Metric Embeddings

Exercises

Week 1

1. Modify Fréchet's embedding to prove that any n-point metric space embeds isometrically into ℓ_∞^{n-1} for $n \geq 2$.

2. Use Schoenberg's theorem to find a 4-point subset of ℓ_p^2 that does not embed isometrically into Hilbert space for any $p \neq 2$.

3. Prove that an n-point metric space $(\{x_1, \ldots, x_n\}, d)$ embeds isometrically into Hilbert space if and only if for every $t \geq 0$, the matrix

$$A = \left(e^{-td(x_i,x_j)^2}\right)_{i,j=1,\ldots,n}$$

is positive semidefinite.

4. Prove that ℓ_2 embeds linearly and isometrically into L_p for any $p \in [1, \infty)$.

Hint: use a sequence of independent standard Gaussian random variables.

5. [1] It is a classical theorem in probability that for every $p \in (0, 2)$, there exist random variables ξ_p whose characteristic function is

$$\forall\ t \in \mathbb{R}, \qquad \mathbb{E}e^{it\xi_p} = e^{-|t|^p}.$$

a. Prove that if ξ_p' is an independent copy of ξ_p, then

$$\forall\ a, b \in \mathbb{R}, \qquad a\xi_p + b\xi_p' \overset{d}{\sim} \left(|a|^p + |b|^p\right)^{1/p}\xi_p,$$

where $\overset{d}{\sim}$ denotes equality in distribution.

b. Prove that the ξ_p are symmetric, that is, $\xi_p \overset{d}{\sim} -\xi_p$.

These random variables are called *standard symmetric p-stable random variables.*

c. It can be shown that $\mathbb{E}|\xi_p|^q < \infty$ if and only if $q < p$. Use this fact to prove that ℓ_p embeds linearly and isometrically into L_q for every $1 \leq q < p < 2$.

6. Prove that for every $d \in \mathbb{N}$, ℓ_1^d embeds linearly and isometrically into $\ell_\infty^{2^d}$.

7. Let (X, d) be a finite metric space and $Y \subseteq X$ an n-point subset of X. Prove that there exists a 1-Lipschitz function $f : X \to \ell_2$ such that $\mathrm{dist}(f|_Y) \lesssim \log n$.

8. Modify Bourgain's embedding to prove that if X is an n-point metric space, then

$$c_p(X) \lesssim \frac{\log n}{p}$$

for every $p \geq 1$.

Hint: replace the probability $\frac{1}{2}$ in the distribution π_j over subsets of X by a parameter $q \in (0, 1)$ and optimize over q.

[1] This exercise requires basic background in theoretical probability.

9. * Modify Bourgain's embedding to show that any n-point metric space embeds with bi-Lipschitz distortion $O(\log n)$ into $\ell_2^{O(\log^2 n)}$. What can be said about embeddings into ℓ_p?

Hint: replace the distribution π_j over subsets of X by an empirical version of it, that is, m independent samples distributed according to π_j. Prove that if $m \gtrsim \log n$ then with high probability this embedding has the required properties.

10. Prove that if (X, d) is any n-point metric space, then its θ-snowflake (X, d^θ) embeds into Hilbert space with distortion $O(\log^\theta n)$. Is this bound sharp?

11. Prove that if $\{G_n\}_{n\geq 1}$ is a sequence of 3-regular expander graphs, then for every $p \in [1, 2]$,

$$\forall\, n \in \mathbb{N}, \qquad c_p(G_n) \asymp \log \big|V(G_n)\big|.$$

12. * Fix $p \in [1, \infty)$ and an n-point metric space (X, d). Prove that $c_p(X)$ is the largest $D \geq 1$ for which there exist two measures α, β on $X \times X$ with the following properties:

(i) Every function $f : X \to \mathbb{R}$ satisfies

$$\iint_{X\times X} \big|f(x) - f(y)\big|^p \, \mathrm{d}\alpha(x, y) \leq \iint_{X\times X} \big|f(x) - f(y)\big|^p \, \mathrm{d}\beta(x, y).$$

(ii) However,

$$\iint_{X\times X} d(x, y)^p \, \mathrm{d}\alpha(x, y) > D^p \iint_{X\times X} d(x, y)^p \, \mathrm{d}\beta(x, y).$$

This is *Rabinovich's lemma* showing the universality of the method of Poincaré inequalities for nonembeddability into L_p spaces.

Hint: if such measures exist, it is clear that $c_p(X) \geq D$. For the opposite inequality, apply the Hahn–Banach separation theorem to the cone $\mathcal{C}_p$ which we encountered in the proof of Ball's isometric embedding theorem.

Week 2

13. Let σ be the uniform probability measure on $\mathbb{S}^{n-1}$ and τ be the Haar probability measure on the unitary group $O(n)$. For $k \in \{1, \ldots, n\}$, denote by $Q_k : \mathbb{R}^n \to \mathbb{R}^n$ the projection

$$\forall\, x \in \mathbb{R}^n, \qquad Q_k(x) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

A *random projection* of rank k is a random linear operator $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ of the form $\Gamma = O^* Q_k O$, where $O \in O(n)$ is distributed according to τ.

a. Prove that for every fixed $u \in \mathbb{S}^{n-1}$ and every Borel subset $A \subseteq [0, 1]$,

$$\mathbb{P}\big\{\|\Gamma(u)\|_2^2 \in A\big\} = \sigma\big\{\theta \in \mathbb{S}^{n-1} : \ \|Q_k(\theta)\|_2^2 \in A\big\}.$$

b. Prove that if θ is distributed according to σ, then $\|Q_k(\theta)\|_2^2$ is distributed according to $B(\frac{k}{2}, \frac{n-k}{2})$ and satisfies $\mathbb{E}\|Q_k(\theta)\|_2^2 = \frac{k}{n}$.

Hint: use that if $(g_1, \ldots, g_n)$ is a standard Gaussian random vector in $\mathbb{R}^n$, then $\|Q_k(\theta)\|_2^2$ has the same distribution as a ratio of χ^2 distributions $(g_1^2 + \cdots + g_k^2)/(g_1^2 + \cdots + g_n^2)$.

c. * Prove that $\|Q_k(\theta)\|_2$ satisfies the concentration inequality

$$\forall\, t > 0, \qquad \sigma\Big\{\theta \in \mathbb{S}^{n-1} : \ \Big|\|Q_k(\theta)\|_2 - \sqrt{\tfrac{k}{n}}\Big| \geq t\Big\} \leq 2e^{-cnt^2}$$

for some universal constant $c > 0$. This can also be proven as a consequence of the *spherical isoperimetric inequality*.

d. Use the concentration inequality above to give a different proof of the Johnson–Lindenstrauss lemma.

e. * Use random projections to prove the following embedding theorem of Matoušek which extends the Johnson–Lindenstrauss lemma: if S is an n-point subset of ℓ_2 and $k \in \{2, \dots, \lceil \log n \rceil\}$, then S embeds into ℓ_2^k with distortion $O(n^{2/k}\sqrt{\log n/k})$.

f. If $k \in \{2, \dots, \lceil \log n \rceil\}$, prove that any n-point metric space embeds into ℓ_2^k with distortion $O(n^{2/k}(\log n)^{3/2}/\sqrt{k})$.

14. The purpose of this exercise is to prove a lower bound for the dimension required in the Johnson–Lindenstrauss lemma that is due to Alon.

a. Consider the $n+1$ points $0, e_1, \dots, e_n \in \ell_2^n$, where e_i are the coordinate vectors. Prove that if these points embed into ℓ_2^k with distortion at most $1+\varepsilon$, then there exist unit vectors $v_1, \dots, v_n \in \mathbb{R}^k$ such that $|\langle v_i, v_j \rangle| \leq 100\varepsilon$ for every $i \neq j$.

b. Let A be an $n \times n$ symmetric matrix with $a_{ii} = 1$ for every i and $|a_{ij}| \leq n^{-1/2}$ for every $i \neq j$. Prove that the rank of A is at least $n/2$.

Hint: use (and prove) the inequality $(\mathrm{tr}A)^2 \leq \mathrm{rank}(A)\mathrm{tr}(A^2)$.

c. Let A be an $n \times n$ real matrix of rank d and let B be the matrix with $b_{ij} = a_{ij}^k$. Prove that the rank of B is at most $\binom{k+d}{k}$.

Hint: think of $\mathrm{rank}(C)$ as the dimension of the column space of C.

d. Prove that if the $n+1$ points from **a.** embed with distortion at most $1+\varepsilon$ into ℓ_2^k for some $100n^{-1/2} \leq \varepsilon \leq 1/2$, then

$$k \gtrsim \frac{\log k}{\varepsilon^2 \log(1/\varepsilon)}.$$

Hint: consider the matrix A with $a_{ij} = \langle v_i, v_j \rangle$ and show that it has rank at most k.

15. The purpose of this exercise is to prove a result of Lee, Mendel and Naor, showing that there cannot be a dimension reduction lemma in ℓ_p for $p \neq 2$ with linear operators.

a. * Prove that for every $p \neq 2$ and arbitrarily large $n \in \mathbb{N}$, there exists an n-point subset $S_{n,p}$ of ℓ_p such that if $T : \ell_p \to \ell_2$ is a linear operator and $f = T|_{S_{n,p}}$, then the distortion of f is at least $\left(\frac{n-1}{2}\right)^{|1/p-1/2|}$.

Hint: for $k \in \mathbb{N}$, $A \subseteq \{1, \dots, k\}$ and $y \in \{-1,1\}^k$, let $w_A : \{-1,1\}^k \to \{-1,1\}$ be the Walsh function $w_A(x) = \prod_{i \in A} x_i$ and $e_y : \{-1,1\}^k \to \{0,1\}$ be the indicator function $e_y(x) = \mathbf{1}_{y=x}$. Consider the set $S_{n,p} = \{0\} \cup \{e_y : y \in \{-1,1\}^k\} \cup \{w_A : A \subseteq \{1, \dots, k\}\}$ as a subset of $L_p(\{-1,1\}^k)$.

b. Deduce that for every linear operator $P : \ell_p \to \ell_p^d$, the function $g = P|_{S_{n,p}}$ has distortion at least $\left(\frac{n-1}{2d}\right)^{|1/p-1/2|}$. In other words, $S_{n,p}$ cannot be bi-Lipschitzly embedded into ℓ_p^d with constant distortion via a linear operator unless $d \asymp n$.

16. Prove that any n-point metric space embeds with distortion $O(\log n)$ into $\ell_\infty^{O(\log^2 n)}$.

17. Let K_n be the complete graph equipped with its shortest path metric. Prove that, if K_n embeds with distortion $D \geq 1$ into some k-dimensional normed space, then $D = \Omega(n^{1/k})$.

18. The *Matoušek extrapolation principle* implies that if $\{G_n\}_{n \geq 1}$ is a sequence of 3-regular expanders, then for every $1 \leq p < \infty$,

$$c_p(G_n) \asymp \frac{\log |G_n|}{p},$$

thus showing that the upper bound of Exercise 8 is sharp. Use this result to show that if G_n embeds into ℓ_∞^d with distortion at most D, then $d \geq n^{\Omega(1/D)}$. This is a matching lower bound for the Matoušek embedding theorem proven in class.

19. Prove that there are no planar 6-regular graphs.

20. Let G be a planar 3-regular graph on n vertices. Prove that its spectral gap satisfies

$$1 - \lambda_2(A_G) \lesssim \frac{1}{\log n}.$$

21. Fix $\Delta > 0$ and let (X, d) be a finite metric space which admits an (ε, δ)-padded distribution over Δ-bounded partitions. Prove that there exists a map $\Phi_\Delta : X \to \ell_2$ which is $O(1)$-Lipschitz, satisfies $\|\Phi_\Delta(x)\|_2 \leq \Delta$ for every $x \in X$, and finally

$$d(x, y) > \Delta \quad \Longrightarrow \quad \big\|\Phi_\Delta(x) - \Phi_\Delta(y)\big\|_2 \gtrsim \varepsilon\sqrt{\delta}\Delta.$$

22. Let (X, d) be a finite metric space and suppose that for each $\Delta > 0$, there exist maps $\Phi_\Delta : X \to \ell_2$ satisfying the conclusions of the previous exercise. Denote also by $\Phi = \frac{\max_{x\neq y} d(x,y)}{\min_{x\neq y} d(x,y)}$ the aspect ratio of X. Prove that the Euclidean distortion of X satisfies

$$c_2(X) \lesssim \frac{\sqrt{\log \Phi}}{\varepsilon\sqrt{\delta}}.$$

23. Let G be a connected planar graph on n vertices and $1 \leq p < \infty$. Prove that

$$c_p(G) \lesssim (\log n)^{\min\{1/2, 1/p\}}.$$

Week 3

24. Prove that every d-dimensional normed space is 5^d-doubling.

25. Prove that for arbitrarily large n, there exist n-point metric spaces X_n with the following property: if X_n admits a distribution over Δ-bounded partitions that is $(\varepsilon_n, 0.1)$-padded for every $\Delta > 0$, then $\varepsilon_n \lesssim (\log n)^{-1}$.

26. Prove that the conclusion of Rao's theorem remains true for weighted planar graphs.

Hint: imitate the gluing argument in the proof of the Gupta–Krauthgamer–Lee theorem.

27. * In their paper, Gupta, Krauthgamer and Lee sketch an alternative proof of their main result (relying on an adaptation of Bourgain's embedding method), which however yields worse bounds with respect to the doubling constant. Complete the missing details.

> **Theorem 4.1.** *For any n-point doubling metric (X, d), the distortion required to embed X into ℓ_2 is $c_2(X) = O(\sqrt{\log n})$.*
>
> *Proof (sketch).* The idea of the embedding is as follows. For every scale 2^k, $k \in \mathbb{Z}$, we construct a map $\varphi_k : X \to \ell_2$ given by $\varphi_k(x) = \sqrt{1/2^{|Y_k|}}\,(d(x, A))_{A \in Y_k}$, where Y_k is the set of all subsets of some 2^k-net in X. The final map is basically the normalized direct sum of all such maps, i.e. $\oplus_k 2^{-\frac{1}{2}|Y_k|}\varphi_k$, though standard considerations must be made to achieve a distortion which depends only on n and not the aspect ratio of X. □

28. Prove the following result of Lee, Mendel and Naor: if (G, d) is a finite connected planar graph, then for every $\varepsilon \in (0, 1)$, the $(1-\varepsilon)$-snowflake $(G, d^{1-\varepsilon})$ of G embeds into ℓ_2 with distortion $O(1/\sqrt{\varepsilon})$.

Week 4

29. Let $\mathcal{Q}_n$ be the hypercube $\{0,1\}^n$ equipped with the Hamming metric

$$\forall\, x,y \in \mathcal{Q}_n, \qquad d_H(x,y) = |\{i:\ x_i \neq y_i\}.$$

Prove that for any weighted tree T, we have $c_T(\mathcal{Q}_n) \gtrsim n$. Is this bound attained?

30. * Let $G = (\{1,\ldots,n\}^2, E)$ be the two-dimensional grid equipped with the edges parallel to the coordinate axes. Prove that if T is a tree and $f : X \to T$ is an injective mapping such that $\|x-y\|_1 \leq d_T(f(x), f(y))$ for every $x, y \in X$, then

$$\frac{1}{n} \sum_{\{x,y\}\in E} d_T\big(f(x), f(y)\big) \gtrsim \log n.$$

31. * Use Gupta's restriction theorem to give a different proof of the lower bound of Rabinovich and Raz for the distortion of the n-cycle into a tree.

32. Prove that the restriction of a tree metric to its leaves is an ultrametric space.

33. Construct an explicit isometric embedding of an ultrametric space into L_p without passing through L_2.

Week 5

34. Prove that if $1 < p_1 \leq p_2 \leq 2$ and $2 \leq q_2 \leq q_1 < \infty$, then for every normed space X,

$$S_{p_1}(X) \leq S_{p_2(X)} \qquad \text{and} \qquad K_{q_1}(X) \leq K_{q_2}(X).$$

35. Prove Lindenstrauss' duality formulas: if X is a normed space, and $p \in (1,2]$, $q \in [2,\infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then

$$K_q(X^*) = S_p(X) \qquad \text{and} \qquad S_p(X^*) = K_q(X).$$

36. Let X be a p-uniformly smooth normed space, where $p \in (1,2]$. Prove that for every $x \in X \setminus \{0\}$, there exists a unique $\xi_x \in X^*$ with $\|\xi_x\|_{X^*} = 1$ such that $\xi_x(x) = \|x\|_X$.

37. Given a function $f : \{-1,1\}^n \to \mathbb{R}$ and $\delta \in (0,1]$, consider the function $T_\delta f : \{-1,1\}^n \to \mathbb{R}$ given by

$$\forall\, x \in \{-1,1\}^n, \qquad T_\delta f(x) = \mathbb{E}\big[f\big(x_1\xi_1(\delta), \ldots, x_n\xi_n(\delta)\big)\big],$$

where $\xi_1(\delta), \ldots, \xi_n(\delta)$ are i.i.d. random variables satisfying

$$\mathbb{P}\big\{\xi_i(\delta) = 1\big\} = \frac{1+\delta}{2} \qquad \text{and} \qquad \mathbb{P}\big\{\xi_i(\delta) = -1\big\} = \frac{1-\delta}{2}.$$

Prove that if $p \in (1,2]$, then

$$\big\|T_{\sqrt{p-1}} f\big\|_2 \leq \|f\|_p,$$

where we denote by $\|\cdot\|_r$ the L_r norm with respect to the uniform probability measure on the hypercube $\{-1,1\}^n$.

Hint: use induction on the dimension n.

38. Prove that if $T : \ell_1^d \to \ell_\infty^k$ is a linear operator satisfying

$$\forall\, x \in \ell_1^d, \qquad \|x\|_1 \leq \|Tx\|_\infty \leq 2\|x\|_1,$$

then $k = 2^{\Omega(d)}$. Compare with the isometric embedding of Exercise 6.

Week 6

39. Prove that the k-th diamond graph D_k satisfies

$$\forall\, p \in (1,\infty), \qquad c_p(D_k) \asymp_p k^{\min\{1/2,1/p\}}.$$

40. Let L_k be the k-th Laakso graph.

a. Prove that L_k is 100-doubling.

b. * Prove that

$$\forall\, p \in (1,\infty), \qquad c_p(L_k) \asymp_p k^{\min\{1/2,1/p\}}$$

and conclude that the embedding theorem of Gupta, Krauthgamer and Lee is sharp.

c. Prove that $c_1(L_k) \leq 1000$.

d. Conclude that for arbitrarily large n, there exists an n-point 100-doubling subset S_n of L_1 such that if S_n embeds into ℓ_1^d with distortion at most D, then $d \geq n^{\Omega(1/D^2)}$.

e. * Let $\varepsilon \in (0,1)$. Prove that if for every $k \in \mathbb{N}$, there exists a snowflake embedding $f_k : V(L_k) \to \ell_2$ such that

$$\forall\, u,v \in V(L_k), \qquad d_{L_k}(u,v)^{1-\varepsilon} \leq \big\|f_k(u) - f_k(v)\big\|_2 \leq D d_{L_k}(u,v)^{1-\varepsilon},$$

then $D \gtrsim 1/\sqrt{\varepsilon}$. Compare with the snowflake embedding of Exercise 28.

41. * Prove the following result of Gupta, Newman, Rabinovich and Sinclair: if the k-th diamond graph D_k embeds into a distribution over dominating trees with distortion at most D, then $D \gtrsim k$.

42. A map $f : (X,d) \to (Y,\rho)$ is a *coarse embedding* if there exist two moduli $\alpha, \beta : [0,\infty) \to [0,\infty)$ such that

$$\forall\, x,y \in X, \qquad \alpha\big(d(x,y)\big) \leq \rho\big(f(x),f(y)\big) \leq \beta\big(d(x,y)\big)$$

and moreover $\lim_{t\to\infty} \alpha(t) = \infty$. Prove that ℓ_∞ does not admit a coarse embedding into L_p for any $p \in [1,\infty)$.

Hint: use metric cotype.

43. Prove that for every $m,n \in \mathbb{N}$, there exists an embedding $h : \mathbb{Z}_m^n \to \{0,\ldots,2m\}^{2n}$ such that for every $q \geq 2$,

$$\forall\, x,y \in \mathbb{Z}_m^n, \qquad m\|x-y\|_{\ell_q^n(\mathbb{C})} \leq \|h(x)-h(y)\|_{\ell_q^{2n}} \leq 3m\|x-y\|_{\ell_q^n(\mathbb{C})},$$

where we identify $\mathbb{Z}_m$ with $\{\exp(2\pi ki/m) : \ k = 1,\ldots,m\} \subseteq \mathbb{C}$.

44. Prove the following result of Linial and Magen: if C_n is the n-cycle, then

$$c_2(C_n) = \tfrac{n}{2}\sin\left(\tfrac{\pi}{n}\right).$$