4 carries over to the anisotropic Banach space $W^{s,t,q}_p$.

Lemma 3.8 (Compactness)**.** *Let $p \in [1,\infty]$, let $s \leqslant q \leqslant t$ such that $\max\{0,t\} - \min\{0,s\} < r-1$ and let $s' < s, t' < t$ and $q' < q$ such that $\max\{0, \min\{s',t',q'\}\} - \min\{0,s',t',q'\} < r-1$. Then there exist cone ensembles Θ_ω, $\omega \in \Omega$, such that the inclusion*

$$W^{s,t,q}_p \subseteq W^{s',t',q'}_p$$

is compact.

Proof. We let $s' < s, t' < t, q' < q$. Let $U \subseteq W_p^{s,t,q}$ be a bounded set in the norm of $W_p^{s,t,q}$. In order to show the compact inclusion we proceed analogous to the proof in Lemma 3.4. To this end we let $\varphi_m \in U$, $m \in \mathbb{N}$, be a sequence, satisfying the analog bound in (41). Suppose now that there has to be some fixed $\omega \in \Omega$ and some fixed $\alpha \geqslant 0$ such that there exists $C_1 > 0$ such that for all $m > 0$

$$\sum_{\omega \in \Omega} \left\| \left(\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha,\phi_\alpha} \varphi_m \circ \kappa_\omega^{-1} \right) \right) \right\|_{W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} \leqslant C_1.$$

and that there exists some $\epsilon > 0$ such that for all $m_1 > m_2$ (up to some subsequence)

$$\left\| \vartheta_\omega \cdot \left(\mathcal{L}_{\alpha,\phi_\alpha} \left(\varphi_{m_1} - \varphi_{m_2} \right) \right) \circ \kappa_\omega^{-1} \right\|_{W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} > \epsilon. \tag{45}$$

Since $\left(\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha,\phi_\alpha} \varphi_m \right) \circ \kappa_\omega^{-1} \right) \in W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$ we find a Cauchy subsequence, using the statement on the compact inclusion in Lemma 3.4. Note that by the Mean Value Theorem there exist non-fixed $\alpha = \alpha(m)$ and $\alpha = \alpha(m_1, m_2)$ which satisfy these inequalities. In particular, we wish to find a Cauchy subsequence for the left-hand side in the inequality (45) for the choice $\alpha = \alpha(m_1, m_2)$. Suppose $0 \leqslant \alpha' \leqslant \alpha_0$. We have

$$\begin{aligned} &\left\| \vartheta_\omega \cdot \left(\mathcal{L}_{2\alpha_0,\phi_{2\alpha_0}} \varphi_m \right) \circ \kappa_\omega^{-1} \right\|_{W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} \leqslant \\ &\sum_{\omega' \in \Omega} \left\| \vartheta_\omega \cdot \left(\mathcal{L}_{2\alpha_0 - \alpha', \phi_{2\alpha_0 - \alpha'}} \left(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha',\phi_{\alpha'}} \varphi_m \right) \right) \circ \kappa_\omega^{-1} \right\|_{W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}}. \end{aligned}$$

By Lemma 2.3 there exists cone ensembles $(\Theta_\omega, \Theta_\omega^\circ)$, $\omega \in \Omega$, satisfying the condition (38) in Lemma 3.4 such that the local diffeomorphism of $g_{-\alpha}$ with $\alpha \geqslant \alpha_0$ is cone hyperbolic. Then, using Lemma 3.3 and $s \leqslant q \leqslant t$, $\max\{0,t\} - \min\{0,s\} < r-1$ and taking $\alpha' = \alpha(m)$, we bound this sequence in m uniformly from above. Let $s'' = \min\{s', t', q'\}$. Then, using Lemma 3.3, recalling that it holds $\max\{0, \min\{s', t', q'\}\} - \min\{0, s', t', q'\} < r - 1$, we find (abusing the notation $\mathcal{L}_{\alpha,\phi_\alpha}$ with negative α)

(46)

$$\begin{aligned} &\left\| \vartheta_\omega \cdot \left(\mathcal{L}_{\alpha',\phi_{\alpha'}} \left(\varphi_{m_1} - \varphi_{m_2} \right) \right) \circ \kappa_\omega^{-1} \right\|_{W^{s'',s'',s''}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} \leqslant \\ &\sum_{\omega' \in \Omega} \left\| \left(\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha' - 2\alpha_0, \phi_{\alpha' - 2\alpha_0}} \left(\vartheta_{\omega'} \cdot \mathcal{L}_{2\alpha_0,\phi_{2\alpha_0}} \left(\varphi_{m_1} - \varphi_{m_2} \right) \right) \right) \circ \kappa_\omega^{-1} \right) \right\|_{W^{s'',s'',s''}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} \\ &\leqslant C_2 \sum_{\omega \in \Omega} \left\| \vartheta_\omega \cdot \left(\mathcal{L}_{2\alpha_0,\phi_{2\alpha_0}} \left(\varphi_{m_1} - \varphi_{m_2} \right) \right) \circ \kappa_\omega^{-1} \right\|_{W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}}. \end{aligned}$$

for some constant $C_2 > 0$ independent of the choice of α' and of m_1, m_2. Now we take $\alpha' = \alpha(m_1, m_2)$ and let the right-hand side vanish in (46) as $(m_1, m_2) \to \infty$. Then from the left-hand side for all σ, n

$$\left\| \Psi^{\mathrm{Op}}_{\sigma,n} \left(\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha(m_1,m_2),\phi_{\alpha(m_1,m_2)}} \left(\varphi_{m_1} - \varphi_{m_2} \right) \right) \circ \kappa_\omega^{-1} \right) \right\|_{L_p} \to 0.$$

By uniform boundedness in $W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$ (analogous to the proof of Lemma 3.4), the lower bound in (45) comes from a finite number of terms

$$\left\| \Psi^{\mathrm{Op}}_{\sigma,n} \left(\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha(m_1,m_2),\phi_{\alpha(m_1,m_2)}} \left(\varphi_{m_1} - \varphi_{m_2} \right) \right) \circ \kappa_\omega^{-1} \right) \right\|_{L_p}$$

with $n \leqslant C_3 = C_3(\epsilon)$. Hence we found a Cauchy subsequence for the left-hand side in the inequality (45) for the choice $\alpha = \alpha(m_1, m_2)$. $\square$

4. Properties of the transfer operator, the generator and its resolvent

4.1. Bounds on the transfer operator.

We introduce a local transfer operator in (47) below and state a local norm estimate for this operator in Lemma 4.1. We then give a norm estimate for the transfer operator family (18) in Lemma 4.2, making use of Lemma 4.1.

Let $K \subset \mathbb{R}^d$ be an open set. Let $f\colon \mathbb{R}^d \to \mathbb{C}$ be a $C_0^{r-1}(K)$-map and let $F\colon K \to F(K)$ be a (Θ°, Θ)-cone hyperbolic C^r-diffeomorphism on K (recall Definition 2.2). The f-weighted local transfer operator is defined by

$$L_{F,f}\colon C_0^{r-1}(F(K)) \to C_0^{r-1}(K) : \varphi \mapsto f \cdot \varphi \circ F. \tag{47}$$

Recalling $\widetilde{\Phi}_-^\circ, \widetilde{\Phi}_+^\circ, \widetilde{\Phi}_0^\circ$ from (24), we put for every subset $I \subseteq K$

$$\|F\|_{-,I} \coloneqq \inf_{\substack{y\in I\\ 0\neq\eta\in\operatorname{supp}\widetilde{\Phi}_-^\circ}} \frac{\left|(\mathrm{D}_y F)^{\mathrm{tr}}\eta\right|}{|\eta|},\quad \|F\|_{+,I} \coloneqq \sup_{\substack{y\in I\\ 0\neq\eta\in\operatorname{supp}\widetilde{\Phi}_+^\circ}} \frac{\left|(\mathrm{D}_y F)^{\mathrm{tr}}\eta\right|}{|\eta|},$$

$$\|F\|_{0,I} \coloneqq \sup_{\substack{y\in I\\ 0\neq\eta\in\operatorname{supp}\widetilde{\Phi}_0^\circ}} \frac{\left|(\mathrm{D}_y F)^{\mathrm{tr}}\eta\right|}{|\eta|}.$$

Lemma 4.1 (Upper bound for local transfer operator)**.** *Let $\{\mathcal{W}\}$ denote the connected components of* $\operatorname{supp} f$. *Let $p \in [1,\infty]$. Let*

$$s' < s < 0 < q \leqslant t < r-1+s',\quad q' < q,\quad t' < t.$$

Then for every $\varphi \in W^{s,t,q}_{p,\Theta,F(K)}$ it holds

$$\|L_{F,f}\varphi\|_{W^{s,t,q}_{p,\Theta,K}} \leqslant C_0 \|\varphi\|_{W^{s',t',q'}_{p,\Theta^\circ,F(K)}} + C_1 \|\varphi\|_{W^{s',t',q}_{p,\Theta^\circ,F(K)}} + C_2 \|\varphi\|_{W^{s,t,q}_{p,\Theta^\circ,F(K)}},$$

where, for some constants $C > 0$ and $k > 0$, it holds

$C_0 \leqslant C \sum_{\mathcal{W}} \max\left\{1, \|F\|_{-,\mathcal{W}}^{1-r}, \|F\|_{0,\mathcal{W}}^{1-r}\right\} \|\mathrm{D} F\|_{C^{r-1}(\mathcal{W})}^k \|f\|_{C^{r-1}(\mathcal{W})} \left\||\det \mathrm{D} F|^{-\frac{1}{p}}\right\|_{L_\infty(\mathcal{W})}$,

$C_1 \leqslant C \sup_{\mathcal{W}} \left\|f\,|\det \mathrm{D} F|^{-\frac{1}{p}}\right\|_{L_\infty(\mathcal{W})} \max\left\{1, \|F\|_{0,\mathcal{W}}^q\right\}$ *and*

$C_2 \leqslant C \sup_{\mathcal{W}} \left\|f\,|\det \mathrm{D} F|^{-\frac{1}{p}}\right\|_{L_\infty(\mathcal{W})} \max\left\{\|F\|_{+,\mathcal{W}}^t, \|F\|_{-,\mathcal{W}}^s\right\}$.

Lemma 4.1 is proven in Section 4.4. For every $s,t,\alpha \in \mathbb{R}$ and every $x \in M$ we set

$$\lambda^{(t,s,\alpha)}(x) \coloneqq \max\left\{\left\|(\mathrm{D}\, g_\alpha)^{\mathrm{tr}}_{|E^*_{+,g_{-\alpha}(x)}}\right\|^{-t}, \left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}_{|E^*_{-,x}}\right\|^{s}\right\}. \tag{48}$$

For $s < 0 < t$ this quantity decreases exponentially fast to 0 as $\alpha \to \infty$, which is a consequence of the Anosov property given in (3) of the flow g_α.

Lemma 4.2 (Bound on the transfer operator)**.** *Let $p \in [1,\infty]$. Let*

$$s' < s < 0 < q \leqslant t < r-1+s' \quad\text{and}\quad t' < t.$$

There exist $\alpha_0 > 0$, cone ensembles Θ_ω, $\omega \in \Omega$, and constants $A > 0$ and $C > 0$, such that for all $\varphi \in W_p^{s,t,q}$ with $\|\varphi\|_{W_p^{s,t,q}} = 1$ and all $\alpha \geqslant 0$ it holds

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|_{W_p^{s,t,q}} \leqslant Ce^{A\alpha}\|\varphi\|_{W_p^{s',t',q}} + C(\alpha+1)\left\|\phi_\alpha\,|\det \mathrm{D}\, g_{-\alpha}|^{-\frac{1}{p}} \cdot \lambda^{(t,s,\alpha)}\right\|_{L_\infty}.$$

Proof. We recall the map $F_{-\alpha,\omega\omega'}$ and the set $V_{\alpha,\omega\omega'}$ defined in Lemma 2.3 for all $\alpha \geqslant 0$ and all $\omega', \omega \in \Omega$. By Lemma 2.3 there exist cone ensembles $\Theta^\circ_{\omega'}, \Theta_\omega$ such that the map $F_{-\alpha,\omega\omega'}$ is $(\Theta^\circ_{\omega'}, \Theta_\omega)$-cone hyperbolic. We recall the partition of unity ϑ_ω (see Definition 3.6). We let

$$V_{\alpha,\omega\omega'} \subseteq \widetilde{V}_{\alpha,\omega\omega'} \subseteq V_\omega$$

such that $F_{-\alpha,\omega\omega'}$ is also $(\Theta^\circ_{\omega'}, \Theta_\omega)$-cone hyperbolic on $\kappa_\omega\left(\widetilde{V}_{\alpha,\omega\omega'}\right)$. This is possible due to the compact inclusion of cones as required in the cone-hyperbolicity definition. We let

$$\vartheta_{\alpha,\omega\omega'} \colon \widetilde{V}_{\alpha,\omega\omega'} \to [0,1]$$

be a C_0^{r-1} map such that

$$\vartheta_{\alpha,\omega\omega'|V_{\alpha,\omega\omega'}} \equiv \vartheta_{\omega|V_{\alpha,\omega\omega'}}.$$

For all $z \in \kappa_\omega\left(V_\omega\right)$ we have

$$\vartheta_\omega \circ \kappa_\omega^{-1}(z) \cdot \vartheta_{\omega'} \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'}(z) = \vartheta_{\alpha,\omega\omega'} \circ \kappa_\omega^{-1}(z) \cdot \vartheta_{\omega'} \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'}(z).$$

Note that $\|\vartheta_{\alpha,\omega\omega'}\|_{C^{r-1}}$ is controlled by the rate of expansion of $F_{-\alpha,\omega\omega'}$. Let $\varphi \in W_p^{s,t,q}$ and put $W_\omega := W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$ and $W_{\omega\omega'} := W^{s,t,q}_{p,\Theta_\omega,\widetilde{V}_{\alpha,\omega\omega'}}$. For all $\alpha \geqslant \alpha_0$, for some $C \geqslant 1$, we estimate for every $p \in [1,\infty]$

$$\begin{aligned}
\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|^2_{W_p^{s,t,q}} &\leqslant C \max_{\omega\in\Omega} \int_0^{\alpha_0} \left\|\left(\vartheta_\omega \cdot \left(\phi_{\alpha'} \cdot \mathcal{L}_{\alpha,\phi_\alpha}\varphi \circ g_{-\alpha'}\right)\right) \circ \kappa_\omega^{-1}\right\|^2_{W_\omega} \mathrm{d}\,\alpha' \\
&= C \max_{\omega\in\Omega} \int_0^{\alpha_0} \left\| \left(\vartheta_\omega \cdot \phi_\alpha\right) \circ \kappa_\omega^{-1} \cdot \sum_{\omega'\in\Omega} \left(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right) \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'} \right\|^2_{W_\omega} \mathrm{d}\,\alpha' \\
&= C \max_{\omega\in\Omega} \int_0^{\alpha_0} \left\| \sum_{\omega'\in\Omega} \left(\vartheta_{\alpha,\omega\omega'} \cdot \phi_\alpha\right) \circ \kappa_\omega^{-1} \cdot \left(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right) \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'} \right\|^2_{W_\omega} \mathrm{d}\,\alpha' \\
&\leqslant C^2 \max_{\omega,\omega'\in\Omega} \int_0^{\alpha_0} \left\| \left(\vartheta_{\alpha,\omega\omega'} \cdot \phi_\alpha\right) \circ \kappa_\omega^{-1} \cdot \left(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right) \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'} \right\|^2_{W_\omega} \mathrm{d}\,\alpha' \\
&= C^2 \max_{\omega,\omega'\in\Omega} \int_0^{\alpha_0} \left\| L_{F_{-\alpha,\omega\omega'},\left(\vartheta_{\alpha,\omega\omega'}\cdot\phi_\alpha\right)\circ\kappa_\omega^{-1}} \left(\left(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right) \circ \kappa_{\omega'}^{-1}\right) \right\|^2_{W_{\omega\omega'}} \mathrm{d}\,\alpha'.
\end{aligned} \tag{49}$$

We used in the last step the definition of the weighted local transfer operator (see (47)) in which we take $F := F_{-\alpha,\omega\omega'}$ and as the C_0^{r-1}-weight $f := \left(\vartheta_{\alpha,\omega\omega'} \cdot \phi_\alpha\right) \circ \kappa_\omega^{-1}$. We now show the claimed upper bound for $\mathcal{L}_{\alpha,\phi_\alpha}$. We recall that

$$\operatorname{supp} f = \kappa_\omega\left(\widetilde{V}_{\alpha,\omega\omega'}\right) = \bigsqcup \kappa_\omega\left(\mathcal{W}\right),$$

where the disjoint union is over all the finitely many connected components $\mathcal{W}$ of $\widetilde{V}_{\alpha,\omega\omega'}$. The inclusion $W^{s,t,q}_{p,\Theta^\circ,\kappa_{\omega'}(V_{\omega'})} \subseteq W_{\omega'}$ is continuous by Lemma 3.4. Together with the bound given by Lemma 4.1 this yields the upper bound

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|_{W_p^{s,t,q}} \leqslant \widetilde{C}_1 \|\varphi\|_{W_p^{s',t',q}} + \widetilde{C}_2 \|\varphi\|_{W_p^{s,t,q}},$$

where

$$\widetilde{C}_1 \leqslant C^2 \max_{\omega,\omega'\in\Omega} C_0\left(F_{-\alpha,\omega\omega'}, f\right) + C_1\left(F_{-\alpha,\omega\omega'}, f\right),\ \widetilde{C}_2 \leqslant C^2 \max_{\omega,\omega'\in\Omega} C_2\left(F_{-\alpha,\omega\omega'}, f\right),$$

and C_0, C_1, C_2 are the constants from Lemma 4.1. We claim for some constant $C_4 > 0$ the following bound

$$\left\|\phi_\alpha \left|\det \mathrm{D}\, g_{-\alpha}\right|^{-\frac{1}{p}}\right\|_{L_\infty(\mathcal{W})} \leqslant C_4 \inf_{x\in\mathcal{W}} \left|\phi_\alpha \left|\det \mathrm{D}\, g_{-\alpha}\right|^{-\frac{1}{p}}\right|(x) = C_4 \left\|\left(\phi_{-\alpha} \left|\det \mathrm{D}\, g_{\alpha}\right|^{-\frac{1}{p}}\right)\circ g_{-\alpha}\right\|_{L_\infty(\mathcal{W})}. \tag{50}$$

Due to the construction of $\widetilde{V}_{\alpha,\omega\omega'}$, all points in a connected component $\mathcal{W}$ stay close under iterates by $g_{\alpha'}$ for all $0 \leqslant \alpha' \leqslant \alpha$. Then in the case of hyperbolic maps the bound in (50) follows, using [32, Proposition 20.2.6.]. However for Anosov flows the distance between two points $x_1, x_2 \in \mathcal{W}$ may never be sufficiently contracted under iterates by $g_{\alpha'}$, e.g. if x_1, x_2 belong to a same orbit of $g_{\alpha'}$. We split (along the flow direction X in charts) each $\mathcal{W}$ into parts $\mathcal{W}_j$, $1 \leqslant j \leqslant \lfloor\alpha\rfloor + 1$, in which now two points are no more than $\sim (\lfloor\alpha\rfloor + 1)^{-1}$ apart. We set $\widetilde{\mathcal{W}} := \{\mathcal{W}_j\}$ for all $1 \leqslant j \leqslant \lfloor\alpha\rfloor + 1$. Then it holds the bound in (50) with $\mathcal{W}$ replaced by $\mathcal{W}_j$. We modify $\vartheta_{\alpha,\omega\omega'}$, taking a sufficiently small neighborhood U_j containing $\mathcal{W}_j$, such that $\vartheta_{\alpha,\omega\omega'|U_j}$ is C_0^r. Then passing to this new weights $\vartheta_{\alpha,\omega\omega'|U_j}$ and summing over j we obtain an additional factor $\sim (\alpha+1)$ in the right-hand side in (49). We recall $\lambda^{(t,s,\alpha)}(x)$ from (48) and $\|F\|_{-,I}$, $\|F\|_{+,I}$ introduced below (47) in which we take $I = \mathcal{W}_j$ and $F = F_{-\alpha,\omega\omega'}$. In addition note $F^{-1}_{-\alpha,\omega\omega'} = F_{\alpha,\omega'\omega}$. Then we write

$$\|F_{-\alpha,\omega\omega'}\|_{+,\kappa_\omega(\mathcal{W}_j)} = \left(\inf_{\substack{y\in\kappa_\omega(\mathcal{W}_j)\\ 0\neq\eta\in\left(\mathrm{D}_y\, F_{-\alpha,\omega\omega'}\right)^{\mathrm{tr}}\operatorname{supp}\widetilde{\Phi}^\circ_{+,\omega'}}} \frac{\left|\left(\mathrm{D}_{F_{-\alpha,\omega\omega'}(y)}\, F_{\alpha,\omega'\omega}\right)^{\mathrm{tr}}\eta\right|}{|\eta|}\right)^{-1}.$$

We recall the construction in (16) of the C^+-cones in the proof of Lemma 2.3. We find a compactly embedded cone

$$C^+_{\gamma_+,\omega} \Subset \left(\mathrm{D}_y\, F_{-\alpha,\omega\omega'}\right)^{\mathrm{tr}} \operatorname{supp}\widetilde{\Phi}^\circ_{+,\omega'},$$

which is transversal to another cone $C^-_{\gamma_-,\omega}$. Hence the unstable distribution E_- (in charts) stays away from $\left(\mathrm{D}_y\, F_{-\alpha,\omega\omega'}\right)^{\mathrm{tr}} \operatorname{supp}\widetilde{\Phi}^\circ_{+,\omega'}$ by some positive angle. Replacing the inf with the sup, it holds for some constant $C_5 > 0$

$$\|F_{-\alpha,\omega\omega'}\|_{+,\kappa_\omega(\mathcal{W}_j)} \leqslant C_5 \left(\sup_{x\in\mathcal{W}_j} \left\|(\mathrm{D}\, g_\alpha)^{\mathrm{tr}}_{|E^*_{+,g_{-\alpha}(x)}}\right\|\right)^{-1}. \tag{51}$$

By analogous reasoning we conclude similar for $\|F_{-\alpha,\omega\omega'}\|_{-,\kappa_\omega(\mathcal{W}_j)}$. We estimate for some constants $C_6, \ldots, C_9 > 0$, using the bounds in (51) and (50),

$$\begin{aligned}
\widetilde{C}_2 &\leqslant C_6 \max_{\widetilde{\mathcal{W}}, \mathcal{W}_j\in\widetilde{\mathcal{W}}} \left\|\phi_\alpha \left|\det \mathrm{D}\, g_{-\alpha}\right|^{-\frac{1}{p}}\right\|_{L_\infty(\mathcal{W}_j)} \left\|\lambda^{(t,s,\alpha)}\right\|_{L_\infty(\mathcal{W}_j)} \\
&\leqslant \alpha C_7 \max_{\widetilde{\mathcal{W}}, \mathcal{W}_j\in\widetilde{\mathcal{W}}} \left\|\phi_\alpha \left|\det \mathrm{D}\, g_{-\alpha}\right|^{-\frac{1}{p}}\right\|_{L_\infty(\mathcal{W}_j)} \left\|\lambda^{(t,s,\alpha)}\right\|_{L_\infty(\mathcal{W}_j)} \\
&\leqslant \alpha C_8 \max_{\widetilde{\mathcal{W}}, \mathcal{W}_j\in\widetilde{\mathcal{W}}} \left\|\left(\phi_{-\alpha} \left|\det \mathrm{D}\, g_{\alpha}\right|^{-\frac{1}{p}}\right)\circ g_{-\alpha}\right\|^{-1}_{L_\infty(\mathcal{W}_j)} \left\|\lambda^{(t,s,\alpha)}\right\|_{L_\infty(\mathcal{W}_j)} \\
&\leqslant \alpha C_9 \max_{\widetilde{\mathcal{W}}, \mathcal{W}_j\in\widetilde{\mathcal{W}}} \left\|\phi_\alpha \left|\det \mathrm{D}\, g_{-\alpha}\right|^{-\frac{1}{p}} \lambda^{(t,s,\alpha)}\right\|_{L_\infty(\mathcal{W}_j)}.
\end{aligned}$$

Inspecting the constant $\widetilde{C}_1$, all terms depending on F and f are bounded by the maximal expansion of $F_{-\alpha,\omega\omega'}$ and ϕ_α, respectively, which grow at most exponentially in α. Hence, there is $A > 0$ and $C_{10} \geqslant 1$ such that $\widetilde{C}_1 \leqslant C_{11}e^{A\alpha}$. If $\alpha < \alpha_0$ we split $\int_0^{\alpha_0} = \int_0^{\alpha_0-\alpha} + \int_{\alpha_0-\alpha}^{\alpha_0}$. Hence it holds $\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|_{W_p^{s,t,q}} \leqslant \|\varphi\|_{W_p^{s,t,q}} + \left\|\mathcal{L}_{\alpha_0,\phi_{\alpha_0}}\varphi\right\|_{W_p^{s,t,q}}$. The latter term is estimated as in the case $\alpha \geqslant \alpha_0$. Since $\alpha \leqslant \alpha_0$, we combine here the upper bound of $\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|_{W_p^{s,t,q}}$ with the second term of our desired estimate, increasing the constant C_{11}. □

Remark 4.3. *A weaker upper bound for the transfer operator, e.g.*

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\|_{W_p^{s,t,q}\to W_p^{s,t,q}} \leqslant C_1 \exp\left(C_2\alpha\right)$$

for all $\alpha \geqslant 0$ and for some constants $C_1, C_2 \geqslant 1$ independent of α, can be obtained for a wider choice of s, t, q, e.g. for some $s > 0$ (and this carries over to Lemma 4.4 below as well). However, we are interested in the parameter range as assumed in Lemma 4.2 which allows us to show the Lasota–Yorke inequality for the resolvent given in Theorem 4.5 below. See also Lemma 5.17 in the next section below for such a bound in the case of a special weight.

We recall that the family $\left\{\mathcal{L}_{\alpha,\phi_\alpha}\colon W_p^{s,t,q} \to W_p^{s,t,q} \mid \alpha \geqslant 0\right\}$ forms a strongly continuous semigroup if and only if $\lim_{\alpha\to 0^+} \|\mathcal{L}_{\alpha,\phi_\alpha}\varphi - \varphi\|_{W_p^{s,t,q}} = 0$ for all $\varphi \in W_p^{s,t,q}$ (e.g. see [33, Proposition I.1.3]).

Lemma 4.4 (Strongly continuous semigroup)**.** *Let $p \in [1,\infty]$ and let $s < 0 < q \leqslant t < r - 1 + s$. Then the transfer operator family*

$$\left\{\mathcal{L}_{\alpha,\phi_\alpha}\colon W_p^{s,t,q} \to W_p^{s,t,q} \mid \alpha \geqslant 0\right\}$$

forms a strongly continuous semigroup.

Proof. Let $\varphi \in W_p^{s,t,q}$. For fixed $s < 0 < q \leqslant t$ such that $t - s < r - 1$ there is $\delta > 0$ such that $t - s < r - 1 - \delta$. We set $s' := s - \delta$ and let $t' < t$. Then s, t, q, s' and t' satisfy the assumptions of Lemma 4.2. Using Lemma 4.2, we bound the transfer operator for all small $\alpha \geqslant 0$

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|_{W_p^{s,t,q}} \leqslant C_1 \|\varphi\|_{W_p^{s',t',q}} + C_2 \|\varphi\|_{W_p^{s,t,q}} \leqslant (C_1 + C_2) \|\varphi\|_{W_p^{s,t,q}}\,, \tag{52}$$

for some constants $C_1, C_2 > 0$ independent of α. By density, for every $\epsilon > 0$ there is $\widetilde{\varphi} \in C_X^{r-1}(M)$ such that

$$\|\varphi - \widetilde{\varphi}\|_{W_p^{s,t,q}} \leqslant \epsilon. \tag{53}$$

Using first the triangle inequality and then the bounds (52)-(53), we estimate

$$\begin{aligned} \|\mathcal{L}_{\alpha,\phi_\alpha}\varphi - \varphi\|_{W_p^{s,t,q}} &\leqslant \|\mathcal{L}_{\alpha,\phi_\alpha}(\varphi - \widetilde{\varphi})\|_{W_p^{s,t,q}} + \|\varphi - \widetilde{\varphi}\|_{W_p^{s,t,q}} + \|\mathcal{L}_{\alpha,\phi_\alpha}\widetilde{\varphi} - \widetilde{\varphi}\|_{W_p^{s,t,q}} \\ &\leqslant C_3\epsilon + \|\mathcal{L}_{\alpha,\phi_\alpha}\widetilde{\varphi} - \widetilde{\varphi}\|_{W_p^{s,t,q}}\,, \end{aligned} \tag{54}$$

for some constant $C_3 > 0$ independent of ϵ and α. Since $\varphi \in C_X^{r-1}(M)$ we have

$$\mathcal{L}_{\alpha,\phi_\alpha}\widetilde{\varphi} - \widetilde{\varphi} = \alpha \int_0^1 \left(\partial_{\alpha'}\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)_{|\alpha'=h\alpha} \mathrm{d}\,h.$$

Since $\left(\partial_{\alpha'}\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)_{|\alpha'=h\alpha} \in C^{(r-1)}(M)$ the norm $\left\|\left(\partial_{\alpha'}\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)_{|\alpha'=h\alpha}\right\|_{W_p^{s,t,q}}$ is finite for all $0 \leqslant h \leqslant 1$. Hence for some constant $C_4(\varphi) = C_4 > 0$ we bound

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\widetilde{\varphi} - \widetilde{\varphi}\|_{W_p^{s,t,q}} \leqslant \alpha \sup_{0\leqslant h\leqslant 1}\left\|\left(\partial_{\alpha'}\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)_{|\alpha'=h\alpha}\right\|_{W_p^{s,t,q}} \leqslant C_4\alpha. \tag{55}$$

We conclude by a combination of the estimates (54)-(55). □

4.2. **Lasota–Yorke inequality for the resolvent.** We use Lemma 4.2 to prove Theorem 4.5 below. We use in addition that the resolvent improves regularity in the flow direction. We set, recalling $\lambda^{(t,s,\alpha)}$ in (48),

$$\lambda_{\min} = \lambda_{\min}(s,t,p) := \lim_{\alpha\to\infty}\frac{1}{\alpha}\log\left\|\phi_\alpha\left|\det \mathrm{D}\, g_{-\alpha}\right|^{-\frac{1}{p}}\lambda^{(t,s,\alpha)}\right\|_{L_\infty(M)}. \tag{56}$$

The following theorem will allow us to show that $\lambda_{\min}(s,t)$ plays the role of the essential spectral bound of $X+V$:

Theorem 4.5 (Lasota–Yorke inequality for the resolvent)**.** *Let $p \in [1,\infty]$ and let*

$$s' < s < 0 < q \leqslant t < r-1+s', \quad q-1 \leqslant q' < q, \quad t' < t.$$

There exist $\alpha_0 > 0$, $A_0 > \lambda_{min}$, cone ensembles Θ_ω, $\omega \in \Omega$, and a constant $C > 0$ such that for every $\varphi \in W_p^{s,t,q}$ with $\|\varphi\|_{W_p^{s,t,q}} = 1$, for every $z \in \mathbb{C}$ with $\Re z > A_0$ and for every $n \in \mathbb{N}$ it holds

$$\left\|\mathcal{R}_z^{n+1}\varphi\right\|_{W_p^{s,t,q}} \leqslant C\frac{|z|+1+(\Re z - A_0)}{(\Re z - A_0)^{n+1}}\|\varphi\|_{W_p^{s',t',q'}} + \frac{Cn\left(\Re z - \lambda_{min}\right)^{-1} + C}{(\Re z - A_0)\left(\Re z - \lambda_{min}\right)^n}.$$

Proof. Since $\lambda^{(s,t,\alpha)}$ grows at most exponentially as $\alpha \to \infty$, the constant $\lambda_{\min}$ is finite by a result on superadditive functions [26, Theorem 7.6.1]. We let $A_0 > \lambda_{\min}$. By Lemma 4.4 the transfer operator family (18) forms a strongly continuous semigroup with a well-defined generator $X+V$. We estimate powers of the resolvent $\mathcal{R}_z$ defined in (20). To this end we work with the integral representation of powers of the resolvent defined in (57) below (see [33, Corollary II.1.11]). We recall the constant A given in Lemma 4.2 and let $A_0 > A$. We set for every $z \in \mathbb{C}$ such that $\Re z > A_0$ and every $n \in \mathbb{N}$

$$\mathcal{R}_z^n\varphi := \int_0^\infty \frac{\alpha^{n-1}e^{-z\alpha}}{(n-1)!}\mathcal{L}_{\alpha,\phi_\alpha}\varphi\,\mathrm{d}\,\alpha, \quad \varphi \in W_p^{s,t,q}. \tag{57}$$

We have directly from (57) for all $\alpha \geqslant 0$

$$\mathcal{R}_z^n\mathcal{L}_{\alpha,\phi_\alpha}\varphi = \mathcal{L}_{\alpha,\phi_\alpha}\mathcal{R}_z^n\varphi. \tag{58}$$

Using Lemma 4.2, we estimate for some constant $C_1 > 0$

$$\begin{aligned}\left\|\mathcal{R}_z^{n+1}\varphi\right\|_{W_p^{s,t,q}} &\leqslant \int_0^\infty \frac{\alpha^{n-1}e^{-\Re z\alpha}}{(n-1)!}\|\mathcal{L}_{\alpha,\phi_\alpha}\mathcal{R}_z\varphi\|_{W_p^{s,t,q}}\,\mathrm{d}\,\alpha \\ &\leqslant \frac{C_1}{(\Re z - A_0)^n}\|\mathcal{R}_z\varphi\|_{W_p^{s',t',q}} + \frac{C_1\left(n+(\Re z - \lambda_{\min})\right)}{(\Re z - \lambda_{\min})^{n+1}}\|\mathcal{R}_z\varphi\|_{W_p^{s,t,q}}. \end{aligned} \tag{59}$$

Using Lemma 4.2, we get boundedness for some constant $C_2 > 0$

$$\|\mathcal{R}_z\varphi\|_{W_p^{s,t,q}} \leqslant \frac{C_2}{\Re z - A_0}\|\varphi\|_{W_p^{s,t,q}}. \tag{60}$$

Therefore the second term in the right-hand side in (59) is bounded as claimed. We bound now the first term in the right-hand side in (59). Inverting the flowbox condition (10), we find $\mathrm{D}\,\kappa_\omega^{-1}\partial_{x_d} = X_{|V_\omega}$. Hence it holds

$$
\begin{aligned}
&\partial_{x_d}\left(\vartheta_\omega\cdot\varphi\right)\circ\kappa_\omega^{-1}(x) = \mathrm{D}\left(\vartheta_\omega\cdot\varphi\right)\circ\kappa_\omega^{-1}(x)\,\mathrm{D}_x\,\kappa_\omega^{-1}\partial_{x_d}\\
&= \left(\mathrm{D}\left(\vartheta_\omega\cdot\varphi\right)X_{|V_\omega}\right)\circ\kappa_\omega^{-1}(x) = \left(\left(X\vartheta_\omega\right)\cdot\varphi+\vartheta_\omega\cdot\left(X\varphi\right)\right)\circ\kappa_\omega^{-1}(x)\,. \qquad (61)
\end{aligned}
$$

We set $W_\omega^q := W^{s',t',q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$, $\omega\in\Omega$. We estimate the local norms inside the norm $\|\mathcal{R}_z\varphi\|_{W_p^{s',t',q}}$, using the equality in (61), then Lemma 3.5 and the equality in (58), for some constant $C_3>0$:

$$
\begin{aligned}
&\left\|\left(\vartheta_\omega\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^q} \leqslant C_3\left\|\left(\vartheta_\omega\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}} \qquad (62)\\
&+C_3\left\|\left(\left(X\vartheta_\omega\right)\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}+C_3\left\|\left(\vartheta_\omega\cdot X\mathcal{R}_z\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}\,.
\end{aligned}
$$

We note that $(X\vartheta_\omega)\circ\kappa_\omega^{-1}\in C_0^{r-1}\left(\kappa_\omega\left(V_\omega\right)\right)$ and $t-s<r-1$. Using Lemma 3.3, we bound for some constant $C_4(X)=C_4>0$

$$
\left\|\left(\left(X\vartheta_\omega\right)\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}} \leqslant C_4\sup_{\omega\in\Omega}\left\|\left(\vartheta_\omega\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}\,. \qquad (63)
$$

Using the equality

$$
X\mathcal{R}_z\varphi = z\mathcal{R}_z\varphi - V\mathcal{R}_z\varphi-\varphi,
$$

together with the equality in (58), we find

$$
\begin{aligned}
&\left\|\left(\vartheta_\omega\cdot X\mathcal{R}_z\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}} \leqslant |z|\left\|\left(\vartheta_\omega\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}} \qquad (64)\\
&+\left\|\left(\vartheta_\omega\cdot V\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}+\left\|\left(\vartheta_\omega\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}\,.
\end{aligned}
$$

Recalling that $V\in C^{r-1}(M)$, we bound the term which contains the factor $(\vartheta_\omega\cdot V)\circ\kappa_\omega^{-1}$ in the right-hand side in (64) analogous as in the estimate in (63). The final estimate follows by a combination of the bounds (59)-(60) and (62)-(64), together with the trivial continuous inclusion $W_\omega^{q'}\subseteq W_\omega^{q-1}$. □

A direct consequence of Theorem 4.5 is the bound on the essential spectral radius of the resolvent:

Corollary 4.6 (Essential spectral radius)**.** *Under the assumptions of Theorem 4.5 (including the choices for $p,s,t,q\in\mathbb{R}$), letting A_0 and $\lambda_{min}=\lambda_{min}(s,t,p)$ be the constants from that theorem, the essential spectral radius of the resolvent $\mathcal{R}_z\colon W_p^{s,t,q}\to W_p^{s,t,q}$ is bounded by $|\Re z-\lambda_{min}|^{-1}$ for all $z\in\mathbb{C}$ with $\Re z>A_0$.*

Proof. Let $s'<s, t'<t$ and $q'<q$. The inclusion $W_p^{s,t,q}\subseteq W_p^{s',t',q'}$ is compact by Lemma 3.8. Then, together with a result of Hennion [25, Corollaire 1] and Theorem 4.5 we find the claimed bound on the essential spectral radius of the resolvent. □

We recall $\lambda_{\min}$ defined in (56) and ϕ_α in (18).

Lemma 4.7. *Let $d=3$ and let $|\det \mathrm{D}\,g_\alpha|\equiv 1$. Set $\widetilde{t}:=\min\{-t,s\}$. Then it holds*

$$
\lambda_{min} = \lim_{\alpha\to\infty}\frac{1}{\alpha}\log\left\|\phi_\alpha\left|\det\left(\mathrm{D}\,g_{-\alpha}\right)^{\mathrm{tr}}{}_{|E_-^*}\right|^{\widetilde{t}}\right\|_{L_\infty(M)}.
$$

Proof. Since the flow is volume preserving, we have

$$\left|\det\left(\mathrm{D}\, g_{-\alpha}\right)^{\mathrm{tr}}{}_{|E_+^*}\right|^{-1} = \left|\det\left(\mathrm{D}_{g_{-\alpha}}\, g_\alpha\right)^{\mathrm{tr}}{}_{|E_0^*}\right| \left|\det\left(\mathrm{D}_{g_{-\alpha}}\, g_\alpha\right)^{\mathrm{tr}}{}_{|E_-^*}\right|.$$

Since $d_- = 1 = d-2$ we can replace $\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}_{|E^*_{-,x}}\right\|$ in $\lambda^{(s,t,\alpha)}$ by $\left|\det\left(\mathrm{D}\, g_{-\alpha}\right)^{\mathrm{tr}}{}_{|E^*_{-,x}}\right|$ and $\left\|(\mathrm{D}\, g_{\alpha})^{\mathrm{tr}}_{|E^*_{+,g_{-\alpha}(x)}}\right\|$ by $\left|\det\left(\mathrm{D}\, g_{-\alpha}\right)^{\mathrm{tr}}{}_{|E^*_{+,x}}\right|^{-1}$. Moreover $\left|\det\left(\mathrm{D}_{g_{-\alpha}}\, g_\alpha\right)^{\mathrm{tr}}{}_{|E_0^*}\right|$ is bounded from above and below and we conclude. □

Remark 4.8. *Note that Lemma 4.7 holds in the particular case of a contact Anosov flow if $d = 3$. Clearly, if $|\phi_\alpha| \leqslant \left|\det\left(\mathrm{D}\, g_{-\alpha}\right)^{\mathrm{tr}}{}_{|E_-^*}\right|^{-\tilde{t}}$ for all $\alpha > 0$ then $\lambda_{min} \leqslant 0 < h_{top}$.*

4.3. **Spectral properties of the generator.** All spectral properties of the generator $X + V$ are with respect to its domain $D(X + V) = D(X + V)_{|W_p^{s,t,q}}$ for admissible choices $p, s, t, q \in \mathbb{R}$ which is discussed in the following lemma.

Lemma 4.9 (Domain of the generator)**.** *Let $p, s, q, t \in \mathbb{R}$ satisfy the assumptions of Lemma 4.4. Then the family $\left\{\mathcal{L}_{\alpha,\phi_\alpha} \colon W_p^{s,t,q} \to W_p^{s,t,q} \mid \alpha \geqslant 0\right\}$ admits a generator*

$$X + V \colon D(X + V) \to W_p^{s,t,q},$$

which is a closed operator on its domain $D(X + V)$. Moreover, the inclusion

$$D(X + V) \subseteq W_p^{s,t,q}$$

is dense and the inclusion

$$C_X^{r-1}(M) \subseteq D(X + V)$$

is dense for the graph norm $\|\cdot\|_{W_p^{s,t,q}} + \|(X + V)(\cdot)\|_{W_p^{s,t,q}}$.

Proof. Using Lemma 4.4, the statement about $X + V$ being a densely (in $W_p^{s,t,q}$) defined closed operator is [33, Theorem II.1.4]. Suppose now $\mathcal{L}_{\alpha,\phi_\alpha}\left(C_X^{r-1}(M)\right) \subseteq C_X^{r-1}(M)$. Then the inclusion statement $C_X^{r-1}(M) \subseteq D(X+V)$ is [33, Proposition II.1.7], using [33, Definition II.1.6]. We let $\varphi \in C_X^{r-1}(M)$. It holds $X\varphi, \mathcal{L}_{\alpha,\phi_\alpha}\varphi \in C^{r-1}(M)$ since the flow is C^r. Recalling the weight ϕ_α of the transfer operator in (18), with generating function $f \in C^{r-1}(M)$, we calculate and conclude:

$$X\mathcal{L}_{\alpha,\phi_\alpha}\varphi = (X\phi_\alpha)\cdot\varphi\circ g_{-\alpha} + \phi_\alpha\cdot(X\varphi)\circ g_{-\alpha} = (f \circ g_{-\alpha} - f)\cdot\mathcal{L}_{\alpha,\phi_\alpha}\varphi + \phi_\alpha\cdot(X\varphi)\circ g_{-\alpha}.$$

□

We set as the maximal spectral bound of the generator

$$\lambda_{\max} = \lambda_{\max}(s, t, q, p) := \sup \Re\sigma\,(X + V)\,|_{W_p^{s,t,q}}. \tag{65}$$

Lemma 4.10 (Discrete spectrum)**.** *Under the assumptions of Theorem 4.5 (including the choices for $p, s, t, q \in \mathbb{R}$), the set*

$$\left\{\lambda \in \sigma\,(X + V)\,|_{W_p^{s,t,q}} \mid \Re\lambda > \lambda_{min}\right\}$$

consists of isolated eigenvalues of finite multiplicity.

The discrete spectrum described in the previous lemma if $\lambda_{\max} > \lambda_{\min}$, is sometimes referred to as (Ruelle-Pollicott) resonances of $X + V$. In principle, the resonances depend on the choices p, s, t, and q of the space $W_p^{s,t,q}$. We shall not enter into details here, but note that our main result in the next section shows that this dependence is mild, in particular, for the choice of V there, $\lambda_{\max}$ is independent of p, t, s, and q.

Proof. Using Corollary 4.6, spectral radius of the resolvent is bounded from above by $|\Re z - \lambda_{\min}|^{-1}$. Assume $\lambda \in \sigma(X + V)|_{W_p^{s,t,q}}$ such that $\Re\lambda > \lambda_{\min}$. It follows from the Spectral Theorem for the Resolvent [33, Theorem V.1.13] that there exists $z \in \mathbb{C}$ (e.g. with $\Im z = \Im\lambda$) in the resolvent set of $X + V$ such that the spectral radius of the resolvent $\mathcal{R}_z$ has a lower bound given by

$$|z - \lambda|^{-1} = (\Re z - \Re\lambda)^{-1} > (\Re z - \lambda_{\min})^{-1}.$$

Since $\Im\lambda$ was arbitrary we conclude. □

The following notation associated to the eigenvalue spectrum is needed in Section 5 for the statement and proof of Theorem 5.7. We assume for the rest of this subsection

$$\lambda_{\max} = \lambda_{\max}(s, t, q, p) > \lambda_{\min}(s, t, p) = \lambda_{\min},$$

for any fixed choice $p \in [1, \infty]$ and $-s < 0 < q \leqslant t < r - 1 + s$. By Lemma 4.10 each $\lambda \in \sigma(X + V)|_{W_p^{s,t,q}}$ such that $\Re\lambda > \lambda_{\min}$ has a finite geometric multiplicity $n_\lambda \in \mathbb{N}$ and finite algebraic multiplicities $m_{\lambda,i} \in \mathbb{N}$, $1 \leqslant i \leqslant n_\lambda$, with generalized eigenstates

$$\mathcal{D}_{(\lambda,i,j)} \in D(X + V), \quad 1 \leqslant j \leqslant m_{\lambda,i},$$

satisfying

$$(X + V - \lambda)^j \mathcal{D}_{(\lambda,i,j)} = 0 \quad \text{and} \quad \text{if } j > 1\colon\ (X + V - \lambda)^{j-1} \mathcal{D}_{(\lambda,i,j)} \neq 0.$$

Moreover, to each geometric eigenvector there is associated a projector $\Pi_{\lambda,i}$ and a nil-potent operator $\mathcal{N}_{\lambda,i}$ of finite ranks such that

$$\Pi_{\lambda_1,i_1}\Pi_{\lambda_2,i_2} \equiv 0, \quad \mathcal{N}_{\lambda_1,i_1}\mathcal{N}_{\lambda_2,i_2} \equiv 0 \quad \text{if} \quad \lambda_1 \neq \lambda_2 \quad \text{or} \quad i_1 \neq i_2, \tag{66}$$

$$\Pi_{\lambda_1,i_1}\mathcal{N}_{\lambda_2,i_2} = \mathcal{N}_{\lambda_2,i_2}\Pi_{\lambda_1,i_1} = \begin{cases} \mathcal{N}_{\lambda_2,i_2} & \text{if } \lambda_1 = \lambda_2 \text{ and } i_1 = i_2 \\ 0 & \text{if } \lambda_1 \neq \lambda_2 \text{ or } i_1 \neq i_2 \end{cases},$$

$$\mathcal{N}_{\lambda,i}^{m_{\lambda,i}-1} \equiv 0.$$

Note that the projector $\Pi_{\lambda,i}$ can be written as a finite rank operator

$$\Pi_{\lambda,i} = \sum_{j=1}^{n_{\lambda,i}} \mathcal{D}_{(\lambda,i,j)} \otimes \mathcal{O}_{(\lambda,i,j)}, \tag{67}$$

where the dual vectors $\mathcal{O}_{(\lambda,i,j)} \in D(X + V)'$ satisfy

$$\mathcal{O}_{(\lambda_1,i_1,j_1)}\left(\mathcal{D}_{(\lambda_2,i_2,j_2)}\right) = \begin{cases} 1, & \text{if } (\lambda_1, i_1, j_1) = (\lambda_2, i_2, j_2) \\ 0, & \text{otherwise.} \end{cases}$$

We shall use the following Dolgopyat-type condition, adapted from [13, Assumption 3A], on the resolvent $\mathcal{R}_z = (z - X - V)^{-1}$, to control the remainder term $\mathcal{E}_{T,x}$ in (1) in Theorem 5.7 (to reduce to the case studied by Butterley, consider the renormalized semi-group $e^{-\lambda_{\max}\alpha}\mathcal{L}_{\alpha,\phi_\alpha}$ with generator $X+V-\lambda_{\max}$ and resolvent $\mathcal{R}_{z+\lambda_{\max}}$)[4]:

Condition 4.11 (Spectral gap with (Dolgopyat) bounds)**.** *There exists*

$$\delta \in (\lambda_{min}(s,t,p), \lambda_{max}(s,t,q,p))$$

so that the following holds: For some $a>0$, $b>0$, $C>0$, *some*

$$\gamma \in (0, 1/\log(1+(\lambda_{max}-\delta)/a)),$$

and for all $z\in\mathbb{C}$ *with* $\Re z = a$ *and* $|\Im z| \geqslant b$, *we have*

$$\left\|\mathcal{R}^{\tilde{n}}_{z+\lambda_{max}}\right\|_{W_p^{s,t,q}} \leqslant C^{\tilde{n}} \left|\Re z + (\lambda_{max}-\delta)\right|^{-\tilde{n}}, \quad \textit{where } \tilde{n} = \lceil \gamma \log|\Im z| \rceil .$$

It is well known that if $\|\mathcal{L}_{\alpha,\phi_\alpha}\|_{W_p^{s,t,q}\to W_p^{s,t,q}} \leqslant Ce^{\lambda_{\max}\alpha}$ for all α and if $\mathcal{R}_z$ enjoys Lasota–Yorke estimates for $\lambda_{\min}(s,t)$ on $W_p^{s,t,q}$, in the sense of Theorem 4.5, then Condition 4.11 for some constant δ implies a *spectral gap* for the same δ, in the sense that

$$\sigma\left(X+V\right)|_{W_p^{s,t,q}} \cap \{\Re\lambda > \delta\} \text{ is a finite set,} \tag{68}$$

see e.g. [13, Theorem 1]. (Note that [13, Assumption 1] follows from the facts that $W_p^{s,t,q} \subset W_p^{s,t,q-1}$, $\|(X+V)\,\varphi\|_{W_p^{s,t,q-1}} \leqslant C\|\varphi\|_{W_p^{s,t,q}}$ for some constant $C>0$, using Lemma 3.3 and Lemma 3.5, and

$$e^{-\lambda_{\max}\alpha}\mathcal{L}_{\tilde{\alpha},\phi_{\tilde{\alpha}}}\varphi - \varphi = (X+V-\lambda_{\max})\int_0^{\alpha} e^{-\lambda_{\max}\tilde{\alpha}}\mathcal{L}_{\tilde{\alpha},\phi_{\tilde{\alpha}}}\varphi \,\mathrm{d}\,\tilde{\alpha} \tag{69}$$

for all $\varphi \in W_p^{s,t,q}$.)
Beware that even when $W_p^{s,t,q}$ is a Hilbert space, the operator $X+V$ is not self-adjoint a priori, so the existence of a spectral gap for $X+V$ with δ does *n*ot imply a spectral gap with bounds on the resolvent in general. (In the self-adjoint case, classical bounds on the iterated resolvent $\mathcal{R}_z^n$ in terms of the distance between z and the spectrum give bounds stronger than Condition 4.11.)
See also Remark 5.11 for a further discussion of Condition 4.11.

4.4. **Proof of Lemma 4.1.**

We need some preparations. We recall the quantities $\|F\|_{-,I}, \|F\|_{+,I}, \|F\|_{0,I}$ given below (47). We introduce an arrow relation as used by Baladi and Tsujii in [6, p.16].

Definition 4.12 (Arrow relation)**.** *Let* $n,\ell \in \mathbb{Z}_{\geqslant 0}$ *and* $\sigma,\tau\in\{-,+,0\}$. *We write*

$$(\tau,\ell) \hookrightarrow_I (\sigma,n) \Leftrightarrow \begin{cases} \tau = + \text{ and } 2^{n-\ell} \leqslant 2^4\|F\|_{+,I} \\ \tau=\sigma=- \text{ and } 2^{n-\ell} \geqslant 2^{-4}\|F\|_{-,I} \end{cases},$$

$$\ell \hookrightarrow_I (\sigma,n) \Leftrightarrow \begin{cases} \tau=\sigma=0 \text{ and } 2^{n-\ell} \leqslant 2^4\|F\|_{0,I} \\ \sigma = - \text{ and } \tau = 0 \end{cases},$$

and $(\tau,\ell) \not\hookrightarrow_I (\sigma,n)$ *in the other cases.*

---

[4]Note the iterated constant $C^{\tilde{n}}$ contrary to C in [13, Assumption 3A]. This change was made to avoid a conflict in the proof of [13, Lemma 4.4], involving in there the constant C_6, and was communicated with Butterley [14].

We recall the function c defined in (29). We let c' be analogously defined for $s' \leqslant s$, $t' \leqslant t$, $q' \leqslant q$. We have for some constant $C > 0$, for all fixed $\tau \in \{-,+,0\}$, $\ell \in \mathbb{Z}_{\geqslant 0}$

$$\sum_{(\tau,\ell)\hookrightarrow_I(\sigma,n)} 2^{c(\sigma)n-c(\tau)\ell} = \sum_{(\tau,\ell)\hookrightarrow_I(\sigma,n)} 2^{(c(\sigma)-c(\tau))n+c(\tau)(n-\ell)} \leqslant \sum_{(\tau,\ell)\hookrightarrow_I(\sigma,n)} 2^{c(\tau)(n-\ell)}$$

$$\leqslant C \max\left\{\|F\|_{+,I}^{t}, \|F\|_{-,I}^{s}\right\}. \tag{70}$$

An analogous estimate holds for all fixed σ, n. Similarly, we find either for all fixed ℓ or for all fixed σ, n

$$\sum_{\ell\hookrightarrow_I(\sigma,n)} 2^{c(\sigma)n-q\ell} \leqslant C \max\left\{1, \|F\|_{0,I}^{q}\right\}. \tag{71}$$

We recall the norm of the Hilbert space ℓ_2^{c} (and analogously $\ell_2^{c'}$) given in (30). Clearly, we have the inclusion $\ell_2^{c} \subseteq \ell_2^{c'}$. We recall the definitions of $\Psi_{\sigma,n}^{\mathrm{Op}}$ in (23). We let given a family of pairwise disjoint sets

$$\mathcal{I} := \{I \subseteq K\}.$$

For every $(a_{\tau,\ell}) = a \in L_p\left(\mathbb{R}^d, \ell_2^{c'}\right)$ we set

$$\begin{aligned}\left(Q_{\hookrightarrow_{\mathcal{I}}}^{\mathrm{Op}} a\right)_{\sigma,n} &:= \Psi_{\sigma,n}^{\mathrm{Op}} \textstyle\sum_{I\in\mathcal{I}} \sum_{(\tau,\ell)\hookrightarrow_I(\sigma,n)} 1_{|I} a_{\tau,\ell},\\ \left(Q_{\hookrightarrow_{0,\mathcal{I}}}^{\mathrm{Op}} a\right)_{\sigma,n} &:= \Psi_{\sigma,n}^{\mathrm{Op}} \textstyle\sum_{I\in\mathcal{I}} \sum_{l\hookrightarrow_I(\sigma,n)} 1_{|I} a_{0,l}.\end{aligned} \tag{72}$$

Lemma 4.13 (Boundedness I). *For all $p \in [1,\infty]$ the map*

$$Q_{\hookrightarrow_{\mathcal{I}}}^{\mathrm{Op}} : L_p\left(\mathbb{R}^d, \ell_2^{c}\right) \to L_p\left(\mathbb{R}^d, \ell_2^{c}\right)$$

is a bounded linear operator. Moreover, for some constant $C \geqslant 0$, for every $f \in L_\infty\left(\mathbb{R}^d, \mathbb{R}\backslash\{0\}\right)$ and every $a \in L_p\left(\mathbb{R}^d, \ell_2^{c}\right)$, it holds

$$\left\|Q_{\hookrightarrow_{\mathcal{I}}}^{\mathrm{Op}} a\right\|_{L_p(\mathbb{R}^d,\ell_2^{c})} \leqslant C \sup_{I\in\mathcal{I}} \max\left\{\|F\|_{+,I}^{t}, \|F\|_{-,I}^{s}\right\} \left\|f_{|I}\right\|_{L_\infty} \left\|\sum_{I\in\mathcal{I}} \frac{1}{f_{|I}} \|a\|_{\ell_2^{c}}\right\|_{L_p}.$$

Let $c'(0) = c(0)$. Then for all $p \in [1,\infty]$ the map

$$Q_{\hookrightarrow_{0,\mathcal{I}}}^{\mathrm{Op}} : L_p\left(\mathbb{R}^d, \ell_2^{c'}\right) \to L_p\left(\mathbb{R}^d, \ell_2^{c}\right)$$

is a bounded linear operator. Moreover, for every $a \in L_p\left(\mathbb{R}^d, \ell_2^{c'}\right)$ it holds

$$\left\|Q_{\hookrightarrow_{0,\mathcal{I}}}^{\mathrm{Op}} a\right\|_{L_p(\mathbb{R}^d,\ell_2^{c})} \leqslant C \sup_{I\in\mathcal{I}} \max\left\{1, \|F\|_{0,I}^{q}\right\} \left\|f_{|I}\right\|_{L_\infty} \left\|\sum_{I\in\mathcal{I}} \frac{1}{f_{|I}} \|a\|_{\ell_2^{c'}}\right\|_{L_p}.$$

Proof. For every $b \in \ell_2^{c}$ we set

$$(Qb)_{\sigma,n} := \Psi_{\sigma,n} b_{\sigma,n},$$

and for every $a \in L_p\left(\mathbb{R}^d, \ell_2^{c}\right)$ we set

$$(\mathcal{K}a)_{\sigma,n} := \sum_{I\in\mathcal{I}} \sum_{(\tau,\ell)\hookrightarrow_I(\sigma,n)} 1_{|I} a_{\tau,\ell}.$$

We let Q^{Op} be the operator in (27) associated to Q. We note that

$$Q_{\hookrightarrow_I}^{\mathrm{Op}} = Q^{\mathrm{Op}} \circ \mathcal{K}.$$

Using Lemma 3.1, we bound for some constant $C_1 > 0$

$$\left\| Q^{\mathrm{Op}} \mathcal{K} a \right\|_{L_p(\mathbb{R}^d, \ell_2^c)} \leqslant C_1 \left\| \mathcal{K} a \right\|_{L_p(\mathbb{R}^d, \ell_2^c)} .$$

We estimate with constants $C_2, C_3 > 0$, using pairwise disjointness of elements $I \in \mathcal{I}$, Cauchy–Schwarz and the bound in (70),

$$\begin{aligned}
&\left\| \mathcal{K} a \right\|_{L_p(\mathbb{R}^d, \ell_2^c)} = \left\| \left(\sum_{\sigma,n} 4^{c(\sigma)n} \left(\sum_{I \in \mathcal{I}} \sum_{(\tau,\ell) \hookrightarrow_I (\sigma,n)} 1_{|I} a_{\tau,\ell} \right)^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\
&\leqslant \left\| \left(\sum_{\sigma,n} \sum_{I \in \mathcal{I}} \left(\sum_{(\tau,\ell) \hookrightarrow_I (\sigma,n)} 2^{c(\sigma)n - c(\tau)\ell} \right) \sum_{(\tau,\ell) \hookrightarrow_I (\sigma,n)} 2^{c(\sigma)n + c(\tau)\ell} \left| 1_{|I} a_{\tau,\ell} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\
&\leqslant C_2 \left\| \left(\sum_{I \in \mathcal{I}} \max \left\{ \|F\|_{+,I}^t , \|F\|_{-,I}^s \right\} \sum_{\sigma,n} \sum_{(\tau,\ell) \hookrightarrow_I (\sigma,n)} 2^{c(\sigma)n + c(\tau)\ell} \left| 1_{|I} a_{\tau,\ell} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\
&= C_2 \left\| \left(\sum_{I \in \mathcal{I}} \max \left\{ \|F\|_{+,I}^t , \|F\|_{-,I}^s \right\} \sum_{\tau,\ell} 2^{2c(\tau)\ell} \left| 1_{|I} a_{\tau,\ell} \right|^2 \sum_{(\tau,\ell) \hookrightarrow_I (\sigma,n)} 2^{c(\sigma)n - c(\tau)\ell} \right)^{\frac{1}{2}} \right\|_{L_p} \\
&\leqslant C_3 \left\| \left(\sum_{I \in \mathcal{I}} \max \left\{ \|F\|_{+,I}^{2t} , \|F\|_{-,I}^{2s} \right\} \left\| f_{|I} \right\|_{L_\infty}^2 \sum_{\tau,\ell} 2^{2c(\tau)\ell} \left| \frac{1}{f_{|I}} a_{\tau,\ell} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_\infty} \\
&\leqslant C_3 \sup_{I \in \mathcal{I}} \max \left\{ \|F\|_{+,I}^t , \|F\|_{-,I}^s \right\} \left\| f_{|I} \right\|_{L_\infty} \left\| \left(\sum_{I \in \mathcal{I}} \sum_{\tau,\ell} 2^{2c(\tau)\ell} \left| \frac{1}{f_{|I}} a_{\tau,\ell} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} .
\end{aligned}$$

The statement about $Q^{\mathrm{Op}}_{\hookrightarrow_{0,\mathcal{I}}}$ follows analogously, using (71). □

We recall (see above (47)) that F is assumed to be (Θ°, Θ)-hyperbolic on K and recall the maps $\widetilde{\Phi}^\circ_-, \widetilde{\Phi}^\circ_0$ assumed in Definition 2.2 which we use to construct $\widetilde{\Psi}^\circ_{\sigma,n}$ defined in (24). We set

$$\mathcal{J} := \begin{matrix} \{(-,0,0,0), (-,0,+,0), (0,0,+,0), (+,0,-,0)\} \cup \\ \{(+,\ell,+,0), (+,\ell,0,0), (0,\ell,0,0) \mid \ell \geqslant 0\} \cup \{(-,0,-,n) \mid n \geqslant 0\} \end{matrix} . \tag{73}$$

We recall the arrow notation f from Definition 4.12.

Lemma 4.14 (Directional inequality)**.** *Let $(\tau,\ell) \not\hookrightarrow_I (n,\sigma)$ and $(\tau,\ell,\sigma,n) \notin \mathcal{J}$. Let $\eta \in \operatorname{supp} \widetilde{\Psi}^\circ_{\tau,\ell}$ and $\xi \in \operatorname{supp} \Psi_{\sigma,n}$. Set*

$$m(\tau) := \begin{cases} \max\{n,\ell\}, & \text{if } \tau \in \{-,0\} \\ n, & \text{if } \tau = + \end{cases} . \tag{74}$$

Then, for some $C > 0$ and for all $y \in I$ it holds

$$\left| (\mathrm{D}_y F)^{\mathrm{tr}} \eta - \xi \right| \geqslant C 2^{m(\tau)} \min \left\{ 1, \|F\|_{-,I} , \|F\|_{0,I} \right\} .$$

Proof. This can be seen case-by-case for admissible σ, τ as follows. We recall the set $\mathcal{J}$ defined in (73). We let $(\tau, \ell) \not\hookrightarrow_I (n, \sigma)$ such that $(\tau, \ell, \sigma, n) \notin \mathcal{J}$. Due to the construction of $\Psi_{\sigma,n}$ and $\widetilde{\Psi}^{\circ}_{\tau,\ell}$, respectively, if $n \geqslant 1$ then $2^{n-1} \leqslant |\xi| \leqslant 2^{n+1}$ and if $\ell \geqslant 1$ then $2^{\ell-2} \leqslant |\eta| \leqslant 2^{\ell+2}$. We assume first $c(\sigma) \leqslant c(\tau)$. Let $\tau = +$. Then $2^{n-\ell} > 2^4 \|F\|_{+,I}$ and moreover, the exclusion of $\mathcal{J}$ implies $n \geqslant 1$. Using the triangle inequality, we find

$$\left|(\mathrm{D}_y F)^{\mathrm{tr}} \eta - \xi\right| \geqslant |\xi| - \left|(\mathrm{D}_y F)^{\mathrm{tr}} \eta\right| \geqslant 2^{n-1} - \|F\|_{+,I} 2^{\ell+2} \geqslant 2^{n-1} - 2^{n-2} \geqslant 2^{n-2}.$$

The case $\tau = 0$ is analogous. Just note that we have also the estimate

$$2^{n-1} - \|F\|_{0,I} 2^{\ell+2} \geqslant \|F\|_{0,I} 2^{l+3} - \|F\|_{0,I} 2^{\ell+2}.$$

If $\tau = -$ it holds $2^{n-\ell} < 2^{-4} \|F\|_{-,I}$. The exclusion of $\mathcal{J}$ implies $l \geqslant 1$. Using the triangle inequality, we find

$$\left|(\mathrm{D}_y F)^{\mathrm{tr}} \eta - \xi\right| \geqslant \left|(\mathrm{D}_y F)^{\mathrm{tr}} \eta\right| - |\xi| \geqslant \|F\|_{-,I} 2^{\ell-2} - 2^{n+1} > 2^{n+2} - 2^{n+1} \geqslant 2^{n+1}.$$

On the other hand we have also the estimate

$$\|F\|_{-,I} 2^{\ell-2} - 2^{n+1} > \|F\|_{-,I} 2^{\ell-2} - \|F\|_{-,I} 2^{l-3}.$$

Now we assume $c(\sigma) > c(\tau)$. We assume first $\tau = -$. Then $\sigma \in \{0, +\}$. We recall that F is cone-hyperbolic (see Definition 2.2). The exclusion of $\mathcal{J}$ implies $n \neq 0$ or $l \neq 0$. Together with the first compact inclusion in (13) we conclude that the angle between $(\mathrm{D}_y F)^{\mathrm{tr}} \eta$ and ξ is bounded from below. This implies a lower bound $\geqslant C 2^{\max\{n,l\}}$ for the distance in both cases where $C > 0$ is some constant. We assume now $\tau = 0$ which implies $\sigma = +$. The reasoning is analogous as for $\tau = -$, using the second compact inclusion in (13) to bound the angle between $(\mathrm{D}_y F)^{\mathrm{tr}} \eta$ and ξ from below. □

Lemma 4.15. *Let $p \in [1, \infty]$, $b \in L_p$ and let $(\tau, \ell) \not\hookrightarrow_I (\sigma, n)$ and $(\tau, n, \sigma, \ell) \neq \mathcal{J}$. It holds for the local transfer operator $L_{F, f_{|I}}$*

$$\left\| \Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f_{|I}} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} b \right\|_{L_p} \leqslant C_3(F, f_{|I}) 2^{-(r-1)m(\tau)} \|b\|_{L_p},$$

where for some $C \geqslant 1$ it holds
$C_3(F, f_{|I}) \leqslant C \max\left\{1, \|F\|^{1-r}_{-,I}, \|F\|^{1-r}_{0,I}\right\} \|\mathrm{D}\, F\|^k_{C^{r-1}} \|f\|_{C^{r-1}} \sup_{y \in K} |\det \mathrm{D}_y F|^{-\frac{1}{p}}$.

Proof. This is analogous to the proof of Lemma 3.3, except that we have to deal with the additional composition operation by the map F. We set $f := f_{|I}$. We expand the convolution and inverse Fourier transform

$$\begin{aligned}
\Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} b(x) &= C \int_{\mathbb{R}^{4d}} e^{\mathrm{i}\,\eta(F(z)-y)} e^{\mathrm{i}\,\xi(x-z)} f(z) b(y) \Psi_{\sigma,n}(\xi) \widetilde{\Psi}^{\circ}_{\tau,\ell}(\eta) \mathrm{d}\eta \mathrm{d}\xi \mathrm{d}z \mathrm{d}y \\
&= C \int_{\mathbb{R}^d} V^{\tau,\ell}_{\sigma,n}(x, y) b(F(y)) \left|\det \mathrm{D}\, F(y)\right| \mathrm{d}y,
\end{aligned}$$

for some constant $C > 0$ and where we set

$$V^{\tau,\ell}_{\sigma,n}(x, y) := \int_{\mathbb{R}^{3d}} e^{-\mathrm{i}\,\eta F(y)} e^{\mathrm{i}\,\xi x} \Psi_{\sigma,n}(\xi) \widetilde{\Psi}^{\circ}_{\tau,\ell}(\eta) e^{\mathrm{i}(\eta F(z) - \xi z)} f(z) \mathrm{d}z \mathrm{d}\eta \mathrm{d}\xi. \tag{75}$$

We transform (75), first integrating by parts $\lfloor r \rceil - 1$-times in z (see Lemma A.3 with function $G(z) := \eta F(z) - \xi z$ which has a gradient bounded from below by Lemma 4.14). Therefore we replace $f(z)$ in (75) with another function $V_{\lfloor r \rceil - 1}(z, \eta, \xi)$ which satisfies the iterative construction given in Lemma A.3 (121). Using Lemma 4.14 and Lemma A.3 (122), we estimate for some constant $C \geqslant 1$

$$\left\|V_{\lfloor r-1\rfloor}\right\|_{C^0} \leqslant C\widetilde{C}_1 2^{-m(\tau)\lfloor r-1\rfloor} \max\left\{1, \|F\|_{-,I}^{-\lfloor r-1\rfloor}, \|F\|_{0,I}^{-\lfloor r-1\rfloor}\right\} \|f\|_{C^{\lfloor r-1\rfloor}},$$

where $\widetilde{C}_1 := \sup_{(z,\eta,\xi)\in\operatorname{supp} f} \max_{0\leqslant|\gamma|\leqslant\lfloor r-1\rfloor} \left|\left|(\mathrm{D}_z\, F)^{\mathrm{tr}}\,\eta-\xi\right|\partial_z^\gamma \frac{(\mathrm{D}_z\, F)^{\mathrm{tr}}\eta-\xi}{\left|(\mathrm{D}_z\, F)^{\mathrm{tr}}\eta-\xi\right|^2}\right|^{\lfloor r-1\rfloor}$.
Moreover, this function is a $C^{\tilde{r}}$-map for $\tilde{r} := r - \lfloor r\rfloor$. Using Lemma A.5 (in there we take $\epsilon = L^{-1} = 2^{-m(\tau)}$), we proceed with a regularized integration by parts in z. This yields

$$V_{\sigma,n}^{\tau,\ell}(x,y) = \int_{\mathbb{R}^{3d}} e^{\mathrm{i}\,\eta(F(z)-F(y))} e^{\mathrm{i}\,\xi(x-z)} \Psi_{\sigma,n}(\xi)\widetilde{\Psi}^\circ_{\tau,\ell}(\eta) V_{r-1}(z,\eta,\xi)\mathrm{d}z\mathrm{d}\eta\mathrm{d}\xi, \tag{76}$$

where V_{r-1} is given in (124) in Lemma A.5 with bound

$$\|V_{r-1}\|_{C^0} \leqslant C\widetilde{C}_2 2^{-m(\tau)\tilde{r}} \max\left\{1, \|F\|_{-,I}^{-\tilde{r}}, \|F\|_{0,I}^{-\tilde{r}}\right\}, \tag{77}$$

where $\widetilde{C}_2 := \sup_{(z,\eta,\xi)\in\operatorname{supp} f} \left(1 + \left|\left((\mathrm{D}_z\, F)^{\mathrm{tr}}\,\eta-\xi\right)\right|\right) \left\|\frac{\left((\mathrm{D}_z\, F)^{\mathrm{tr}}\eta-\xi\right)V_{\lfloor r-1\rfloor}(\cdot,\eta,\xi)}{\left|(\mathrm{D}_z\, F)^{\mathrm{tr}}\eta-\xi\right|^2}\right\|_{C^{\tilde{r}}}$.
We now substitute $\xi \to 2^\ell\xi'$ and $\eta \to 2^n\eta'$ in (76). By construction the function $V_{r-1}\left(z, 2^n\eta', 2^\ell\xi'\right)$ is uniformly bounded in n and ℓ in the C^∞-norm with respect to η' and ξ'. We transform (76), integrating by parts $d+1$-times in ξ' if $|2^n(z-x)| > 1$, and $d+1$-times in η' if $\left|2^\ell(F(z)-F(y))\right| > 1$, which yields for some constant $C_1 > 0$

$$V_{\sigma,n}^{\tau,\ell}(x,y) = C_1 \int_{\mathbb{R}^{3d}} \frac{e^{\mathrm{i}\,2^\ell\eta'(F(z)-F(y))}}{u_\ell(F(z)-F(y))} \frac{e^{\mathrm{i}\,2^n\xi'(x-z)}}{u_n(x-z)} 2^{d(\ell+n)} \widetilde{V}_{\sigma,n}^{\tau,\ell}(z,\eta',\xi')\mathrm{d}z\mathrm{d}\eta'\mathrm{d}\xi',$$

where $\widetilde{V}_{\sigma,n}^{\tau,\ell}(z,\eta',\xi')$ together with $u_n\colon \mathbb{R}^d \to (0,1] : x \mapsto \begin{cases} 1 & \text{, if } |2^n x| \leqslant 1 \\ |2^n x|^{d+1} & \text{, else} \end{cases}$ replaces $\Psi_{\sigma,n}(2^\ell\xi')\widetilde{\Psi}^\circ_{\tau,\ell}(2^n\eta')V_{r-1}\left(z, 2^n\eta', 2^\ell\xi'\right)$ in (76). Since we only derived V_{r-1} with respect to η' and ξ', respectively, the C^0-norm of $\widetilde{V}_{\sigma,n}^{\tau,\ell}(z,\eta',\xi')$ is controlled by the upper bound given in (77). We recall that ξ', η' are uniformly bounded. We estimate trivially for some constant $C_2 \geqslant 1$

$$\begin{aligned}\left\|\Psi_{\sigma,n}^{\mathrm{Op}} L_{F,f} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} b\right\|_{L_p} &\leqslant C \left\|\widetilde{V}_{\sigma,n}^{\tau,\ell}\right\|_{C^0} 2^{d(n+\ell)} \left\|\frac{1}{u_n} * \left((b * \frac{1}{u_\ell}) \circ F\right)\right\|_{L_p} \\ &\leqslant C_2 C_3(F,f) 2^{-m(\tau)(r-1)} \|b\|_{L_p},\end{aligned}$$

where we used twice Young's inequality in the last step.

□

We set for all $n \in \mathbb{Z}_{\geqslant 0}$, for all $\sigma \in \{-,+,0\}$ and for all $I \in \mathcal{I}$

$$\left(Q_{\nleftrightarrow,I}^{\mathrm{Op}} a\right)_{\sigma,n} := \Psi_{\sigma,n}^{\mathrm{Op}} \sum_{(\tau,\ell)\nleftrightarrow_I(\sigma,n)} a_{\tau,\ell}, \quad \left(Q^{\mathrm{Op}} a\right)_{\sigma,n} := \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\sigma,n} a_{\sigma,n}. \tag{78}$$

Lemma 4.16 (Boundedness II)**.** *Let $c(+) - c'(-) < r-1$. Then for all $p \in [1,\infty]$ the map $Q_{\nleftrightarrow,I}^{\mathrm{Op}} L_{F,f_{|I}} Q^{\mathrm{Op}}\colon L_p\left(\mathbb{R}^d, \ell_2^{c'}\right) \to L_p\left(\mathbb{R}^d, \ell_2^{c}\right)$ is a bounded linear operator. In particular, it holds*

$$\left\|Q_{\nleftrightarrow,I}^{\mathrm{Op}} L_{F,f_{|I}} Q^{\mathrm{Op}}\right\|_{L_p\left(\mathbb{R}^d,\ell_2^{c'}\right)\to L_p\left(\mathbb{R}^d,\ell_2^{c}\right)} \leqslant CC_4(F, f_{|I}),$$

where for some $C \geqslant 1$ *and some* $k \geqslant 0$

$$C_4(F,f) \leqslant C \max\left\{1, \|F\|_{-,I}^{1-r}, \|F\|_{0,I}^{1-r}\right\} \max\left\{1, \|\mathrm{D}\,F\|_{C^{r-1}}^{k}\right\} \|f\|_{C^{r-1}} \sup_{y\in K} |\det \mathrm{D}_y\, F|^{-\frac{1}{p}}$$

Proof. Let $(a_{\tau,\ell}) = a \in L_p\left(\mathbb{R}^d, \ell_2^{c'}\right)$. We have

$$\left\|Q^{\mathrm{Op}}_{\leftrightarrow,I} L_{F,f_{|I}} Q^{\mathrm{Op}} a\right\|_{L_p(\mathbb{R}^d,\ell_2^c)} = \left\| \left(\sum_{\sigma,n} 4^{c(\sigma)n} \left| \sum_{(\tau,\ell)\leftrightarrow_I(\sigma,n)} \Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f_{|I}} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} a_{\tau,\ell} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p}$$

$$\leqslant \sum_{(\tau,\ell)\leftrightarrow_I(\sigma,n)} 2^{c(\sigma)n} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f_{|I}} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} a_{\tau,\ell} \right\|_{L_p}. \tag{79}$$

We recall the set of indices $\mathcal{J}$ in (73). We assume $(\tau,l,\sigma,n) \in \mathcal{J}$. Now we make three distinctions in the estimate of the corresponding part of the sum in (79). If $\tau \in \{+,0\}$ then $n = 0$ and $l \geqslant 0$. Then, using Young's inequality, for some $C \geqslant 1$

$$\sum_{l=0}^{\infty} \left\| \Psi^{\mathrm{Op}}_{\sigma,0} L_{F,f_{|I}} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} a_{\tau,\ell} \right\|_{L_p} \leqslant C \sup_{z\in I} |f(z)|\, |\det \mathrm{D}_z\, F|^{-\frac{1}{p}} \sum_{l=0}^{\infty} 2^{-c'(\tau)l} \sup_{l} 2^{c'(\tau)l} \|a_{\tau,\ell}\|_{L_p}$$
$$\leqslant C \sup_{z\in I} |f(z)|\, |\det \mathrm{D}_z\, F|^{-\frac{1}{p}} \|a\|_{L_p\left(\mathbb{R}^d,\ell_2^{c'}\right)}.$$

If $\tau = \sigma = -$ then $n \geqslant 0$ and $l = 0$. Recall that $s < 0$. Then, using Young's inequality,

$$\sum_{n=0}^{\infty} 2^{sn} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f_{|I}} \widetilde{\Psi}'^{\,\mathrm{Op}}_{-,0} a_{-,0} \right\|_{L_p} \leqslant C \sup_{z\in I} |f(z)|\, |\det \mathrm{D}_z\, F|^{-\frac{1}{p}} \|a\|_{L_p\left(\mathbb{R}^d,\ell_2^{c'}\right)}.$$

In the three remaining cases $n = l = 0$ we estimate analogously, using Young's inequality. Now we assume $(\tau,l,\sigma,n) \notin \mathcal{J}$. We recall $m(\tau)$ defined in (74) in Lemma 4.14 and the constant $C_3(F,f)$ in Lemma 4.15. Using Lemma 4.15, we estimate the remaining part of the sum in (79)

$$\sum_{\substack{(\tau,\ell)\leftrightarrow_I(\sigma,n)\\(\tau,l,\sigma,n)\notin\mathcal{J}}} 2^{c(\sigma)n} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f_{|I}} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} a_{\tau,\ell} \right\|_{L_p}$$
$$\leqslant C C_3(F,f) \sum_{\substack{(\tau,\ell)\leftrightarrow_I(\sigma,n)\\(\tau,l,\sigma,n)\notin\mathcal{J}}} 2^{c(\sigma)n - c'(\tau)l - m(\tau)(r-1)} \sup_{\tau,\ell} 2^{c'(\tau)l} \|a_{\tau,\ell}\|_{L_p} \tag{80}$$
$$\leqslant C C_3(F,f) \|a\|_{L_p\left(\mathbb{R}^d,\ell_2^{c'}\right)},$$

where the sums in n, l, respectively, in the right-hand side in (80) are bounded by geometric sums, using the assumption $c'(-) > r - 1 - c(+)$. In particular, we find for (the worst-case since $0 < c(0) \leqslant c(+) < r-1$) $\tau = -$, $\sigma = +$, if $l \geqslant n$ for all small enough $\epsilon > 0$

$$(c(+)+\epsilon)\,n - c'(-)l - m(-)(r-1) \leqslant \left(c(+)+\epsilon - c'(-)\right) l - m(-)(r-1)$$
$$= \left(c(+)+\epsilon - c'(-) - r + 1\right) l < 0,$$

and an analogous estimate holds for $l < n$. We note

$$\sup_{z\in I} |f(z)|\, |\det \mathrm{D}_z\, F|^{-\frac{1}{p}} \leqslant C \|f\|_{C^{r-1}} \sup_{z\in I} |\det \mathrm{D}_z\, F|^{-\frac{1}{p}}.$$

We set $C_4(F, f_{|I}) \coloneqq CC_3(F, f_{|I})$. Combining the estimates for all the parts of the sum (79), we conclude. □

Proof of Lemma 4.1. Let s, q, t, p satisfy the hypotheses in Lemma 4.1. That is $s' < s < 0 < q \leqslant t < r - 1 + s'$, $q' < q$, $t' < t$ and $p \in [1, \infty]$. We put $c(-) \coloneqq s$, $c(+) \coloneqq t$, $c(0) \coloneqq c'(0) \coloneqq q$ and $c'(-) \coloneqq c''(-) \coloneqq s'$, $c'(+) \coloneqq c''(+) \coloneqq t'$ and $c''(0) \coloneqq q'$. Then c,c',c'' satisfy (29), respectively, while c, c' satisfy the hypotheses in Lemma 4.13, and c, c'' that of Lemma 4.16. Let $\varphi \in W^{s,t,q}_{p,\Theta,F(K)}$. We set

$$a_{\tau,\ell} \coloneqq L_{F,f} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi.$$

We have $a \in L_p\left(\mathbb{R}^d, \ell_2^c\right) \subseteq L_p\left(\mathbb{R}^d, \ell_2^{c'}\right)$ because

$$\begin{aligned}\|a\|_{L_p(\mathbb{R}^d,\ell_2^c)} &= \left\| \left(\sum_{\tau,\ell} 4^{c(\tau)\ell} \left|a_{\tau,\ell}\right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} = \left\| \left(\sum_{\tau,\ell} 4^{c(\tau)\ell} \left| f \cdot \left(\Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right) \circ F \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\leqslant \left\| f \left|\det \mathrm{D}\, F\right|^{-\frac{1}{p}} \right\|_{L_\infty(K)} \left\| \left(\sum_{\tau,\ell} 4^{c(\tau)\ell} \left| \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p(F(K))} \qquad (81) \\ &\leqslant \left\| f \left|\det \mathrm{D}\, F\right|^{-\frac{1}{p}} \right\|_{L_\infty} \|\varphi\|_{W^{s,t,q}_{p,\Theta^\circ,F(K)}}.\end{aligned}$$

We set $b_{\tau,\ell} \coloneqq \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi$. By the first statement in Lemma 3.4, it holds $\varphi \in W^{s',t',q'}_{p,\Theta^\circ,F(K)}$ hence $(b_{\tau,\ell} \mid \tau \in \{-,+,0\}, \ell \in \mathbb{Z}_{\geqslant 0}) \eqqcolon b \in L_p\left(\mathbb{R}^d, \ell_2^{c''}\right)$. By assumption on K, we can decompose $K = \sqcup \mathcal{W}$ into finitely many open sets $\mathcal{W}$. For each component $\mathcal{W}$ we set

$$a_{|\mathcal{W},\tau,\ell} \coloneqq L_{F,f_{|\mathcal{W}}} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \quad \text{and} \quad a \coloneqq \sum_{\mathcal{W}} a_{|\mathcal{W}}.$$

By construction (see above (78)), it holds $\widetilde{\Psi}^\circ_{\tau,\ell|\,\mathrm{supp}\,\Psi'_{\tau,\ell}} \equiv 1$ hence $\Psi^{\circ\,\mathrm{Op}}_{\tau,\ell}\varphi = \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi$. For each $\mathcal{W}$ there is a corresponding arrow relation given by Definition 4.12 and the restriction $f_{|\mathcal{W}}$ is also C^{r-1}. We rewrite

$$\begin{aligned}\Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f_{|\mathcal{W}}} \varphi = \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{\tau,\ell} a_{|\mathcal{W},\tau,\ell} &= \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{(\tau,\ell) \hookrightarrow_{\mathcal{W}} (\sigma,n)} a_{|\mathcal{W},\tau,\ell} + \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{l \hookrightarrow_{\mathcal{W}} (\sigma,n)} a_{|\mathcal{W},0,\ell} \\ &\quad + \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{(\tau,\ell) \not\hookrightarrow_{\mathcal{W}} (\sigma,n)} L_{F,f_{|\mathcal{W}}} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi. \qquad (82)\end{aligned}$$

We recall the definitions of the operators $Q^{\mathrm{Op}}_{\hookrightarrow,\{\mathcal{W}\}}$, $Q^{\mathrm{Op}}_{\hookrightarrow_0,\{\mathcal{W}\}}$, $Q^{\mathrm{Op}}_{\not\hookrightarrow,\mathcal{W}}$ given in (72) and in (78), respectively (in which we take $\mathcal{I} = \{\mathcal{W}\}$ and $I = \mathcal{W}$). We estimate, using the decomposition given in (82),

$$\begin{aligned}\|L_{F,f}\varphi\|_{W^{s,t,q}_{p,\Theta,K}} &= \left\| \sum_{\mathcal{W}} L_{F,f_{|\mathcal{W}}} \varphi \right\|_{W^{s,t,q}_{p,\Theta,K}} = \left\| \left(\sum_{\sigma,n} 4^{c(\sigma)n} \left| \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{\mathcal{W}} L_{F,f_{|\mathcal{W}}} \varphi \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\leqslant 5 \left\| Q^{\mathrm{Op}}_{\hookrightarrow,\{\mathcal{W}\}} a \right\|_{L_p(\mathbb{R},\ell_2^c)} + 5 \left\| Q^{\mathrm{Op}}_{\hookrightarrow_0,\{\mathcal{W}\}} a \right\|_{L_p(\mathbb{R},\ell_2^c)} + 5 \sum_{\mathcal{W}} \left\| Q^{\mathrm{Op}}_{\not\hookrightarrow,\mathcal{W}} L_{F,f_{|\mathcal{W}}} Q^{\mathrm{Op}} b \right\|_{L_p(\mathbb{R},\ell_2^c)}.\end{aligned}$$

We conclude, using Lemma 4.13 and Lemma 4.16 together with the estimate given in (81). □

5. Asymptotics of horocycle averages

In this section, we assume $r \geqslant 2$ and topological mixing of the Anosov flow g_α. (Contact Anosov flows are topologically mixing [30, Theorem 3.6] and hence serve as examples for such Anosov flows g_α in the case $d = 3$.) In order to define the horocycle flow in Definition 5.1 below we assume that the stable dimension $d_- = 1$ and that the strong-stable distribution E_- is orientable. The stable manifolds of M with respect to the flow g_α are those (non-compact) Riemannian submanifolds which are tangent to E_-. As consequence of topological mixing, each of those stable manifolds is dense in M [35, p. 84].

5.1. Horocycle flows and integrals and main results (Theorem 5.7).

Definition 5.1 (Horocycle flow)**.** *A flow $h_\rho \colon M \to M$ in $\rho \in \mathbb{R}$ is called a stable horocycle flow if and only if for all $\rho \in \mathbb{R}$*

$$\partial_\rho h_\rho \in E_- \setminus \{0\}\,.$$

Remark 5.2 (Unit speed parametrization)**.** *By the Stable Manifold Theorem (see e.g. [31, Theorem 8.12]), there exists a parametrization of stable manifolds by the arc-length induced by the Riemannian metric on M. Since we assumed that E_- is orientable, this yields the unit speed parametrization of the horocycle flow (i.e. $|\partial_\rho h_\rho| \equiv 1$).*

Our main result, Theorem 5.7 provides a decomposition giving the T-asymptotics of the following horocycle integral:

Definition 5.3 (Horocycle integral)**.** *For all $\varphi \in C_X^{r-1}(M)$, for all $x \in M$ let*

$$\gamma_x(\varphi, T) := \int_0^T \varphi \circ h_\rho(x)\,\mathrm{d}\,\rho. \tag{83}$$

denote the horocycle integral of the horocycle flow h_ρ for the observable φ at base point x.

In Theorem 5.7 we reveal its connection to the eigendistributions of a weighted transfer operator for the Anosov flow $g_{-\alpha}$ introduced in Section 3.1, using renormalization dynamics to connect the stable flow with the Anosov flow. Results can be obtained for an unstable horocycle flow in an analogous way.

Definition 5.4 (Pointwise renormalization time)**.** *A map $\tau \colon \mathbb{R}^2 \times M \to \mathbb{R}$ which satisfies*

$$g_\alpha \circ h_\rho(x) = h_{\tau(\rho,\alpha,x)} \circ g_\alpha(x)\,, \qquad \forall \rho, \alpha \in \mathbb{R}\,, \forall x \in M, \tag{84}$$

is called a pointwise renormalization time for the stable flow h_ρ.

Remark 5.5. *This definition of the renormalization time τ is the same as used by Marcus (denoted by s^* in his notation) in [35, p.83] to study ergodic properties of the horocycle flow.*

Lemma 5.6 (Existence and uniqueness)**.** *A pointwise renormalization time exists and is unique.*

Proof. For every $x \in M$ and for every $\rho, \alpha \in \mathbb{R}$ we set $h_{\alpha,\rho}(x) := g_\alpha \circ h_\rho \circ g_{-\alpha}(x)$. By Definition 5.1 and the invariant splitting (2), we find $\partial_\rho h_{\alpha,\rho} \in E_{-,x} \setminus \{0\}$. Hence $h_{\alpha,\rho}(x)$ parametrizes the same stable manifold with respect to ρ as $h_\rho(x)$. If there were two different pointwise renormalization times τ, there would be $\rho_1 < \rho_2 \in \mathbb{R}$ such that $h_{\alpha,\rho}(x) = h_{\rho_1}(x) = h_{\rho_2}(x)$. By density of stable leaves and non-singularity of the flow h_ρ, there are no periodic points of h_ρ hence $\rho_1 = \rho_2$. □

Further properties of the renormalization time τ are given in Proposition 5.13 below. Assuming $\partial_\rho \tau(0, -\alpha, \cdot) \in C^{r-1}(M)$ for all $\alpha \geqslant 0$, we will consider the potential V defined by

$$V \equiv -\partial_\alpha \partial_\rho \tau(0, 0, \cdot) . \tag{85}$$

Then ϕ_α defined in (18) is just

$$\phi_\alpha := \partial_\rho \tau(0, -\alpha, \cdot). \tag{86}$$

It follows from (iv) in Lemma 5.18 below that for any $p \in [1, \infty]$, $t - s < r - 1$ and $s < 0 < q \leqslant t$ the spectral bound $\lambda_{\max} = \sup \Re\sigma(X + V)|_{W_p^{s,t,q}}$ for the generator satisfies

$$\lambda_{\max} = h_{\text{top}}. \tag{87}$$

In the special case of unit speed horocycle flow (see Remark 5.2) it holds (using Proposition 5.13 (viii) below)

$$\phi_\alpha = \det \mathrm{D}\, g_{-\alpha|E^-}.$$

Hence if the strong stable distribution E_- is C^1 (see Proposition 5.10 where this holds true if $d = 3$ under the contact assumption) and $r \geqslant 2$ then we find $\partial_\rho \tau(0, -\alpha, \cdot) \in C^1_X(M)$. In particular, our results apply to all C^1 time reparametrizations of the unit speed horocycle flow h_ρ (this is analogous to [22, Remark 2.4]).

The following theorem will be proved at the end of Section 5.3:

Theorem 5.7 (Expansion of horocycle integrals). *Let g_α be a topologically mixing C^r-Anosov flow, with $r \geqslant 2$, such that E_- is orientable and $d_- = 1$. Let μ be the unique Borel measure which is invariant by the horocycle flow h_ρ. Assume for all $\alpha \geqslant 0$*

$$\phi_\alpha := \partial_\rho \tau(0, -\alpha, \cdot) \in C^{r-1}(M).$$

Assume further that there exist $p \in [1, \infty]$, and $s < 0 < q \leqslant t$ with $t - s < r - 1$ such that, for the corresponding anisotropic space $W_p^{s,t,q}$ it holds $\lambda_{min} < \lambda_{max} = h_{top}$, with $\lambda_{min} = \lambda_{min}(t, s, p)$ from (56). Then, for all $x \in M$ and $T \geqslant 0$ there exist, for each $\lambda \in \sigma(X + V)|_{W_p^{s,t,q}}$ with $\Re\lambda > \lambda_{min}$, constants $c_{(\lambda,i,j)}(T, x) \in \mathbb{C}$ with

$$\sup_{T>0,\, x\in M} |c_{(\lambda,i,j)}(T, x)| < \infty\,, \ \forall 1 \leqslant i \leqslant n_\lambda\,,\ 1 \leqslant j \leqslant m_{\lambda,i},$$

such that, for any $\delta \in \mathbb{R}$ with

$$\max\{\lambda_{min}, 0\} \leqslant \delta < h_{top}$$

and any finite [5] subset Λ_δ of

$$\Sigma_\delta := \sigma(X + V)|_{W_p^{s,t,q}} \cap \{\lambda \in \mathbb{C} \mid \Re\lambda > \delta\}\,,$$

[5]Note that Lemma 4.10 and our choice of δ ensure that for any finite $b > 0$ the spectral box $\Lambda_\sigma(b) = \sigma(X + V)|_{W_p^{s,t,q}} \cap \{\Re\lambda > \delta,\ |\Im\lambda| \leqslant b\}$ is a finite set.

such that for all $\varphi \in C_X^{r-1}(M)$ *and all* $T \geqslant e$

$$\int_0^T \varphi \circ h_\rho(x)\,\mathrm{d}\rho = \gamma_x\left(\mathcal{D}_{(h_{top},1,1)}, T\right)\mu(\varphi)$$
$$+ \sum_{\substack{\lambda\in\Lambda_\delta \\ \Re\lambda<h_{top}}} \sum_{\substack{1\leqslant i\leqslant n_\lambda \\ 1\leqslant j\leqslant m_{\lambda,i}}} T^{\frac{\lambda}{h_{top}}} (\log T)^{j-1} c_{(\lambda,i,j)}(T,x)\,\mathcal{O}_{(\lambda,i,j)}(\varphi) + \mathcal{E}_{T,x,\Lambda_\delta}(\varphi),$$

where the dual eigendistributions $\mathcal{O}_{(\lambda,i,j)} \in D(X+V)'$ *are associated to the eigenvalue* λ *by Lemma 4.10 (see (67)), and where*

$$\lim_{T\to\infty} \frac{\gamma_x\left(\mathcal{D}_{(h_{top},1,1)}, T\right)}{T} = 1 \quad \textit{and} \quad \lim_{T\to\infty} \frac{\mathcal{E}_{T,x,\Lambda_\delta}(\varphi)}{T} = 0.$$

Moreover, if for some $c > 0$ *and some constant* $C_1 = C_1(\varphi, c, \Lambda_\delta) \geqslant 0$ *for all* $\alpha \geqslant 0$

$$\left\| \mathcal{L}_{\alpha,\phi_\alpha}\left(\sum_{\lambda\in\Lambda_\delta} \sum_{i=1}^{n_\lambda} \Pi_{\lambda,i}\varphi - \varphi \right) \right\|_{W_p^{s,t,q}} \leqslant C_1 e^{c\alpha}, \tag{88}$$

then there exists $C_2 > 0$ *such that*

$$\sup_{x\in M} \left|\mathcal{E}_{T,x,\Lambda_\delta}(\varphi)\right| \leqslant C_2\left(C_1 T^{\frac{c}{h_{top}}} + \|\varphi\|_{C^0} + 1\right).$$

If, in addition, Condition 4.11 holds for δ*, then* Σ_δ *is finite and, taking* $\Lambda_\delta = \Sigma_\delta$ *and assuming*

$$t - r + 2 \leqslant 0 < r - 2,$$

it holds $c = \delta + \epsilon$ *in (88) for all* $\epsilon > 0$ *and all* $\varphi \in W_p^{s,t,q}$*.*

Recall that if $\mathcal{D}_{(\lambda,i,j)} \in W_p^{s,t,q}$, for some λ with $\Re\lambda > \delta$, is a generalized eigenvector of the generator $X+V$ then for all $\Re\tilde{\lambda} > \delta$ we have that $\mathcal{O}_{\tilde{\lambda},\tilde{\imath},\tilde{\jmath}}(\mathcal{D}_{\lambda,i,j}) = 1$ if $\lambda = \tilde{\lambda}$, $i = \tilde{\imath}$, and $j = \tilde{\jmath}$, while $\mathcal{O}_{\tilde{\lambda},\tilde{\imath},\tilde{\jmath}}(\mathcal{D}_{\lambda,i,j})$ vanishes otherwise.

Remark 5.8. *The condition* $\lambda_{max} = h_{top}$ *is superficial although we show only* $\lambda_{max} = h_{top}$ *and unique simplicity under an additional vanishing assumption in Section 5.3. The proof of Theorem 5.7 however shows that the horocycle expansion sees only the part of the spectrum with real part below* h_{top} *and the eigendistribution* μ *which is associated to* h_{top}*.*
Recalling Remark 4.8, we find always $\lambda_{min} < h_{top}$ *if* $-s$ *and* t *can be taken to be* $1-\epsilon$ *for all* $\epsilon > 0$*. This is the case if the geodesic flow is* $C^{3-\epsilon}$ *for all* $\epsilon > 0$ *(e.g. the flow is of Zygmund type). If one knows then that the weight is* $C^{2-\epsilon}$ *the basic assumptions of Theorem 5.7 are all satisfied (an example is given in Proposition 5.10 below for* C^3 *contact Anosov flows when* $d = 3$*).*
Note that $\gamma_x\left(\mathcal{D}_{(h_{top},1,1)}, T\right)$ *is well-defined in the sense of distributions is part of the theorem. By unique ergodicity the expected principal term* $T\mu(\varphi)$ *is hidden by the term* $\gamma_x\left(\mathcal{D}_{(h_{top},1,1)}, T\right)\mu(\varphi)$ *as we ordered the expansion by the distributions* $\mathcal{O}_{\dots}(\varphi)$*. We can always write*

$$T\mu(\varphi) = \gamma_x(1,T)\,\mu(\varphi)$$

and use the expansion result on $\gamma_x(1,T)$ *again which shows that the leading order term is indeed what we expect. The other terms are modified by the contributions of* $\mathcal{O}_{\dots}(1)\,\mu(\varphi)$*. We make use of this in the following corollary.*

Assuming all conditions in the above theorem, this gives polynomial convergence of horocycle averages to the ergodic mean:

Corollary 5.9 (Polynomial convergence)**.** *Under the assumptions of Theorem 5.7 (including Condition 4.11 for δ and $t-r+2 \leqslant 0 < r-2$) then there exists $\epsilon > 0$ such that for all $\varphi \in C_X^{r-1}(M)$ there exists $C > 0$ such that*

$$\left| \frac{1}{T} \int_0^T \varphi \circ h_\rho (x) \,\mathrm{d}\rho - \mu(\varphi) \right| \leqslant C T^{-\epsilon},$$

where μ is the unique Borel measure which is invariant by the horocycle flow h_ρ.

Proof. We apply Theorem 5.7, using the assumption that Condition 4.11 holds for δ and that $t-r+2 \leqslant 0 < r-2$. Then there are only finitely many eigenvalues $\lambda \in \sigma(X+V)$ such that $\Re\lambda > \delta$ and the remainder term $\mathcal{E}_{T,x,\Lambda_\delta}(\varphi)$ is bounded from above by $T^{\frac{\delta}{h_{\mathrm{top}}}+\epsilon}$ for all $\epsilon > 0$. Hence all but one term in the expansion of the ergodic average decay like $T^{-\epsilon}$ for some $\epsilon > 0$. We finally bound the leading term in the expansion

$$\frac{1}{T}\gamma_x\left(\mathcal{D}_{(h_{\mathrm{top}},1,1)}, T\right)\mu(\varphi) - \mu(\varphi) = \frac{1}{T}\gamma_x\left(\mathcal{D}_{(h_{\mathrm{top}},1,1)} - 1, T\right)\mu(\varphi),$$

using again Theorem 5.7 as before, noting that $\mu(1) = \mu\left(\mathcal{D}_{h_{\mathrm{top}},1,1}\right) = 1$. ☐

We next discuss the assumptions of our main theorem and the corollary above. We first give sufficient conditions ensuring that $\partial_\rho \tau(0,-\alpha,\cdot) \in C^{r-1}$ and that there exist parameters in our anisotropic space giving $\lambda_{\min}(s,t,p) < h_{\mathrm{top}}$:

Proposition 5.10. *Let g_α be a C^3 contact Anosov flow on a closed Riemannian manifold M of dimension $d=3$ preserving a C^1 contact form and let the strong-stable distribution E_- be orientable. Then there exists a horocycle flow h_ρ such that $\partial_\rho \tau(0,-\alpha,\cdot) \in C^{r-1}$ for every $\alpha \geqslant 0$ and for any $r \in [2,3)$.*
Setting $-s = t = \frac{r-1}{2} - \frac{\epsilon}{2}$ for suitable $0 < \epsilon < \frac{r-1}{2}$, the constant $\lambda_{min}(s,t,p)$ is independent of p and can be taken arbitrary close to 0^+ while $t-r+2 \leqslant 0 < r-2$.

Proof. The contact assumption means that there is an invariant 1-form $\eta \in T^*M$ such that $\mu := \eta \wedge \mathrm{d}\eta \neq 0$ everywhere. By assumption η is C^1. Moreover η is annihilated on $E_+ + E_-$ and $\mu \in \wedge^3 T^*M$ is preserved by the flow. We use [29, Theorem 3.1] together with the comment on the relation between Zygmund and Hölder regularity to infer that the strong-stable distribution is C^{r-1} for all $r \in [2,3)$ if $d=3$. Hence for the horocycle flow given by the unit speed parametrization (and more general all of its C^{r-1} reparametrizations) we find $\partial_\rho \tau(0,-\alpha,\cdot) \in C^{r-1}$. By assumption the flow g_α preserves volume and $d=3$.
To see a gap between $\lambda_{\min} = \lambda_{\min}(s,t,p)$ and h_{top}, we may assume the unit speed parametrization of the horocycle flow h_ρ. It follows by Proposition 5.13 (viii) and Lemma 3.3 that for all C^{r-1} reparametrizations the resulting transfer operators are conjugate to each other.
Then it follows from Lemma 4.7 together with Proposition 5.13 (viii) that $\lambda_{\min}$ is independent of p and is arbitrary close to 0^+ for a suitable choice of s,t and r. Moreover, if we assume $0 < t \leqslant \frac{r-3}{2} + \epsilon$ we satisfy $t-r+2 \leqslant 0 < r-2$ since $\epsilon < \frac{r-1}{2}$. ☐

Second, we discuss Condition 4.11:

Remark 5.11. *Condition 4.11 was inspired by estimates of Dolgopyat [17], who was working with operators acting on symbolic spaces. This condition, replacing however our* $W_p^{s,t,q}$ *by other anisotropic Banach spaces, was proved by several authors [34, 43, 5, 23] for the generator* $X+V$*, associated to contact Anosov flows and* $V=0$ *the trivial potential, for which they also obtained the additional condition in Corollary 5.9.*

In the case of geodesic flows on compact surfaces of constant negative curvature, we find that V *is a constant, but the fact that our Banach space is different makes it difficult to apply the results of [34, 43, 5, 23] directly in order to establish Condition 4.11. We expect however that the condition holds and (as pointed out by Liverani and Butterley) can be obtained by exploiting e.g. [13, Remark 2.6].*

For non-constant potential V*, since Dolgopyat [17] obtained exponential decay of correlations for Gibbs measures with Hölder potentials, we expect that Condition 4.11 indeed holds also in our setting, in particular for compact surfaces of variable negative curvature (e.g. using an argument similar as for the proof in [18, Proposition 3.4]). (We warn the reader that the value of* δ *given by Dolgopyat-type arguments is usually very close to* λ_{max}*.)*

We end this subsection by a comparison of our main theorem and the results of Flaminio and Forni [21]: Let M be the unit tangent bundle of a compact hyperbolic Riemann surface. Let g_α be its unit speed geodesic flow and let vol be the canonical (invariant) volume form on M (which is also a measure of maximal entropy) and consider the unit speed horocycle flow which leaves vol invariant as well (hence $\mu = \mathrm{vol}$). Then $h_{\mathrm{top}} = 1$ because $\tau(\rho,\alpha,x) = \rho\exp(-\alpha)$ and $\mathcal{D}_{(1,1,1)} \equiv 1$ (hence $\gamma_x\left(\mathcal{D}_{(1,1,1)}, T\right) = T$). In the setting of Riemann surfaces, the possible Jordan blocks are known [21, Theorem 1.5]. In particular, the eigenvalue $h_{\mathrm{top}} = 1$ is simple, there are no other eigenvalues of real part equal to one, all eigenvalues with $\Re\lambda > 0$ are semi-simple, and there are only finitely many eigenvalues with $\Re\lambda > \frac{1}{2}$. Moreover, since the vector fields are constant, the regularity parameters $-s, t$ can be taken large enough such that $\lambda_{\min} < 0$. Hence we can take any $\delta \geqslant 0$ in Theorem 5.7, and we find, for any finite subset of Σ_δ containing 1,

$$\int_0^T \varphi\circ h_\rho(x)\,\mathrm{d}\,\rho = T\,\mathrm{vol}(\varphi) + \sum_{\lambda\in\Lambda_\delta\setminus\{1\}}\sum_{i=1}^{n_\lambda} T^\lambda c_{(\lambda,i,1)}(T,x)\,\mathcal{O}_{(\lambda,i,1)}(\varphi) + \mathcal{E}_{T,x,\Lambda_\delta}(\varphi),$$

where we can take $\Lambda_\delta = \Sigma_\delta$ if $\delta \geqslant \frac{1}{2}$, and where $c_{(\lambda,i,1)}$ and $\mathcal{E}_{T,x,\Lambda_\delta}$ satisfy the claims of Theorem 5.7 (with an additional $\log T$-factor if $\lambda = \frac{1}{2}$). In particular, if Condition 4.11 holds for some $\delta > \frac{1}{2}$ (see Remark 5.11) there exists $C > 0$ such that for all $\epsilon > 0$

$$\left|\mathcal{E}_{T,x,\Sigma_\delta}(\varphi)\right| \leqslant C T^{\delta+\epsilon}.$$

Note that we required $\delta \geqslant 0$ because for $\delta < 0$ we find no improvement of the remainder term (this comes the local bounds in Lemma 5.14). An analogous behavior is seen in the corresponding expansion of Flaminio–Forni in [21, Theorem 1.5]. However they are not limited to finite sets Λ_δ of eigenvalues (Faure–Tsujii do not seem to be limited either in [19]). Our methods, however, do not seem to allow to go beyond the first vertical line with infinitely many resonances in $\sigma(X+V)|_{W_p^{s,t,q}}$ in the expansion of the horocycle integral. (This could be a natural limitation, as discussed in [43, p.1497, below Theorem 1.1].)

5.2. **Weighted horocycle integrals, properties of τ, local bounds.** In order to use a smooth cutoff trick of Giulietti–Liverani to decompose $\gamma_x(\cdot, T)$ in Lemma 5.14 below, we need to consider weighted horocycle integrals: For all $\varphi \in C_X^{r-1}(M)$, for all compactly supported $w \in C(\mathbb{R}, \mathbb{C})$ and for all $x \in M$, let

$$\gamma_{w,x}(\varphi) := \int_{\mathbb{R}} w(\rho) \cdot (\varphi \circ h_\rho(x)) \, \mathrm{d}\rho. \tag{89}$$

denote the horocycle integral of the horocycle flow h_ρ for the observable φ at base point x with weight w.

For further purposes, it is useful to view $\gamma_{w,x}$ as a functional in the topological dual space of $W_p^{s,t,q}$ for weights w with compact support and sufficient differentiability:

Lemma 5.12. *Let $p \in [1, \infty]$ and let $0 < q \leqslant t < r-1$ and let $-r < s < 0$. Let $x \in M$. Then for some $C > 0$, for all C^{-s} maps $w \colon \mathbb{R} \to \mathbb{C}$ with compact support it holds*

$$\|\gamma_{w,x}\|_{W_p^{s,t,q} \to \mathbb{C}} \leqslant C \, |\mathrm{supp}\, w| \, \|w\|_{C^{-s}}.$$

Proof. We recall the partition of unity ϑ_ϖ and chart maps κ_ϖ, $\varpi \in \Omega$ (see Definition 3.6). We set for all $x \in M$, for all $\alpha \geqslant 0$ and for all $\varphi \in C_X^{r-1}(M)$

$$\begin{aligned} &y_{x,\varpi_1,\alpha}(\rho) := \kappa_{\varpi_1} \circ g_\alpha \circ h_\rho(x), \\ &\varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z) := (\vartheta_{\varpi_2} \cdot \phi_{-\alpha} \circ g_{-\alpha}) \circ \kappa_{\varpi_1}^{-1}(z) \cdot \int_{-\infty}^{\infty} w(\rho) \, \delta(z - y_{x,\varpi,\alpha}(\rho)) \, \mathrm{d}\rho, \\ &\varphi_{\varpi_1,\alpha}(z) := (\vartheta_{\varpi_1} \cdot \mathcal{L}_{\alpha,\phi_\alpha} \varphi_1) \circ \kappa_{\varpi_1}^{-1}(z). \end{aligned} \tag{90}$$

With this notation, recalling the weighted horocycle integral associated to Definition 5.1, we express for all $\alpha \geqslant 0$

$$\gamma_{w,x}(\varphi) = \sum_{\varpi_1,\varpi_2 \in \Omega} \int_{\mathbb{R}^d} \varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z) \cdot \varphi_{\varpi_1,\alpha}(z) \, \mathrm{d}z. \tag{91}$$

We set

$$c'(+) := -s, \quad c'(0) := -t, \quad c'(-) := -t.$$

We recall $\widetilde{\Psi}_{\sigma,n}$ defined in (24). We bound, using Plancherel's Theorem, Cauchy–Schwarz for the sum in σ and n, and twice Hölder's inequality with respect to z and α, respectively, for some constant $C > 0$

$$\begin{aligned} \alpha_0 \left|\gamma_{w,x}(\varphi)\right| &= \int_0^{\alpha_0} \left| \sum_{\varpi_1,\varpi_2} \int_{\mathbb{R}^d} \varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z) \cdot \varphi_{\varpi_1,\alpha}(z) \, \mathrm{d}z \right| \mathrm{d}\alpha \\ &\leqslant \int_0^{\alpha_0} \sum_{\varpi_1,\varpi_2} \left| \int_{\mathbb{R}^d} \sum_{\sigma,n} 2^{-c(\sigma)n} \widetilde{\Psi}_{\sigma,n}^{\mathrm{Op}} \varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z) \, 2^{c(\sigma)n} \Psi_{\sigma,n}^{\mathrm{Op}} \varphi_{\varpi_1,\alpha}(z) \, \mathrm{d}z \right| \mathrm{d}\alpha \\ &\leqslant C \sup_{\alpha,\varpi_1,\varpi_2} \left\| \left(\sum_{\sigma,n} 4^{c'(\sigma)n} \left| \widetilde{\Psi}_{\sigma,n}^{\mathrm{Op}} \varphi_{w,x,\varpi_1,\varpi_2,\alpha} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_{p^*}} \|\varphi\|_{W_p^{s,t,q}}, \end{aligned} \tag{92}$$

where $p^* := 1 - \frac{1}{p}$ is the Hölder conjugate of p. To conclude, it is enough to establish the following upper bounds for $\left\| \widetilde{\Psi}_{\sigma,n}^{\mathrm{Op}} \varphi_{w,x,\varpi_1,\varpi_2,\alpha} \right\|_{L_{p^*}}$:

(i) There exists a constant $C_1 > 0$ such that for every C^0 map $w\colon \mathbb{R} \to \mathbb{R}$, every $p \in [1,\infty]$, every $\sigma \in \{-,+,0\}$, $n \in \mathbb{N}$, every $\varpi_1, \varpi_2 \in \Omega$ it holds

$$\left\| \widetilde{\Psi}^{\mathrm{Op}}_{\sigma,n} \varphi_{w,x,\varpi_1,\varpi_2,\alpha} \right\|_{L_p} \leqslant C_1 \left|\operatorname{supp} w\right| \|w\|_{L_\infty}, \ \forall x \in M, \ \forall 0 \leqslant \alpha \leqslant \alpha_0.$$

(ii) There exists a constant $C_2 > 0$ such that for every $-r < s < 0$, for every $C^{|s|}$ map $w\colon \mathbb{R} \to \mathbb{R}$ with compact support, every $p \in [1,\infty]$, every $n \in \mathbb{N}$, every $\varpi_1, \varpi_2 \in \Omega$ it holds

$$\left\| \widetilde{\Psi}^{\mathrm{Op}}_{-,n} \varphi_{w,x,\varpi_1,\varpi_2,\alpha} \right\|_{L_p} \leqslant C_2 2^{sn} \left|\operatorname{supp} w\right| \|w\|_{C^{-s}}, \ \forall x \in M, \ \forall 0 \leqslant \alpha \leqslant \alpha_0.$$

We first show claim (i). We fix $w, \sigma, n, \varpi_1, \varpi_2, x$ and α. We let $J \subseteq \operatorname{supp} w$ be the maximal subset such that $y_{x,\varpi_1,\alpha|J}$ is well-defined. We note that J decomposes into a finite disjoint union, e.g. $J = \bigsqcup_{k=1}^{N} I_k$ for some $N \in \mathbb{N}$ and some real intervals I_k. In particular, since the flow h_ρ is non-singular and, in addition the manifold M is compact and each stable leaf is dense in M and $0 \leqslant \alpha \leqslant \alpha_0$, for some constant $C_1 > 0$, we have $|I_k| \leqslant C_1 \operatorname{diam} V_{\varpi_1}$ and $N \leqslant C_1 \frac{|\operatorname{supp} w|}{\operatorname{diam} V_{\varpi_1}}$. For every $z \in \mathbb{R}^d$ we estimate for some constants $C_2, \ldots, C_4 > 0$

$$\begin{aligned}
&\left|\varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z)\right| \leqslant C_2 \left| \int_J w(\rho)\, \delta\left(z - y_{x,\varpi_1,\alpha}(\rho)\right) \mathrm{d}\rho \right| \\
&= \left| \sum_{k=1}^{N} \int_{I_k} w(\rho)\, \delta\left(z - y_{x,\varpi_1,\alpha}(\rho)\right) \mathrm{d}\rho \right| = \left| \sum_{k=1}^{N} \sum_{\rho \in y^{-1}(z) \cap I_k} w(\rho) \left|\partial_\rho y_{x,\varpi_1,\alpha}(\rho)\right|^{-1} \right| \\
&\leqslant C_3 N \max_{\rho \in \operatorname{supp} w} \left| w(\rho) \left|\partial_\rho y_{x,\varpi_1,\alpha}(\rho)\right|^{-1} \right| \leqslant C_4 \left|\operatorname{supp} w\right| \|w\|_{L_\infty},
\end{aligned} \tag{93}$$

where we used in the last step non-singularity of h_ρ and $0 \leqslant \alpha \leqslant \alpha_0$. We conclude, using Young's inequality on $\left\| \widetilde{\Psi}^{\mathrm{Op}}_{\sigma,n} \varphi_{w,x,\varpi_1,\varpi_2,\alpha} \right\|_{L_p}$ together with the bound in (93).

We now show claim (ii). Again we fix $w, \sigma, n, \varpi_1, \varpi_2, x$ and α and set $y := y_{x,\varpi_1,\alpha}$. Analogously as in the proof of (i), we let $I_k \subseteq \mathbb{R}$, $1 \leqslant k \leqslant N$, be the N connected components of J for some $N \in \mathbb{N}$. For every $z \in \mathbb{R}^d$ we expand

$$\widetilde{\Psi}^{\mathrm{Op}}_{-,n} \varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z) = \frac{2^{dn}}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{\Psi}_{-,1}(\xi)\, e^{\mathrm{i}\, 2^n \xi (z - \widetilde{y})} \varphi(\widetilde{y})\, \mathrm{d}\xi\, \mathrm{d}\widetilde{y},$$

where we set

$$\varphi(\widetilde{y}) := \varphi_{w,x,\varpi_1,\varpi_2,\alpha}(\widetilde{y}).$$

We note that $\operatorname{supp} \varphi \subseteq y(J)$. In particular, we reparametrize $\widetilde{y} \in \operatorname{supp} \varphi$ by $\widetilde{y} = \widetilde{z}(\widetilde{\rho})$ for some diffeomorphism $z \in C^r$ and $\widetilde{\rho} \in \mathbb{R}$. We set $\mathrm{D}_{\widetilde{\rho}}(\cdot) := \mathrm{i}\, \partial_{\widetilde{\rho}} \frac{(\cdot)}{\partial_{\widetilde{\rho}} \xi \widetilde{z}}$. Since $\widetilde{z}(\mathbb{R})$ is a piece of a stable manifold in charts there exists a constant $C_3 > 0$ such that we have $|\partial_{\widetilde{\rho}} \xi \widetilde{z}(\widetilde{\rho})| \geqslant C_1 2^n$ for all ξ in $\operatorname{supp} \widetilde{\Psi}_{-,n}$ is essentially part of an unstable cone in charts by construction. We note that $\varphi \circ z$ is C^r. Using $\lfloor -s \rfloor$-times integration by parts (see Lemma A.3), followed by a regularized integration by parts with respect to $\widetilde{\rho}$ if $-s \notin \mathbb{N}$, respectively (see Lemma A.5 in which we take $d = 1$, $G = y$ and $L^{-1} = \epsilon = 2^{-n}$), this yields

$$\widetilde{\Psi}^{\mathrm{Op}}_{-,n} \varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z) = \frac{2^{((s+d)n)}}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \widetilde{\Psi}_{+,1}(\xi)\, e^{\mathrm{i}\, 2^n \xi (z - \widetilde{z}(\widetilde{\rho}))} \widetilde{\mathrm{D}}_{\widetilde{\rho}}^{-s} \widetilde{\varphi}(\widetilde{\rho})\, \mathrm{d}\xi\, \mathrm{d}\widetilde{\rho},$$

where

$$\widetilde{\mathrm{D}}_{\widetilde{\rho}}^{-s}\widetilde{\varphi} := \begin{cases} \mathrm{D}_{\widetilde{\rho}}^{-s}\left(\widetilde{\varphi}\circ\widetilde{z}\cdot\partial_{\widetilde{\rho}}\widetilde{z}\right), & \text{if } -s\in\mathbb{N} \\ 2^{(-s+\lfloor -s\rfloor)n}\left(\partial_{\widetilde{\rho}}\left(\frac{1}{2^n\xi\widetilde{z}}\widetilde{\widetilde{\varphi}}\right)_\epsilon + \partial_{\widetilde{\rho}}\xi\widetilde{z}\cdot\left(\frac{1}{\xi\widetilde{z}}\widetilde{\widetilde{\varphi}} - \left(\frac{1}{\xi\widetilde{z}}\widetilde{\widetilde{\varphi}}\right)_\epsilon\right)\right), & \text{if } -s\notin\mathbb{N} \end{cases},$$

$$\widetilde{\widetilde{\varphi}} := \mathrm{D}_{\widetilde{\rho}}^{\lfloor -s\rfloor}\left(\widetilde{\varphi}\circ\widetilde{z}\cdot\partial_{\widetilde{\rho}}\widetilde{z}\right),$$

and the ϵ-term is just the convolution $\left(\frac{1}{\xi y}\mathrm{D}_{\widetilde{\rho}}^{\lfloor -s\rfloor}\widetilde{\varphi}\right) * \nu_\epsilon$ with a C^∞ map ν_ϵ with $\operatorname{supp}\nu_\epsilon \subseteq (-\epsilon,\epsilon)$. Note that all derivatives of $\widetilde{\mathrm{D}}_{\widetilde{\rho}}^{-s}\widetilde{\varphi}$ in ξ are bounded in n, using Lemma A.5 and non-singularity of h_ρ and $0 \leqslant \alpha \leqslant \alpha_0$. We proceed analogously as in the proof of Lemma 3.3, integrating $(d+1)$-times by parts in ξ if $2^n\left|z-\widetilde{z}\left(\widetilde{\rho}\right)\right| > 1$ and conclude, using that $\operatorname{supp}\widetilde{\mathrm{D}}_{\widetilde{\rho}}^{-s}\widetilde{\varphi} \subseteq \operatorname{supp}\widetilde{\varphi}$ is bounded. □

We group below some properties of the pointwise renormalization. (Note in particular that Claim (xi) in Proposition 5.13, which will follow from [23, Remark C.4] of Giulietti–Liverani–Pollicott, will play a key part to estimate the spectral bound of $X+V$. Also, Claim (viii) in Proposition 5.13 shows that $\phi_\alpha = \partial_\rho\tau(0,-\alpha,\cdot)$ differs from the unit speed parametrization function by a multiplicative 1-coboundary.)

Proposition 5.13 (Properties of pointwise renormalization)**.** *Let τ be the renormalization time of a stable horocycle flow. For all $\rho,\alpha\in\mathbb{R}$ and for all $x\in M$ it holds:*

- *(i)* $\tau(0,\alpha,x) = 0$,
- *(ii)* $\tau(\rho,0,x) = \rho$,
- *(iii)* $\tau(\rho,\alpha_1+\alpha_2,x) = \tau(\tau(\rho,\alpha_2,x),\alpha_1,g_{\alpha_2}(x))$, *for all* $\alpha_1,\alpha_2\in\mathbb{R}$,
- *(iv)* $\tau(\rho_1+\rho_2,\alpha,x) = \tau\left(\rho_1,\alpha,h_{\rho_2}(x)\right) + \tau(\rho_2,\alpha,x)$, *for all* $\rho_1,\rho_2\in\mathbb{R}$,
- *(v)* $\partial_\rho\tau(\rho,\alpha,x) = \partial_\rho\tau(0,\alpha,h_\rho(x))$,
- *(vi)* $\partial_\rho\tau(0,\alpha_1,g_{\alpha_2}(x))\partial_\rho\tau(0,\alpha_2,x) = \partial_\rho\tau(0,\alpha_1+\alpha_2,x)$, *for all* $\alpha_1,\alpha_2\in\mathbb{R}$,
- *(vii)* $\tau(\rho,\alpha,x) = \gamma_x\left(\partial_\rho\tau(0,\alpha,\cdot),\rho\right)$,
- *(viii)* $\partial_\rho\tau(0,\alpha,\cdot) = \det\mathrm{D}\,g_{\alpha|E_-}\frac{(\partial_\rho h_0)^*(\partial_\rho h_0)}{(\partial_\rho h_0\circ g_\alpha)^*(\partial_\rho h_0\circ g_\alpha)}$ *where* $\partial_\rho h_0(x) := \partial_\rho h_{\rho|\rho=0}(x)$,
- *(ix)* $0 < \partial_\rho\tau(0,\alpha,x) < \infty$,
- *(x) if* $\alpha \geqslant 0$ *there exist* $C_1 > 0$ *and* $0<\theta<1$ *both independent of* α *and* x *such that* $\left\|\partial_\rho\tau\left(\cdot,\alpha,x\right)\right\|_{C^{r-1}} \leqslant C_1\theta^\alpha$,
- *(xi) if* $|\rho|\geqslant 1$ *and* $\alpha\geqslant 0$ *there exists* $C_2 \geqslant 1$ *independent of* ρ,α *and* x *such that*

$$C_2^{-1}\left|\rho\right|e^{h_{top}\alpha} \leqslant \left|\tau\left(\rho,-\alpha,x\right)\right| \leqslant C_2\left|\rho\right|e^{h_{top}\alpha}.$$

- *(xii) if* $\alpha\geqslant 0$ *such that* $\left|\tau\left(\rho,\alpha,x\right)\right| = c$ *for some* $c\geqslant 1$ *then there exists* $C_3\geqslant 1$ *independent of* ρ,α *and* x *such that*

$$C_3^{-1}ce^{h_{top}\alpha} \leqslant |\rho| \leqslant C_3ce^{h_{top}\alpha}.$$

Proof. We note that by Definition 5.1, Definition 5.4 and Lemma 5.6 the renormalization time is differentiable in ρ. Every stable leaf is dense in M hence together with non-singularity of the flow h_ρ it follows $h_{\rho_1}(x) = h_{\rho_2}(x) \Rightarrow \rho_1 = \rho_2$. Then Claim (i)-(ii) follow directly from (84). We deduce from (84)

$$\begin{aligned} h_{\tau(\rho,\alpha_1+\alpha_2,x)}\circ g_{\alpha_1+\alpha_2}(x) &= g_{\alpha_1+\alpha_2}\circ h_\rho(x) = g_{\alpha_1}\circ h_{\tau(\rho,\alpha_2,x)}\circ g_{\alpha_2}(x) \\ &= h_{\tau\left(\tau(\rho,\alpha_2,x),\alpha_1,g_{\alpha_2}(x)\right)}\circ g_{\alpha_1+\alpha_2}(x). \end{aligned}$$

This yields Claim (iii). Also from (84) we find

$$h_{\tau(\rho_1+\rho_2,\alpha,x)} \circ g_\alpha(x) = g_\alpha \circ h_{\rho_1+\rho_2}(x) = g_\alpha \circ h_{\rho_1} \circ g_{-\alpha} \circ g_\alpha \circ h_{\rho_2} \circ g_{-\alpha} \circ g_\alpha(x)$$
$$= h_{\tau(\rho_1,\alpha,h_{\rho_2}(x))} \circ h_{\tau(\rho_2,\alpha,x)} \circ g_\alpha(x).$$

This yields Claim (iv). Claim (v) and (vi), using Claim (i), follow by differentiating both sides in (iv) and (iii) at $\rho_1 = 0$ and $\rho = 0$, respectively.
Claim (vii) follows from (83) and (v).
To show Claim (viii), we take derivatives on both sides of (84) with respect to ρ

$$\mathrm{D}\, g_\alpha \partial_\rho h_\rho(x) = \partial_\rho \tau(\rho,\alpha,x) \cdot (\partial_\rho h_0) \circ h_{\tau(\rho,\alpha,x)} \circ g_\alpha(x). \tag{94}$$

Now we let $(\partial_\rho h_0)^* \in E_-^*$ be the canonical dual of $\partial_\rho h_0$. We calculate

$$(\partial_\rho h_0 \circ g_\alpha)^* (\mathrm{D}\, g_\alpha \partial_\rho h_0) = (\partial_\rho h_0 \circ g_\alpha)^* \left((g_\alpha)_* \partial_\rho h_0\right) = (g_\alpha)^* (\partial_\rho h_0 \circ g_\alpha)^* (\partial_\rho h_0)$$
$$= \det \left(\mathrm{D}\, g_{\alpha|E_-}\right)^* (\partial_\rho h_0)^* (\partial_\rho h_0) = \det \mathrm{D}\, g_{\alpha|E_-} (\partial_\rho h_0)^* (\partial_\rho h_0)\,. \tag{95}$$

We set $\rho = 0$ in (94) and conclude, using (95) and non-singularity of the horocycle flow.
Claim (ix) follows from (viii) together with the fact $\lim_{\alpha\to 0} \det \mathrm{D}\, g_{\alpha|E_-} = 1$ and compactness of M.
In order to show (x), we note first, since $r \geqslant 2$, using Claim (v) and the cocycle property (vi),

$$\partial_\rho \tau(\rho,\alpha,x) = \partial_\rho \tau(0,\alpha,h_\rho(x)) = \exp - \int_0^\alpha V \circ g_{\tilde\alpha} \circ h_\rho(x)\, \mathrm{d}\,\tilde\alpha,$$

where $V \coloneqq -\partial_\alpha \partial_\rho \tau(0,0,\cdot) \in C^{r-1}$. Therefore it holds, using the equality in (84),

$$\partial_\rho^2 \tau(\rho,\alpha,x) = -\partial_\rho \tau(\rho,\alpha,x) \cdot \partial_\rho \int_0^\alpha V \circ g_{\tilde\alpha} \circ h_\rho(x)\, \mathrm{d}\,\tilde\alpha$$
$$= -\partial_\rho \tau(\rho,\alpha,x) \cdot \int_0^\alpha \partial_\rho \tau(\rho,\tilde\alpha,x) \cdot (\mathrm{D}\, V \partial_\rho h_0) \circ g_{\tilde\alpha} \circ h_\rho(x)\, \mathrm{d}\,\tilde\alpha, \tag{96}$$

where $|\partial_\rho \tau(\rho,\alpha,x)| \leqslant C\theta^\alpha$ for some $0 < \theta < 1$ and $C_1 > 0$ both independent of α, ρ and x by (viii). Hence there is $C_2 = C_2(V) > 0$ such that $\left|\partial_\rho^2 \tau(\rho,\alpha,x)\right| \leqslant C_2\theta^\alpha$. By induction, all derivatives $\partial_\rho^k \tau(\rho,\alpha,x)$, where $k \in \mathbb{N}$, depend only on $\partial_\rho \tau(\rho,\alpha,x)$ (and k and derivatives of V which are independent of α) and so does the Hoelder norm $\|\partial_\rho \tau(\rho,\alpha,x)\|_{C^{r-1}}$. Since $r \geqslant 2$ the Hoelder coefficicent of $\partial_\rho \tau(\cdot,\alpha,x)$ is bounded by $\left\|\partial_\rho^2 \tau(\cdot,\alpha,x)\right\|_{C^0}$ and we conclude.
Claim (xi) for $\rho \geqslant 1$ and $\alpha \leqslant 0$ follows from [23, Lemma C.1] and [23, Remark C.4] (recall that g_α is transitive) in which we replace W with a manifold which contracts in forward time. To this end we set $W_x \coloneqq h_{[0,1]}(x)$ for every $x \in M$. Since the stable flow is non-singular, the stable manifold W_x is of bounded length (from above and below) for all $x \in M$. We estimate, using Proposition 5.13 (viii) for the first and [23, Remark C.4] for the last inequality, with constants $C_3, \dots, C_6 > 0$ independent of ρ, α, x

$$\tau(\rho,-\alpha,x) \leqslant C_3 \int_0^\rho \det \mathrm{D}\, g_{-\alpha|E_-} \circ h_\rho(x)\, \mathrm{d}\,\rho \leqslant C_4 \rho \int \det \mathrm{D}\, g_{-\alpha|E_-}\, \mathrm{d}\, W_x$$
$$\leqslant C_5 \rho \operatorname{vol}(g_{-\alpha}(W_x)) \leqslant C_6 \rho e^{h_{\mathrm{top}}\alpha}.$$

A lower bound for $\tau(\rho,-\alpha,x)$ is obtained in an analogous way, using the last statement in [23, Lemma C.1]. We conclude for all $|\rho| \geqslant 1$, noting that $\tau(-\rho,\alpha,x) =$

$-\tau(\rho,\alpha,h_{-\rho}(x))$, using Claims (iv) and (i).
Claim (xii) follows from Proposition 5.13 (viii), and the following equality which follows from Claim (iii)

$$\rho = \tau(\tau(\rho,\alpha,x), -\alpha, g_\alpha(x)) = \tau(c,-\alpha,g_\alpha(x)).$$

□

We shall use in the next two lemmas the following key identity for the horocyle integral (89)

$$\gamma_{w,x}(\varphi) = \gamma_{w\circ\tau(\cdot,-\alpha,g_\alpha(x)),g_\alpha(x)}\left(\mathcal{L}_{\alpha,\partial_\rho\tau(0,-\alpha,\cdot)}\varphi\right), \ \forall\alpha \geqslant 0. \tag{97}$$

To check the above identity, using (84) and Proposition 5.13 (iii), (v)-(vi), just notice that for all $\alpha \in \mathbb{R}$

$$\begin{aligned}
\gamma_{w,x}(\varphi) &= \int_{-\infty}^{\infty} w(\rho)\cdot\varphi\circ g_{-\alpha}\circ g_\alpha\circ h_\rho(x)\,\mathrm{d}\rho \\
&= \int_{-\infty}^{\infty} w(\rho)\cdot\varphi\circ g_{-\alpha}\circ h_{\tau(\rho,\alpha,x)}\circ g_\alpha(x)\,\mathrm{d}\rho \\
&= \int_{-\infty}^{\infty} w(\tau(\rho,-\alpha,g_\alpha(x)))\cdot\varphi\circ g_{-\alpha}\circ h_\rho\circ g_\alpha(x)\cdot\partial_\rho\tau(\rho,-\alpha,g_\alpha(x))\,\mathrm{d}\rho \\
&= \int_{-\infty}^{\infty} w(\tau(\rho,-\alpha,g_\alpha(x)))\cdot(\partial_\rho\tau(0,-\alpha,\cdot)\cdot\varphi\circ g_{-\alpha})\circ h_\rho\circ g_\alpha(x)\,\mathrm{d}\rho \\
&= \gamma_{w\circ\tau(\cdot,-\alpha,g_\alpha(x)),g_\alpha(x)}(\partial_\rho\tau(0,-\alpha,\cdot)\cdot\varphi\circ g_{-\alpha}) \qquad (98)\\
&= \gamma_{w\circ\tau(\cdot,-\alpha,g_\alpha(x)),g_\alpha(x)}\left(\mathcal{L}_{\alpha,\partial_\rho\tau(0,-\alpha,\cdot)}\varphi\right), \quad \text{if } \alpha \geqslant 0.
\end{aligned}$$

We now state upper bounds for $|\gamma_x(\varphi,T)|$ similar to the results in [21, Lemma 5.16]. The prove uses the analogue of the smooth cutoff used by Giulietti–Liverani [22] but uses a different construction of the local decomposition of $\gamma_x(\varphi,T)$.

Lemma 5.14 (Local bounds)**.** *For every $T>0$ and for every $x \in M$ there exists $w \in C^{-s}(\mathbb{R},[0,1])$ such that for every $\varphi \in W_p^{s,t,q}$, where $p \in [1,\infty]$ and $s<0<q\leqslant t<r-1+s$, the following holds:*

(i) There exists $C_1>0$ independent of T, x and φ such that

$$|\gamma_x(\varphi,T) - \gamma_{w,x}(\varphi)| \leqslant C_1\|\varphi\|_{C^0}.$$

Moreover, if $-s < \frac{\theta_{min}}{\theta_{max}}$, where for some $0<\theta_{min}<\theta_{max}$, for some $C_0 \geqslant 1$ and for all $\alpha \geqslant 0$ it holds

$$C_0^{-1}e^{-\theta_{max}\alpha} \leqslant \inf_{x\in M}\partial_\rho\tau(\cdot,\alpha,x) \leqslant \sup_{x\in M}\partial_\rho\tau(\cdot,\alpha,x) \leqslant C_0e^{-\theta_{min}\alpha}, \tag{99}$$

then for some $C_2>0$ independent of T,x and φ it holds

$$|\gamma_x(\varphi,T)| \leqslant C_2\max\left\{T, T^{\frac{\theta_{min}}{\theta_{max}}+s\frac{\theta_{max}}{\theta_{min}}}\right\}\|\varphi\|_{W_p^{s,t,q}}.$$

(ii) If for some $\tilde\varphi \in W_p^{s,t,q}$ it holds for all $\alpha \geqslant 0$, for some $\lambda \in \mathbb{R}$, $c \geqslant 0$ and some $C = C(\lambda,c,\tilde\varphi) > 0$

$$\|\partial_\rho\tau(0,-\alpha,\cdot)\cdot\tilde\varphi\circ g_{-\alpha}\|_{W_p^{s,t,q}} \leqslant Ce^{\lambda\alpha}\max\{1,|\alpha|^c\}, \tag{100}$$

then there exists $C_3 = C_3(\lambda, c) > 0$ independent of T, x and $\widetilde{\varphi}$ such that

$$|\gamma_{w,x}(\widetilde{\varphi})| \leqslant CC_3 \begin{cases} T^{\frac{\lambda}{h_{top}}} \left(\max\{1, \log T\}\right)^c, & \text{if } \lambda > 0 \\ \min\{1,T\} \left(\max\{1, \log T\}\right)^{c+1}, & \text{if } \lambda = 0 \\ \min\{1,T\} \left(\max\{1, \log T\}\right)^c, & \text{if } \lambda < 0 \end{cases}.$$

Moreover, if the bound in (100) holds for all $\alpha \in \mathbb{R}$ with $\lambda > 0$ then

$$|\gamma_x(\widetilde{\varphi}, T) - \gamma_{w,x}(\widetilde{\varphi})| \leqslant CC_3.$$

Proof. Let $x \in M$, $T > 0$, $0 < \epsilon \leqslant \frac{1}{4}$. We define $\beta_k^+, \beta_k^- \in \mathbb{R}$ for every $k \in \mathbb{N}$ by

$$\tau\left(T, \beta_0^+, x\right) = \frac{1}{\epsilon} \quad \text{and} \quad \tau\left(\tau\left(\frac{1}{\epsilon}, -\beta_k^+, g_{\beta_k^+}(x)\right), \beta_{k-1}^+, x\right) = 1,$$

$$\beta_0^- := \beta_0^+ \quad \text{and} \quad \tau\left(\tau\left(-\frac{1}{\epsilon}, -\beta_k^-, g_{\beta_k^-} \circ h_T(x)\right), \beta_{k-1}^-, h_T(x)\right) = -1. \tag{101}$$

If $T > 1$ we assume $\beta_0^+ > 0$, if $T = 1$ we assume $\beta_0^+ = 0$ and if $T < 1$ we assume $\beta_0^+ < 0$. This is justified since $\tau(T, 0, x) = T$ and by Proposition 5.13 (xi). Since $\epsilon < 1$ we may assume without loss of generality for all $k \in \mathbb{N}$

$$\beta_k^+ < \beta_{k-1}^+ \quad \text{and} \quad \beta_k^- < \beta_{k-1}^-.$$

Combining the definitions in (101) with (iii) and (xi) in Proposition 5.13, we find $C_1 \geqslant 1$ independent of ϵ, x, k and T such that for all $k \in \mathbb{N}$ it holds

$$C_1^{-1}\epsilon \leqslant e^{h_{\text{top}}\left(\beta_k^+ - \beta_{k-1}^+\right)}, e^{h_{\text{top}}\left(\beta_k^- - \beta_{k-1}^-\right)} \leqslant C_1\epsilon. \tag{102}$$

If $\beta_0^+ \geqslant 0$ it follows for all $k \in \mathbb{Z}_{\geqslant 0}$, using the upper bounds in (102) and Proposition 5.13 (xii) on $\tau\left(T, \beta_0^+, x\right) = \frac{1}{\epsilon}$,

$$T\left(C_1^{-1}\epsilon\right)^{k+1} \leqslant e^{h_{\text{top}}\beta_k^+} \leqslant T\left(C_1\epsilon\right)^{k+1}. \tag{103}$$

If $\beta_0^+ < 0$ it holds for all $k \in \mathbb{Z}_{\geqslant 0}$

$$\left(C_0^{-1}T\right)^{\frac{h_{\text{top}}}{\theta_{\min}}} \left(C_1^{-1}\epsilon\right)^k \leqslant e^{h_{\text{top}}\beta_k^+} \leqslant \left(C_0 T\right)^{\frac{h_{\text{top}}}{\theta_{\max}}} \left(C_1\epsilon\right)^k, \tag{104}$$

where C_0, $\theta_{\min}$ and $\theta_{\max}$ are from the assumptions in (99). By symmetry we obtain analogous bounds for β_k^-. We let $w^+, w^- \in C^\infty(\mathbb{R}, [0,1])$ such that

$$w^- = w^+ \circ (T - \cdot), \quad w^+_{|\left(\frac{1}{2\epsilon}, \infty\right)} \equiv 1 \quad \text{and} \quad w^+_{|\left(-\infty, \frac{1}{4\epsilon}\right)} \equiv 0.$$

We set

$$w_0 := w^+ \circ \tau\left(\cdot, \beta_0^+, x\right) \cdot w^- \circ \left(T + \tau\left(\cdot - T, \beta_0^-, h_T(x)\right)\right),$$

and we set for all $k \in \mathbb{N}$

$$\begin{aligned} w_k^+ &:= w^+ \circ \tau\left(\cdot, \beta_k^+, x\right) - w^+ \circ \tau\left(\cdot, \beta_{k-1}^+, x\right), \\ w_k^- &:= w^- \circ \left(T + \tau\left(\cdot - T, \beta_k^-, h_T(x)\right)\right) - w^- \circ \left(T + \tau\left(\cdot - T, \beta_{k-1}^-, h_T(x)\right)\right), \\ w_k &:= w_k^+ + w_k^-. \end{aligned}$$

We let $N \in \mathbb{Z}$ for now be arbitrary. If $N \geqslant 0$ we set

$$\begin{aligned} w := \sum_{k=0}^N w_k = w_0 &+ w^+ \circ \tau\left(\cdot, \beta_N^+, x\right) - w^+ \circ \tau\left(\cdot, \beta_0^+, x\right) \\ &+ w^- \circ \left(T + \tau\left(\cdot - T, \beta_N^-, h_T(x)\right)\right) - w^- \circ \left(T + \tau\left(\cdot - T, \beta_0^-, h_T(x)\right)\right). \end{aligned}$$

If $N < 0$ we put $w \equiv 0$. Since $\epsilon \leqslant \frac{1}{4}$, it follows directly from the definitions of β_k^+ and β_k^- in (101) that for all $k \in \mathbb{N}$

$$\tau\left(\frac{1}{2\epsilon}, -\beta_k^+, g_{\beta_k^+}(x)\right) \leqslant \tau\left(\frac{1}{4\epsilon}, -\beta_{k-1}^+, g_{\beta_{k-1}^+}(x)\right) \quad \text{and}$$
$$\tau\left(\frac{1}{-2\epsilon}, -\beta_k^-, g_{\beta_k^-} \circ h_T(x)\right) \geqslant \tau\left(-\frac{1}{4\epsilon}, -\beta_{k-1}^-, g_{\beta_{k-1}^+} \circ h_T(x)\right).$$

Together with the assumptions on the supports of w^+ and w^-, we find if $N \geqslant 0$

$$\begin{aligned} \operatorname{supp}\left(1_{|[0,T]} - w\right) \subseteq & \left(0, \tau\left(\frac{1}{2\epsilon}, -\beta_N^+, g_{\beta_N^+}(x)\right)\right) \\ & \cup \left(T + \tau\left(-\frac{1}{2\epsilon}, -\beta_N^-, g_{\beta_N^-} \circ h_T(x)\right), T\right). \end{aligned}$$

We put

$$N := \left\lfloor -\log\left(C_1^{-1} T\right) / \log\left(C_1 \epsilon\right)\right\rfloor.$$

Hence if $N < 0$ then T is bounded and if $N \geqslant 0$ then $\beta_N^+, \beta_N^- \geqslant 0$. The latter follows from the lower bounds in (103). Therefore the first statement in Claim (i) follows immediately, using in addition Proposition 5.13 (xii) and the upper bounds in (103).

Using Proposition 5.13 (iii) and also Proposition 5.13 (iv) in the last equality for $\widetilde{w}_0$, we find for all $k \in \mathbb{N}$

$$\begin{aligned} \widetilde{w}_k^+ :=& w_k^+ \circ \tau\left(\cdot, -\beta_k^+, g_{\beta_k^+}(x)\right) = w^+ - w^+ \circ \tau\left(\cdot, \beta_{k-1}^+ - \beta_k^+, g_{\beta_k^+}(x)\right), \\ \widetilde{w}_k^- :=& w_k^-(\cdot + T) \circ \tau\left(\cdot, -\beta_k^-, g_{\beta_k^-} \circ h_T(x)\right) \\ =& w^-(T + \cdot) - w^- \circ \left(T + \tau\left(\cdot, \beta_{k-1}^- - \beta_k^-, g_{\beta_k^-} \circ h_T(x)\right)\right), \\ \widetilde{w}_0 :=& w_0 \circ \tau\left(\cdot, -\beta_0^+, g_{\beta_0^+}(x)\right) \\ =& w^+ \cdot w^- \circ \left(T + \tau\left(\tau\left(\cdot, -\beta_0^+, g_{\beta_0^+}(x)\right) - T, \beta_0^+, h_T(x)\right)\right) \\ =& w^+ \cdot w^- \circ \left(T - \frac{1}{\epsilon} + \cdot\right). \end{aligned}$$

For this construction it holds for all $k \in \mathbb{N}$

$$\operatorname{supp} \widetilde{w}_k^+, -\operatorname{supp} \widetilde{w}_k^-, \operatorname{supp} \widetilde{w}_0 \subseteq \left[0, \frac{C_1^2}{2} 1/\epsilon^2\right]. \tag{105}$$

Since $\partial_\rho \tau(\cdot, \alpha, x) \in C^{r-1}$ for all $\alpha \in \mathbb{R}$ and all $x \in M$ it follows for some constant $C_3 > 0$ for all $k \in \mathbb{N}$, all $x \in M$ and all $T > 0$, using Proposition 5.13 (viii) and the bounds in (102),

$$\left\|\widetilde{w}_k^+\right\|_{C^r}, \left\|\widetilde{w}_k^-\right\|_{C^r}, \left\|\widetilde{w}_0\right\|_{C^r} \leqslant C_3. \tag{106}$$

We note

$$\gamma_{w,x}(\varphi) = \gamma_{w \circ (\cdot + T), h_T(x)}(\varphi).$$

Assuming $N \geqslant 0$, together with the equality in (97), we find the local decomposition[6] for all $\varphi \in W_p^{s,t,q}$

$$\gamma_{w,x}(\varphi) = \gamma_{\tilde{w}_0, g_{\beta_0^+}(x)}\left(\mathcal{L}_{\beta_0^+, \phi_{\beta_0^+}}\varphi\right)$$

$$+ \sum_{k=1}^{N} \gamma_{\tilde{w}_k^+, g_{\beta_k^+}(x)}\left(\mathcal{L}_{\beta_k^+, \phi_{\beta_k^+}}\varphi\right) + \gamma_{\tilde{w}_k^-, g_{\beta_k^-}\circ h_T(x)}\left(\mathcal{L}_{\beta_k^-, \phi_{\beta_k^-}}\varphi\right). \tag{107}$$

Using the bound in Lemma 5.12 with the bounds in (106), and using the assumption in (100) for some $\tilde{\varphi} \in W_p^{s,t,q}$

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\tilde{\varphi}\|_{W_p^{s,t,q}} = \|\partial_\rho \tau(0,-\alpha,\cdot)\cdot \tilde{\varphi}\circ g_{-\alpha}\|_{W_p^{s,t,q}} \leqslant Ce^{\lambda\alpha}\max\{1,|\alpha|^c\}, \tag{108}$$

and the bounds in (103), we estimate the right-hand side in the decomposition in (107) for some constant $C_4 = C_4(c) > 0$, recalling that T is uniformly bounded from below if $N \geqslant 0$, and conclude the first statement in Claim (ii):

$$|\gamma_{w,x}(\varphi)| \leqslant CC_4 T^{\frac{\lambda}{h_{\text{top}}}} \sum_{k=0}^{N} (C_1\epsilon)^{k\frac{\lambda}{h_{\text{top}}}} \left(\max\{1,(k+1)|\log(C_1\epsilon)|,\log T\}\right)^c. \tag{109}$$

If $N < 0$ then $w \equiv 0$ and T is uniformly bounded from above and we conclude as well. To see the second statements in Claims (i)-(ii), we recall that the construction of the functions w_k is valid for every $T > 0$ and hence

$$\sum_{k=0}^{\infty} w_k = 1_{|(0,T)}.$$

Since for all $\varphi \in C_X^{r-1}(M)$ it holds

$$\gamma_x(\varphi,T) - \gamma_{\sum_{k=0}^{\infty} w_k, x}(\varphi) = 0,$$

we find by density for all $\varphi \in W_p^{s,t,q}$

$$\gamma_x(\varphi,T) = \gamma_{\sum_{k=0}^{\infty} w_k, x}(\varphi).$$

It holds

$$\operatorname{supp} w_0 \subseteq [0,T].$$

Comparing with the supports in (105), together with the bounds in (103) and (104), we find some $C_5 > 0$ independent of k, T, x and φ such that for all $k \in \mathbb{N}$, if $\beta_k^+ \geqslant 0$ and $\beta_k^- \geqslant 0$,

$$\left(T - \operatorname{supp} w_k^-\right), \operatorname{supp} w_k^+ \subseteq \left[0, C_5 (C_1\epsilon)^k T\right],$$

respectively, and if $\beta_k^+ < 0$ and $\beta_k^- < 0$,

$$\left(T - \operatorname{supp} w_k^-\right), \operatorname{supp} w_k^+ \subseteq \left[0, C_5 T^{\frac{\theta_{\min}}{\theta_{\max}}} \left((C_1\epsilon)^k\right)^{\frac{\theta_{\min}}{h_{\text{top}}}}\right].$$

Moreover, we find for some constant $C_5 \geqslant 1$ for all $\rho_1, \rho_2 \in \mathbb{R}$, all $\alpha, -s \geqslant 0$ and all $x \in M$, using Proposition 5.13 (iv) and the assumption of the upper bound for $\partial_\rho \tau$ in (99),

$$(\tau(\rho_1,\alpha,x) - \tau(\rho_2,\alpha,x))^{-s} = \tau(\rho_1-\rho_2,\alpha,h_{\rho_2}(x))^{-s} \leqslant C_5^{-s}(\rho_1-\rho_2)^{-s} e^{s\theta_{\min}\alpha}.$$

[6] This is analogous to the decomposition in [23, Lemma 3.1]. The main difference to our decomposition is that we use a more explicit construction of the smoothing functions.

If $\alpha \leqslant 0$ it holds analogously, now using the lower bound for $\partial_\rho \tau$ in (99),

$$\left(\tau\left(\rho_1, \alpha, x\right) - \tau\left(\rho_2, \alpha, x\right)\right)^{-s} = \tau\left(\rho_1 - \rho_2, \alpha, h_{\rho_2}\left(x\right)\right)^{-s} \leqslant C_5^{-s}\left(\rho_1 - \rho_2\right)^{-s} e^{s\theta_{\max}\alpha}.$$

Since $0 < -s < 1$ and $0 < \theta_{\min} \leqslant \theta_{\max}$ it holds for some constant $C_6 > 0$ independent of T, x and φ, using the lower bounds in (103)-(104), for all $k \in \mathbb{N}$ and for all $T > 0$,

$$\left\|w_k\right\|_{C^{-s}} \leqslant C_6 \max\left\{1, T^{s\frac{\theta_{\max}}{\theta_{\min}}}\left(\left(C_1\epsilon\right)^k\right)^{s\frac{\theta_{\max}}{h_{\text{top}}}}\right\}.$$

Then we estimate for every $\varphi \in W_p^{s,t,q}$, using Lemma 5.12 and $-s < \frac{\theta_{\min}}{\theta_{\max}}$, for some constants $C_7, C_8 > 0$ independent of T, x, w and φ

$$\begin{aligned}
&\left|\gamma_x\left(\varphi, T\right)\right| \leqslant \lim_{n\to\infty}\left|\sum_{k=0}^{n}\gamma_{w_k,x}\left(\varphi\right)\right| \\
&\leqslant C_7\sum_{k=0}^{\infty}\max\left\{T, T^{\frac{\theta_{\min}}{\theta_{\max}}+s\frac{\theta_{\max}}{\theta_{\min}}}\right\}\left(\left(C_1\epsilon\right)^k\right)^{\frac{\theta_{\min}}{h_{\text{top}}}}\max\left\{1, \left(\left(C_1\epsilon\right)^k\right)^{s\frac{\theta_{\max}}{h_{\text{top}}}}\right\}\left\|\varphi\right\|_{W_p^{s,t,q}} \\
&\leqslant C_8\max\left\{T, T^{\frac{\theta_{\min}}{\theta_{\max}}+s\frac{\theta_{\max}}{\theta_{\min}}}\right\}\left\|\varphi\right\|_{W_p^{s,t,q}}.
\end{aligned}$$

This yields the second statement in Claim (i). On the other hand, using the equality in (98) and assuming $N > 0$, we find,

$$\begin{aligned}
&\gamma_x\left(\tilde{\varphi}, T\right) - \sum_{k=0}^{N}\gamma_{w_k,x}\left(\tilde{\varphi}\right) = \sum_{k=N+1}^{\infty}\gamma_{\tilde{w}_k^+, g_{\beta_k^+}(x)}\left(\partial_\rho\tau\left(0, -\beta_k^+, \cdot\right)\cdot\tilde{\varphi}\circ g_{-\beta_k^+}\right) \\
&+ \sum_{N+1}^{\infty}\gamma_{\tilde{w}_k^-, g_{\beta_k^-}\circ h_T(x)}\left(\partial_\rho\tau\left(0, -\beta_k^-, \cdot\right)\cdot\tilde{\varphi}\circ g_{-\beta_k^-}\right).
\end{aligned}$$

Then we proceed analogously as for the bound in (109), now using the upper bounds in (103) and the assumption in (108) for all $\alpha \in \mathbb{R}$ and some $\lambda > 0$, $c \geqslant 0$ (recall that $T\left(C_1\epsilon\right)^N$ is bounded from above). If $N \leqslant 1$ then $T^\lambda\left|\log T\right|^c$ is bounded from above and we conclude as well, now using the upper bounds in (104). □

Remark 5.15. *The second statement in Lemma 5.14 (i) can be used to avoid the $\|\varphi\|_{C^0}$-term in the bound of the error term in Theorem 5.7. However the required range for s may not be very large (except in the case of constant vector fields). The second statement in Lemma 5.14 (ii) is free from an additional condition on s. We use it in the following subsection in the proof of Lemma 5.18 (v) and Theorem 5.7. Both statements give also bounds for all values $T > 0$ which seems to be new.*

5.3. **Showing $\lambda_{\max} = h_{\text{top}}$ and Theorem 5.7.** In this subsection we shall prove Theorem 5.7. First, we state and prove two lemmas which will imply that $\lambda_{max} = h_{\text{top}}$, assuming $\lambda_{\min} < \lambda_{\max}$, is a simple eigenvalue and that λ_{max} is uniquely attained.

We remind the reader that uniqueness and simplicity of the spectral bound is known to hold (see [16, Lemma 5.1], [15]) for the spectrum of mixing Anosov flows (which are not necessarily contact), but for different anisotropic spaces, and only for the potential V given by the Jacobian of the flow (and associated to the SRB measure).

For the sake of the next two lemmas we have to introduce the following condition[7]:

Condition 5.16 (Strong vanishing)**.** *Let $0 < t, q, -s < r-1$ and let $p \in [1, \infty]$. Let $\varphi_\alpha \in W_p^{s,t,q}$ for all $\alpha \geqslant 0$ such that $\|\varphi_\alpha\|_{W_p^{s,t,q}} = 1$ and*

$$\limsup_{\alpha \to \infty} e^{-h_{top}\alpha} \|\mathcal{L}_{\alpha,\phi_\alpha} \varphi_\alpha\|_{W_p^{s,t,q}} > 0.$$

If for some $-s \leqslant -s'$ for all $x \in M$ and all $w \in C_0^{-s'}(\mathbb{R})$

$$\lim_{\alpha \to \infty} \gamma_{w,x} \left(\frac{\mathcal{L}_{\alpha,\phi_\alpha} \varphi_\alpha}{\|\mathcal{L}_{\alpha,\phi_\alpha} \varphi_\alpha\|_{W_p^{s,t,q}}} \right) = 0$$

then $\lim_{\alpha \to \infty} \|\varphi_\alpha\|_{W_p^{s,t,q}} = 0$.

We give the upper bound on the spectral radius:

Lemma 5.17 (Upper bound on the spectral radius)**.** *Let $0 < t, q, -s < r-1$ and let $p \in [1, \infty]$. For all $x \in M$ and all $\alpha \geqslant 0$ let $\partial_\rho \tau(\cdot, -\alpha, x) \in C^{r-1}(\mathbb{R}, M)$. Under Condition 5.16, With the choice $\phi_\alpha = \partial_\rho \tau(0, -\alpha, \cdot)$ for some constant $C > 0$ it holds for all $\alpha \geqslant 0$*

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\|_{W_p^{s,t,q} \to W_p^{s,t,q}} \leqslant C e^{h_{top}\alpha}.$$

Proof. We show the claim on $\|\mathcal{L}_{\alpha,\phi_\alpha}\|_{W_p^{s,t,q} \to W_p^{s,t,q}}$ by contradiction. Suppose

$$e^{-h_{\mathrm{top}}\alpha} \|\mathcal{L}_{\alpha,\phi_\alpha}\|_{W_p^{s,t,q} \to W_p^{s,t,q}} \to \infty \quad \text{as} \quad \alpha \to \infty.$$

Then there exists $\varphi_\alpha \in W_p^{s,t,q}$ such that $\|\varphi_\alpha\|_{W_p^{s,t,q}} = 1$ and

$$\left\|e^{-h_{\mathrm{top}}\alpha} \mathcal{L}_{\alpha,\phi_\alpha} \varphi_\alpha\right\|_{W_p^{s,t,q}} \to \infty \qquad \text{as} \qquad \alpha \to \infty. \tag{110}$$

We assume for some $w \in C_0^s(\mathbb{R})$ and some $x \in M$

$$\liminf_{\alpha \to \infty} \left| \gamma_{w,x} \left(\frac{\mathcal{L}_{\alpha,\phi_\alpha} \varphi_\alpha}{\|\mathcal{L}_{\alpha,\phi_\alpha} \varphi_\alpha\|_{W_p^{s,t,q}}} \right) \right| > 0. \tag{111}$$

This assumption is justified, assuming Condition 5.16. We choose $T \geqslant 1$ and $\alpha \geqslant 0$ such that

$$\tau(T, \alpha, x) = 1.$$

Then, using Proposition 5.13 (xii), we find for some constant $C \geqslant 1$

$$C^{-1} e^{h_{\mathrm{top}}\alpha} \leqslant T \leqslant C e^{h_{\mathrm{top}}\alpha}. \tag{112}$$

We have, using the equality given in (97),

$$\gamma_{w \circ \tau(\cdot,\alpha,x),x}(\varphi_\alpha) = \gamma_{w,g_\alpha(x)}(\mathcal{L}_{\alpha,\phi_\alpha} \varphi_\alpha). \tag{113}$$

We recall $-s < r-1$. Therefore the norm $\|w \circ \tau(\cdot, \alpha, x)\|_{C^{-s}}$ is bounded as $\alpha \to \infty$, using Proposition 5.13 (x). By Lemma 5.12, the linear functionals $\gamma_{w\circ\tau(\cdot,\alpha,x),x}$ and $\gamma_{w,g_\alpha(x)}$ which appear in (113) are continuous on $W_p^{s,t,q}$. Hence the left-hand side in (113) grows at most by T as $\alpha \to \infty$ uniformly in x.
Then, comparing with the estimates for T in (112), using the assumption in (111), this contradicts the assumption in (110) and we conclude. ☐

[7]This is introduced ad hoc as it was pointed out by Colin Guillarmou and Giovanni Forni that for the weak-vanishing to imply strong vanishing is not obvious here. In some sense one would expect even a stronger statement. Namely that for every eigendistribution $\mathcal{D}$ in the expansion of Theorem 5.7 at least for one piece of horocycle orbit w around $x \in M$ one has $|\gamma_{w,x}(\mathcal{D})| > 0$.

We next show the lower bound (and uniqueness and simplicity of the spectral bound $\lambda_{\max}$):

Lemma 5.18 (Invariant measure and spectral bound)**.** *Let μ be the unique Borel probability measure which is invariant by the horocycle flow h_ρ. Let $p \in [1, \infty]$ and let $s < 0 < q \leqslant t$ such that $t - s < r - 1$. It holds:*

(i) $\mu \in \left(W_p^{s,t,q}\right)'$,
(ii) $\mathcal{L}'_{\alpha,\phi_\alpha}\mu = e^{h_{top}\alpha}\mu$ *($\mathcal{L}'_{\alpha,\phi_\alpha}$ denotes the adjoint operator of $\mathcal{L}_{\alpha,\phi_\alpha}$),*
(iii) $h_{top} \in \sigma\left(X + V\right)|_{W_p^{s,t,q}}$.
Moreover, assuming Condition 5.16, it holds:
(iv) $\lambda_{max} = h_{top}$.
(v) The spectral bound λ_{max} is uniquely attained by the simple eigenvalue h_{top}, assuming $\lambda_{min} < \lambda_{max}$.

The vector μ is also invariant by the adjoint horocycle flow since the time average converges to the (unique) ergodic mean (a result by Marcus [36]). This is in analogy to [23, Lemma 2.11].

Proof. We note for every $\varphi \in C_X^{r-1}(M)$, using [35, Theorem 2.1] for the first, the equality in (97) for the second and [35, Lemma 3.1] for the third equality, for some $\lambda > 0$, for every $\alpha \geqslant 0$

$$\mu(\varphi) = \lim_{T\to\infty}\frac{1}{T}\gamma_x(\varphi, T) = \lim_{T\to\infty}\frac{\tau(T,\alpha,x)}{T}\frac{1}{\tau(T,\alpha,x)}\gamma_{g_{\alpha(x)}}(\mathcal{L}_{\alpha,\phi_\alpha}\varphi, \tau(T,\alpha,x))$$
$$= \lambda^{-\alpha}\mu\left(\mathcal{L}_{\alpha,\phi_\alpha}\varphi\right). \tag{114}$$

To see $\lambda = e^{h_{\rm top}}$ we refer to [35, p.84] (alternatively use Proposition 5.13 (xi)). Using Claims (i)-(ii) with $\lambda = h_{\rm top}$ in Lemma 5.14 together with the bound given by Lemma 5.17, there is $w \in C^r(\mathbb{R})$ and a constant $C_1 > 0$ such that for all $\varphi \in C_X^{r-1}(M)$

$$|\mu(\varphi)| \leqslant \lim_{T\to\infty}\left|\frac{1}{T}\gamma_{w,x}(\varphi)\right| \leqslant C_1\|\varphi\|_{W_p^{s,t,q}}. \tag{115}$$

Claim (iii) follows from $\sigma\left((X+V)'\right)|_{\left(W_p^{s,t,q}\right)'} = \sigma\left(X+V\right)|_{W_p^{s,t,q}}$, using [33, Section II.2.5]. Claim (iv) follows from (iii) together with Lemma 5.17. To see Claim (v), first we note that all $\lambda \in \sigma(X+V)$ such that $\Re\lambda = h_{\rm top}$ are eigenvalues, using Lemma 4.10 together with the assumption $\lambda_{\min} < \lambda_{\max}$. Using Claim (iii), there exists $\mathcal{D}_1 \in W_p^{s,t,q}$ such that $\mathcal{L}_{\alpha,\phi_\alpha}\mathcal{D}_1 = e^{h_{\rm top}\alpha}\mathcal{D}_1$ for all $\alpha \geqslant 0$. We let $\mathcal{D}_1 \neq \mathcal{D}_2 \in W_p^{s,t,q}\setminus\{0\}$ such that $\mathcal{L}_{\alpha,\phi_\alpha}\mathcal{D}_2 = e^{\lambda\alpha}\mathcal{D}_2$ for all $\alpha \geqslant 0$, where $\lambda \in \mathbb{C}$ and $\Re\lambda = h_{\rm top}$. Then it holds, using Claim (ii) for the last equality,

$$e^{\lambda\alpha}\mu(\mathcal{D}_2) = \mu(\mathcal{L}_{\alpha,\phi_\alpha}\mathcal{D}_2) = e^{h_{\rm top}\alpha}\mu(\mathcal{D}_2).$$

Since $\lambda \neq h_{\rm top}$ it holds $\mu(\mathcal{D}_2) = 0$. In fact, by same reasoning we can always assume $\mu(\mathcal{D}_2) = 0$ if $\lambda \neq h_{\rm top}$. And if $\lambda = h_{\rm top}$ there are only finitely many such $\mathcal{D}_2$ and we can again assume $\mu(\mathcal{D}_2) = 0$ by a change of basis. The upshot is that the following reasoning works always if $\Re\lambda \geqslant h_{\rm top}$ and $\mu(\mathcal{D}_2) = 0$.
Then, using Claim (i) and the equality in (114), for every $\epsilon > 0$ there is $\varphi \in C_X^{r-1}(M)$ such that for all $\alpha \in \mathbb{R}$ and for all $x \in M$

$$\lim_{T\to\infty}\left|\frac{1}{T}\gamma_{g_{-\alpha}(x)}(\varphi, T)\right| = |\mu(\varphi)| \leqslant \epsilon. \tag{116}$$

Using Lemma 5.14 (i), for all $T > 0$, for all $x \in M$ and for all $\alpha \in \mathbb{R}$ there exists $w \in C^r$ such that

$$\lim_{T\to\infty} \left| \frac{1}{T} \gamma_{g_{-\alpha}(x)} (\varphi, T) \right| = \lim_{T\to\infty} \left| \frac{1}{T} \gamma_{w, g_{-\alpha}(x)} (\varphi) \right|. \tag{117}$$

Since $\partial_\rho \tau (0, -\alpha, \cdot) \in C^{r-1}$ for all $\alpha \in \mathbb{R}$, using Lemma 3.3, we find $\mathcal{L}_{\alpha,\phi_\alpha} \mathcal{D}_2 = e^{\lambda\alpha} \mathcal{D}_2$ for all $\alpha \in \mathbb{R}$ which matches the condition (100) in Lemma 5.14 (ii).
Then, using Lemma 5.14 (ii) for the upper bound and the equality in (97) for the last step, we find for some constant $C_2 > 0$ independent of x, α and φ, for all $\alpha \in \mathbb{R}$

$$\begin{aligned}
\lim_{T\to\infty} \left| \frac{1}{T} \gamma_{w,g_{-\alpha}(x)} (\varphi) \right| &= \lim_{T\to\infty} \left| \frac{1}{T} \gamma_{w,g_{-\alpha}(x)} (\mathcal{D}_2) + \frac{1}{T} \gamma_{w,g_{-\alpha}(x)} (\varphi - \mathcal{D}_2) \right| \\
&\geqslant \lim_{T\to\infty} \left| \frac{1}{T} \gamma_{g_{-\alpha}(x)} (\mathcal{D}_2, T) \right| - C_2 \left\| \varphi - \mathcal{D}_2 \right\|_{W_p^{s,t,q}} \\
&= \lim_{T\to\infty} \left| \frac{e^{h_{\mathrm{top}}\alpha}}{T} \gamma_x (\mathcal{D}_2, \tau(T, \alpha, x)) \right| - C_2 \left\| \varphi - \mathcal{D}_2 \right\|_{W_p^{s,t,q}}. \qquad (118)
\end{aligned}$$

By density of $C_X^{r-1}(M)$ in $W_p^{s,t,q}$ we assume

$$\left\| \varphi - \mathcal{D}_2 \right\|_{W_p^{s,t,q}} \leqslant \epsilon.$$

For every $T \geqslant T_0 \geqslant 1$ we let $\alpha \geqslant 0$ such that $\tau(T, \alpha, x) = T_0$. By Proposition 5.13 there is $C_3 \geqslant 1$ independent of T and x such that $e^{h_{\mathrm{top}}\alpha} \geqslant C_3^{-1} \frac{T}{T_0}$. Since $\epsilon > 0$ was arbitrary we conclude for all $T_0 \geqslant 1$ and all $x \in M$, using the estimates (116)-(118),

$$\gamma_x (\mathcal{D}_2, T_0) = 0.$$

On the other hand we find for all $T_1, T_0 \geqslant 1$

$$\gamma_x (\mathcal{D}_2, T_0 - T_1) = \gamma_{h_{-T_1}(x)} (\mathcal{D}_2, T_0) - \gamma_{h_{-T_1}(x)} (\mathcal{D}_2, T_1) = 0.$$

Hence it holds $\gamma_x (\mathcal{D}_2, T) = 0$ for every $T \in \mathbb{R}$ and every $x \in M$. Then for every $w \in C_0^{s+1}$ we find, using integration by parts,

$$\gamma_{w,x} (\mathcal{D}_2) = -\int_{\mathbb{R}} (\partial_\rho w)(\rho) \cdot \gamma_x (\mathcal{D}_2, \rho) \, \mathrm{d}\rho.$$

Since $\gamma_x (\mathcal{D}_2, \rho) = 0$ for all $\rho \geqslant 0$ we conclude $\gamma_{w,x} (\mathcal{D}_2) = 0$. Then, using Condition 5.16, we find $\mathcal{D}_2 \equiv 0$ but we assumed $\mathcal{D}_2 \not\equiv 0$. ☐

Proof of Theorem 5.7. By assumption

$$\max \{\lambda_{\min}, 0\} < \delta \leqslant \lambda_{\max} = h_{\mathrm{top}}.$$

We note that we have always $\lambda_{\max} = h_{\mathrm{top}}$ and uniqueness and simplicity of $\lambda_{\max}$ under Condition 5.16. Using the equality in (67) for the projectors $\Pi_{\lambda,i}$, we have for all $1 \leqslant i \leqslant n_\lambda$

$$\Pi_{\lambda,i} \varphi = \sum_{j=1}^{m_{\lambda,i}} \mathcal{O}_{(\lambda,i,j)} (\varphi) \, \mathcal{D}_{(\lambda,i,j)}.$$

Recalling the nil-potent operators $\mathcal{N}_{\lambda,i}$ of finite rank (e.g. see in (66)), using the formula for the matrix action $\mathcal{L}_{\alpha,\phi_\alpha} \Pi_{\lambda,i} = \exp(\lambda\alpha) \exp(\mathcal{N}_{\lambda,i}\alpha) \Pi_{\lambda,i}$ for all $\alpha \geqslant 0$ and

$$\exp(-\lambda\alpha) \exp(-\mathcal{N}_{\lambda,i}\alpha) \mathcal{L}_{\alpha,\phi_\alpha} \Pi_{\lambda,i} = \Pi_{\lambda,i},$$

we find for some constant $C_1 = C_1(\lambda, i, j)$ for all $\alpha \in \mathbb{R}$

$$\left\|\partial_\rho(0, -\alpha, \cdot) \cdot \mathcal{D}_{(\lambda,i,j)} \circ g_{-\alpha}\right\|_{W_p^{s,t,q}} \leqslant C_1 \exp(\Re\lambda\alpha) \max\left\{1, |\alpha|^{j-1}\right\}.$$

Hence $\mathcal{D}_{(\lambda,i,j)}$ satisfies the upper bound in (100) for all $\alpha \in \mathbb{R}$ if $\Re\lambda > 0$.
Inspecting the end of the proof of Lemma 5.18, we notice that all eigendistributions $\mathcal{D}_{(\lambda,i,j)}$ associated to some eigenvalue λ with $\Re\lambda \geqslant h_{\text{top}}$ do not contribute to the expansion except $\mathcal{D}_{(h_{\text{top}},1,1)}$. This follows, if $j = 1$ using that $\gamma_x\left(\mathcal{D}_{(\lambda,i,1)}, T\right) = 0$ for all $T \geqslant 0$ and all $x \in M$. If $j > 1$ we arrive at the same conclusion, using in the estimate in (118) for all $\alpha \geqslant 0$

$$\mathcal{L}_{\alpha,\phi_\alpha}\mathcal{D}_{(\lambda,i,j)} = \exp(\lambda\alpha)\exp(\mathcal{N}_{\lambda,i}\alpha)\,\mathcal{D}_{(\lambda,i,j)}.$$

Let $\lambda \in \Sigma_\delta = \sigma(X+V)|_{W_p^{s,t,q}} \cap \{z \in \mathbb{C} \mid \Re z \geqslant \delta\}$. For every $T \geqslant 0$ and every $x \in M$ we set, using $w \in C^r$ given in Lemma 5.14,

$$c_{(\lambda,i,j)} = c_{(\lambda,i,j)}(T, x) := T^{-\frac{\lambda}{h_{\text{top}}}} \max\left\{1, |\log T|^{1-j}\right\} \gamma_{w,x}\left(\mathcal{D}_{(\lambda,i,j)}\right).$$

Then, using the first statement in Lemma 5.14 (ii), the coefficients $c_{(\lambda,i,j)}$ are bounded independently of T and x. It holds

$$\begin{aligned}\gamma_{w,x}(\Pi_{\lambda,i}\varphi) &= \sum_{j=1}^{m_{\lambda,i}} \mathcal{O}_{(\lambda,i,j)}(\varphi)\,\gamma_{w,x}\left(\mathcal{D}_{(\lambda,i,j)}\right)\\ &= \sum_{j=1}^{m_{\lambda,i}} c_{(\lambda,i,j)} T^{\frac{\lambda}{h_{\text{top}}}} \max\left\{1, |\log T|^{j-1}\right\} \mathcal{O}_{(\lambda,i,j)}(\varphi).\end{aligned}$$

We let μ as given in Lemma 5.18. Using Lemma 5.18 (v), and assuming $T \geqslant e$, we find for every finite subset $\Lambda_\delta \subseteq \Sigma_\delta$

$$\gamma_x(\cdot, T) = \gamma_x\left(\mathcal{D}_{(h_{\text{top}},1,1)}, T\right)\mu + \sum_{\substack{\lambda\in\Lambda_\delta\\ \Re\lambda < h_{\text{top}}}} \sum_{i=1}^{n_\lambda}\sum_{j=1}^{m_{\lambda,i}} c_{(\lambda,i,j)} T^{\frac{\lambda}{h_{\text{top}}}} (\log T)^{j-1}\,\mathcal{O}_{(\lambda,i,j)} + \mathcal{E}_{T,x,\Lambda_\delta},$$

where the remainder term is

$$\begin{aligned}\mathcal{E}_{T,x,\Lambda_\delta} :=& \left(\gamma_{w,x}\left(\mathcal{D}_{h_{\text{top}},1,1}\right) - \gamma_x\left(\mathcal{D}_{(h_{\text{top}},1,1)}, T\right)\right)\mu\\ &+ \gamma_{w,x}\left(\text{id} - \sum_{\lambda\in\Lambda_\delta}\sum_{i=1}^{n_\lambda}\Pi_{\lambda,i}\right) + \left(\gamma_x(\cdot, T) - \gamma_{w,x}\right).\end{aligned} \tag{119}$$

The existence of the limit $\lim_{T\to\infty} T^{-1}\gamma_x\left(\mathcal{D}_{(h_{\text{top}},1,1)}, T\right)$ is shown by analogue estimates (116)-(118). Then the statement on the limit $\lim_{T\to\infty} T^{-1}\mathcal{E}_{T,x,\Lambda_\delta}(\varphi)$ follows, using unique ergodicity of the horocycle flow [35, Theorem 2.1] and finiteness of Λ_δ. We bound $|\mathcal{E}_{T,x,\Lambda_\delta}(\varphi)|$ as required, using the first statement in Lemma 5.14 (i) and the full statement in Lemma 5.14 (ii) together with the assumed upper bound in (88).
The additional claims under Condition 4.11 can be seen as follows (see also the remarks above and below Condition 4.11): The finiteness of Σ_δ follows from [13, Theorem 1]. To this end we have to show that [13, Assumption 1-3A] are satisfied for the renormalized semigroup $e^{-h_{\text{top}}\alpha}\mathcal{L}_{\alpha,\phi_\alpha}: W_p^{s,t,q} \to W_p^{s,t,q}$. In fact Condition 4.11 yields just a reformulation of [13, Assumption 3A] for the resolvent of the generator $X + V - h_{\text{top}}$. Now [13, Assumption 1] states that for some Banach space

$W_p^{s,t,q} \subset \mathcal{B}$ it holds

$$\sup_{\alpha \geqslant 0} \frac{1}{\alpha} \left\| \mathrm{id} - e^{-h_{\mathrm{top}}\alpha} \mathcal{L}_{\alpha,\phi_\alpha} \right\|_{W_p^{s,t,q} \to \mathcal{B}} < \infty. \tag{120}$$

We set $\mathcal{B} := W_p^{s,t,q-1}$. We bound the left-hand side in (120), using the equality in (69) together with Lemma 3.3, Lemma 3.5 and Lemma 5.17. Now [13, Assumption 2] just states that the essential spectral bound of $X + V - h_{\mathrm{top}}$ is bounded by some $\lambda < 0$, where $V = -\partial_\alpha \partial_\rho \tau(0, 0, \cdot)$. By assumption it holds $\lambda \leqslant \lambda_{\min} - h_{\mathrm{top}} < 0$. Finally, the claimed choice $c = \delta + \epsilon$ for all $\epsilon > 0$ follows from [13, Theorem 1] as well. In particular, this choice for c follows if for all $\alpha \geqslant 0$ and for all $\epsilon > 0$ there exists $C_2 = C_2(\delta, \epsilon, \varphi)$ such that

$$\left\| \mathcal{L}_{\alpha,\phi_\alpha} \left(\mathrm{id} - \sum_{\lambda \in \Sigma_\delta} \sum_{i=1}^{n_\lambda} \Pi_{\lambda,i} \right) \varphi \right\|_{W_p^{s,t,q}} \leqslant C_2 e^{(\delta+\epsilon)\alpha}.$$

We set $\varphi_\delta := \varphi - \sum_{\lambda \in \Sigma_\delta} \sum_{i=1}^{n_\lambda} \Pi_{\lambda,i}\varphi$. If $t - q + 1 < r - 1$ it follows, using Lemma 3.3, Lemma 3.5 and [13, Theorem 1], for some constants $C_3, C_4 = C_4(\epsilon) > 0$

$$\begin{aligned} \left\| \mathcal{L}_{\alpha,\phi_\alpha} \varphi_\delta \right\|_{W_p^{s,t,q}} &\leqslant C_3 \left\| \mathcal{L}_{\alpha,\phi_\alpha} (X + V) \varphi_\delta \right\|_{W_p^{s,t,q-1}} + C_3 \left\| \mathcal{L}_{\alpha,\phi_\alpha} \varphi_\delta \right\|_{W_p^{s,t,q-1}} \\ &\leqslant C_3 \left\| \mathcal{L}_{\alpha,\phi_\alpha} (X + V - h_{\mathrm{top}}) \varphi_\delta \right\|_{W_p^{s,t,q-1}} + (C_3 + h_{\mathrm{top}}) \left\| \mathcal{L}_{\alpha,\phi_\alpha} \varphi_\delta \right\|_{W_p^{s,t,q-1}} \\ &\leqslant C_4 e^{(\delta+\epsilon)\alpha} \left(\left\| (X + V - h_{\mathrm{top}})^2 \varphi \right\|_{W_p^{s,t,q}} + \left\| (X + V - h_{\mathrm{top}}) \varphi \right\|_{W_p^{s,t,q}} \right). \end{aligned}$$

Boundedness of the last estimate follows if $q < r - 2$ because then $\|X(V\varphi)\|_{W_p^{s,t,q}}$ and $\left\|X^2\varphi\right\|_{W_p^{s,t,q}}$ are bounded, recalling $\varphi \in C_X^{r-1}(M)$ and $V \in C^{r-1}$. Combining the required bounds for q yields

$$t - r + 2 < q < r - 2.$$

Since we required $q > 0$ it is enough to require $t - r + 2 \leqslant 0$ and $0 < r - 2$ which yields the additional condition on t and r. □

Appendix

We check the expansion and contraction properties of the cones claimed in Section 2:

Lemma A.1. *Let C and θ be the constants from (2). Let $x \in M$ and $0 < \gamma < 1$ and recall the cones $C_\gamma^-(x)$ and $C_\gamma^+(x)$ defined in (8). Let $\alpha > 0$ and $\gamma' > 0$ such that $C^2\theta^\alpha\gamma < \gamma' \leqslant 1$. Then it holds:*

(i) $(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}} C_\gamma^-(x) \Subset C_{\gamma'}^-(g_\alpha(x))$,

(ii) $(\mathrm{D}\, g_\alpha)^{\mathrm{tr}} C_\gamma^+(x) \Subset C_{\gamma'}^+(g_{-\alpha}(x))$.

In particular, there exists $\gamma' > 0$ such that for all large enough $\alpha > 0$ it holds $\gamma' < \gamma$.

Proof. First we note that a fixed choice $\gamma' < \gamma$ is possible for all large $\alpha > 0$ because $\theta < 0$. We show claim (i). Claim (ii) is shown analogously. We let $v^- + v^+ + v^0 = v \in C_\gamma^-(x)$. We estimate (assuming $\frac{1}{C}\left\|v^0\right\| \leqslant \left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}} v^0\right\| \leqslant C\left\|v^0\right\|$)

$$\begin{aligned} \left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}} v^+\right\| + \left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}} v^0\right\| &\leqslant C\left(\left\|v^+\right\| + \left\|v^0\right\|\right) \leqslant C\gamma\left\|v^-\right\| \\ &\leqslant C^2\theta^\alpha\gamma\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}} v^-\right\|. \end{aligned}$$

It follows that $(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v \in C^{-}_{\gamma'}\,(g_\alpha\,(x))$ if $\gamma' \geqslant C^2\theta^\alpha\gamma$. Since $C^{-}_{\gamma'}\,(g_\alpha\,(x)) \Subset C^{-}_{\gamma'+\epsilon}\,(g_\alpha\,(x))$ for all $\epsilon > 0$ we conclude. □

Lemma A.2. *Assuming the constants C and θ from (2), let $\gamma > 0$, $x \in M$ and suppose that $C^2\theta^\alpha\gamma < 1$. Then for all $C^2\theta^\alpha\gamma < \gamma' < 1$ it holds:*

(i) *If $v \in C^{-}_{\gamma}\,(x)$ then $\frac{\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}} v\right\|}{\|v\|} \geqslant C\frac{1+\gamma}{1+\gamma'}\theta^{-\alpha}$.*

(ii) *If $v \in (\mathrm{D}\, g_\alpha)^{\mathrm{tr}}\, C^{+}_{\gamma}\,(x)$ then $\frac{\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}} v\right\|}{\|v\|} \leqslant C\frac{1+\gamma}{1-\gamma'}\theta^{\alpha}$.*

Proof. Let $v \in T^*_x M$. We recall

$$v = v^- + v^+ + v^0, \quad v^\sigma \in E_{\sigma,x}, \sigma \in \{-,+,0\}\,.$$

If $v \in C^{-}_{\gamma}\,(x)$ then by (9), for all $\lambda \geqslant 0$ it holds

$$\begin{aligned}
\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v\right\| &\geqslant \left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v_-\right\| - \left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v_+\right\| - \left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v_0\right\| \\
&\geqslant \frac{1}{C}\theta^{-\alpha}\,\|v_-\| + C\,(\|v_+\| + \|v_0\|) \\
&\geqslant C\theta^{-\alpha}\,(1-\gamma')\,\|v^-\| = C\theta^{-\alpha}\,\big((1-\gamma'-\lambda)\,\|v^-\| + \lambda\,\|v^-\|\big) \\
&\geqslant C\theta^{-\alpha}\left((1+\gamma-\lambda)\,\|v^-\| + \frac{\lambda}{\gamma'}\,\big(\|v^0\| + \|v^+\|\big)\right).
\end{aligned}$$

The choice $\lambda = \frac{1+\gamma}{\frac{1}{\gamma'}+1}$ yields $\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v\right\| \geqslant C\frac{1+\gamma}{1+\gamma'}\theta^{-\alpha}\,\|v\|$.

If $v \in (\mathrm{D}\, g_\alpha)^{\mathrm{tr}}\, C^{+}_{\gamma}(x)$ then by (9), for all $\lambda \geqslant 0$ it holds

$$\begin{aligned}
\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v\right\| &\leqslant C\theta^{\alpha}\,(1+\gamma)\,\|v^+\| = C\theta^{\alpha}\,\big((1+\gamma+\lambda)\,\|v^+\| - \lambda\,\|v^+\|\big) \\
&\leqslant C\theta^{\alpha}\left((1+\gamma+\lambda)\,\|v^+\| - \frac{\lambda}{\gamma'}\,\big(\|v^0\| + \|v^-\|\big)\right).
\end{aligned}$$

The choice $\lambda = \frac{1+\gamma}{\frac{1}{\gamma'}-1}$ yields $\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v\right\| \leqslant C\frac{1+\gamma}{1-\gamma'}\theta^{\alpha}\,\|v\|$. □

We let ∇_z be the gradient and ∇_z^{tr} the divergence with respect to $z \in \mathbb{R}^d$.

Lemma A.3 (Integration by parts (cf. [6, p.10]))**.** *Let $\mathcal{B}$ be a Banach space and let $f\colon \mathbb{R}^d \to \mathcal{B}$ be C^1 such that*

$$\|f\,(z)\|_{\mathcal{B}} \to 0 \quad \textit{as} \quad |z| \to \infty.$$

Let $G\colon \mathbb{R}^d \to \mathbb{R}$ be C^2 and assume that $|\nabla_z G(z)| > 0$ for every $z \in \operatorname{supp} f$. Then it holds

$$\int_{\mathbb{R}^d} e^{\mathrm{i}\,G(z)} f(z)\,\mathrm{d}\,z = \mathrm{i}\int_{\mathbb{R}^d} e^{\mathrm{i}\,G(z)}\nabla_z^{\mathrm{tr}}\frac{\nabla_z G(z) f(z)}{|\nabla_z G(z)|^2}\,\mathrm{d}\,z.$$

We understand the above transformation as integration by parts. Repeated application leads to the following iteration pattern.

Lemma A.4. *Let $f(z,\eta,\xi)$ and $\nabla_z G\,(z,\eta,\xi)$ be complex and real valued functions, respectively, both C^{r_1}, C^{r_2}, C^{r_3} in $z,\eta,\xi \in \mathbb{R}^d$ for some $r_1, r_2, r_3 > 0$, respectively. Let $V_0(z,\eta,\xi) \coloneqq f(z,\eta,\xi)$ and*

$$V_k(z,\eta,\xi) \coloneqq \nabla_z^{\mathrm{tr}}\frac{\nabla_z G(z,\eta,\xi) V_{k-1}(z,\eta,\xi)}{|\nabla_z G(z,\eta,\xi)|^2}, \textit{ where } k = 1,\dots,\lfloor r_1\rfloor\,. \tag{121}$$

If $|\nabla_z G(x,\eta,\xi)| > 0$ *then it holds*

$$V_k(z,\eta,\xi) = |\nabla_z G(z,\eta,\xi)|^{-k} f_k(z,\eta,\xi),$$

where $f_k(z,\eta,\xi)$ *is* C^{r_1-k} *in* z*,* C^{r_2}*,* C^{r_3} *in* η,ξ*, respectively and* $\operatorname{supp} f = \operatorname{supp} f_k$*. Moreover, it holds for some constant* $C \geqslant 1$

(122)

$$\|f_k\|_{C^0} \leqslant C \sup_{(z,\eta,\xi)\in \operatorname{supp} f} \max_{0\leqslant|\gamma|\leqslant k} \left| |\nabla_z G(z,\eta,\xi)|\, \partial_z^\gamma \frac{\nabla_z G(z,\eta,\xi)}{|\nabla_z G(z,\eta,\xi)|^2} \right|^k \|f(\cdot,\eta,\xi)\|_{C^k}.$$

Proof. We prove this by induction. For $V_0 = V_0(z,\eta,\xi)$ the hypothesis holds. We assume the hypothesis to hold for $V_k = V_k(z,\eta,\xi)$ up to some $0 \leqslant k \leqslant \lfloor r_1 \rfloor - 1$. We have therefore

$$V_{k+1} = \nabla_z^{\mathrm{tr}} \frac{\frac{\nabla_z G}{|\nabla_z G|} f_k}{|\nabla_z G|^{k+1}} = \frac{\nabla_z^{\mathrm{tr}} \left(\frac{\nabla_z G}{|\nabla_z G|} f_k \right)}{|\nabla_z G|^{k+1}} - (k+1) \frac{\frac{\nabla_z^{\mathrm{tr}} G}{|\nabla_z G|^2} f_k \nabla_z |\nabla_z G|}{|\nabla_z G|^{k+1}}. \tag{123}$$

Hence we can write $V_{k+1} = |\nabla_z G|^{-k-1} f_{k+1}$, where $f_{k+1} = f_{k+1}(z,\eta,\xi)$ is regular as required by the lower bound on $|\nabla_z G|$. In (123) one sees that $\operatorname{supp} f_{k+1} \subseteq \operatorname{supp} f_k$. From (123) one finds

$$f_{k+1} = |\nabla_z G|^{k+1} \nabla_z^{\mathrm{tr}} \left(\frac{\nabla_z G}{|\nabla_z G|^2} \frac{f_k}{|\nabla_z G|^k} \right).$$

We recursively expand f_k into this equation and estimate by the worst term which yields the upper bound (122). □

A regularized version of integration by parts is used if the involved maps are only Hölder continuous. A form of Lemma A.5 below appeared in a work of Baladi-Tsujii [6, p.12, Equation 3.4]. We let $\nu\colon \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ be C^∞, supported on the unit ball such that $\int_{\mathbb{R}^d} \nu(x)\,\mathrm{d}\,x = 1$. For every $\epsilon > 0$ we set $\nu_\epsilon(x) = \frac{1}{\epsilon^d}\nu\left(\frac{x}{\epsilon}\right)$.

Lemma A.5 (Regularized integration by parts)**.** *Let* $0 < \delta < 1$*. Let* $f\colon \mathbb{R}^d \to \mathbb{C}$ *be a compactly supported* C^δ*-map and let* $G\colon \mathbb{R}^d \to \mathbb{R}$ *be* $C^{1+\delta}$ *and assume that* $|\nabla_z G| > 0$ *for every* $z \in \operatorname{supp} f$*. Set* $h(z) := \frac{\nabla_z G(z) f(z)}{|\nabla_z G(z)|^2}$ *and* $h_\epsilon := h * \nu_\epsilon$*. For every* $L \geqslant 1$ *it holds*

$$\begin{aligned} \int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} f(z)\,\mathrm{d}\,z =& \frac{\mathrm{i}}{L} \int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} \nabla_z^{\mathrm{tr}} h_\epsilon(z)\,\mathrm{d}\,z \\ &+ \int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} \nabla_z^{\mathrm{tr}} G(z)\,(h(z) - h_\epsilon(z))\,\mathrm{d}\,z. \end{aligned} \tag{124}$$

In particular, for some constant $C \geqslant 1$*, it holds* $\|\nabla_z h_\epsilon\|_{L_\infty} \leqslant C \|h\|_{C^\delta} \epsilon^{\delta-1}$ *and* $\|h - h_\epsilon\|_{L_\infty} \leqslant C \|h\|_{C^\delta} \epsilon^\delta$*.*

Proof. Since G is $C^{1+\delta}$ and $|\nabla_z G| > 0$, the map h is C^δ. We have $\nabla_z^{\mathrm{tr}} G(z) h(z) = f(z)$ and we write

$$\int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} f(z)\,\mathrm{d}\,z = \int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} \left(\nabla_z^{\mathrm{tr}} G(z) h_\epsilon(z) + \nabla_z^{\mathrm{tr}} G(z)\,(h(z) - h_\epsilon(z)) \right) \mathrm{d}\,z.$$

And since h_ϵ is compactly supported we have, using integration by parts,

$$\int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} \nabla_z^{\mathrm{tr}} G(z) h_\epsilon(z)\,\mathrm{d}\,z = -\frac{1}{\mathrm{i}\, L} \int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} \nabla_z^{\mathrm{tr}} h_\epsilon(z)\,\mathrm{d}\,z.$$

To see the norm estimates, we have

$$\begin{aligned}|h(z)-h_\epsilon(z)| &= \left|\epsilon^{-d}\int_{\mathbb{R}^d}\big(h(z)-h(z-z')\big)\,\nu\left(\frac{z'}{\epsilon}\right)\mathrm{d}\,z'\right| \\ &= \left|\int_{\mathbb{R}^d}\big(h(z)-h(z-\epsilon z')\big)\,\nu\left(z'\right)\mathrm{d}\,z'\right| \leqslant \|h\|_{C^\delta}\,\epsilon^\delta.\end{aligned}$$

Since $\operatorname{supp} h$ is compact, for every $z\in\mathbb{R}^d$ there exists $\bar{z}\in\mathbb{R}^d$ such that $h(z-\bar{z})=0$. We estimate, for some constant $C\geqslant 1$, using 1-Lipschitz continuity of the norm,

$$\begin{aligned}\left|\int_{\mathbb{R}^d} h(z')\,(\nabla_z\nu)\left(\frac{z-z'}{\epsilon}\right)\mathrm{d}\,z'\right| &= \left|\int_{\mathbb{R}^d}\big(h(z-z')-h(z-\bar{z})\big)\,(\nabla_z\nu)\left(\frac{z'}{\epsilon}\right)\mathrm{d}\,z'\right| \\ &\leqslant C\,\|h\|_{C^\delta}\int_{\mathbb{R}^d}\left|z'-\bar{z}\right|^\delta\left|(\nabla_z\nu)\left(\frac{z'}{\epsilon}\right)\right|\mathrm{d}\,z' \leqslant C\,\|h\|_{C^\delta}\,\epsilon^{\delta+d}.\end{aligned}$$

□

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